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ARTICLE

Covering and tiling hypergraphs with tight cycles

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Abstract

A k-uniform tight cycle C^k_s is a hypergraph on $s > k$ vertices with a cyclic ordering such that every k consecutive vertices under this ordering form an edge. The pair (k, s) is admissible if $\gcd(k, s) = 1$ or $k/\gcd(k, s)$ is even. We prove that if $s \geqslant 2k^2$ and H is a k-uniform hypergraph with minimum codegree at least $(1/2 + o(1))|V(H)|$, then every vertex is covered by a copy of C^k_s. The bound is asymptotically sharp if (k, s) is admissible. Our main tool allows us to arbitrarily rearrange the order in which a tight path wraps around a complete k-partite k-uniform hypergraph, which may be of independent interest.

For hypergraphs F and H, a perfect F-tiling in H is a spanning collection of vertex-disjoint copies of F. For $k \geqslant 3$, there are currently only a handful of known F-tiling results when F is k-uniform but not k-partite. If $s \not\equiv 0 \bmod k$, then C^k_s is not k-partite. Here we prove an F-tiling result for a family of non-k-partite k-uniform hypergraphs F. Namely, for $s \geqslant 5k^2$, every k-uniform hypergraph H with minimum codegree at least $(1/2 + 1/(2s) + o(1))|V(H)|$ has a perfect C^k_s-tiling. Moreover, the bound is asymptotically sharp if k is even and (k, s) is admissible.

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1. Introduction

Let H and F be graphs. An F-*tiling* in H is a set of vertex-disjoint copies of F. An F-tiling is *perfect* if it spans the vertex set of H. Note that a perfect F-tiling is also known as an F-*factor* or a *perfect* F-*matching*. The following question in extremal graph theory has a long and rich story: given F and n such that $|V(F)|$ divides n, what is the maximum δ such that there exists a graph H on n vertices with minimum degree at least δ without a perfect F-tiling? We call such δ the *tiling degree threshold* for F and denote it by $t(n, F)$.

A first result in the study of tiling thresholds in graphs comes from the celebrated theorem of Dirac [8] on Hamiltonian cycles, which easily shows that $t(n, K_2) = n/2 - 1$. Corrádi and Hajnal [5] proved that $t(n, K_3) = 2n/3 - 1$, and Hajnal and Szemerédi [13] generalized this result for complete graphs of any size, showing that $t(n, K_t) = (1 - 1/t)n - 1$. For a general graph F, Kühn and Osthus [22] determined $t(n, F)$ up to an additive constant depending only on F. This

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improved previous results due to Alon and Yuster [3], Komlós, Sárközy and Szemerédi [19] and Komlós [18].

We study tilings in the setting of k-graphs, *i.e.* hypergraphs where every edge has exactly k vertices, for some $k \geqslant 2$. We focus on tilings using 'tight cycles', which are k-graphs that generalize the usual notion of cycles in graphs. We also study the related problem of finding F-coverings in a hypergraph H, *i.e.* finding copies of F, not necessarily vertex-disjoint, which together cover every vertex of H. After choosing a notion of 'minimum degree' for k-uniform hypergraphs, both tilings and coverings give rise to corresponding questions in extremal hypergraph theory, which generalize the 'tiling thresholds' in graphs to the setting of hypergraphs. In what follows, we describe precisely all of the problems under consideration.

1.1 Tiling thresholds

A *hypergraph* $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and an edge set $E(H)$, where each edge $e \in E(H)$ is a subset of $V(H)$. We will simply write V and E for $V(H)$ and $E(H)$, respectively, if it is clear from the context. Given a set V and a positive integer k, $\binom{V}{k}$ denotes the set of subsets of V with size exactly k. We say that H is a *k-uniform hypergraph* or *k-graph*, for short, if $E \subseteq \binom{V}{k}$. Note that 2-graphs are usually known simply as *graphs*.

Given a hypergraph H and a set $S \subseteq V$, let the *neighbourhood $N_H(S)$ of S* be the set $\{T \subseteq V \setminus S \colon T \cup S \in E\}$ and let $\deg_H(S) = |N_H(S)|$ denote the number of edges of H containing S. If $w \in V$, then we also write $N_H(w)$ for $N_H(\{w\})$. We will omit the subscript if H is clear from the context. We let $\delta_i(H)$ denote the *minimum i-degree of H*, *i.e.* the minimum of $\deg_H(S)$ over all i-element sets $S \in \binom{V}{i}$. Note that $\delta_0(H)$ is equal to the number of edges of H. Given a k-graph H, $\delta_{k-1}(H)$ and $\delta_1(H)$ are referred to as the *minimum codegree* and the *minimum vertex degree* of H, respectively.

For k-graphs H and F, an *F-tiling* in H is a set of vertex-disjoint copies of F, and an F-tiling is *perfect* if it spans the vertex set of H. For a k-graph F, define the *codegree tiling threshold $t(n, F)$* to be the maximum of $\delta_{k-1}(H)$ over all k-graphs H on n vertices without a perfect F-tiling. We will implicitly assume $n \equiv 0 \bmod |V(F)|$ whenever we discuss $t(n, F)$ (as otherwise a k-graph on n vertices cannot have a perfect F-tiling at all, so this case is not interesting).

We describe known results on tiling thresholds for k-graphs, when $k \geqslant 3$. Let K_t^k denote the complete k-graph on t vertices. For $k \geqslant 3$, Kühn and Osthus [22] determined $t(n, K_k^k)$ asymptotically; the exact value was determined by Rödl, Ruciński and Szemerédi [25] for sufficiently large n. Lo and Markström [23] determined $t(n, K_4^3)$ asymptotically, and independently, Keevash and Mycroft [17] determined $t(n, K_4^3)$ exactly for sufficiently large n.

We say that a k-graph H is *t-partite* (or that H is a *(k, t)-graph*, for short) if V has a partition $\{V_1, \ldots, V_t\}$ such that $|e \cap V_i| \leqslant 1$ for all edges $e \in E$ and all $1 \leqslant i \leqslant t$. A (k, t)-graph H is *complete* if E consists of all k-sets e such that $|e \cap V_i| \leqslant 1$, for all $1 \leqslant i \leqslant t$. Recently, Mycroft [24] determined the asymptotic value of $t(n, K)$ for all complete (k, k)-graphs K. However, much less is known for non-k-partite k-graphs. For more results on tiling thresholds for k-graphs, see the survey of Zhao [29].

1.2 Covering thresholds

Given a k-graph F, an *F-covering in H* is a spanning set of copies of F. Similarly to $t(n, F)$, define the *codegree covering threshold $c(n, F)$* of F to be the maximum of $\delta_{k-1}(H)$ over all k-graphs H on n vertices not containing an F-covering.

Trivially, a perfect F-tiling is an F-covering, and an F-covering has a copy of F. Thus

$$\mathrm{ex}_{k-1}(n, F) \leqslant c(n, F) \leqslant t(n, F),$$

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where $\mathrm{ex}_{k-1}(n, F)$ is *codegree Turán threshold*, *i.e.* the maximum of $\delta_{k-1}(H)$ over all F-free k-graphs H on n vertices. In this sense, the covering problem is an intermediate problem between the Turán and the tiling problems.

As for results on covering thresholds, for any non-empty (2-)graph F, we have

$$c(n, F) = \left(\frac{\chi(F) - 2}{\chi(F) - 1} + o(1) \right) n$$

(see [14]), where $\chi(F)$ is the chromatic number of F. Han, Zang and Zhao [14] studied the vertex-degree variant of the covering problem, for complete (3, 3)-graphs K. Falgas-Ravry and Zhao [11] studied $c(n, F)$ when F is K_4^3, K_4^3 with one edge removed, K_5^3 with one edge removed and other 3-graphs.

1.3 Cycles in hypergraphs

Given $1 \leqslant \ell < k$, we say that a k-graph on more than k vertices is an *ℓ-cycle* if every vertex lies in some edge and there is a cyclic ordering of the vertices such that under this ordering, every edge consists of k consecutive vertices and two consecutive edges intersect in exactly ℓ vertices. Note that an ℓ-cycle on s vertices can exist only if $k - \ell$ divides s. If $\ell = 1$ we call the cycle *loose*, and if $\ell = k - 1$ we call the cycle *tight*. We write C_s^k for the k-uniform tight cycle on s vertices.

When $k = 2$, ℓ-cycles reduce to the usual notion of cycles in graphs. Corrádi and Hajnal [5] determined $t(n, C_3^2)$ and Wang [27, 28] determined $t(n, C_4^2)$ and $t(n, C_5^2)$. In fact, El-Zahar [9] gave the following conjecture on cycle tilings.

Conjecture 1.1 (El-Zahar [9]). *Let G be a graph on n vertices and let $n_1, \ldots, n_r \geqslant 3$ be integers such that $n_1 + \cdots + n_r = n$. If $\delta(G) \geqslant \sum_{i=1}^{r} \lceil n_i/2 \rceil$, then G contains r vertex-disjoint cycles of lengths $n_1, \ldots, n_r$ respectively.*

The bound on the minimum degree, if true, would be best possible. In particular, the conjecture would imply that $t(n, C_s^2) = \lceil s/2 \rceil n/s - 1$. The conjecture was verified for $r = 2$ by El-Zahar, and a proof (for large n) was announced by Abbasi [1] as well as by Abbasi, Khan, Sárközy and Szemerédi (see [26]).

Given integers ℓ, k such that $1 \leqslant \ell \leqslant (k-1)/2$, it is easy to see that a k-uniform ℓ-cycle on s vertices C satisfies $c(n, C) \leqslant s - (k-1) + 1 = s - k + 2$ (by constructing C greedily). If $s \equiv 0 \bmod k$, then the tight cycle C_s^k is k-partite. For all $t \geqslant 1$, let $K^k(t)$ denote the complete (k, k)-graph whose vertex classes each have size t. Note that C_s^k is a spanning subgraph of $K^k(s/k)$. Erdős [10] proved the following result, which implies an upper bound on the Turán number of C_s^k.

Theorem 1.2 (Erdős [10]). *For all $k \geqslant 2$ and $s > 1$, there exists $n_0 = n_0(k, s)$ such that*

$$\mathrm{ex}\,(n, K^k(s)) < n^{k-1/s^{k-1}} \quad \textit{for all } n \geqslant n_0.$$

Our first result is a sublinear upper bound for $c(n, C_s^k)$ when $s \equiv 0 \bmod k$.

Proposition 1.1. *For all $2 \leqslant k \leqslant s$ with $s \equiv 0 \bmod k$, there exist $n_0(k, s)$ and $c = c(k, s)$ such that $c(n, C_s^k) \leqslant cn^{1-1/s^{k-1}}$ for all $n \geqslant n_0$.*

There are some previously known results for tiling problems regarding ℓ-cycles. Whenever C is a 3-uniform loose cycle, $t(n, C)$ was determined exactly by Czygrinow [6]. For general loose cycles C in k-graphs, $t(n, C)$ was determined asymptotically by Mycroft [24] and exactly by Gao, Han and Zhao [12]. For tight cycles C_s^k with $s \equiv 0 \bmod k$, Mycroft [24] proved that $t(n, C_s^k) = (1/2 + o(1))n$. Note that all mentioned cycle tiling results correspond to cases where the cycles

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are k-partite (since k-uniform loose cycles are k-partite for $k \geqslant 3$). In contrast, we will investigate the covering and tiling problems for the tight cycle C_s^k in some cases where C_s^k is not necessarily a (k, k)-graph.

We show that a minimum codegree of $(1/2 + o(1))n$ suffices to find a C_s^k-covering.

Theorem 1.3. *Let $k, s \in \mathbb{N}$ with $k \geqslant 3$ and $s \geqslant 2k^2$. For all $\gamma > 0$, there exists $n_0 = n_0(k, s, \gamma)$ such that for all $n \geqslant n_0$, $c(n, C_s^k) \leqslant (1/2 + \gamma)n$.*

Moreover, this result is asymptotically tight if k and s satisfy the following divisibility conditions. Let $2 \leqslant k < s$ and let $d = \gcd(k, s)$. We say that the pair (k, s) is *admissible* if $d = 1$ or k/d is even. Note that an admissible pair (k, s) satisfies $s \not\equiv 0 \bmod k$.

Proposition 1.2. *Let $3 \leqslant k < s$ be such that (k, s) is admissible. Then $c(n, C_s^k) \geqslant \lfloor n/2 \rfloor - k + 1$. Moreover, if k is even, then $\mathrm{ex}_{k-1}(n, C_s^k) \geqslant \lfloor n/2 \rfloor - k + 1$.*

Note that if (k, s) is admissible, $k \geqslant 3$ is even and $s \geqslant 2k^2$, then Theorem 1.3 and Proposition 1.2 imply that $\mathrm{ex}_{k-1}(n, C_s^k) = (1/2 + o(1))n$.

We also study the tiling problem corresponding to C_s^k. We give some lower bounds on $t(n, C_s^k)$. Note that the bound is significantly higher if (k, s) is admissible.

Proposition 1.3. *Let $2 \leqslant k < s \leqslant n$ with n divisible by s. Then $t(n, C_s^k) \geqslant \lfloor n/2 \rfloor - k$. Moreover, if (k, s) is admissible, then*

$$t(n, C_s^k) \geqslant \begin{cases} \left\lfloor \left(\frac{1}{2} + \frac{1}{2s}\right) n \right\rfloor - k & \text{if } k \text{ is even,} \\ \left\lfloor \left(\frac{1}{2} + \frac{k}{4s(k-1) + 2k}\right) n \right\rfloor - k & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand, recall that the case $s \equiv 0 \bmod k$ was solved asymptotically by Mycroft [24], so we study the complementary case. We prove an upper bound on $t(n, C_s^k)$ which is valid whenever $s \not\equiv 0 \bmod k$ and $s \geqslant 5k^2$. Note that the bound is asymptotically sharp if k is even and (k, s) is admissible.

Theorem 1.4. *Let $3 \leqslant k < s$ be such that $s \geqslant 5k^2$ and $s \not\equiv 0 \bmod k$. Then, for all $\gamma > 0$, there exists $n_0 = n_0(k, s, \gamma)$ such that for all $n \geqslant n_0$ with $n \equiv 0 \bmod s$,*

$$t(n, C_s^k) \leqslant \left(\frac{1}{2} + \frac{1}{2s} + \gamma\right) n.$$

1.4 Organization of the paper

In Section 2 we set up basic notation and give sketches of the proofs of our main results, Theorems 1.3 and 1.4.

In Section 3 we give constructions which imply lower bounds for the Turán numbers and covering and tiling thresholds of tight cycles, thus proving Propositions 1.2 and 1.3.

In the next two sections we study the covering problem. In Section 4 we describe a family of gadgets which will be useful during the proofs of Proposition 1.1 and Theorem 1.3. Those proofs are done in Section 5.

Sections 6–9 are dedicated to investigating the tiling problem. Our aim is the proof of Theorem 1.4, *i.e.* bounding $t(n, C_s^k)$ from above. Our proof uses the absorbing method, first introduced in a systematic way by Rödl, Ruciński and Szemerédi [25] to tackle problems of finding spanning structures in hypergraphs. In Section 6 we review the absorption method for tilings,

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which we use in Section 7 to prove Theorem 1.4 under the assumption that we can find an almost perfect C_s^k-tiling (Lemma 7.1). We prove Lemma 7.1 in the next two sections: in Section 8 we review tools of hypergraph regularity and in Section 9 we introduce various auxiliary tilings that we use to finish the proof.

We conclude with some remarks and open problems in Section 10.

2. Notation and sketches of proofs

For a hypergraph H and $S \subseteq V$, we denote $H[S]$ to be the subgraph of H induced on S, *i.e.* $V(H[S]) = S$ and $E(H[S]) = \{e \in E\colon e \subseteq S\}$. Let $H \setminus S = H[V \setminus S]$. For hypergraphs H and G, let $H - G$ be the subgraph of H obtained by removing all edges in $E(H) \cap E(G)$.

Given a, b, c reals with $c > 0$, by $a = b \pm c$ we mean that $b - c \leqslant a \leqslant b + c$. We write $x \ll y$ to mean that for all $y \in (0, 1]$ there exists an $x_0 \in (0, 1)$ such that for all $x \leqslant x_0$ the subsequent statement holds. Hierarchies with more constants are defined in a similar way and are to be read from right to left. We will always assume that the constants in our hierarchies are reals in $(0, 1]$. Moreover, if $1/x$ appears in a hierarchy, this implicitly means that x is a natural number.

For all k-graphs H and all $x \in V$, define the *link* $(k-1)$*-graph* $H(x)$ *of* x *in* H to be the $(k-1)$-graph with $V(H(x)) = V \setminus \{x\}$ and $E(H(x)) = N_H(x)$. Given integers $a_1, \ldots, a_t \geqslant 1$, let $K^k(a_1, \ldots, a_t)$ denote the complete (k, t)-graph with vertex partition $V_1, \ldots, V_t$ such that $|V_i| = a_i$ for all $1 \leqslant i \leqslant t$.

For a family $\mathcal{F}$ of k-graphs, an $\mathcal{F}$*-tiling* is a set of vertex-disjoint copies of (not necessarily identical) members of $\mathcal{F}$.

For a sequence of distinct vertices $v_1, \ldots, v_s$ in a k-graph H, we say $P = v_1 \cdots v_s$ is a *tight path* if all k consecutive vertices form an edge. Note that all tight paths have an associated ordering of vertices. Hence $v_1 \cdots v_s$ and $v_s \cdots v_1$ are assumed to be different tight paths, even if the corresponding subgraphs they define are the same.

Suppose that $P_1 = v_1 \cdots v_s$ and $P_2 = w_1 \cdots w_{s'}$ are two vertex-disjoint tight paths in a k-graph H. If it happens that $v_1 \cdots v_s w_1 \cdots w_{s'}$ is also a tight path in H, then we will denote it by $P_1 P_2$. We sometimes refer to $P_1 P_2$ as the *concatenation of* P_1 *and* P_2. Note that $P_1 P_2$ has more edges than $P_1 \cup P_2$. We naturally extend this definition (whenever it makes sense) to the concatenation of a sequence of paths $P_1, \ldots, P_r$, and we denote the resulting path by $P_1 \cdots P_r$. For two tight paths P_1 and P_2, we say that P_2 *extends* P_1 if $P_2 = P_1 P'$ for some tight path P' (where we may have $|V(P')| < k$, *i.e.* P' contains no edge). Also, we may define a tight cycle C by writing $C = v_1 \cdots v_s$, whenever $v_i \cdots v_s v_1 \cdots v_{i-1}$ is a tight path for all $1 \leqslant i \leqslant s$.

For all $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. Let S_k be the symmetric group of all permutations of the set $[k]$, with the composition of functions as the group operation. Let $\mathrm{id} \in S_k$ be the *identity function* that fixes all elements in $[k]$. Given distinct $i_1, \ldots, i_r \in [k]$, the *cyclic permutation* $(i_1 i_2 \cdots i_r) \in S_k$ is the permutation that maps i_j to i_{j+1} for all $1 \leqslant j < r$ and i_r to i_1, and fixes all the other elements; we say that such a cyclic permutation has *length* r. All permutations $\sigma \in S_k$ can be written as a composition of cyclic permutations $\sigma_1 \cdots \sigma_t$ such that these cyclic permutations are *disjoint*, meaning that there are no common elements between all pairs of these different cyclic permutations.

Let H be a k-graph, let $V_1, \ldots, V_k$ be disjoint vertex sets of V and let $\sigma \in S_k$. We say that a tight path $P = v_1 \cdots v_\ell$ in H has *end-type* σ *with respect to* $V_1, \ldots, V_k$ if, for all $2 \leqslant i \leqslant k$, $v_{\ell-k+i} \in V_{\sigma(i)}$. Similarly, we say P has *start-type* σ *with respect to* $V_1, \ldots, V_k$ if $v_i \in V_{\sigma(i)}$ for all $1 \leqslant i \leqslant k-1$. If H and $V_1, \ldots, V_k$ are clear from the context, we simply say that P has *end-type* σ and *start-type* σ, respectively. Note that one could define start-type and end-type in terms of $(k-1)$-tuples in $[k]$ instead. However, for our purposes, it is more convenient to define start-types and end-types in terms of permutations of $[k]$.

2.1 Sketches of proofs of Theorems 1.3 and 1.4

We now sketch the proof of Theorem 1.3. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2+\gamma)n$. Consider any vertex $x \in V(H)$. We can show that, for some appropriate value of t, x is contained in some copy K of $K_k^k(t)$ with vertex classes $V_1, \ldots, V_k$. Suppose that $s \equiv r \not\equiv 0 \bmod k$ with $1 \leqslant r < k$. Suppose $P = v_1 \cdots v_k$ is a tight path in K such that $v_i \in V_i$ for all $1 \leqslant i \leqslant k$ and $v_1 = x$. By wrapping around K, we may find a tight path $P_2 = v_1 \cdots v_\ell$ which extends P_1, but if we only use vertices and edges of K, then we have $v_j \in V_\ell$ where $j \equiv \ell \bmod k$, for all $j \in [\ell]$. To break this pattern, we will use some gadgets (see Section 4 for a formal definition). Roughly speaking, a gadget is a k-graph on $V(K)$ and some extra vertices of H. Using these gadgets we can extend P to a tight path P' with end-type σ, for an arbitrary $\sigma \in S_k$ (see Lemma 4.2). Having done that (and choosing σ appropriately), then it is easy to extend P' into a copy of C_s^k by wrapping around $V_1, \ldots, V_k$.

The proof of Theorem 1.4 uses the absorbing method, introduced by Rödl, Ruciński and Szemerédi [25]. We first find a small vertex set $U \subseteq V(H)$ such that $H[U \cup W]$ has a perfect C_s^k-tiling for all small sets W with $|U| + |W| \equiv 0 \bmod s$. Thus the problem of finding a perfect C_s^k-tiling is reduced to finding a C_s^k-tiling in $H \setminus U$ covering almost all of the remaining vertices. However, we do not find such C_s^k-tiling directly. First we show that there exists a k-graph F_s on s vertices containing a C_s^k which has a particularly useful structure: F_s is obtained from a complete (k, k)-graph by adding a few extra vertices. So finding an almost perfect F_s-tiling suffices. Instead, we show that there exists an $\{F_s, E_s\}$-tiling $\mathcal{T}$ for some suitable k-graph E_s, subject to the minimization of some objective function $\phi(\mathcal{T})$. We do so by considering its fractional relaxation, which we call a weighted fractional $\{F_s^*, K_s^*\}$-tiling (see Section 9.1). Further, we use the hypergraph regularity lemma in the form of 'regular slice lemma' of Allen, Böttcher, Cooley and Mycroft [2].

3. Lower bounds

In this section we construct k-graphs which give lower bounds for the codegree Turán numbers and covering and tiling thresholds for tight cycles. These constructions will imply Propositions 1.2 and 1.3. We remark that the bounds obtained here can be improved by an additive constant via careful calculations and case distinctions, which we omit for the sake of giving a clear presentation.

Let A and B be disjoint vertex sets. Define $H_0^k = H_0^k(A, B)$ to be the k-graph on $A \cup B$ such that the edges of H_0^k are exactly the k-sets e of vertices that satisfy $|e \cap B| \equiv 1 \bmod 2$. Note that $\delta_{k-1}(H_0^k) \geqslant \min\{|A|, |B|\} - k + 1$.

Proposition 3.1. *Let $3 \leqslant k \leqslant s$ and $d = \gcd(k, s)$. Let A and B be disjoint vertex sets. Suppose that $H_0^k(A, B)$ contains a tight cycle C_s^k on s vertices with $V(C_s^k) \cap A \neq \emptyset$. Then $|V(C_s^k) \cap A| \equiv 0 \bmod s/d$ and (k, s) is not an admissible pair.*

Proof. Let $C_s^k = v_1 \cdots v_s$. For all $1 \leqslant i \leqslant s$, let $\phi_i \in \{A, B\}$ be such that $v_i \in \phi_i$ and let $\phi_{s+i} = \phi_i$. If two edges e and e' in $E(H_0^k(A, B))$ satisfy $|e \cap e'| = k-1$, then $|e \cap A| = |e' \cap A|$ by construction. Thus $\phi_{i+k} = \phi_i$ for all $1 \leqslant i \leqslant s$. Therefore $\phi_{i+d} = \phi_i$ for all $1 \leqslant i \leqslant s$. Hence $|V(C_s^k) \cap A| \equiv 0 \bmod s/d$.

Let

$$r = |\{v_1, \ldots, v_k\} \cap A| = |\{i : 1 \leqslant i \leqslant k, \phi_i = A\}|.$$

Note that $r > 0$ and $r \in \{k/d, 2k/d, \ldots, k\}$. Since $\{v_1, \ldots, v_k\}$ is an edge in $H_0^k(A, B)$, it follows that $k - r \equiv 1 \bmod 2$ and so $r \not\equiv k \bmod 2$. This implies $d \geqslant 2$ and k/d is odd, *i.e.* (k, s) is not an admissible pair. □

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Now we use Proposition 3.1 to prove Propositions 1.2 and 1.3.

Proof of Proposition 1.2. Let A and B be disjoint vertex sets of sizes $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$. Consider the k-graph $H_0 = H_0^k(A, B)$. By Proposition 3.1, no vertex of A can be covered with a copy of C_s^k. Then $c(n, C_s^k) \geqslant \delta_{k-1}(H_0) \geqslant \lfloor n/2 \rfloor - k + 1$.

Moreover, if k is even, then $H_0^k(A, B) = H_0^k(B, A)$. So no vertex of B can be covered by a copy of C_s^k. Hence H_0 is C_s^k-free. Therefore $\mathrm{ex}_{k-1}(n, C_s^k) \geqslant \delta_{k-1}(H_0) \geqslant \lfloor n/2 \rfloor - k + 1$. □

Proof of Proposition 1.3. To see the first part of the statement, let $d := \gcd(k, s)$ and $s' := s/d$. Note that $d \leqslant k < s$, thus $s' > 1$. Let A and B be disjoint vertex sets chosen such that $|A| + |B| = n$, $\big||A| - |B|\big| \leqslant 2$ and $|A| \not\equiv 0 \bmod s'$. Consider the k-graph $H_0 = H_0^k(A, B)$ and note that $\delta_{k-1}(H_0) \geqslant \min\{|A|, |B|\} - k + 1 \geqslant \lfloor n/2 \rfloor - k$. Proposition 3.1 implies that all copies C of C_s^k in H_0 satisfy $|V(C) \cap A| \equiv 0 \bmod s'$. Since $|A| \not\equiv 0 \bmod s'$, it is impossible to cover all vertices in A with vertex-disjoint copies of C_s^k. This proves that $t(n, C_s^k) \geqslant \delta_{k-1}(H_0) \geqslant \lfloor n/2 \rfloor - k$ as desired.

Now suppose that (k, s) is an admissible pair. Let H be the k-graph on n vertices with a vertex partition $\{A, B, T\}$ with $|A| = \lceil (n - |T|)/2 \rceil$ and $|B| = \lfloor (n - |T|)/2 \rfloor$, where $|T|$ will be specified later. The edge set of H consists of all k-sets e such that $|e \cap B| \equiv 1 \bmod 2$ or $e \cap T \neq \emptyset$. Note that

$$\delta_{k-1}(H) \geqslant \min\{|A|, |B|\} + |T| - (k-1) \geqslant \lfloor (n + |T|)/2 \rfloor - k + 1.$$

We separate the analysis into two cases depending on the parity of k.

Case 1: k even. Since $H[A \cup B] = H_0^k(A, B) = H_0^k(B, A)$, by Proposition 3.1, $H[A \cup B]$ is C_s^k-free. Thus all copies of C_s^k in H must intersect T in at least one vertex. Hence all C_s^k-tilings have at most $|T|$ vertex-disjoint copies of C_s^k. Taking $|T| = n/s - 1$ ensures that H does not contain a perfect C_s^k-tiling. This implies that $t(n, C_s^k) \geqslant \lfloor (1/2 + 1/(2s))n \rfloor - k$.

Case 2: k odd. Since $H[A \cup B] = H_0^k(A, B)$, by Proposition 3.1 no vertex in A can be covered by a copy of C_s^k. Hence all copies of C_s^k in H with non-empty intersection with A must also have non-empty intersection with T. Moreover, all edges in H intersect A in at most $k - 1$ vertices, so all copies of C_s^k in H intersect A in at most $s(k-1)/k$ vertices. Thus a perfect C_s^k-tiling would contain at most $|T|$ and at least $k|A|/(s(k-1))$ cycles intersecting A. Let $|T| = \lceil nk/(2s(k-1) + k) \rceil - 1$. Since $|T| < nk/(2s(k-1) + k)$ and $|A| \geqslant (n - |T|)/2$,

$$\frac{k|A|}{s(k-1)} \geqslant \frac{k(n - |T|)}{2s(k-1)} > \frac{nk}{2s(k-1)}\left(1 - \frac{k}{2s(k-1) + k}\right) > |T|,$$

and thus a perfect C_s^k-tiling in H cannot exist. This implies

$$t(n, C_s^k) \geqslant \delta_{k-1}(H) \geqslant \left\lfloor \frac{n + |T|}{2} \right\rfloor - k + 1 \geqslant \left\lfloor \left(\frac{1}{2} + \frac{k}{4s(k-1) + 2k}\right)n \right\rfloor - k,$$

as desired. □

4. *G*-gadgets

Throughout this section, let $\tau = (123 \cdots k) \in S_k$. Let H be a k-graph, and let K be a complete (k, k)-graph in H with its natural vertex partition $\{V_1, \ldots, V_k\}$. Knowing the end-types and start-types of paths with respect to $V_1, \ldots, V_k$ will help us to concatenate them and form longer paths which contain them both. For instance, if P_1 and P_2 are vertex-disjoint tight paths, P_1 has end-type π and P_2 has start-type π, then we can concatenate the paths and obtain P_1P_2.

Let P be a tight path in H with end-type $\pi \in S_k$. For $x \in V_{\pi(1)} \setminus V(P)$, Px is a tight path of H with end-type $\pi\tau$. We call such an extension a *simple extension of P*. By repeatedly applying r simple

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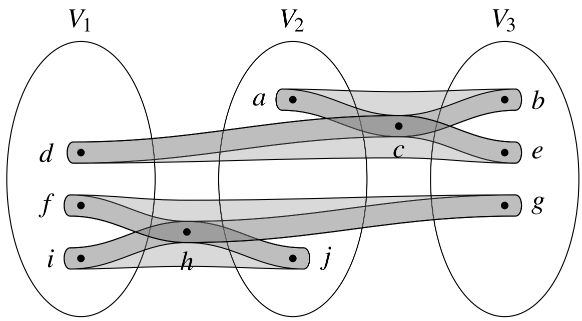

Figure 1. An example of a G-gadget in a 3-graph H. Let G be the graph on [3] consisting of the edges 12 and 23. K is the complete (3, 3)-graph with vertex partition V_1, V_2, V_3 (edges not shown) and W_G consists of the union of $W_{12} = \{a, b, c, d, e\}$ and $W_{23} = \{f, g, h, i, j\}$, including the edges $\{abc, bcd, cde, ace, fgh, ghi, hij, fhj\}$. The coloured edges are in $H \setminus K$. We show an example of (W3). Let $\sigma = \mathrm{id}$, and note that $\sigma(1) = 1$. In $H[W_{12}]$ we find the tight path $abcde$ on $5 = 2k - 1$ vertices, whose start-type is $\sigma(123) = (123)$ and end-type is $(12)\sigma = (12)$. This means the first two vertices of $abcde$ are in clusters V_2, V_3, and its last two vertices are in clusters V_1, V_3, respectively.

extensions (which is possible as long as there are available vertices), we may obtain an extension $Px_1 \cdots x_r$ of P with end-type $\pi\tau^r$, using r extra vertices and edges in K.

In the same spirit, observe that if P_1 has end-type π and P_2 has start-type $\pi\tau$, then the sequence of ordered clusters corresponding to the last $k-1$ vertices of P_1 coincides with the corresponding sequence of the first $k-1$ vertices of P_2. Thus, by using one extra vertex $x \in V_{\pi(1)} \setminus (V(P_1) \cup V(P_2))$ we can join these paths by considering the $P_1 x P_2$.

If P is a path with end-type π, we would like to find a path P' that extends P such that $|V(P')| \equiv |V(P)| \bmod k$ and P' has end-type σ, for arbitrary $\sigma \in S_k$. The goal of this section is to define and study 'G-gadgets', a tool which will allow us to do precisely that.

Let G be a 2-graph on $[k]$ and $S \subseteq V(H)$. We say $W_G \subseteq V(H)$ is a *G-gadget for K avoiding S* if there exists a family of pairwise-disjoint sets $\{W_{ij} \colon ij \in E(G)\}$ such that $W_G = \bigcup_{ij \in E(G)} W_{ij}$ and, for all $ij \in E(G)$,

(W1) $|W_{ij}| = 2k - 1$,
(W2) $|W_{ij} \setminus V(K)| = 1$, $W_{ij} \cap S = \emptyset$ and, for all $1 \leqslant i' \leqslant k$,

$$|W_{ij} \cap V_{i'}| = \begin{cases} 1 & \text{if } i' \in \{i, j\}, \\ 2 & \text{otherwise,} \end{cases}$$

(W3) for all $\sigma \in S_k$ with $\sigma(1) \in \{i, j\}$, $H[W_{ij}]$ contains a spanning tight path with start-type $\sigma\tau$ and end-type $(ij)\sigma$.

If K is clear from the context, we will just say a 'G-gadget avoiding S'. For all edges $ij \in E(G)$, we write w_{ij} for the unique vertex in $W_{ij} \setminus V(K)$.

We emphasize that (W3) is the key property that allows us to obtain an extension of a path at the same time we perform a change in the end-type. In words, (W3) says that given any $k-1$ ordered clusters that miss V_i, there exists a tight path with vertex set W_{ij}, which starts with the same ordered $k-1$ clusters and ends with the same ordered $k-1$ clusters but with V_j replaced by V_i. In other words, W_{ij} allows us to 'switch' the type of a path by replacing i with j. See Figure 1 for an example.

Suppose P is a tight path with end-type π and σ is a cyclic permutation. In the next lemma we show how to extend P into a tight path with end-type $\sigma\pi$ using a G-gadget, where G is a path.

Lemma 4.1. *Let $k \geqslant 3$ and $r \geqslant 2$. Let $\sigma = (i_1 i_2 \cdots i_r) \in S_k$ be a cyclic permutation. Let G be a 2-graph on $[k]$ containing the path $Q = i_1 i_2 \cdots i_r$. Let H be a k-graph containing a complete (k, k)-graph K with vertex partition $V_1, \ldots, V_k$. Suppose P is a tight path in H with end-type $\pi \in S_k$ such*

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that $\pi(1) = i_r$. Suppose W_G is a G-gadget avoiding $V(P)$ and $|V_{i_j} \setminus V(P)| \geqslant 2|E(G)|$ for all $1 \leqslant j \leqslant r$. Then there exists an extension P' of P with end-type $\sigma\pi$ such that

(i) $|V(P')| = |V(P)| + 2k(r-1)$,
(ii) *for all* $1 \leqslant i \leqslant k$,

$$|V_i \cap (V(P') \setminus V(P))| = \begin{cases} 2(r-1)-1 & \text{if } i \in \{i_1, i_2, \ldots, i_{r-1}\}, \\ 2(r-1) & \text{otherwise,} \end{cases}$$

(iii) *there exists a $(G-Q)$-gadget W_{G-Q} for K avoiding $V(P')$ and*
(iv) $V(P') \setminus V(P \cup K) = \{w_{i_j i_{j+1}} : 1 \leqslant j < r\}$.

Proof. We proceed by induction on r. First suppose that $r=2$ and so $\sigma = (i_1 i_2)$. Consider a G-gadget W_G avoiding $V(P)$. Since $i_1 i_2 \in E(G)$, there exists a set $W_{i_1 i_2} \subseteq W_G$ disjoint from $V(P)$ such that $|W_{i_1 i_2}| = 2k-1$ and $H[W_{i_1 i_2}]$ contains a spanning tight path P'' with start-type $\pi\tau$ and end-type $(i_1 i_2)\pi = \sigma\pi$. Note that $|V_{i_2} \cap W_G| \leqslant 2|E(G)| - 1$, as $|V_{i_2} \cap W_{i_1 i_2}| = 1$. Hence $V_{i_2} \setminus (V(P) \cup W_G) \neq \emptyset$. Take an arbitrary vertex $x_{i_2} \in V_{i_2} \setminus (V(P) \cup W_G)$ and set $P' = P x_{i_2} P''$. Since $\pi(1) = i_2$, it follows that P' is a tight path with end-type $\sigma\pi$, and P' satisfies properties (i), (ii) and (iv). Set $W_{G - i_1 i_2} = W_G \setminus W_{i_1 i_2}$. Then $W_{G-i_1 i_2}$ is a $(G - i_1 i_2)$-gadget for K avoiding $V(P')$, so P' satisfies property (iii), as desired.

Next, suppose $r > 2$. Define $\sigma' = (i_2 i_3 \cdots i_r)$ and note that $\sigma = (i_1 i_2)\sigma'$. Then σ' is a cyclic permutation of length $r-1$, with $\pi(1) = i_r$ and the path $Q' = i_2 \cdots i_{r-1} i_r$ is a subgraph of G. By the induction hypothesis, there exists an extension P'' of P with end-type $\sigma'\pi$ such that $|V(P'')| = |V(P)| + 2k(r-2)$ and, for all $1 \leqslant i \leqslant k$,

$$|V_i \cap (V(P'') \setminus V(P))| = \begin{cases} 2(r-2)-1 & \text{if } i \in \{i_2, i_3, \ldots, i_{r-1}\}, \\ 2(r-2) & \text{otherwise.} \end{cases}$$

Moreover, there exists a $(G-Q')$-gadget $W_{G-Q'}$ avoiding

$$V(P'') \quad \text{and} \quad V(P'') \setminus V(P \cup K) = \{w_{i_j i_{j+1}} : 2 \leqslant j < r\}.$$

Note that $\sigma'\pi(1) = \sigma'(i_r) = i_2$ and $i_1 i_2 \in E(G - Q')$. For all $1 \leqslant i \leqslant r$,

$$|V_i \setminus V(P')| \geqslant 2|E(G-Q')|.$$

Again by the induction hypothesis, there exists an extension P' of P'' with end-type $(i_1 i_2)\sigma'\pi = \sigma\pi$ such that

$$|V(P')| = |V(P'')| + 2k = |V(P)| + 2k(r-1)$$

and, for all $1 \leqslant i \leqslant k$,

$$|V_i \cap (V(P') \setminus V(P''))| = \begin{cases} 1 & \text{if } i = i_1, \\ 2 & \text{otherwise,} \end{cases}$$

and $V(P') \setminus (V(P'' \cup K)) = \{w_{i_1 i_2}\}$, so P' satisfies properties (i), (ii) and (iv). Furthermore, set

$$W_{G-Q} = W_G - \bigcup_{j=1}^{r-1} W_{i_j i_{j+1}}.$$

Then W_{G-Q} is a $(G-Q)$-gadget for K avoiding $V(P')$, so P' satisfies property (iii) as well. □

In the next lemma, we show how to extend a path with end-type id to one with an arbitrary end-type. We will need the following definitions. Consider an arbitrary $\sigma \in S_k \setminus \{\text{id}\}$. Write σ in its cyclic decomposition

$$\sigma = (i_{1,1}i_{1,2}\cdots i_{1,r_1})(i_{2,1}i_{2,2}\cdots i_{2,r_2})\cdots(i_{t,1}i_{t,2}\cdots i_{t,r_t}),$$

where σ is a product of $t=t(\sigma)$ disjoint cyclic permutations of respective lengths $r_1,\ldots,r_t$ so that $r_j\geqslant 2$ and $i_{j,r_j}=\min\{i_{j,r'}\colon 1\leqslant r'\leqslant r_j\}$ for all $1\leqslant j\leqslant t$, and $i_{1,r_1}<i_{2,r_2}<\cdots<i_{t,r_t}$. Define $m(\sigma)=i_{t,r_t}$. On the other hand, if $\sigma=\mathrm{id}$, then define $t(\sigma)=0$ and $m(\sigma)=1$. Define G_σ to be the 2-graph on $[k]$ consisting precisely of the (vertex-disjoint) paths $Q_j=i_{j,1}i_{j,2}\cdots i_{j,r_j}$ for all $1\leqslant j\leqslant t(\sigma)$. So G_{id} is an empty 2-graph. Note that for all σ,

$$2|E(G_\sigma)|+t(\sigma)=2\sum_{j=1}^{t(\sigma)}r_j-t(\sigma)\leqslant 2k-1. \tag{4.1}$$

For $1\leqslant i\leqslant k$ and $\sigma\in S_k\setminus\{\mathrm{id}\}$, set $X_{i,\sigma}=1$ if $i\in\{i_{t',1},\ldots,i_{t',r_{t'}-1}\}$ for some $1\leqslant t'\leqslant t$, and $X_{i,\sigma}=0$ otherwise. Also, for $1\leqslant i\leqslant k$, set $Y_{i,\sigma}=1$ if $i\in\{\sigma(j)\colon 1\leqslant j<m(\sigma)\}$ and $Y_{i,\sigma}=0$ otherwise. If $\sigma=\mathrm{id}$, then define $X_{i,\sigma}=Y_{i,\sigma}=0$ for all $1\leqslant i\leqslant k$.

Lemma 4.2. *Let $k\geqslant 3$. Let H be a k-graph containing a complete (k,k)-graph K with vertex partition $V_1,\ldots,V_k$ and a tight path P with end-type* id. *Let $\sigma\in S_k$ and let G be a 2-graph on $[k]$ containing G_σ. Suppose that K has a G-gadget W_G avoiding $V(P)$, and $|V_i\setminus V(P)|\geqslant 2|E(G)|+2$. Then there exists an extension P' of P with end-type $\sigma\tau^{m(\sigma)-1}$ such that*

(i) $|V(P')|=|V(P)|+2k|E(G_\sigma)|+m(\sigma)-1$,
(ii) *for all* $1\leqslant i\leqslant k$, $|V_i\cap(V(P')\setminus V(P))|=2|E(G_\sigma)|-X_{i,\sigma}+Y_{i,\sigma}$,
(iii) *K has a $(G-G_\sigma)$-gadget avoiding $V(P')$ and*
(iv) $V(P')\setminus V(P\cup K)=\{w_{ij}\colon ij\in E(G_\sigma)\}$.

Proof. Let

$$\sigma = (i_{1,1}i_{1,2}\cdots i_{1,r_1})(i_{2,1}i_{2,2}\cdots i_{2,r_2})\cdots(i_{t,1}i_{t,2}\cdots i_{t,r_t})$$

as defined above. We proceed by induction on $t=t(\sigma)$. If $t=0$, then $\sigma=\mathrm{id}$ and $m(\sigma)=1$, so the lemma holds by setting $P'=P$. Now suppose that $t\geqslant 1$ and the lemma is true for all $\sigma'\in S_k$ with $t(\sigma')<t$. Let

$$\sigma_1 = (i_{1,1}i_{1,2}\cdots i_{1,r_1})(i_{2,1}i_{2,2}\cdots i_{2,r_2})\cdots(i_{t-1,1}i_{t-1,2}\cdots i_{t-1,r_{t-1}})$$

and $\sigma_2=(i_{t,1}i_{t,2}\cdots i_{t,r_t})$, so $\sigma_1\sigma_2=\sigma_2\sigma_1=\sigma$. For $1\leqslant i\leqslant 2$, let $G_i=G_{\sigma_i}$ and $m_i=m(\sigma_i)$. Note that $G_\sigma=G_1\cup G_2$. Let $G'=G-G_1$. Since $t(\sigma_1)=t-1$, by the induction hypothesis, there exists a path P_1 that extends P with end-type $\sigma_1\tau^{m_1-1}$ such that

(i′) $|V(P_1)|=|V(P)|+2k|E(G_1)|+m_1-1$,
(ii′) for all $1\leqslant i\leqslant k$, $|V_i\cap(V(P_1)\setminus V(P))|=2|E(G_1)|-X_{i,\sigma_1}+Y_{i,\sigma_1}$,
(iii′) K has a G'-gadget $W_{G'}$ avoiding $V(P_1)$ and
(iv′) $V(P_1)\setminus V(P\cup K)=\{w_{ij}\colon ij\in E(G_1)\}$.

Note that for all $1\leqslant i\leqslant k$,

$$|V_i\setminus(V(P_1)\cup W_{G'})|\geqslant 2|E(G)|+2-(2|E(G_1)|+1)-2|E(G')|=1.$$

We extend P_1 using $m_2-m_1>0$ simple extensions, avoiding the set $V(P_1)\cup W_{G'}$ in each step, to obtain an extension P_2 of P_1 with end-type $\sigma_1\tau^{m_1-1}\tau^{m_2-m_1}=\sigma_1\tau^{m_2-1}$ such that

$$|V(P_2)|=|V(P_1)|+m_2-m_1=|V(P)|+2k|E(G_1)|+m_2-1$$

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and $W_{G'}$ is a G'-gadget for K that avoids $V(P_2)$. As P_1 has end-type $\sigma_1\tau^{m_1-1}$, $V(P_2)\setminus V(P_1)$ contains precisely one vertex in V_i for all

$$i\in\{\sigma_1\tau^{m_1-1}(j)\colon 1\leqslant j\leqslant m_2-m_1\}=\{\sigma_1(m_1),\dots,\sigma_1(m_2-1)\}.$$

Since $\sigma_1(i)=\sigma(i)$ for all $m_1\leqslant i<m_2$ and $m_2=i_{t,r_t}$, together with (ii′) we deduce that

$$|V_i\cap(V(P_2)\setminus V(P))|=2|E(G_1)|-X_{i,\sigma_1}+Y_{i,\sigma}. \tag{4.2}$$

Note that $\sigma_1\tau^{m_2-1}(1)=\sigma_1(m_2)=\sigma_1(i_{t,r_t})=i_{t,r_t}$. Since G' contains G_2, by Lemma 4.1 there exists an extension P' of P_2 with $|V(P')|=|V(P_2)|+2k|E(G_2)|$ and P' has end-type $\sigma_2\sigma_1\tau^{m_2-1}=\sigma\tau^{m(\sigma)-1}$, as $m_2=m(\sigma)$. Moreover, as $G'-G_2=G-G_\sigma$, K has a $(G-G_\sigma)$-gadget avoiding $V(P')$, implying (iii). Similarly, (iv) holds. Note that

$$|V(P')|=|V(P_2)|+2k|E(G_2)|=|V(P)|+2k|E(G_\sigma)|+m(\sigma)-1$$

implying (i). Finally, for all $1\leqslant i\leqslant k$, we have

$$|V_i\cap(V(P')\setminus V(P_2))|=\begin{cases}2|E(G_2)|-1 & \text{if } i\in\{i_{t,1},\dots,i_{t,r_t-1}\},\\ 2|E(G_2)| & \text{otherwise.}\end{cases}$$

So

$$|V_i\cap(V(P')\setminus V(P_2))|=2|E(G_2)|-X_{i,\sigma_2}.$$

Note that $X_{i,\sigma}=X_{i,\sigma_1}+X_{i,\sigma_2}$ because σ_1 and σ_2 are disjoint. Thus, together with (4.2), (ii) holds. □

Now we want to use the above lemmas to find tight cycles of a given length. Let P be a tight path with start-type σ and end-type π. If $\pi=\sigma$, then there exists a tight cycle C containing P with $V(C)=V(P)$. Similarly, if $\pi=\sigma\tau^{-r}$, then (by using r simple extensions) there exists a tight cycle C on $|V(P)|+r$ vertices containing P. In general, in order to extend P into a tight cycle we use Lemma 4.2 to first extend P to a path P' with end-type $\sigma\tau^{-r}$ for some suitable r, using the edges of K and a suitable G-gadget. The next lemma formalizes the aforementioned construction of the tight cycle C containing P and gives us precise bounds on the sizes of $V_i\cap(V(C)\setminus V(P))$ in the case where $\sigma=\pi$, which will be useful during Section 9.

Lemma 4.3. *Let $k\geqslant 3$. Let $\sigma,\pi\in S_k$ and $0\leqslant r<k$. Then there exists a 2-graph $G:=G(\sigma,\pi,r)$ on $[k]$ consisting of a vertex-disjoint union of paths such that the following holds for all $s\geqslant k(2k-1)$ with $s\equiv r \bmod k$. Let H be a k-graph containing a complete (k,k)-graph K with vertex partition $V_1,\dots,V_k$, and let P be a tight path with start-type σ and end-type π. Suppose W_G is a G-gadget for K avoiding $V(P)$ and $|V_i\setminus V(P)|\geqslant\lfloor s/k\rfloor+1$. Then there exists a tight cycle C on $|V(P)|+s$ vertices containing P, such that*

$$V(C)\setminus(V(P\cup K))=\{w_{ij}\colon ij\in E(G)\}.$$

Moreover, if $\sigma=\pi$, then for all $1\leqslant i,j\leqslant k$,

$$\big||V_i\cap(V(C)\setminus V(P))|-|V_j\cap(V(C)\setminus V(P))|\big|\leqslant 1.$$

Proof. Without loss of generality, we may assume that $\pi=\mathrm{id}$. Define $\sigma'=\sigma\tau^{-r}\in S_k$. Let $G=G_{\sigma'}$. Note that $|E(G)|\leqslant k-1$, $t(\sigma')\leqslant k/2$ and $2|E(G)|+t(\sigma')\leqslant 2k-1$ by (4.1). Let H,K,P be as defined in the lemma. By Lemma 4.2, there exists an extension P' of P with end-type $\sigma'\tau^{m(\sigma')-1}$ such that $|V(P')|=|V(P)|+2k|E(G)|+m(\sigma')-1$ for all $1\leqslant i\leqslant k$,

$$|V_i\cap(V(P')\setminus V(P))|=2|E(G)|-X_{i,\sigma'}+Y_{i,\sigma'}$$

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and

$$V(P') \setminus (V(P \cup K)) = \{w_{ij} \colon ij \in E(G)\}.$$

We use $k - m(\sigma') + 1$ simple extensions to get an extension P'' of P' of order

$$|V(P'')| = |V(P')| + (k - m(\sigma') + 1) = |V(P)| + 2k|E(G)| + k.$$

Note that $V(P'') \setminus V(P')$ uses precisely one vertex in each of the clusters V_i for all

$$i \in \{\sigma' \tau^{m(\sigma')-1}(j) \colon 1 \leqslant j \leqslant k - m(\sigma') + 1\} = \{\sigma'(j) \colon m(\sigma') \leqslant j \leqslant k\} = \{j \colon Y_{j,\sigma'} = 0\}.$$

It follows that for all $1 \leqslant i \leqslant k$,

$$|V_i \cap (V(P'') \setminus V(P))| = 2|E(G)| + 1 - X_{i,\sigma'}.$$

Note that P'' has end-type $\sigma' \tau^{m(\sigma')-1} \tau^{k-m(\sigma')+1} = \sigma' = \sigma \tau^{-r}$. For all $1 \leqslant i \leqslant k$ and $0 \leqslant r < k$, set $Z_{i,\sigma,r} = 1$ if $i \in \{\sigma(j) \colon k - r + 1 \leqslant j \leqslant k\}$ and set $Z_{i,\sigma,r} = 0$ otherwise. We use r more simple extensions to get an extension P''' of P with end-type $\sigma \tau^{-r} \tau^{r} = \sigma$ of order

$$|V(P''')| = |V(P'')| + r = |V(P)| + 2k|E(G)| + k + r$$

such that, for all $1 \leqslant i \leqslant k$,

$$|V_i \cap (V(P''') \setminus V(P))| = 2|E(G)| + 1 + Z_{i,\sigma,r} - X_{i,\sigma'}.$$

Since $|E(G)| \leqslant k - 1$ and $s \equiv r \bmod k$, it follows that $|V(P''')| \leqslant |V(P)| + s$. Also, $|V(P''') \setminus V(P)| \equiv s \bmod k$. For all $1 \leqslant i \leqslant k$,

$$\begin{aligned} |V_i \setminus V(P''')| &\geqslant |V_i \setminus V(P)| - 2|E(G)| - 1 + X_{i,\sigma'} - Z_{i,\sigma,r} \\ &\geqslant \lfloor s/k \rfloor - 2|E(G)| - 1 \\ &= \frac{1}{k}(k\lfloor s/k \rfloor - 2k|E(G)| - k) \\ &= \frac{1}{k}(s - r - 2k|E(G)| - k) \\ &= \frac{1}{k}(s - (|V(P''')| - |V(P)|)). \end{aligned}$$

Since P''' has start-type σ and end-type σ, we can easily extend P''' (using simple extensions) to a tight cycle C on $|V(P)| + s$ vertices. Note that

$$V(C) \setminus (V(P \cup K)) = \{w_{ij} \colon ij \in E(G)\},$$

as desired.

Moreover, for all $1 \leqslant i, j \leqslant k$,

$$\begin{aligned} &\big| |V_i \cap (V(C) \setminus V(P))| - |V_j \cap (V(C) \setminus V(P))| \big| \\ &= \big| |V_i \cap (V(P''') \setminus V(P))| - |V_j \cap (V(P''') \setminus V(P))| \big| \\ &= |(Z_{i,\sigma,r} - X_{i,\sigma'}) - (Z_{j,\sigma,r} - X_{j,\sigma'})|. \end{aligned}$$

Suppose now that $\sigma = \pi = \mathrm{id}$. We will show that $-1 \leqslant Z_{i,\sigma,r} - X_{i,\sigma'} \leqslant 0$ for all $1 \leqslant i \leqslant k$, implying that for all $1 \leqslant i, j \leqslant k$,

$$\big| |V_i \cap (V(C) \setminus V(P))| - |V_j \cap (V(C) \setminus V(P))| \big| \leqslant 1.$$

It suffices to show that if $Z_{i,\sigma,r} = 1$, then $X_{i,\sigma'} = 1$. If $r = 0$, then $Z_{i,\sigma,0} = 0$ for all $1 \leqslant i \leqslant k$, thus we may suppose from now on that $1 \leqslant r < k$. Let $1 \leqslant i \leqslant k$ such that $Z_{i,\sigma,r} = 1$. Since $\sigma = \pi = \mathrm{id}$, then $\sigma' = \tau^{-r}$. So if $Z_{i,\sigma,r} = 1$, then $k - r + 1 \leqslant i \leqslant k$. To show that $X_{i,\tau^{-r}} = 1$, we need to show

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that i is not the minimal element in the cycle that it belongs to in the cyclic decomposition of τ^{-r}, that is, there exists $m < i$ such that i is in the orbit of m under τ^{-r}. Let $d = \gcd(r, k)$. Choose $1 \leqslant m \leqslant d$ such that $m \equiv i \bmod d$. The order of τ^{-r} is exactly k/d and the orbit of m has exactly k/d elements. There are exactly k/d elements i' satisfying $1 \leqslant i' \leqslant k$ and $i' \equiv m \bmod d$, and all elements i' in the orbit of m also satisfy $i' \equiv m \bmod d$, so it follows that i is in the orbit of m under τ^{-r}. Finally, $m \leqslant d \leqslant k - r < i$. This proves that $X_{i,\tau^{-r}} = 1$, as desired. □

4.1 Finding G-gadgets in k-graphs with large codegree

We now turn our attention to the existence of G-gadgets. We prove that all large complete (k, k)-graphs contained in a k-graph H with $\delta_{k-1}(H)$ large have a G-gadget, for an arbitrary 2-graph G on $[k]$.

Lemma 4.4. *Let $0 < 1/n, 1/t_0 \ll \gamma, 1/k$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2 + \gamma)n$ containing a complete (k, k)-graph K with vertex partition $V_1, \ldots, V_k$. Let $S \subseteq V(H)$ be a set of vertices such that $|V(K) \cup S| \leqslant \gamma n/2$ and $|V_i \setminus S| \geqslant t_0$ for all $1 \leqslant i \leqslant k$. Let G be a 2-graph on $[k]$. Then there exists a G-gadget for K avoiding S.*

Proof. Choose $0 < 1/t \ll \gamma, 1/k$ and let $t_0 = t + k^2$. Suppose that $ij \in E(G)$ and $|V_\ell \setminus S| \geqslant t + 2|E(G)|$ for all $1 \leqslant \ell \leqslant k$. Let $U_\ell \subseteq V_\ell \setminus S$ with $|U_\ell| = t$ for all $1 \leqslant \ell \leqslant k$ and let $R = [k] \setminus \{i, j\}$. Let $U = \bigcup_{1 \leqslant \ell \leqslant k} U_\ell$ and

$$T = \left\{ A \in \binom{U}{k-1} : |A \cap U_r| = 1 \text{ for all } r \in R \text{ and } |A \cap (U_i \cup U_j)| = 1 \right\}.$$

Then T has size $2t^{k-1}$. By the codegree condition, all members of T have $(1/2 + \gamma)n - |V(K) \cup S| \geqslant (1/2 + \gamma/2)n$ neighbours outside of $V(K) \cup S$ and by an averaging argument, there exists a vertex $w \notin V(K) \cup S$ such that $H(w)$ satisfies $|H(w) \cap T| \geqslant (1 + \gamma)t^{k-1}$. For all $u \in U_i \cup U_j$, $N_{H(w)\cap T}(u)$ is a family of $(k-2)$-sets of $\bigcup_{r \in R} U_r$. We have that

$$\begin{aligned}
\sum_{(u_i,u_j) \in U_i \times U_j} |N_{H(w)\cap T}(u_i) \cap N_{H(w)\cap T}(u_j)| &\geqslant \sum_{(u_i,u_j) \in U_i \times U_j} (d_{H(w)\cap T}(u_i) + d_{H(w)\cap T}(u_j) - t^{k-2}) \\
&= t|H(w) \cap T| - t^k \\
&\geqslant t^k(1 + \gamma) - t^k \\
&= \gamma t^k,
\end{aligned}$$

and by an averaging argument, there exists a pair $(x_i^*, x_j^*) \in U_i \times U_j$ such that

$$|N_{H(w)\cap T}(x_i^*) \cap N_{H(w)\cap T}(x_j^*)| \geqslant \gamma t^{k-2}.$$

By the choice of t and by Theorem 1.2, we have that $N_{H(w)\cap T}(x_i^*) \cap N_{H(w)\cap T}(x_j^*)$ contains a copy K' of $K_{k-2}^{k-2}(2)$. Define $W_{ij} = V(K') \cup \{w, x_i^*, x_j^*\}$ and note that $|W_{ij}| = 2(k-2) + 3 = 2k - 1$.

We now check that (W3) holds for W_{ij}. Recall that, informally, this means that given any $k-1$ ordered clusters that miss V_i, there exists a tight path with vertex set W_{ij}, which starts with the same ordered $k-1$ clusters and ends with the same ordered $k-1$ clusters but with V_j replaced by V_i. For all $r \in R$, let $U_r \cap V(K') = \{x_r, x'_r\}$. Consider an arbitrary $\sigma \in S_k$ with $\sigma(1) = i$ and $\sigma(j') = j$. By construction, we have that

$$x_{\sigma(2)}x_{\sigma(3)} \cdots x_{\sigma(j'-1)}x_j^* x_{\sigma(j'+1)}x_{\sigma(j'+2)} \cdots x_{\sigma(k)} w x'_{\sigma(2)}x'_{\sigma(3)} \cdots x'_{\sigma(j'-1)}x_i^* x'_{\sigma(j'+1)}x'_{\sigma(j'+2)} \cdots x'_{\sigma(k)}$$

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is a spanning tight path in $H[W_{ij}]$, of start-type $\sigma\tau$ and end-type $(ij)\sigma$. Clearly W_{ij} is an ij-gadget avoiding S.

Set $S' = S \cup W_{ij}$ and $G' = G - ij$. Repeating this construction for all edges in $E(G - ij)$ and using that $t_0 = t + k^2$, it is possible to conclude that K has a G-gadget avoiding S. □

4.2 Auxiliary k-graphs F_s

Given a tight cycle C_s^k, we would like to find a k-graph F_s such that $C_s^k \subseteq F_s$ and F_s is obtained from a complete (k, k)-graph by adding 'few' extra vertices. This will be useful in Section 9.

Let K be a (k, k)-graph with vertex partition $V_1, \ldots, V_k$. Let G be an arbitrary 2-graph on $[k]$ which has exactly ℓ edges. Consider an arbitrary enumeration of the edges of $E(G)$, and for each $1 \leqslant i \leqslant \ell$, let $j_i, j_i' \in V(G)$ be such that $j_i j_i'$ is the ith edge of G. Let $y_1, \ldots, y_\ell$ be a set of ℓ vertices disjoint from $V(K)$. Let $W_G := \{y_1, \ldots, y_\ell\}$. We define the *G-augmentation of K* to be the k-graph $F = F(K, G)$ such that

$$V(F) = V(K) \cup W_G \quad \text{and}$$
$$E(F) = E(K) \cup \bigcup_{1\leqslant i\leqslant \ell} (E(H(y_i, j_i)) \cup E(H(y_i, j_i'))),$$

where $H(v, j)$ is a complete (k, k)-graph with partition $V_1, V_2, \ldots, V_{j-1}, \{v\}, V_{j+1}, \ldots, V_k$.

The easy (but crucial) observation is that if $|V_i| \geqslant 2\ell$ for all $1 \leqslant i \leqslant k$, then the G-augmentation of K contains a G-gadget for K avoiding $\emptyset$. Using that, we can prove the following.

Proposition 4.5. *Let $k \geqslant 3$, $s \geqslant 2k^2$ and $s \not\equiv 0 \bmod k$. Then there exists a 2-graph G_s on $[k]$ that is a disjoint union of paths, and $a_{s,1}, \ldots, a_{s,k}, \ell \in \mathbb{N}$ such that $|a_{s,i} - a_{s,j}| \leqslant 1$ for all $i, j \in [k]$, $\ell = |E(G_s)| \leqslant k - 1$, and if $K = K^k(a_{s,1}, \ldots, a_{s,k})$, then F_s, the G_s-augmentation of K, contains a spanning C_s^k and $|V(F_s) \setminus V(K)| = \ell$.*

Proof. Let $r \in \{1, \ldots, k-1\}$ be such that $s \equiv r \bmod k$. Let G_s be the 2-graph obtained from Lemma 4.3 (with parameters $\sigma = \pi = \mathrm{id}$ and r). Note that G_s is a disjoint union of paths and thus $\ell = E(G_s) \leqslant k - 1$.

Suppose that $V_1, \ldots, V_k$ are disjoint sets of size $\lfloor s/k \rfloor + 1$ and let K' be the complete (k, k)-graph with partition $\{V_1, \ldots, V_k\}$. For all $i \in [k]$ let $v_i \in V_i$ and consider the tight path $P = v_1 \cdots v_k$. Note that P has both start-type and end-type id. Let F' be the G_s-augmentation of K'. It is easily checked that $|V_i \setminus V(P)| \geqslant 2(k-1) \geqslant 2\ell$ and therefore there is a G_s-gadget for K' in F' avoiding $V(P)$. By the choice of G_s, F' contains a tight cycle C on s vertices containing P such that $V(C) \setminus V(K) = V(F') \setminus V(K') = W_{G_s}$ and, over the range $i \in [k]$, the values $|V(C) \cap V_i|$ differ at most by 1. It is easily checked that letting $a_{s,i} := |V(C) \cap V_i|$ we obtain the desired properties. □

5. Covering thresholds for tight cycles

In this section we prove the upper bounds for the covering codegree threshold for tight cycles, proving Proposition 1.1 and Theorem 1.3. We first prove Proposition 5.1, which immediately implies Proposition 1.1 since $K^k(s)$ contains a $C_{s'}^k$-covering for all $s' \equiv 0 \bmod k$ with $s' \leqslant sk$. We will use the following classic result of Kővári, Sós and Turán [20].

Theorem 5.1 (Kővári, Sós and Turán [20]). *Let $z(m, n; s, t)$ denote the maximum possible number of edges in a bipartite 2-graph G with parts U and V for which $|U| = m$ and $|V| = n$, which does not contain a $K_{s,t}$ subgraph with s vertices in U and t vertices in V. Then*

$$z(m, n; s, t) < (s-1)^{1/t}(n - t + 1)m^{1-1/t} + (t-1)m.$$

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Proposition 5.1. *For all $k \geqslant 3$ and $s \geqslant 1$, let $n, c \geqslant 2$ be such that $1/n, 1/c \ll 1/k, 1/s$. Then $c(n, K^k(s)) \leqslant cn^{1-1/s^{k-1}}$.*

Proof. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant cn^{1-1/s^{k-1}}$. For brevity, define $m := n^{k-1-1/s^{k-1}}$. Fix a vertex $x \in V(H)$ and consider the link $(k-1)$-graph $H(x)$ of x. Let $U_1 := E(H(x))$. Note that

$$|U_1| \geqslant \frac{\binom{n-1}{k-2}\delta_{k-1}(H)}{k-1} \geqslant c^{1/2}m. \tag{5.1}$$

Let $U_2 := V(H) \setminus \{x\}$. Consider the bipartite 2-graph B with parts U_1 and U_2, where $e \in U_1$ is joined to $u \in U_2$ if and only if $e \cup \{u\} \in E(H)$. By the codegree condition of H, all $(k-1)$-sets $e \in U_1$ have degree at least $\delta_{k-1}(H) - 1$ in B. Hence

$$|E(B)| \geqslant |U_1|(\delta_{k-1}(H) - 1) \geqslant |U_1|(cn^{1-1/s^{k-1}} - 1). \tag{5.2}$$

We claim that there is a $K_{m,s-1}$ as a subgraph in B, with m vertices in U_1 and $s-1$ vertices in U_2. Suppose not. Then, by Theorem 5.1,

$$\begin{aligned}
|E(B)| &\leqslant z(|U_1|, n-1; m, s-1) \\
&< m^{1/(s-1)} n |U_1|^{1-1/(s-1)} + (s-1)|U_1| \\
&= |U_1|\left(n\left(\frac{m}{|U_1|}\right)^{1/(s-1)} + s - 1\right) \\
&\overset{(5.1)}{\leqslant} |U_1|(c^{-1/(2(s-1))} n^{1-1/(s^{k-1})} + s - 1) \\
&< |U_1| n^{1-1/(s^{k-1})}.
\end{aligned}$$

This contradicts (5.2).

Let K be a copy of $K_{m,s-1}$ in B. Let

$$W := V(K) \cap U_1 \quad \text{and} \quad X := \{x_1, \ldots, x_{s-1}\} = V(K) \cap U_2.$$

Since $|W| = m = n^{k-1-1/s^{k-2}}$ and $1/n \ll 1/k, 1/s$, by Theorem 1.2, W contains a copy K' of $K^{k-1}(s)$. By construction, for all $y \in \{x\} \cup X$ and all $e \in E(K')$, $\{y\} \cup e \in E(H)$. Hence $H[\{x\} \cup X \cup V(K')]$ contains a $K^k(s)$ covering x, as desired. □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $t \in \mathbb{N}$ be such that $1/n_0 \ll 1/t \ll \gamma, 1/s$. Let H be a k-graph on $n \geqslant n_0$ vertices with $\delta_{k-1}(H) \geqslant (1/2 + \gamma)n$. Fix a vertex x and a copy K of $K_k^k(t)$ containing x, which exists by Proposition 5.1. Let $V_1, \ldots, V_k$ be the vertex partition of K with $x \in V_1$. By the choice of t, $|V_i| \geqslant \max\{2k^2 + 2, \lfloor s/k \rfloor + 2\}$ for all $1 \leqslant i \leqslant k$.

Let $x_1 = x$ and select arbitrarily vertices $x_i \in V_i$ for $2 \leqslant i \leqslant k$. Now $P = x_1 \cdots x_k$ is a tight path on k vertices with both start-type and end-type id. Let G be a complete 2-graph on $[k]$. By Lemma 4.4, there exists a G-gadget for K avoiding $V(P)$. Thus, by Lemma 4.3, there exists a tight cycle in $V(H)$ on s vertices containing P, and in turn, x. □

6. Absorption

We need the following 'absorbing lemma', which is a special case of a lemma of Lo and Markström [23, Lemma 1.1].

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Lemma 6.1 **([23, Lemma 1.1]).** *Let $s \geqslant k \geqslant 3$ and $0 < 1/n \ll \eta, 1/s$ and $0 < \alpha \ll \mu \ll \eta, 1/s$. Suppose that H is a k-graph on n vertices and for all distinct vertices $x, y \in V(H)$ there exist ηn^{s-1} sets S of size $s-1$ such that $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ contain a spanning C_s^k. Then there exists $U \subseteq V(H)$ of size $|U| \leqslant \mu n$ with $|U| \equiv 0 \bmod s$ such that there exists a perfect C_s^k-tiling in $H[U \cup W]$ for all $W \subseteq V(H) \setminus U$ of size $|W| \leqslant \alpha n$ with $|W| \equiv 0 \bmod s$.*

Thus, to find an absorbing set U, it is enough to find many $(s-1)$-sets S as above for each pair $x, y \in V(H)$. First we show that we can find one such S.

Lemma 6.2. *Let $s \geqslant 5k^2$ with $s \not\equiv 0 \bmod k$. Let $1/n \ll \gamma, 1/s$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2 + \gamma)n$. Then, for all pair of distinct vertices $x, y \in V(H)$, there exists $S \subseteq V(H) \setminus \{x, y\}$ such that $|S| = s-1$ and both $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ contain a spanning C_s^k.*

Proof. Let $1/n \ll 1/t \ll \gamma, 1/s$. Consider the k-graph H_{xy} with vertex set $V(H_{xy}) = (V(H) \setminus \{x, y\}) \cup \{z\}$ (for some $z \notin V(H)$) and edge set

$$E(H_{xy}) = E(H \setminus \{x, y\}) \cup \{\{z\} \cup S \colon S \in N_H(x) \cap N_H(y)\}.$$

Note that $|V(H_{xy})| = n-1$ and $\delta_{k-1}(H_{xy}) \geqslant \gamma |V(H_{xy})|$. By Proposition 5.1, H_{xy} contains a copy K of $K_k^k(t)$ containing z. Let $V_1, \ldots, V_k$ be the vertex partition of K with $z \in V_1$.

Arbitrarily select vertices $v_i \in V_i$ for $2 \leqslant i \leqslant k$. Let $H' = H_{xy} \setminus \{z, v_2, \ldots, v_k\}$ and $K' = K \setminus \{z, v_2, \ldots, v_k\}$. Note that $\delta_{k-1}(H') \geqslant (1/2 + \gamma/2)|V(H')|$ and $K' \subseteq H'$. By Lemma 4.4 with H' and K' playing the roles of H and K respectively, there exists a K_k-gadget for K' in H'. Hence there exists a K_k-gadget for K in H_{xy} avoiding $\{z, v_2, \ldots, v_k\}$.

Now we construct a copy of C_s^k in H_{xy} containing z. Note that $P = zv_2 \cdots v_k$ is a tight path on k vertices with start-type and end-type id. Since there exists a K_k-gadget for K avoiding $V(P)$, by Lemma 4.3 H_{xy} contains a copy C of C_s^k containing z.

Finally, let $S = V(C) \setminus \{z\} \subseteq V(H)$. By construction, $|S| = s-1$ and both $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ contain a spanning C_s^k in H, as desired. □

We now apply the standard supersaturation trick to find many sets S.

Lemma 6.3. *Let $k \geqslant 3$ and $0 < 1/m \ll \gamma, 1/k$. Let H be a k-graph on $n \geqslant m$ vertices with $\delta_{k-1}(H) \geqslant (1/2 + \gamma)n$. Let $x, y \in V(H)$ be distinct. Then the number of m-sets $R \subseteq V(H) \setminus \{x, y\}$ such that $\delta_{k-1}(H[R \cup \{x, y\}]) \geqslant (1/2 + \gamma/2)(m+2)$ is at least $\binom{n-2}{m}/2$.*

To prove Lemma 6.3, first we recall the following fact about concentration for hypergeometric random variables around their mean (see *e.g.* [16, p. 29]).

Lemma 6.4. *Let $a, \gamma > 0$ with $a + \gamma < 1$. Suppose that $S \subseteq [n]$ and $|S| \geqslant (a + \gamma)n$. Then*

$$\left| \left\{ M \in \binom{[n]}{m} \colon |M \cap S| \leqslant am \right\} \right| \leqslant \binom{n}{m} e^{-\gamma^2 m/(3(a+\gamma))} \leqslant \binom{n}{m} e^{-\gamma^2 m/3}.$$

Proof of Lemma 6.3. Let T be a $(k-1)$-set in $V(H)$. Note that, since $1/n \leqslant 1/m \ll \gamma$,

$$|N_H(T) \setminus \{x, y\}| \geqslant \left(\frac{1}{2} + \gamma\right)n - 2 \geqslant \left(\frac{1}{2} + \frac{2}{3}\gamma\right)(n-2).$$

We call an m-set $R \subseteq V(H) \setminus \{x, y\}$ *bad for T* if $|N_H(T) \cap R| \leqslant (1/2 + 3\gamma/5)m$. An application of Lemma 6.4 (with $1/2 + 3\gamma/5$, $\gamma/15$, $n-2$, $N_H(T) \setminus \{x, y\}$ playing the roles of a, γ, n and S,

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respectively) implies that the number of m-sets which are bad for T is at most

$$\left|\left\{R \in \binom{V(H)\setminus\{x,y\}}{m} : |N_H(T)\cap R| \leqslant (1/2+3\gamma/5)m\right\}\right| \leqslant \binom{n-2}{m} e^{-\gamma^2 m/675}.$$

Say an m-set $R \subseteq V(H)\setminus\{x,y\}$ is *good* if $\delta_{k-1}(R\cup\{x,y\}) > (1/2+3\gamma/5)m$ (and *bad* otherwise). Note that for any good m-set R,

$$\delta_{k-1}(H[R\cup\{x,y\}]) > (1/2+3\gamma/5)m \geqslant (1/2+\gamma/2)(m+2),$$

so it is enough to prove that there are at most $\binom{n-2}{m}/2$ bad m-sets. Note that R is bad if and only if there exists a $(k-1)$-set $T \subseteq R\cup\{x,y\}$ such that R is bad for T. Therefore the number of bad sets is at most

$$\binom{m+2}{k-1}\binom{n-2}{m} e^{-\gamma^2 m/675} \leqslant \frac{1}{2}\binom{n-2}{m},$$

where the inequality follows from the choice of m. □

Lemma 6.5. *Let $k \geqslant 3$ and $s \geqslant 5k^2$. Let $1/n \ll \alpha \ll \mu \ll \gamma, 1/s$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2+\gamma)n$. Then there exists $U \subseteq V(H)$ of size $|U| \leqslant \mu n$ with $|U| \equiv 0 \bmod s$ such that there exists a perfect C_s^k-tiling in $H[U\cup W]$ for all $W \subseteq V(H)\setminus U$ of size $|W| \leqslant \alpha n$ with $|W| \equiv 0 \bmod s$.*

Proof. Let $\mu \ll \eta \ll 1/m \ll \gamma, 1/s$. Let x, y be distinct vertices in $V(H)$. By Lemma 6.3, at least $\binom{n-2}{m}/2$ of the m-sets $R \subseteq V(H)\setminus\{x,y\}$ are such that

$$\delta_{k-1}(H[R\cup\{x,y\}]) \geqslant (1/2+\gamma/2)(m+2).$$

By Lemma 6.2, each one of these subgraphs contains a set $S \subseteq R$ of size $s-1$ such that $H[S\cup\{x\}]$ and $H[S\cup\{y\}]$ have spanning copies of C_s^k. Then the number of these sets S in H is at least

$$\frac{\frac{1}{2}\binom{n-2}{m}}{\binom{n-2-(s-1)}{m-(s-1)}} = \frac{\binom{n-2}{s-1}}{2\binom{m}{s-1}} \geqslant \eta n^{s-1}.$$

Then the result follows from Lemma 6.1. □

7. Tiling thresholds for tight cycles

Now we prove Theorem 1.4 under the assumption that the following 'almost perfect C_s^k-tiling lemma' holds.

Lemma 7.1. *Let $1/n \ll \alpha, \gamma, 1/s$, $k \geqslant 3$ and $s \geqslant 5k^2$ such that $s \not\equiv 0 \bmod k$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2+1/(2s)+\gamma)n$. Then H has a C_s^k-tiling covering at least $(1-\alpha)n$ vertices.*

Assuming Lemma 7.1 is true, we use it to prove Theorem 1.4.

Proof of Theorem 1.4. Choose $1/n \ll \alpha \ll \mu \ll \gamma, 1/k, 1/s$. By Lemma 6.5, there exists $U \subseteq V(H)$ of size $|U| \leqslant \mu n$ with $|U| \equiv 0 \bmod s$ such that there exists a perfect C_s^k-tiling in $H[U\cup W]$ for all $W \subseteq V(H)\setminus U$ of size $|W| \leqslant \alpha n$ with $|W| \equiv 0 \bmod s$.

Define $H' = H\setminus U$. Then

$$\delta_{k-1}(H') \geqslant \delta_{k-1}(H) - |U| \geqslant (1/2+1/(2s)+\gamma/2)|V(H')|.$$

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An application of Lemma 7.1 (with $\gamma/2$, $|V(H')|$ playing the roles of γ, n, respectively, and noting that the hierarchies of constants in both lemmas are consistent) implies that there exists a C_s^k-tiling $\mathcal{T}'$ in H' covering at least $(1-\alpha)|V(H')|$ vertices. Let W be the set of uncovered vertices by $\mathcal{T}'$ in H'. Then $|W| \leqslant \alpha n$ and $|W| \equiv 0 \bmod s$. By the absorbing property of U, there exists a perfect C_s^k-tiling $\mathcal{T}''$ in $H[U \cup W]$. Then $\mathcal{T}' \cup \mathcal{T}''$ is a perfect C_s^k-tiling in H. □

The rest of the paper will be devoted to the proof of Lemma 7.1.

8. Hypergraph regularity and regular slice lemma

To prove Lemma 7.1 we will use the hypergraph regularity lemma, which requires the following definitions.

8.1 Regular complexes

Let $\mathcal{P}$ be a partition of V into vertex classes $V_1, \ldots, V_s$. A subset $S \subseteq V$ is *$\mathcal{P}$-partite* if $|S \cap V_i| \leqslant 1$ for all $1 \leqslant i \leqslant s$. A hypergraph is *$\mathcal{P}$-partite* if all of its edges are $\mathcal{P}$-partite, and it is *s-partite* if it is $\mathcal{P}$-partite for some partition $\mathcal{P}$ with $|\mathcal{P}| = s$.

A hypergraph H is a *complex* if, whenever $e \in E(H)$ and e' is a non-empty subset of e, we have that $e' \in E(H)$. All the complexes considered in this paper have the property that all vertices are contained in an edge. For a positive integer k, a complex H is a *k-complex* if all the edges of H consist of at most k vertices. The edges of size i are called i-edges of H. Given a k-complex H, for all $1 \leqslant i \leqslant k$ we let H_i denote the underlying i-graph of H: the vertices of H_i are those of H and the edges of H_i are the i-edges of H. Given $s \geqslant k$, a *(k, s)-complex* H is an s-partite k-complex.

Let H be a $\mathcal{P}$-partite k-complex. For $i \leqslant k$ and $X \in \binom{\mathcal{P}}{i}$, we write H_X for the subgraph of H_i induced by $\bigcup X$. Note that H_X is an (i, i)-graph. In a similar manner we write $H_{X^<}$ for the hypergraph on the vertex set $\bigcup X$, whose edge set is $\bigcup_{X' \subsetneq X} H_{X'}$. Note that if H is a k-complex and X is a k-set, then $H_{X^<}$ is a $(k-1, k)$-complex.

Given $i \geqslant 2$, consider an (i, i)-graph H_i and an $(i-1, i)$-graph H_{i-1} on the same vertex set, which are i-partite with respect to the same partition $\mathcal{P}$. We write $\mathcal{K}_i(H_{i-1})$ for the family of all $\mathcal{P}$-partite i-sets that form a copy of the complete $(i-1)$-graph K_i^{i-1} in H_{i-1}. We define the *density of H_i with respect to H_{i-1}* to be

$$d(H_i|H_{i-1}) = \frac{|\mathcal{K}_i(H_{i-1}) \cap E(H_i)|}{|\mathcal{K}_i(H_{i-1})|} \quad \text{if } |\mathcal{K}_i(H_{i-1})| > 0,$$

and $d(H_i|H_{i-1}) = 0$ otherwise. More generally, if $\mathbf{Q} = (Q_1, \ldots, Q_r)$ is a collection of r subhypergraphs of H_{i-1}, we define $\mathcal{K}_i(\mathbf{Q}) := \bigcup_{j=1}^{r} \mathcal{K}_i(Q_j)$ and

$$d(H_i|\mathbf{Q}) = \frac{|\mathcal{K}_i(\mathbf{Q}) \cap E(H_i)|}{|\mathcal{K}_i(\mathbf{Q})|} \quad \text{if } |\mathcal{K}_i(\mathbf{Q})| > 0,$$

and $d(H_i|\mathbf{Q}) = 0$ otherwise.

We say that H_i is *(d_i, ε, r)-regular with respect to H_{i-1}* if, for all r-tuples $\mathbf{Q}$ with $|\mathcal{K}_i(\mathbf{Q})| > \varepsilon|\mathcal{K}_i(H_{i-1})|$, we have $d(H_i|\mathbf{Q}) = d_i \pm \varepsilon$. Instead of $(d_i, \varepsilon, 1)$-regularity we simply refer to *(d_i, ε)-regularity*; we also say simply that H_i is (ε, r)-regular with respect to H_{i-1} to mean that there exists some d_i for which H_i is (d_i, ε, r)-regular with respect to H_{i-1}. Given an i-graph G whose vertex set contains that of H_{i-1}, we say that G is *(d_i, ε, r)-regular with respect to H_{i-1}* if the i-partite subgraph of G induced by the vertex classes of H_{i-1} is (d_i, ε, r)-regular with respect to H_{i-1}.

Given $3 \leqslant k \leqslant s$ and a (k, s)-complex H with vertex partition $\mathcal{P}$, we say that H *is $(d_k, d_{k-1}, \ldots, d_2, \varepsilon_k, \varepsilon, r)$-regular* if the following conditions hold.

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(i) For all $2 \leqslant i \leqslant k-1$ and $A \in \binom{\mathcal{P}}{i}$, H_A is (d_i, ε)-regular with respect to $(H_{A^<})_{i-1}$.

(ii) For all $A \in \binom{\mathcal{P}}{k}$, the induced subgraph H_A is (d_k, ε_k, r)-regular with respect to $(H_{A^<})_{i-1}$.

Sometimes we denote $(d_k, \ldots, d_2)$ by $\mathbf{d}$ and write $(\mathbf{d}, \varepsilon_k, \varepsilon, r)$-regular for $(d_k, \ldots, d_2, \varepsilon_k, \varepsilon, r)$-regular.

We will need the following 'regular restriction lemma', which states that the restriction of regular complexes to a sufficiently large set of vertices in each vertex class is still regular, with somewhat degraded regularity properties.

Lemma 8.1. (regular restriction lemma [2, Lemma 24]). *Let $k, m \in \mathbb{N}$ and $\beta, \varepsilon, \varepsilon_k, d_2, \ldots, d_k$ be such that*

$$\frac{1}{m} \ll \varepsilon \ll \varepsilon_k, d_2, \ldots, d_{k-1} \quad \text{and} \quad \varepsilon_k \ll \beta, \frac{1}{k}.$$

Let $r, s \in \mathbb{N}$ and $d_k > 0$. Set $\mathbf{d} = (d_k, \ldots, d_2)$. Let G be a $(\mathbf{d}, \varepsilon_k, \varepsilon, r)$-regular (k, s)-complex with vertex classes $V_1, \ldots, V_s$ each of size m. Let $V_i' \subseteq V_i$ with $|V_i'| \geqslant \beta m$ for all $1 \leqslant i \leqslant s$. Then the induced subcomplex $G[V_1' \cup \cdots \cup V_s']$ is $(\mathbf{d}, \sqrt{\varepsilon_k}, \sqrt{\varepsilon}, r)$-regular.

8.2 Statement of the regular slice lemma

In this section we state the version of the regularity lemma (Theorem 8.1) due to Allen, Böttcher, Cooley and Mycroft [2], which they call the *regular slice lemma*. A similar lemma was previously applied by Haxell, Łuczak, Peng, Rödl, Ruciński and Skokan in the case of 3-graphs [15]. This lemma says that all k-graphs G admit a regular slice $\mathcal{J}$, which is a regular multipartite $(k-1)$-complex whose vertex classes have equal size such that G is regular with respect to $\mathcal{J}$.

Let $t_0, t_1 \in \mathbb{N}$ and $\varepsilon > 0$. We say that a $(k-1)$-complex $\mathcal{J}$ is (t_0, t_1, ε)-*equitable* if it has the following two properties.

(i) There exists a partition $\mathcal{P}$ of $V(\mathcal{J})$ into t parts of equal size, for some $t_0 \leqslant t \leqslant t_1$, such that $\mathcal{J}$ is $\mathcal{P}$-partite. We refer to $\mathcal{P}$ as the *ground partition* of $\mathcal{J}$, and to the parts of $\mathcal{P}$ as the *clusters* of $\mathcal{J}$.

(ii) There exists a *density vector* $\mathbf{d} = (d_{k-1}, \ldots, d_2)$ such that, for all $2 \leqslant i \leqslant k-1$, we have $d_i \geqslant 1/t_1$ and $1/d_i \in \mathbb{N}$, and the $(k-1)$-complex $\mathcal{J}$ is $(\mathbf{d}, \varepsilon, \varepsilon, 1)$-regular.

Let $X \in \binom{\mathcal{P}}{k}$. We write $\hat{\mathcal{J}}_X$ for the $(k-1, k)$-graph $(\mathcal{J}_{X^<})_{k-1}$. A k-graph G on $V(\mathcal{J})$ is (ε_k, r)-*regular with respect to* $\hat{\mathcal{J}}_X$ if there exists some d such that G is (d, ε_k, r)-regular with respect to $\hat{\mathcal{J}}_X$. We also write $d^*_{\mathcal{J},G}(X)$ for the density of G with respect to $\hat{\mathcal{J}}_X$, or simply $d^*(X)$ if $\mathcal{J}$ and G are clear from the context.

Definition 8.1. (regular slice). Given $\varepsilon, \varepsilon_k > 0$, $r, t_0, t_1 \in \mathbb{N}$, a k-graph G and a $(k-1)$-complex $\mathcal{J}$ on $V(G)$, we call $\mathcal{J}$ a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-*regular slice for* G if $\mathcal{J}$ is (t_0, t_1, ε)-equitable and G is (ε_k, r)-regular with respect to all but at most $\varepsilon_k \binom{t}{k}$ of the k-sets of clusters of $\mathcal{J}$, where t is the number of clusters of $\mathcal{J}$.

Given a regular slice $\mathcal{J}$ for a k-graph G, we keep track of the relative densities $d^*(X)$ for k-sets X of clusters of $\mathcal{J}$, which is done via a weighted k-graph.

Definition 8.2. Given a k-graph G and a (t_0, t_1, ε)-equitable $(k-1)$-complex $\mathcal{J}$ on $V(G)$, we let $R_{\mathcal{J}}(G)$ be the complete weighted k-graph whose vertices are the clusters of $\mathcal{J}$, and where each edge X is given weight $d^*(X)$. When $\mathcal{J}$ is clear from the context, we write $R(G)$ instead of $R_{\mathcal{J}}(G)$.

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The regular slice lemma (Theorem 8.1) guarantees the existence of a regular slice $\mathcal{J}$ with respect to which $R(G)$ resembles G in various senses. In particular, $R(G)$ inherits the codegree condition of G in the following sense.

Let G be a k-graph on n vertices. Given a set $S \in \binom{V(G)}{k-1}$, recall that $\deg_G(S)$ is the number of edges of G which contain S. The *relative degree* $\overline{\deg}(S; G)$ *of* S *with respect to* G is defined to be

$$\overline{\deg}(S; G) = \frac{\deg_G(S)}{n-k+1}.$$

Thus $\overline{\deg}(S; G)$ is the proportion of k-sets of vertices in G extending S which are in fact edges of G. To extend this definition to weighted k-graphs G with weight function d^*, we define

$$\overline{\deg}(S; G) = \frac{\sum_{e \in E(G):\, S \subseteq e} d^*(e)}{n-k+1}.$$

Finally, for a collection $\mathcal{S}$ of $(k-1)$-sets in $V(G)$, the *mean relative degree* $\overline{\deg}(\mathcal{S}; G)$ *of* $\mathcal{S}$ *in* G is defined to be the mean of $\overline{\deg}(S; G)$ over all sets $S \in \mathcal{S}$.

We will need an additional property of regular slices. Suppose G is a k-graph, $\mathcal{S}$ is a $(k-1)$-graph on the same vertex set, and $\mathcal{J}$ is a regular slice for G on t clusters. We say $\mathcal{J}$ is $(\eta, \mathcal{S})$-*avoiding* if, for all but at most $\eta\binom{t}{k-1}$ of the $(k-1)$-sets Y of clusters of $\mathcal{J}$, it holds that $|\mathcal{J}_Y \cap \mathcal{S}| \leqslant \eta|\mathcal{J}_Y|$.

We can now state the version of the regular slice lemma that we will use.

Theorem 8.1 (regular slice lemma [2, Lemma 6]). *Let $k \in \mathbb{N}$ with $k \geqslant 3$. For all $t_0 \in \mathbb{N}$, $\varepsilon_k > 0$ and all functions $r\colon \mathbb{N} \to \mathbb{N}$ and $\varepsilon\colon \mathbb{N} \to (0, 1]$, there exist $t_1, n_1 \in \mathbb{N}$ such that the following holds for all $n \geqslant n_1$ which are divisible by $t_1!$. Let G be a k-graph on n vertices, and let $\mathcal{S}$ be a $(k-1)$-graph on the same vertex set with $|E(\mathcal{S})| \leqslant \theta\binom{n}{k-1}$. Then there exists a $(t_0, t_1, \varepsilon(t_1), \varepsilon_k, r(t_1))$-regular slice $\mathcal{J}$ for G such that, for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have $\overline{\deg}(Y; R(G)) = \overline{\deg}(\mathcal{J}_Y; G) \pm \varepsilon_k$, and furthermore $\mathcal{J}$ is $(3\sqrt{\theta}, \mathcal{S})$-avoiding.*

We remark that the original statement of [2, Lemma 6] did not include the 'avoiding' property with respect to a fixed $(k-1)$-graph $\mathcal{S}$. This, however, can be obtained easily from their proof. We sketch this in Appendix A.1.

8.3 The d-reduced k-graph and strong density

Once we have a regular slice $\mathcal{J}$ for a k-graph G, we would like to work within k-tuples of clusters with respect to which G is both regular and dense. To keep track of those tuples, we introduce the following definition.

Definition 8.3 (d-reduced k-graph). Let G be a k-graph and let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G. Then for $d > 0$ we define the *d-reduced k-graph $R_d(G)$ of G* to be the k-graph whose vertices are the clusters of $\mathcal{J}$ and whose edges are all k-sets of clusters X of $\mathcal{J}$ such that G is (ε_k, r)-regular with respect to X and $d^*(X) \geqslant d$. Note that $R_d(G)$ depends on the choice of $\mathcal{J}$ but this will always be clear from the context.

The next lemma states that for regular slices $\mathcal{J}$ as in Theorem 8.1, the codegree conditions are also preserved by $R_d(G)$.

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Lemma 8.2 **([2, Lemma 8]).** *Let $k, r, t_0, t \in \mathbb{N}$ and $\varepsilon, \varepsilon_k > 0$. Let G be a k-graph and let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G. Then, for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have*

$$\overline{\deg}(Y; R_d(G)) \geqslant \overline{\deg}(Y; R(G)) - d - \zeta(Y),$$

where $\zeta(Y)$ is defined to be the proportion of k-sets Z of clusters with $Y \subseteq Z$ that are not (ε_k, r)-regular with respect to G.

For $0 \leqslant \mu, \theta \leqslant 1$, we say that a k-graph H on n vertices is (μ, θ)-*dense* if there exists $\mathcal{S} \subseteq \binom{V(H)}{k-1}$ of size at most $\theta\binom{n}{k-1}$ such that, for all $S \in \binom{V(H)}{k-1} \setminus \mathcal{S}$, we have $\deg_H(S) \geqslant \mu(n-k+1)$. In particular, if H has $\delta_{k-1}(H) \geqslant \mu n$, then it is $(\mu, 0)$-dense.

By using Lemma 8.2, we show that $R_d(G)$ 'inherits' the property of being (μ, θ)-dense.

Lemma 8.3. *Let $1/n \ll 1/t_1 \leqslant 1/t_0 \ll 1/k$ and $\mu, \theta, d, \varepsilon, \varepsilon_k > 0$. Suppose that G is a k-graph on n vertices, that G is (μ, θ)-dense, and let $\mathcal{S}$ be the $(k-1)$-graph on $V(G)$ whose edges are precisely*

$$\left\{S \in \binom{V(G)}{k-1} : \deg_G(S) < \mu(n-k+1)\right\}.$$

Let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G such that for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have $\overline{\deg}(Y; R(G)) = \overline{\deg}(\mathcal{J}_Y; G) \pm \varepsilon_k$, and furthermore $\mathcal{J}$ is $(3\sqrt{\theta}, \mathcal{S})$-avoiding. Then $R_d(G)$ is $((1-3\sqrt{\theta})\mu - d - \varepsilon_k - \sqrt{\varepsilon_k}, 3\sqrt{\theta} + 3\sqrt{\varepsilon_k})$-dense.

Proof. Let $\mathcal{P}$ be the ground partition of $\mathcal{J}$ and $t = |\mathcal{P}|$. Let $m = n/t$. Clearly $|V| = m$ for all $V \in \mathcal{P}$. Let $\mathcal{Y}_1$ be the set of all $Y \in \binom{\mathcal{P}}{k-1}$ such that $|\mathcal{J}_Y \cap \mathcal{S}| \geqslant 3\sqrt{\theta}|\mathcal{J}_Y|$. Since $\mathcal{J}$ is $(3\sqrt{\theta}, \mathcal{S})$-avoiding, $|\mathcal{Y}_1| \leqslant 3\sqrt{\theta}\binom{t}{k-1}$.

For all $Y \in \binom{\mathcal{P}}{k-1}$, let $\zeta(Y)$ be defined as in Lemma 8.2. Let $\mathcal{Y}_2$ be the set of all $Y \in \binom{\mathcal{P}}{k-1}$ with $\zeta(Y) > \sqrt{\varepsilon_k}$. Since G is (ε_k, r)-regular with respect to all but at most $\varepsilon_k\binom{t}{k}$ of the k-sets of clusters of $\mathcal{P}$, it follows that $|\mathcal{Y}_2|\sqrt{\varepsilon_k}(t-k+1)/k \leqslant \varepsilon_k\binom{t}{k}$, namely $|\mathcal{Y}_2| \leqslant \sqrt{\varepsilon_k}\binom{t}{k-1}$.

Then it follows that $|\mathcal{Y}_1 \cup \mathcal{Y}_2| \leqslant 3(\sqrt{\theta} + \sqrt{\varepsilon_k})\binom{t}{k-1}$. We will show that all $Y \in \binom{\mathcal{P}}{k-1} \setminus (\mathcal{Y}_1 \cup \mathcal{Y}_2)$ will have large codegree in $R_d(G)$, thus proving the lemma.

Consider any $Y \in \binom{\mathcal{P}}{k-1} \setminus (\mathcal{Y}_1 \cup \mathcal{Y}_2)$. Since $Y \notin \mathcal{Y}_2$, $\zeta(Y) \leqslant \sqrt{\varepsilon_k}$. By Lemma 8.2, we have

$$\begin{aligned}\overline{\deg}(Y; R_d(G)) &\geqslant \overline{\deg}(Y; R(G)) - d - \zeta(Y)\\ &\geqslant \overline{\deg}(Y; R(G)) - d - \sqrt{\varepsilon_k}\\ &\geqslant \overline{\deg}(\mathcal{J}_Y; G) - \varepsilon_k - d - \sqrt{\varepsilon_k}.\end{aligned}$$

So it suffices to show that $\overline{\deg}(\mathcal{J}_Y; G) \geqslant (1-3\sqrt{\theta})\mu$. Recall that $\overline{\deg}(\mathcal{J}_Y; G)$ is the mean of $\overline{\deg}(S; G)$ over all $S \in \mathcal{J}_Y$. Since $Y \notin \mathcal{Y}_1$, $|\mathcal{J}_Y \cap \mathcal{S}| \leqslant 3\sqrt{\theta}|\mathcal{J}_Y|$. By definition, for all $S \in \mathcal{J}_Y \setminus \mathcal{S}$, $\deg_G(S) \geqslant \mu(n-k+1)$. Thus $\overline{\deg}(\mathcal{J}_Y; G) \geqslant (1-3\sqrt{\theta})\mu$, as required. □

For $0 \leqslant \mu, \theta \leqslant 1$, a k-graph H on n vertices is *strongly* (μ, θ)-*dense* if it is (μ, θ)-dense and, for all edges $e \in E(H)$ and all $(k-1)$-sets $X \subseteq e$, $\deg_H(X) \geqslant \mu(n-k+1)$. We prove that all (μ, θ)-dense k-graphs contain a strongly (μ', θ')-dense subgraph, for some degraded constants μ', θ'.

Lemma 8.4. *Let $n \geqslant 2k$ and $0 < \mu, \theta < 1$. Suppose that H is a k-graph on n vertices that is (μ, θ)-dense. Then there exists a sub-k-graph H' on $V(H)$ that is strongly $(\mu - 2^k\theta^{1/(2k-2)}, \theta + \theta^{1/(2k-2)})$-dense.*

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Proof. Let $\mathcal{S}_1$ be the set of all $S \in \binom{V(H)}{k-1}$ such that $\deg_H(S) < \mu(n-k+1)$. Thus $|\mathcal{S}_1| \leqslant \theta\binom{n}{k-1}$. Let $\beta = \theta^{1/(k-1)}$. Now, for all $j \in \{k-1, k-2, \ldots, 1\}$ in turn we construct $\mathcal{A}_j \subseteq \binom{V(H)}{j}$ in the following way. Initially, let $\mathcal{A}_{k-1} = \mathcal{S}_1$. Given $j > 1$ and $\mathcal{A}_j$, we define $\mathcal{A}_{j-1} \subseteq \binom{V(H)}{j-1}$ to be the set of all $X \in \binom{V(H)}{j-1}$ such that there exist at least $\beta(n-j+1)$ vertices $w \in V(H)$ with $X \cup \{w\} \in \mathcal{A}_j$.

Claim 1. *For all* $1 \leqslant j \leqslant k-1$, $|\mathcal{A}_j| \leqslant \beta^j\binom{n}{j}$.

Proof of the claim. We prove the claim by induction on $k-j$. When $j = k-1$, the result is immediate. Now suppose $2 \leqslant j \leqslant k-1$ and that $|\mathcal{A}_j| \leqslant \beta^j\binom{n}{j}$. By double-counting the number of tuples (X, w) where X is a $(j-1)$-set in $\mathcal{A}_{j-1}$ and $X \cup \{w\} \in \mathcal{A}_j$, we have $|\mathcal{A}_{j-1}|\beta(n-j+1) \leqslant j|\mathcal{A}_j|$. By the induction hypothesis it follows that

$$|\mathcal{A}_{j-1}| \leqslant \frac{j}{\beta(n-j+1)}|\mathcal{A}_j| \leqslant \beta^{j-1}\binom{n}{j-1}. \qquad \square$$

For all $1 \leqslant j \leqslant k-1$, let F_j be the set of edges $e \in E(H)$ such that there exists $S \in \mathcal{A}_j$ with $S \subseteq e$, and let $F = \bigcup_{j=1}^{k-1} F_j$. Define $H' = H - F$. We will show that H' satisfies the desired properties.

For each j-set, there are at most $\binom{n-j}{k-j}$ k-edges containing it. Thus, for all $1 \leqslant j \leqslant k-1$, the claim above implies that

$$|F_j| \leqslant |\mathcal{A}_j|\binom{n-j}{k-j} \leqslant \beta^j\binom{n}{j}\binom{n-j}{k-j} = \beta^j\binom{k}{j}\binom{n}{k}.$$

Therefore

$$|F| \leqslant \sum_{j=1}^{k-1}|F_j| \leqslant \binom{n}{k}\sum_{j=1}^{k-1}\binom{k}{j}\beta^j \leqslant 2^k\beta\binom{n}{k}.$$

Let $\mathcal{S}_2$ be the set of all $S \in \binom{V(H)}{k-1}$ contained in more than $2^k\sqrt{\beta}(n-k+1)$ edges of F. It follows that $|\mathcal{S}_2| \leqslant \sqrt{\beta}\binom{n}{k-1}$. This implies that

$$|\mathcal{S}_1 \cup \mathcal{S}_2| \leqslant (\theta + \sqrt{\beta})\binom{n}{k-1} = (\theta + \theta^{1/(2k-2)})\binom{n}{k-1}.$$

Now consider an arbitrary

$$S \in \binom{V(H)}{k-1} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2).$$

As $S \notin \mathcal{S}_1$, it follows that $\deg_H(S) \geqslant \mu(n-k+1)$. As $S \notin \mathcal{S}_2$, it follows that

$$\deg_{H'}(S) \geqslant \deg_H(S) - 2^k\sqrt{\beta}(n-k+1) \geqslant (\mu - 2^k\theta^{1/(2k-2)})(n-k+1).$$

Therefore H' is $(\mu - 2^k\theta^{1/(2k-2)}, \theta + \theta^{1/(2k-2)})$-dense.

Let $e \in E(H')$ and $X \in \binom{e}{k-1}$. It is enough to prove that $X \notin \mathcal{S}_1 \cup \mathcal{S}_2$. As $e \notin F_{k-1}$, it follows that $X \notin \mathcal{A}_{k-1} = \mathcal{S}_1$. So it is enough to prove that $X \notin \mathcal{S}_2$. Suppose the contrary, that $X \in \mathcal{S}_2$. Then X is contained in more than $2^k\sqrt{\beta}(n-k+1)$ edges $e' \in E(F)$. Let $W = N_F(X)$. For all $w \in W$, fix a set $A_w \in \bigcup_{j=1}^{k-1}\mathcal{A}_j$ such that $A_w \subseteq X \cup \{w\}$ and let $T_w = X \cap A_w$. If $A_w \subseteq X$ then $A_w \subseteq e \in E(H')$, a contradiction. Hence $w \in A_w$ for all $w \in W$, and therefore $|T_w| = |A_w| - 1 \leqslant k-2 < |X|$ for all $w \in W$. We deduce $T_w \neq X$ for all $w \in W$. By the pigeonhole principle, there exists $T \subsetneq X$ and $W_T \subseteq W$ such that for all $w \in W_T$,

$$T_w = T \quad \text{and} \quad |W_T| \geqslant |W|/(2^{k-1}) \geqslant 2\sqrt{\beta}(n-k+1) > \sqrt{\beta}n.$$

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Suppose $|T| = t \geqslant 1$. Then, for all $w \in W_T$, $T \cup \{w\} = A_w \in \mathcal{A}_{t+1}$, so there are at least $\sqrt{\beta}n \geqslant \beta(n-t)$ vertices $w \in V(H)$ such that $T \cup \{w\} \in \mathcal{A}_{t+1}$. Therefore $T \in \mathcal{A}_t$ and $T \subseteq X \subseteq e$, which is a contradiction because $e \notin F_t$. Hence we may assume that $T = \emptyset$. Then, for all $w \in W_T$, $\{w\} \in \mathcal{A}_1$. Therefore $|\mathcal{A}_1| \geqslant |W_T| > \sqrt{\beta}n$, contradicting the claim. □

8.4 The embedding lemma

We will need a version of the 'embedding lemma' which gives sufficient conditions to find a copy of a (k, s)-graph H in a regular (k, s)-complex G.

Suppose that G is a (k, s)-graph with vertex classes $V_1, \ldots, V_s$, which all have size m. Suppose also that H is a (k, s)-graph with vertex classes $X_1, \ldots, X_s$ of size at most m. We say that a copy of H in G is *partition-respecting* if, for all $1 \leqslant i \leqslant s$, the vertices corresponding to those in X_i lie within V_i.

Given a k-graph G and a $(k-1)$-graph J on the same vertex set, we say that G *is supported on* J if, for all $e \in E(G)$ and all $f \in \binom{e}{k-1}$, $f \in E(J)$.

We state the following lemma, which can be easily deduced from a lemma stated by Cooley, Fountoulakis, Kühn and Osthus [4].

Lemma 8.5 (embedding lemma [4, Theorem 2]). *Let $k, s, r, t, m_0 \in \mathbb{N}$ and let $d_2, \ldots, d_{k-1}$, d, ε, $\varepsilon_k > 0$ be such that $1/d_i \in \mathbb{N}$ for all $2 \leqslant i \leqslant k-1$, and*

$$\frac{1}{m_0} \ll \frac{1}{r}, \quad \varepsilon \ll \varepsilon_k, \quad d_2, \ldots, d_{k-1} \quad \text{and} \quad \varepsilon_k \ll d, \frac{1}{t}, \frac{1}{s}.$$

Then the following holds for all $m \geqslant m_0$. Let H be a (k, s)-graph on t vertices with vertex classes $X_1, \ldots, X_s$. Let $\mathcal{J}$ be a $(d_{k-1}, \ldots, d_2, \varepsilon, \varepsilon, 1)$-regular $(k-1, s)$-complex with vertex classes $V_1, \ldots, V_s$ all of size m. Let G be a k-graph on $\bigcup_{1 \leqslant i \leqslant s} V_i$ which is supported on $\mathcal{J}_{k-1}$ such that, for all $e \in E(H)$ intersecting the vertex classes $\{X_{i_j} : 1 \leqslant j \leqslant k\}$, the k-graph G is (d_e, ε_k, r)-regular with respect to the k-set of clusters $\{V_{i_j} : 1 \leqslant j \leqslant k\}$, for some $d_e \geqslant d$ depending on e. Then there exists a partition-respecting copy of H in G.

The differences between Lemma 8.5 and [4, Theorem 2] are discussed in Appendix A.2.

9. Almost perfect C_s^k-tilings

The aim of this section is to prove Lemma 7.1, *i.e.* finding an almost perfect C_s^k-tiling. Throughout this section we fix $k \geqslant 3$ and $s \geqslant 5k^2$ with $s \not\equiv 0 \bmod k$. Let $G_s, W_{G_s}, a_{s,1}, \ldots, a_{s,k}, \ell, F_s$ be given by Proposition 4.5. Recall that F_s contains a spanning C_s^k. Therefore an F_s-tiling in H implies the existence of a C_s^k-tiling in H of the same size.

Here we summarize some useful inequalities that will be used throughout the section. Let $M_s = \max_i a_{s,i}$ and $m_s = \min_i a_{s,i}$. We have

$$\ell + \sum_{i=1}^{k} a_{s,i} = s, \quad M_s \leqslant m_s + 1 \quad \text{and} \quad 1 \leqslant \ell \leqslant k-1. \tag{9.1}$$

From this, we can easily deduce

$$m_s + 1 \geqslant M_s \geqslant \frac{s-\ell}{k} \geqslant \frac{s-k+1}{k}. \tag{9.2}$$

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Define $E_s = K^k(M_s)$, the complete (k, k)-graph with each part of size M_s. Given an $\{F_s, E_s\}$-tiling $\mathcal{T}$ in H, let $\mathcal{F}_\mathcal{T}$ and $\mathcal{E}_\mathcal{T}$ be the set of copies of F_s and E_s in $\mathcal{T}$, respectively. Define

$$\phi(\mathcal{T}) = \frac{1}{n}\left(n - s\left(|\mathcal{F}_\mathcal{T}| + \frac{3}{5}|\mathcal{E}_\mathcal{T}|\right)\right).$$

Note that if $\mathcal{E}_\mathcal{T} = \emptyset$, then $\mathcal{T}$ is an F_s-tiling covering all but $\phi(\mathcal{T})n$ vertices. Let $\phi(H)$ be the minimum of $\phi(\mathcal{T})$ over all $\{F_s, E_s\}$-tilings $\mathcal{T}$ in H. Given $n \geqslant k$ and $0 \leqslant \mu, \theta < 1$, let $\Phi(n, \mu, \theta)$ be the maximum of $\phi(H)$ over all (μ, θ)-dense k-graphs H on n vertices. Note that $\phi(H)$ and $\Phi(n, \mu, \theta)$ depend on k and s but they will be clear from the context.

Lemma 9.1. *Let $k \geqslant 3$ and $s \geqslant 5k^2$ with $s \not\equiv 0 \bmod k$. Let $1/n, \theta \ll \alpha, \gamma, 1/k, 1/s$. Then*

$$\Phi(n, 1/2 + 1/(2s) + \gamma, \theta) \leqslant \alpha.$$

We now show that Lemma 9.1 implies Lemma 7.1.

Proof of Lemma 7.1. Fix $\alpha, \gamma > 0$. Note that $|V(F_s)| = s$ and $|V(E_s)| = kM_s$. Let $\delta = 7/10$. Using $s \geqslant 5k^2$, (9.2) and $k \geqslant 3$, we deduce $kM_s/s \geqslant 1 - (k-1)/(5k^2) \geqslant 43/45$. Hence

$$3s/5 \leqslant 43s\delta/45 \leqslant \delta k M_s. \tag{9.3}$$

Define $\alpha_1 = \alpha(1-\delta)$ and choose some $\theta \ll \alpha, \gamma, 1/k, 1/s$. Since $1/n \ll \alpha, \gamma, 1/k, 1/s$ as well, Lemma 9.1 (with α_1 in place of α) implies that $\Phi(n, 1/2 + 1/(2s) + \gamma, \theta) \leqslant \alpha_1$.

Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2 + 1/(2s) + \gamma)n$. Then

$$\phi(H) \leqslant \Phi(n, 1/2 + 1/(2s) + \gamma, 0) \leqslant \Phi(n, 1/2 + 1/(2s) + \gamma, \theta) \leqslant \alpha_1.$$

Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling in H with $\phi(\mathcal{T}) \leqslant \alpha_1$. Hence

$$1 - \alpha_1 \leqslant 1 - \phi(\mathcal{T}) \leqslant \frac{s}{n}\left(|\mathcal{F}_\mathcal{T}| + \frac{3}{5}|\mathcal{E}_\mathcal{T}|\right) \overset{(9.3)}{\leqslant} \frac{1}{n}(s|\mathcal{F}_\mathcal{T}| + \delta k M_s|\mathcal{E}_\mathcal{T}|).$$

As $\mathcal{T}$ is a tiling, we have that $s|\mathcal{F}_\mathcal{T}| + kM_s|\mathcal{E}_\mathcal{T}| \leqslant n$. Hence $1 - \alpha_1 \leqslant (1-\delta)s|\mathcal{F}_\mathcal{T}|/n + \delta$ and so

$$s|\mathcal{F}_\mathcal{T}| \geqslant \left(1 - \frac{\alpha_1}{1-\delta}\right)n = (1-\alpha)n.$$

Therefore H contains an $\mathcal{F}_s$-tiling $\mathcal{F}_\mathcal{T}$ covering all but at most αn vertices, implying the existence of a C_s^k-tiling of the same size. □

9.1 Weighted fractional tilings

Our strategy for proving Lemma 9.1 is to apply the regular slice lemma (Theorem 8.1). In the reduced k-graph, we find a fractional $\{F_s^*, E_s^*\}$-tiling for some simpler k-graphs F_s^* and E_s^*. By using the regularity methods, this fractional tiling can then be lifted to an actual tiling with copies of F_s, E_s in the original k-graph, which covers a similar proportion of vertices.

To define the k-graphs F_s^* and E_s^*, we use the notion of G-augmentation introduced in Section 4.2. Let K be a k-edge with vertices $\{x_1, \ldots, x_k\}$. Let G_s be the 2-graph on $[k]$ given by Corollary 4.5. Let F_s^* be the G_s-augmentation of K (with respect to the vertex partition $V_i := \{x_i\}$ for all $i \in [k]$). Let $V(F_s^*) = \{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_\ell\}$, where $\ell = |E(G_s)|$. We refer to $c(F_s^*) = \{x_1, \ldots, x_k\}$ as the set of *core vertices of* F_s^* and $p(F_s^*) = \{y_1, \ldots, y_\ell\}$ as the set of *pendant vertices of* F_s^*. Define the function $\alpha\colon V(F_s^*) \to \mathbb{N}$ to be such that for $u \in V(F_s^*)$,

$$\alpha(u) = \begin{cases} a_{s,i} & \text{if } u = x_i, \\ 1 & \text{if } u \in p(F_s^*). \end{cases}$$

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Note that there is a natural k-graph homomorphism θ from F_s to F_s^* such that for all $u \in V(F_s^*)$, $|\theta^{-1}(u)| = \alpha(u)$. Observe that (9.2), $s \geqslant 5k^2$ and $k \geqslant 3$ imply that $\alpha(u) = 1$ if and only if u is a pendant vertex.

Let $\mathcal{F}_s^*(H)$ be the set of copies of F_s^* in H. Given $v \in V(H)$ and $F_s^* \in \mathcal{F}_s^*(H)$, define

$$\alpha_{F_s^*}(v) = \begin{cases} \alpha(u) & \text{if } v \text{ corresponds to vertex } u \in V(F_s^*), \\ 0 & \text{otherwise.} \end{cases}$$

Given $v \in V(H)$ and $e \in E(H)$, define

$$\alpha_e(v) = \begin{cases} M_s & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

We now define a *weighted fractional* $\{F_s^*, E_s^*\}$*-tiling of* H to be a function $\omega^* \colon \mathcal{F}_s^*(H) \cup E(H) \to [0, 1]$ such that, for all vertices $v \in V(H)$,

$$\omega^*(v) = \sum_{F_s^* \in \mathcal{F}_s^*(H)} \omega^*(F_s^*)\alpha_{F_s^*}(v) + \sum_{e \in E(H)} \omega^*(e)\alpha_e(v) \leqslant 1.$$

Note that if (contrary to our assumptions) $a_{s,1} = \cdots = a_{s,k} = 1$, then we have $\alpha_{F_s^*}(v) = \mathbf{1}\{v \in V(F_s^*)\}$ and $\alpha_e(v) = \mathbf{1}\{v \in e\}$ implying that ω^* is the standard fractional $\{F_s, E_s\}$-tiling. Note that the definition depends on k and the functions $\alpha_{F_s^*}$ and α_e, but those will always be clear from the context.

Define the *minimum weight of* ω^* to be

$$\omega^*_{\min} = \min_{\substack{J \in \mathcal{F}_s^*(H) \cup E(H) \\ v \in V(H) \\ \omega^*(J)\alpha_J(v) \neq 0}} \omega^*(J)\alpha_J(v).$$

Analogously to $\phi(\mathcal{T})$, define

$$\phi(\omega^*) = \frac{1}{n}\left(n - s\left(\sum_{F_s^* \in \mathcal{F}_s^*(H)} \omega^*(F_s^*) + \frac{3}{5}\sum_{e \in E(H)} \omega^*(e)\right)\right).$$

Given $c > 0$ and a k-graph H, let $\phi^*(H, c)$ be the minimum of $\phi(\omega^*)$ over all weighted fractional $\{F_s^*, E_s^*\}$-tilings ω^* of H with $\omega^*_{\min} \geqslant c$. Note that $\phi^*(H, c)$ also depends on k, s, $\alpha_{F_s^*}$ and α_e, which will always be clear from the context.

Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling. We say that a vertex v is *saturated under* $\mathcal{T}$ if it is covered by a copy of F_s and v corresponds to a vertex in W_{G_s} under that copy. Let $S(\mathcal{T})$ denote the set of all saturated vertices under $\mathcal{T}$. Define $U(\mathcal{T})$ as the set of all uncovered vertices under $\mathcal{T}$.

Analogously, given a weighted fractional $\{F_s^*, E_s^*\}$-tiling ω^*, we say that a vertex v is *saturated under* ω^* if

$$\sum_{\substack{F_s^* \in \mathcal{F}_s^*(H) \\ \alpha_{F_s^*}(v) = 1}} \omega^*(F_s^*)\alpha_{F_s^*}(v) = 1,$$

that is, $\omega^*(v) = 1$ and all its weight comes from copies of F_s^* such that v corresponds to a pendant vertex. Let $S(\omega^*)$ be the set of all saturated vertices under ω^*. Also, define $U(\omega^*)$ as the set of all vertices $v \in V(H)$ such that $\omega^*(v) = 0$.

Proposition 9.2. *Let $k \geqslant 3$ and $s \geqslant 5k^2$ with $s \not\equiv 0 \bmod k$. Let H be a k-graph on n vertices. Let ω^* be a weighted fractional $\{F_s^*, E_s^*\}$-tiling in H. Then the following holds.*

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(i) *We have*

$$s\sum_{F^*\in\mathcal{F}_s^*}\omega^*(F^*)+kM_s\sum_{e\in E(H)}\omega^*(e)\leqslant n.$$

In particular,

$$\sum_{F^*\in\mathcal{F}_s^*}\omega^*(F^*)\leqslant n/s \quad \text{and} \quad \sum_{e\in E(H)}\omega^*(e)\leqslant n/(kM_s).$$

(ii) $|S(\omega^*)|\leqslant \ell n/s$.

(iii) *If* $S'\subseteq S(\omega^*)$ *with* $|S'|>n/s$, *then there exists* $F^*\in\mathcal{F}_s^*(H)$ *with* $\omega^*(F^*)>0$ *such that* $|p(F^*)\cap S'|\geqslant 2$.

Proof. For (i), note that

$$\begin{aligned} n&\geqslant \sum_{v\in V(H)}\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)\alpha_{F^*}(v)+\sum_{v\in V(H)}\sum_{e\in E(H)}\omega^*(e)\alpha_e(v)\\ &=\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)\sum_{v\in V(H)}\alpha_{F^*}(v)+\sum_{e\in E(H)}\omega^*(e)\sum_{v\in V(H)}\alpha_e(v)\\ &=s\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)+kM_s\sum_{e\in E(H)}\omega^*(e).\end{aligned}$$

To prove (ii), recall that all of the vertices $v\in S(\omega^*)$ only receive weight from pendant vertices, and all copies of $F\in\mathcal{F}_s^*(H)$ have precisely ℓ pendant vertices, and therefore

$$|S(\omega^*)|=\sum_{v\in S(\omega^*)}\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)\alpha_{F^*}(v)\leqslant \ell\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)\overset{\text{(i)}}{\leqslant}\ell n/s.$$

Finally, for (iii), suppose the contrary, that for all $F^*\in\mathcal{F}_s^*(H)$ with $\omega^*(F^*)>0$ we have $\sum_{v\in S'}\alpha_{F^*}(v)=|p(F^*)\cap S'|\leqslant 1$. Then

$$\begin{aligned}|S'|&=\sum_{v\in S'}\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)\alpha_{F^*}(v)\\ &=\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)\sum_{v\in S'}\alpha_{F^*}(v)\\ &\leqslant\sum_{F^*\in\mathcal{F}_s^*(H)}\omega^*(F^*)\\ &\leqslant n/s,\end{aligned}$$

a contradiction. □

Note that F_s admits a natural perfect weighted fractional F_s^*-tiling, defined as follows. Let $a=\prod_{1\leqslant i\leqslant k}a_{s,i}$. Let F be a copy of F_s and suppose that $V(F)=V_1\cup\cdots\cup V_k\cup W$, where $V_1,\ldots,V_k$ forms a complete (k,k)-graph with $|V_i|=a_{s,i}$ for all $1\leqslant i\leqslant k$ and $|W|=\ell$. Note that $a\leqslant M_s^k$. For all $(v_1,\ldots,v_k)\in V_1\times\cdots\times V_k$, the vertices $\{v_1,\ldots,v_k\}\cup W$ span a copy of F_s^*, where we identify $\{v_1,\ldots,v_k\}$ with the core vertices of F_s^* and W with the pendant vertices of F_s^*. Define ω^* by assigning to all such copies the weight $1/a$. A similar method shows that E_s admits a perfect weighted fractional E_s^*-tiling, by setting $\omega^*(e)=M_s^{-k}$ for all $e\in E_s$.

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We can naturally extend these constructions to find a weighted fractional $\{F_s^*, E_s^*\}$-tiling given an $\{F_s, E_s\}$-tiling, by repeating the above procedure over all copies of F_s and E_s. The following proposition (whose proof we omit) collects useful properties of the obtained fractional tiling, for future reference. All of the stated properties are straightforward to check by using the construction outlined above.

Proposition 9.3. *Let $k \geqslant 3$ and $s \geqslant 5k^2$ with $s \not\equiv 0 \bmod k$. Let H be a k-graph and let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling in H. Then there exists a weighted fractional $\{F_s^*, E_s^*\}$-tiling ω^* such that*

(i) $\phi(\mathcal{T}) = \phi(\omega^*)$,
(ii) $|\mathcal{F}_{\mathcal{T}}| = \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*)$,
(iii) $|\mathcal{E}_{\mathcal{T}}| = \sum_{e \in E(H)} \omega^*(e)$,
(iv) $S(\omega^*) = S(\mathcal{T})$ *and* $U(\omega^*) = U(\mathcal{T})$,
(v) *for all* $F^* \in \mathcal{F}_s^*(H)$, $\omega^*(F^*) \in \{0, a^{-1}\}$, *where* $a = \prod_{1 \leqslant i \leqslant k} a_{s,i}$,
(vi) *for all* $e \in E(H)$, $\omega^*(e) \in \{0, M_s^{-k}\}$, *moreover if* $e \in E(E_s)$ *for some* $E_s \in \mathcal{E}_{\mathcal{T}}$, *then* $\omega^*(e) = M_s^{-k}$,
(vii) $\omega^*_{\min} \geqslant M_s^{-k}$, *and*
(viii) $\omega^*(v) \in \{0, 1\}$ *for all* $v \in V(H)$.

The next lemma ensures that if R is a reduced k-graph of H, then $\phi(H)$ is roughly bounded above by $\phi^*(R, c)$.

Lemma 9.4. *Let $k \geqslant 3$ and $s \geqslant 5k^2$ with $s \not\equiv 0 \bmod k$. Let $c \geqslant \beta > 0$ and*

$$1/n \ll \varepsilon, 1/r \ll \varepsilon_k \ll 1/t_1 \leqslant 1/t_0 \ll \beta, c, 1/s, 1/k$$

and

$$\varepsilon_k \ll d, 1/k, 1/s.$$

Let H be a k-graph on n vertices, let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for H and let $R = R_d(H)$ be its d-reduced k-graph obtained from $\mathcal{J}$. Then $\phi(H) \leqslant \phi^*(R, c) + s\beta/c$.

Proof. Let ω^* be a weighted fractional $\{F_s^*, E_s^*\}$-tiling on R such that $\phi(\omega^*) = \phi^*(R, c)$ and $\omega^*_{\min} \geqslant c$. Let $t = |V(R)|$ and let $m = n/t$, so that each cluster in $\mathcal{J}$ has size m. Let n_F^* be the number of $F_s^* \in \mathcal{F}_s^*(R)$ with $\omega^*(F_s^*) > 0$ and let n_E^* be the number of $E \in E(R)$ with $\omega^*(E) > 0$. Note that

$$n_F^* + n_E^* \leqslant t/c.$$

For all clusters $U \in V(R)$, we subdivide U into disjoint sets $\{U_J\}_{J \in \mathcal{F}_s^*(R) \cup E(R)}$ of size $|U_J| = \lfloor \omega^*(J) \alpha_J(U) m \rfloor$.

In the next claim, we show that if $\omega^*(J) > 0$ for some $J \in \mathcal{F}_s^*(R) \cup E(R)$, then we can find a large F_s-tiling or a large E_s-tiling on $\bigcup_{U \in V(J)} U_J$.

Claim 2. *For all $J \in \mathcal{F}_s^*(R) \cup E(R)$ with $\omega^*(J) > 0$, $H[\bigcup_{U \in V(J)} U_J]$ contains*

(i) *an F_s-tiling $\mathcal{F}_J$ with $|\mathcal{F}_J| \geqslant m(\omega^*(J) - \beta)$ if $J \in \mathcal{F}_s^*(R)$, or*
(ii) *an E_s-tiling $\mathcal{E}_J$ with $|\mathcal{E}_J| \geqslant m(\omega^*(J) - \beta)$ if $J \in E(R)$.*

Proof of the claim. We will only consider the case when $J \in \mathcal{F}_s^*(R)$, as the case $J \in E(R)$ is proved similarly.

Suppose $c(J) = \{X_1, \ldots, X_k\}$ and $p(J) = \{Y_1, \ldots, Y_\ell\}$, so $V(J) = c(J) \cup p(J)$. We will first show that if $X_i' \subseteq X_i$ for all $1 \leqslant i \leqslant k$ and $Y_j' \subseteq Y_j$ for all $1 \leqslant j \leqslant \ell$ are such that $|X_i'| = |Y_j'| \geqslant \beta m$, then

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$H[\bigcup_{1\leqslant i\leqslant k} X'_i \cup \bigcup_{1\leqslant j\leqslant \ell} Y'_j]$ contains a copy F of F_s such that $|V(F)\cap X'_i| = a_{s,i}$ for all $1\leqslant i\leqslant k$ and $|V(F)\cap Y'_j| = 1$ for all $1\leqslant j\leqslant \ell$.

Indeed, take X'_i, Y'_j as above and construct the subcomplex H' obtained by restricting H along with $\mathcal{J}$ to the subsets X'_i, Y'_j and then deleting the edges in H not supported in k-tuples of clusters corresponding to edges in $E(J)$. Then H' is a $(k, k+\ell)$-complex. Since $\mathcal{J}$ is (t_0, t_1, ε)-equitable, there exists a density vector $\mathbf{d} = (d_{k-1}, \dots, d_2)$ such that, for all $2\leqslant i\leqslant k-1$, we have $d_i \geqslant 1/t_1$, $1/d_i \in \mathbb{N}$ and $\mathcal{J}$ is $(d_{k-1}, \dots, d_2, \varepsilon, \varepsilon, 1)$-regular. As $J\subseteq R$, all edges e in $E(J)\cap E(R)$ induce k-tuples X_e of clusters in H with $d^*(X_e) = d_e \geqslant d$ and H is (d_e, ε_k, r)-regular with respect to X_e. By Lemma 8.1, the restriction of X_e to the subsets $\{X'_1, \dots, X'_k, Y'_1, \dots, Y'_\ell\}$ is $(d_e, \sqrt{\varepsilon_k}, \sqrt{\varepsilon}, r)$-regular. Hence, by Lemma 8.5, there exists a partition-respecting copy F of F_s in H', that is, F satisfies $|V(F)\cap X'_i| = a_{s,i}$ for all $1\leqslant i\leqslant s$ and $|V(F)\cap Y'_j| = 1$ for all $1\leqslant j\leqslant \ell$, as desired.

Now consider the largest F_s-tiling $\mathcal{F}_J$ in $H[\bigcup_{U\in V(J)} U_J]$ such that all $F\in\mathcal{F}_J$ satisfy $|V(F)\cap X_i| = a_{s,i}$ for all $1\leqslant i\leqslant k$ and $|V(F)\cap Y_j| = 1$ for all $1\leqslant j\leqslant \ell$. Let $V(\mathcal{F}_J) = \bigcup_{F\in\mathcal{F}_J} V(F)$. By the discussion above, we may assume that $|U_J \setminus V(\mathcal{F}_J)| < \beta m$ for some $U\in V(J)$. A simple calculation shows that $|(Y_j)_J \setminus V(\mathcal{F}_J)| < \beta m$ for all $1\leqslant j\leqslant \ell$ and $|(X_i)_J \setminus V(\mathcal{F}_J)| < a_{s,i}\beta m$ for all $1\leqslant i\leqslant k$. Therefore $\mathcal{F}_J$ covers at least $sm(\omega^*(J) - \beta)$ vertices and it follows that $|\mathcal{F}_J| \geqslant m(\omega^*(J) - \beta)$. □

Now consider the $\{F_s, E_s\}$-tiling $\mathcal{T} = \mathcal{F}_\mathcal{T} \cup \mathcal{E}_\mathcal{T}$ in H, where

$$\mathcal{F}_\mathcal{T} = \bigcup_{J\in\mathcal{F}_s^*(R)} \mathcal{F}_J \quad \text{and} \quad \mathcal{E}_\mathcal{T} = \bigcup_{E\in E(R)} \mathcal{E}_J$$

as given by the claim (and we take $\mathcal{F}_J = \mathcal{E}_J = \emptyset$ whenever $\omega^*(J) = 0$). Therefore

$$\begin{aligned}
|\mathcal{F}_\mathcal{T}| + \frac{3}{5}|\mathcal{E}_\mathcal{T}| &\geqslant \sum_{\substack{F_s^*\in\mathcal{F}_s^*(R)\\ \omega^*(F_s^*)>0}} m(\omega^*(F_s^*) - \beta) + \frac{3}{5}\sum_{\substack{E\in E(R)\\ \omega^*(E)>0}} m(\omega^*(E) - \beta)\\
&\geqslant m\left(\sum_{F_s^*\in\mathcal{F}_s^*(R)} \omega^*(F_s^*) + \frac{3}{5}\sum_{E\in E(R)} \omega^*(E) - \beta(n_F^* + n_E^*)\right)\\
&\geqslant m\left(\sum_{F_s^*\in\mathcal{F}_s^*(R)} \omega^*(F_s^*) + \frac{3}{5}\sum_{E\in E(R)} \omega^*(E) - \frac{\beta t}{c}\right)\\
&= mt\left(\frac{1-\phi(\omega^*)}{s} - \frac{\beta}{c}\right)\\
&= \frac{n}{s}\left(1 - \phi(\omega^*) - \frac{\beta s}{c}\right).
\end{aligned}$$

Thus we have

$$\phi(H) \leqslant \phi(\mathcal{T}) \leqslant \phi(\omega^*) + s\beta/c = \phi^*(R, c) + s\beta/c.$$ □

9.2 Proof of Lemma 9.1

We begin with some lemmas before formally proving Lemma 9.1.

Lemma 9.5. *Let $k\geqslant 3$ and $s\geqslant 5k^2$ with $s\not\equiv 0 \bmod k$. Let $\mu + \gamma/3 \leqslant 2/3$. Then*

$$\Phi(n, \mu, \theta) \leqslant \Phi((1+\gamma)n, \mu + \gamma/3, \theta) + s\gamma.$$

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Proof. Let H be a k-graph on n vertices that is (μ,θ)-dense. Consider the k-graph H' on the vertices $V(H)\cup A$ obtained from H by adding a set of $|A|=\gamma n$ vertices and adding all of the k-edges that have non-empty intersection with A. Since

$$\frac{\mu+\gamma}{1+\gamma}\geqslant \mu+\gamma/3$$

as $\mu+\gamma/3\leqslant 2/3$, H' is $(\mu+\gamma/3,\theta)$-dense.

Let $\mathcal{T}'$ be an $\{F_s,E_s\}$-tiling on H' satisfying $\phi(\mathcal{T}')=\phi(H')$. Consider the $\{F_s,E_s\}$-tiling $\mathcal{T}$ in H obtained from $\mathcal{T}'$ by removing all copies of F_s or E_s intersecting with A. It follows that

$$\begin{aligned} 1-\phi(\mathcal{T}) &= \frac{s}{n}\left(|\mathcal{F}_{\mathcal{T}}|+\frac{3}{5}|\mathcal{E}_{\mathcal{T}}|\right)\\ &\geqslant \frac{s}{n}\left(|\mathcal{F}_{\mathcal{T}'}|+\frac{3}{5}|\mathcal{E}_{\mathcal{T}'}|\right)-s\gamma\\ &\geqslant \frac{s}{(1+\gamma)n}\left(|\mathcal{F}_{\mathcal{T}'}|+\frac{3}{5}|\mathcal{E}_{\mathcal{T}'}|\right)-s\gamma\\ &= 1-\phi(\mathcal{T}')-s\gamma. \end{aligned}$$

Hence

$$\phi(H)\leqslant\phi(\mathcal{T})\leqslant\phi(\mathcal{T}')+s\gamma\leqslant\Phi((1+\gamma)n,\mu+\gamma/3,\theta)+s\gamma. \qquad\square$$

The next lemma shows that given an $\{F_s,E_s\}$-tiling $\mathcal{T}$ of a strongly (μ,θ)-dense k-graph H with $\phi(T)$ 'large', we can always find a better weighted fractional $\{F_s^*,E_s^*\}$-tiling in terms of ϕ^*.

Lemma 9.6. *Let $k\geqslant 3$, $s\geqslant 5k^2$ with $s\not\equiv 0 \bmod k$, and $c=s^{-2k}$. For all $\gamma>0$ and $0\leqslant\alpha\leqslant 1$ there exists $n_0=n_0(k,s,\gamma,\alpha)\in\mathbb{N}$ and $\nu=\nu(k,s,\gamma)>0$ and $\theta=\theta(\alpha,k)$ such that the following holds for all $n\geqslant n_0$. Let H be a k-graph on n vertices that is strongly $(1/2+1/(2s)+\gamma,\theta)$-dense and $\phi(H)\geqslant\alpha$. Then $\phi^*(H,c)\leqslant(1-\nu)\phi(H)$.*

We defer the proof of Lemma 9.6 to the next subsection and now we use it to prove Lemma 9.1.

Proof of Lemma 9.1. Consider a fixed $\gamma>0$. Suppose the result is false, that is, there exists $\alpha>0$ such that, for all $n\in\mathbb{N}$ and $\theta^*>0$, there exists $n'>n$ satisfying $\Phi(n',1/2+1/(2s)+\gamma,\theta^*)>\alpha$. Let α_0 be the supremum of all such α. Apply Lemma 9.6 (with parameters $\gamma/2,\alpha_0/2$ playing the roles of γ,α) to obtain $n_0=n_0(k,s,\gamma/2,\alpha_0/2)$, $\nu=\nu(k,s,\gamma/2)$ and $\theta=\theta(\alpha_0/2,k)$. Let

$$0<\eta\ll\nu,\gamma,\alpha_0,1/s.$$

By the definition of α_0, there exists $\theta_1>0$ and $n_1\in\mathbb{N}$ such that for all $n\geqslant n_1$,

$$\Phi(n,1/2+1/(2s)+\gamma,\theta_1)\leqslant\alpha_0+\eta/2. \tag{9.4}$$

Now we prepare the set-up to use the regular slice lemma (Theorem 8.1). Let $\beta,\varepsilon_k,\varepsilon,d,\theta^*,\theta'>0$ and $t_0,t_1,r,n_2\in\mathbb{N}$ be such that

$$\begin{aligned} &1/n_2\ll\varepsilon,\ \ 1/r\ll\varepsilon_k,\ \ 1/t_1\ll 1/t_0\ll\beta\ll\gamma'\ll\eta,\ \ c=s^{-k},\ \ 1/s,\ \ 1/k,\ \ 1/n_0,\ \ 1/n_1,\\ &\varepsilon_k\ll d\ll\gamma',\\ &\varepsilon_k\ll\theta'\ll\theta^*\ll\gamma',\ \ \theta,\theta_1 \end{aligned}$$

and $n_2\equiv 0 \bmod t_1!$.

Let H be a $(1/2+1/(2s)+\gamma,\theta')$-dense k-graph on $n\geqslant n_2$ vertices with

$$\phi(H)>\alpha_0-\eta; \tag{9.5}$$

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such an H exists by the definition of α_0. By removing at most $t_1! - 1$ vertices we get a k-graph H' on at least n_2 vertices such that $|V(H')|$ is divisible by $t_1!$ and H' is $(1/2 + 1/(2s) + \gamma - \gamma', 2\theta')$-dense.

Let $\mathcal{S}$ be the set of $(k-1)$-tuples T of vertices of $V(H')$ such that

$$\deg_{H'}(T) < (1/2 + 1/(2s) + \gamma - \gamma')(|V(H')| - k + 1).$$

Thus $|\mathcal{S}| \leqslant 2\theta' \binom{|V(H')|}{k-1}$. By Theorem 8.1, there exists a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice $\mathcal{J}$ for H' such that, for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have $\overline{\deg}(Y; R(H')) = \overline{\deg}(\mathcal{J}_Y; H') \pm \varepsilon_k$, and furthermore, $\mathcal{J}$ is $(3\sqrt{2\theta'}, \mathcal{S})$-avoiding.

Let $R = R_d(H')$ be the d-reduced k-graph obtained from H' and $\mathcal{J}$. Using $\theta', d, \varepsilon_k \ll \gamma'$ together with $\varepsilon_k \ll \theta'$ and that $\mathcal{J}$ is $(3\sqrt{2\theta'}, \mathcal{S})$-avoiding, we can invoke Lemma 8.3 to get that R is $(1/2 + 1/(2s) + \gamma - 2\gamma', 5\sqrt{\theta'})$-dense. By Lemma 8.4, there exists a subgraph $R' \subseteq R$ on the same vertex set that is strongly $(1/2 + 1/(2s) + \gamma - 3\gamma', \theta^*)$-dense as $\theta' \ll \gamma', 1/k, \theta^*$. Since the vertices of R' are the clusters of $\mathcal{J}$, we have $|V(R')| \geqslant t_0 \geqslant n_1$. By the fact that $\theta^* \leqslant \theta_1$, Lemma 9.5 (with $9\gamma'$ playing the role of γ) and (9.4), we deduce that

$$\begin{aligned}\phi(R') &\leqslant \Phi(|V(R')|, 1/2 + 1/(2s) + \gamma - 3\gamma', \theta^*)\\ &\leqslant \Phi((1+9\gamma')|V(R')|, 1/2 + 1/(2s) + \gamma, \theta^*) + 9\gamma' s\\ &\leqslant \alpha_0 + \eta/2 + 9\gamma' s\\ &\leqslant \alpha_0 + \eta.\end{aligned}$$

We further claim that $\phi^*(R', c) \leqslant \alpha_0 - 2\eta$. Note that $c = s^{-k}$ and $\alpha_0 \geqslant 4\eta$. Therefore, if $\phi(R') < \alpha_0/2$, then the claim holds by Proposition 9.3. Thus we may assume that $\phi(R') \geqslant \alpha_0/2$. Note that $|V(R')| \geqslant t_0 \geqslant n_0$, $\gamma - 3\gamma' \geqslant \gamma/2$ and $\theta^* \leqslant \theta$. By the choice of n_0, ν and θ (given by Lemma 9.6), we have

$$\phi^*(R', c) \leqslant (1-\nu)\phi(R') \leqslant (1-\nu)(\alpha_0 + \eta) \leqslant \alpha_0 - 2\eta,$$

where the last inequality holds since $\eta \ll \nu, \alpha_0$. Finally, recall that $\beta \ll \eta, c$, so an application of Lemma 9.4 implies that

$$\phi(H) \leqslant \phi^*(R, c) + s\beta/c \leqslant \phi^*(R', c) + s\beta/c \leqslant \alpha_0 - \eta,$$

contradicting (9.5). □

9.3 Proof of Lemma 9.6

Before proceeding with the full details of the proof of Lemma 9.6, we first give a rough outline of the proof. Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling of H satisfying $\phi(\mathcal{T}) = \phi(H)$. By Proposition 9.3, we obtain a weighted fractional $\{F_s^*, E_s^*\}$-tiling ω_0^* with $\phi(\omega_0^*) = \phi(\mathcal{T})$, $U(\omega_0^*) = U(\mathcal{T})$ and $(\omega_0^*)_{\min} \geqslant M_s^{-k}$. Our aim is to sequentially define weighted fractional $\{F_s^*, E_s^*\}$-tilings $\omega_1^*, \omega_2^*, \ldots, \omega_t^*$ such that $\phi(\omega_{j-1}^*) - \phi(\omega_j^*) \geqslant \nu_1/n$ for all $j \in [t]$, where ν_1 is a fixed positive constant. We will follow this procedure for $t = \Omega(n)$ steps, and we will show that ω_t^* satisfies the required properties.

Moreover, we will construct ω_{j+1}^* based on ω_j^* by changing the weights of $\mathcal{F}_s(H)$ and $E(H)$ on a small number of vertices, such that no vertex has its weight changed more than once during the whole procedure. Recall that $U(\mathcal{T})$ is the set of uncovered vertices. If $|U(\mathcal{T})|$ is large then we construct ω_{j+1}^* from ω_j^* via assigning weights to edges that contain at least $k-1$ vertices in $U(\mathcal{T})$. Suppose that $|U(\mathcal{T})|$ is small. Since $\phi(\mathcal{T}) \geqslant \alpha$, not all of the weight of ω_0^* can be contributed by copies of F_s^*. Thus there must exist edges $e \in E(H)$ with positive weight under ω_0^*. We use this to find $e \in E(H)$ with $\omega_j^*(e) > 0$. The crucial property is that a copy of F_s^* can be obtained from an edge by adding a few extra vertices to it. We use this to obtain ω_{j+1}^* from ω_j^* by reducing the

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weight on e before assigning weight to some copy of F_s^* which originates from e. More care is needed to ensure that ω_{j+1}^* is indeed a weighted fractional $\{F_s^*, E_s^*\}$-tiling. Ideally we would like the extra vertices added to e to form a copy of F_s^* not to be saturated, if possible.

We summarize and recall the relevant properties of F_s^*, which was originally as defined at the beginning of Section 9.1. There exists a 2-graph G_s on $[k]$ with $\ell \leqslant k-1$ edges which consists of a disjoint union of paths. Suppose $e_1, \dots, e_\ell$ is an enumeration of the edges of G_s and $e_i = j_i j_i'$ for all $1 \leqslant i \leqslant s$. If $X = \{x_1, \dots, x_k\}$, then we may describe F_s^* as having vertices $V(F_s^*) = X \cup \{y_1, \dots, y_\ell\}$, and the edges of F_s^* are X together with $(X \setminus \{x_{j_i}\}) \cup \{y_i\}$ and $(X \setminus \{x_{j_i'}\}) \cup \{y_i\}$ for all $1 \leqslant i \leqslant \ell$. We call $c(F_s^*) = X$ and $p(F_s^*) = \{y_1, \dots, y_\ell\}$ the core and pendant vertices of F_s^*, respectively.

The following two lemmas are needed for the case when $U(\mathcal{T})$ is small. The idea is as follows. Suppose H is a k-graph on n vertices with $\delta_{k-1}(H) \geqslant (1/2 + 1/(2s) + \gamma)n$. If X is a k-edge in H, we would like to extend it to a copy F of F_s^* such that $c(F) = X$. Lemma 9.8 will indicate where we should look for the vertices of $p(F)$.

Lemma 9.7. *Let $k \geqslant 3$, $s \geqslant 2k^2$ and $\ell \leqslant k-1$. Suppose that $N_i \subseteq [n]$ are such that $|N_i| \geqslant (1/2 + 1/(2s) + \gamma)n$ for all $1 \leqslant i \leqslant k$. Let G be a 2-graph on $\{N_1, \dots, N_k\}$ such that $N_i N_j \in E(G)$ if and only if $|N_i \cap N_j| \leqslant (\ell/s + \gamma)n$. Then G is bipartite.*

Proof. We will show that G does not have any cycle of odd length. It suffices to show that $N_{i_1} N_{i_{2j+1}} \notin E(G)$ for all paths $N_{i_1} \cdots N_{i_{2j+1}}$ in G on an odd number of vertices.

For any $S \subseteq [n]$, write $\overline{S} := [n] \setminus S$. First, note that if N_i is adjacent to N_j in G, then

$$|N_i \setminus \overline{N_j}| = |N_i \cap N_j| \leqslant (\ell/s + \gamma)n \quad \text{and} \quad |\overline{N_j} \setminus N_i| \leqslant (n - |N_j|) - (|N_i| - |N_i \cap N_j|) \leqslant (\ell/s - \gamma)n.$$

Hence, if $N_i N_j N_k$ is a path on three vertices in G, then

$$|N_i \setminus N_k| \leqslant |N_i \setminus \overline{N_j}| + |\overline{N_j} \setminus N_k| \leqslant 2\ell n/s.$$

Now consider a path in G on an odd number of vertices. Without loss of generality (after a suitable relabelling), we assume the path is given by $N_1 N_2 \cdots N_{2j+1}$ for some j which necessarily satisfies $2j + 1 \leqslant k$. By using the previous bounds repeatedly, we obtain

$$|N_1 \setminus N_{2j+1}| \leqslant |N_1 \setminus N_3| + |N_3 \setminus N_5| + \cdots + |N_{2j-1} \setminus N_{2j+1}| \leqslant \frac{2\ell j n}{s} \leqslant \frac{\ell(k-1)n}{s}.$$

Since $\ell \leqslant k-1$ and $s > 2k^2$, we obtain

$$|N_1 \cap N_{2j+1}| \geqslant |N_1| - \frac{\ell(k-1)n}{s} \geqslant \left(\frac{1}{2} + \frac{1}{2s} + \gamma\right)n - \frac{(k-1)^2 n}{s} > \left(\frac{\ell}{s} + \gamma\right)n.$$

Hence $N_1 N_{2j+1} \notin E(G)$ as desired. □

Lemma 9.8. *Let $k \geqslant 3$ and $s \geqslant 5k^2$ with $s \not\equiv 0 \bmod k$. Let $1/n \ll \gamma, 1/k$ and $\theta > 0$. Let H be a strongly $(1/2 + 1/(2s) + \gamma, \theta)$-dense k-graph on n vertices. Let $X = \{x_1, \dots, x_k\}$ be an edge of H and let $N_i = N_H(X \setminus \{x_i\})$ for all $1 \leqslant i \leqslant k$. Let $S \subseteq V(H)$ with $|S| \leqslant (\ell/s + \gamma/3)n$ and $y_0 \in N_1 \cap N_2$. Suppose either $|N_1 \cap N_2| < (\ell/s + 2\gamma/3)n$ or $|N_i \cap N_j| \geqslant (\ell/s + 2\gamma/3)n$ for all $1 \leqslant i, j \leqslant k$. Then there exists a copy F^* of F_s^* such that $c(F^*) = X$ and $p(F^*) \cap (S \setminus \{y_0\}) = \emptyset$.*

Proof. Note that

$$|N_i| \geqslant (1/2 + 1/(2s) + \gamma)(n - k + 1) \geqslant (1/2 + 1/(2s) + 2\gamma/3)n \quad \text{for all } 1 \leqslant i \leqslant k.$$

Let G be the 2-graph on $[k]$ such that $ij \in E(G)$ if and only if $|N_i \cap N_j| < (\ell/s + 2\gamma/3)n$. Note that if $ij \notin E(G)$, then $|N_i \cap N_j| \geqslant (\ell/s + 2\gamma/3)n \geqslant |S| + \ell$.

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Recall that G_s, the 2-graph which defines F^*_s, is a disjoint union of paths. By our assumption, either $12 \in E(G)$ or G is empty. By Lemma 9.7, G is bipartite. Thus, in either case, there exists a bijection $\phi\colon V(G_s) \to [k]$ such that $\{\phi(j_i)\phi(j'_i)\colon j_ij'_i \in E(G_s)\} \cap E(G) \subseteq \{12\}$.

Let $e_1, \ldots, e_\ell$ be an enumeration of the edges of $E(G_s)$. Consider $e_i = j_ij'_i \in E(G_s)$. If $\{\phi(j_i), \phi(j'_i)\} = \{1, 2\}$, then let $y_i = y_0$. Otherwise, $\phi(j_i)\phi(j'_i) \notin E(G)$ and therefore $|N_{\phi(j_i)} \cap N_{\phi(j'_i)}| \geqslant |S| + \ell$. Thus we can greedily pick $y_i \in (N_{\phi(j_i)} \cap N_{\phi(j'_i)}) \setminus S$ such that $y_1, \ldots, y_\ell$ are pairwise distinct. Then there exists a copy F^* of F^*_s with $c(F^*) = X$ and $p(F^*) = \{y_1, \ldots, y_\ell\}$, which satisfies the required properties. □

Now we are ready to prove Lemma 9.6.

Proof of Lemma 9.6. We may assume that $\gamma \ll \alpha, 1/k, 1/s$. Recall that our aim is to define a sequence of fractional $\{F^*_s, E^*_s\}$-tilings $\omega^*_0, \ldots, \omega^*_t$ for some $t \geqslant 0$. Let

$$\nu_1 = \frac{s}{25kM_s^k}, \quad \nu_2 = \frac{\gamma}{40k^3s^k} \quad \text{and} \quad \nu = \frac{\nu_1\nu_2}{2}.$$

Choose $\theta \ll \alpha, 1/k$ and $1/n_0 \ll \alpha, \gamma, 1/k, 1/s$. Let H be a strongly $(1/2 + 1/(2s) + \gamma, \theta)$-dense k-graph on $n \geqslant n_0$ vertices with $\phi(H) \geqslant \alpha$. Choose $t = \lfloor \nu_2\phi(H)n \rfloor$.

Recall that G_s, ℓ, F_s, m_s, M_s are given by Proposition 4.5 and they satisfy (9.1) and (9.2). Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling on H with $\phi(\mathcal{T}) = \phi(H)$. Apply Proposition 9.3 and obtain a weighted fractional $\{F^*_s, E^*_s\}$-tiling w^*_0 satisfying all the properties of the proposition.

Given that ω^*_j has been defined for some $0 \leqslant j \leqslant t$, define

$$A_j = \{v \in V(H)\colon \forall J \in \mathcal{F}^*_s(H) \cup E(H),\ \omega^*_j(J)\alpha_J(v) = \omega^*_0(J)\alpha_J(v)\}.$$

So A_j is the set of vertices such that ω^*_j is 'identical to w^*_0'. Note that by Proposition 9.3(viii), for all $v \in A_j$,

$$\omega^*_j(v) = \omega^*_0(v) \in \{0, 1\}. \tag{9.6}$$

Clearly we have $A_0 = V(H)$. Let

$$\mathcal{T}_0^+ = \{J \in \mathcal{F}^*_s(H) \cup E(H)\colon \omega^*_0(J) > 0\}.$$

The set A_j will indicate where we should look for graphs $J \in \mathcal{T}_0^+$ whose weight on ω^*_j is known (by knowing the weight on $J \in \omega^*_0$), and we will modify those to define the subsequent weighting ω^*_{j+1}.

By Proposition 9.3 and (9.1), we have that for all $J \in \mathcal{T}_0^+$, if $V(J) \cap A_j \neq \emptyset$, then $\omega^*_j(J) = \omega^*_0(J)$ and therefore

$$\omega^*_j(J) - \frac{1}{M_s^{-k}} \begin{cases} = 0 & \text{if } J \in E(H) \text{ or } m_s = M_s, \\ \geqslant c & \text{otherwise.} \end{cases} \tag{9.7}$$

Now we turn to the task of making the construction of $\omega^*_1, \ldots, \omega^*_t$ explicit.

Claim 3. *There is a sequence of weighted fractional $\{F^*_s, E^*_s\}$-tilings $\omega^*_1, \ldots, \omega^*_t$ such that for all $1 \leqslant j \leqslant t$,*

- (i) *$A_j \subseteq A_{j-1}$ and $|A_j| \geqslant |A_{j-1}| - 5k^2$,*
- (ii) *$(\omega^*_j)_{\min} \geqslant c$, and*
- (iii) *$\phi(\omega^*_j) \leqslant \phi(\omega^*_{j-1}) - \nu_1/n$.*

Note that Lemma 9.6 follows immediately from Claim 3 as

$$\phi(\omega^*_t) \leqslant \phi(H) - \nu_1 t/n \leqslant (1 - \nu)\phi(H).$$

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Proof of Claim 3. Suppose that for some $0 \leqslant j < t$ we have already defined $\omega_0^*, \omega_1^*, \ldots, \omega_j^*$ satisfying (i)–(iii). We write $U_i = U(\omega_i^*)$, for each $0 \leqslant i \leqslant j$. Observe that $U_0 = U(\mathcal{T})$ by the choice of ω_0^* and Proposition 9.3(iv). Note that (i) implies that

$$|A_j| \geqslant |A_0| - 5k^2 j \geqslant n - 5k^2 \nu_2 \phi(H) n \geqslant (1 - \alpha\gamma/40)n,$$

and therefore

$$|V \setminus A_j| = n - |A_j| \leqslant \frac{\alpha\gamma}{40} n. \tag{9.8}$$

Now our task is to construct ω_{j+1}^*. We will use the following shorthand notation. For all $J \in \mathcal{F}_s^*(H) \cup E(H)$, if we have already specified the values of ω_{j+1}^*, then let

$$\partial(J) = \omega_{j+1}^*(J) - \omega_j^*(J).$$

The proof splits into two cases depending on the size of U_0.

Case 1: $|U_0| \geqslant 3\alpha n/4$. Note that $(U_0 \setminus U_j) \cap A_j = \emptyset$, which implies that $A_j \cap U_0 \subseteq A_j \cap U_j$. By (9.8),

$$|A_j \cap U_j| \geqslant |A_j \cap U_0| \geqslant |U_0| - \alpha\gamma n/40 \geqslant 3\alpha n/4 - \alpha\gamma n/40 \geqslant \alpha n/2.$$

Together with $1/n \ll \alpha$, we get

$$\binom{|U_j \cap A_j|}{k-1} \geqslant \binom{\alpha n/2}{k-1} \geqslant \frac{\alpha^{k-1}}{2^k}\binom{n}{k-1} \geqslant \theta\binom{n}{k-1} + k^2\binom{n}{k-2}$$

as θ, $1/n \ll \alpha$, $1/k$. Since H is strongly $(1/2 + 1/(2s) + \gamma, \theta)$-dense, we can (greedily) find k disjoint $(k-1)$-sets $W_1, \ldots, W_k$ of $U_j \cap A_j$ such that

$$\deg(W_i) \geqslant (1/2 + 1/(2s) + \gamma)(n - k + 1) \quad \text{for all } 1 \leqslant i \leqslant k.$$

Define $N_i = N(W_i) \cap A_j$. Then

$$|N_i| \geqslant \left(\frac{1}{2} + \frac{1}{2s} + \gamma\right)(n - k + 1) - (n - |A_j|) \overset{(9.8)}{\geqslant} \left(\frac{1}{2} + \frac{1}{2s} + \frac{\gamma}{2}\right)n. \tag{9.9}$$

Suppose that for some $1 \leqslant i \leqslant k$, there exists $x \in N_i \cap U_j$. Then $e = \{x\} \cup W_i \in E(H)$, so we can define $\omega_{j+1}^*(e) = 1$ and $\omega_{j+1}^*(J) = \omega_j^*(J)$ for all $J \in (\mathcal{F}_s^*(H) \cup E(H)) \setminus \{e\}$. In this case

$$|A_{j+1}| = |A_j| - k \geqslant |A_j| - 5k^2,$$
$$(\omega_{j+1}^*)_{\min} = (\omega_j^*)_{\min} \geqslant c$$
$$\text{and} \quad \phi(\omega_{j+1}^*) = \phi(\omega_j^*) - 3s/(5n) \leqslant \phi(\omega_j^*) - \nu_1/n,$$

so we are done. Thus we may assume that

$$\bigcup_{1 \leqslant i \leqslant k} N_i \subseteq A_j \setminus U_j. \tag{9.10}$$

For all $F^* \in \mathcal{F}_s^*(H)$ and $e \in E(H)$, define

$$d_{F^*} = \sum_{i=1}^{k} |N_i \cap c(F^*)| \quad \text{and} \quad d_e = \sum_{i=1}^{k} |N_i \cap e|.$$

Case 1.1: there exists $F^* \in \mathcal{F}_s^*(H)$ with $\omega_j^*(F^*) > 0$ and $d_{F^*} \geqslant k + 1$. There exist distinct $i, i' \in \{1, \ldots, k\}$ and distinct $x \in N_i \cap c(F^*)$, $x' \in N_{i'} \cap c(F^*)$ such that both $e_1 = W_i \cup \{x\}$ and $e_2 = W_{i'} \cup$

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$\{x'\}$ are edges in H. Note that since $x \in A_j$, by (9.7) we have $\omega_j^*(F^*) = \omega_0^*(F^*) \geqslant M_s^{-k}$. Also, since $x, x' \in c(F^*)$, $\alpha_{F^*}(x), \alpha_{F^*}(x') \geqslant m_s$. Define ω_{j+1}^* to be such that

$$\partial(J) = \begin{cases} m_s M_s^{-(k+1)} & \text{if } J \in \{e_1, e_2\}, \\ -M_s^{-k} & \text{if } J = F^*, \\ 0 & \text{otherwise.} \end{cases}$$

Then ω_{j+1}^* is a weighted fractional $\{F_s^*, E_s^*\}$-tiling. First, note that

$$|A_{j+1}| = |A_j| - (3k + \ell - 2) \geqslant |A_j| - 5k^2.$$

Secondly, using (9.7) we have that $\omega_j^*(F^*)$ is either 0 or at least c. Thus we obtain

$$(\omega_{j+1}^*)_{\min} \geqslant \min\{(\omega_j^*)_{\min}, M_s\omega_{j+1}^*(e_1), c\} \geqslant \min\{c, m_s M_s^{-k}, c\} \geqslant c.$$

Finally,

$$\phi(\omega_j^*) - \phi(\omega_{j+1}^*) = \frac{s}{n}\left(\partial(F^*) + \frac{3}{5}(\partial(e_1) + \partial(e_2))\right) = \frac{s}{nM_s^k}\left(\frac{6m_s}{5M_s} - 1\right).$$

Using (9.2), $s \geqslant 5k^2$, $\ell \leqslant k-1$ and $k \geqslant 3$, we can lower-bound m_s/M_s by

$$\frac{m_s}{M_s} \geqslant \frac{M_s - 1}{M_s} \geqslant \frac{s - \ell - k}{s - \ell} = 1 - \frac{k}{s - \ell} \geqslant 1 - \frac{k}{5k^2 - k + 1} \geqslant \frac{40}{43}.$$

We deduce $\phi(\omega_j^*) - \phi(\omega_{j+1}^*) \geqslant 5s/(43M_s^k n) \geqslant \nu_1/n$, so we are done in this subcase.

Case 1.2: there exists $e \in E(H)$ with $\omega_j^*(e) > 0$ and $d_e \geqslant k+1$. We prove this case using an argument similar to that used in Case 1.1. There exist distinct $i, i' \in \{1, \ldots, k\}$ and distinct $x, x' \in e$ such that both $e_1 = W_i \cup \{x\}$ and $e_2 = W_{i'} \cup \{x'\}$ are edges in H. Since $x \in A_j$, Proposition 9.3(vi) and (9.7) implies that $\omega_j^*(e) = M_s^{-k}$. Define ω_{j+1}^* to be such that

$$\partial(J) = \begin{cases} -M_s^{-k} & \text{if } J = e, \\ M_s^{-k} & \text{if } J \in \{e_1, e_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then ω_{j+1}^* is a weighted fractional $\{F_s^*, E_s^*\}$-tiling with $|A_{j+1}| = |A_j| - (3k-2) \geqslant |A_j| - 5k^2$. Note that $\omega_{j+1}^*(e) = 0$ and $\omega_{j+1}^*(e_i) > \omega_j^*(e_i)$ for $i \in [2]$, so we have $(\omega_{j+1}^*)_{\min} \geqslant (\omega_j^*)_{\min} \geqslant c$. Note that

$$\phi\left(\omega_j^*\right) - \phi\left(\omega_{j+1}^*\right) = \frac{3s}{5n}(\partial(e_1) + \partial(e_2) + \partial(e)) = \frac{3s}{5M_s^k n} \geqslant \frac{\nu_1}{n},$$

so this finishes the proof of this subcase.

Case 1.3: both Case 1.1 and Case 1.2 do not hold. Thus $d_J \leqslant k$ for all $J \in \mathcal{F}_s^*(H) \cup E(H)$ with $\omega_j^*(J) > 0$. Recall that $\alpha_{F^*}(v) \leqslant M_s$ if $v \in c(F^*)$ and $\alpha_{F^*}(v) = 1$ if $v \in p(F^*)$. Thus, for all $F^* \in \mathcal{F}_s^*(H)$ with $\omega_j^*(F^*) > 0$, we have

$$\begin{aligned}\sum_{i=1}^{k}\sum_{x \in N_i} \alpha_{F^*}(x) &\leqslant \sum_{i=1}^{k} (M_s|N_i \cap c(F^*)| + |N_i \cap p(F^*)|) \\ &= M_s d_{F^*} + \sum_{i=1}^{k} |N_i \cap p(F^*)|\end{aligned}$$

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$$\begin{aligned}&\leqslant k(M_s+\ell)\\&\leqslant s+k^2.\end{aligned}$$

Therefore

$$\sum_{F^*\in\mathcal{F}_s^*}\sum_{i=1}^{k}\sum_{x\in N_i}\omega_0^*(F^*)\alpha_{F^*}(x)\leqslant(s+k^2)\sum_{F^*\in\mathcal{F}_s^*(H)}w_0^*(F^*). \tag{9.11}$$

Similarly, for $e\in E(H)$ with $\omega_j^*(e)>0$, we obtain

$$\sum_{i=1}^{k}\sum_{x\in N_i}\alpha_e(x)=\sum_{i=1}^{k}M_s|e\cap N_i|=M_sd_e\leqslant kM_s.$$

Hence

$$\sum_{e\in E(H)}\sum_{i=1}^{k}\sum_{x\in N_i}\omega_0^*(e)\alpha_e(x)\leqslant kM_s\sum_{e\in E(H)}w_0^*(e). \tag{9.12}$$

Combining everything, we deduce

$$\begin{aligned}\sum_{i=1}^{k}|N_i|&=\sum_{i=1}^{k}\sum_{x\in N_i}1\\
&\overset{(9.10),(9.6)}{=}\sum_{i=1}^{k}\sum_{x\in N_i}\omega_0^*(x)\\
&=\sum_{i=1}^{k}\sum_{x\in N_i}\sum_{J\in\mathcal{F}_s^*(H)\cup E(H)}\omega_0^*(J)\alpha_J(x)\\
&=\sum_{J\in\mathcal{F}_s^*(H)\cup E(H)}\sum_{i=1}^{k}\sum_{x\in N_i}\omega_0^*(J)\alpha_J(x)\\
&\overset{(9.11),(9.12)}{\leqslant}(s+k^2)\sum_{F^*\in\mathcal{F}_s^*(H)}w_0^*(F^*)+kM_s\sum_{e\in E(H)}w_0^*(e)\\
&\overset{\text{Prop. 9.2(i)}}{\leqslant}n+k^2\sum_{F^*\in\mathcal{F}_s^*(H)}w_0^*(F^*)\\
&\leqslant n+\frac{k^2}{s}n\\
&\leqslant\frac{6n}{5},\end{aligned}$$

where the last inequality uses $s\geqslant 5k^2$. This contradicts (9.9) and finishes the proof of Case 1.

Case 2: $|U_0|<3\alpha n/4$. Write $\mathcal{F}$, $\mathcal{E}$ for $\mathcal{F}_{\mathcal{T}}$, $\mathcal{E}_{\mathcal{T}}$, respectively. Note that $n=s|\mathcal{F}|+kM_s|\mathcal{E}|+|U_0|$. Hence

$$\alpha\leqslant\phi(\mathcal{T})\leqslant 1-\frac{s}{n}|\mathcal{F}|\leqslant\frac{1}{n}(kM_s|\mathcal{E}|+|U_0|)\leqslant\frac{kM_s|\mathcal{E}|}{n}+\frac{3\alpha}{4}.$$

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Using that $s \geqslant 5k^2$, that $k \geqslant 3$, that $1/n \ll \alpha, \gamma \leqslant 1$ and (9.8), we have

$$|\mathcal{E}| \geqslant \frac{\alpha n}{4kM_s} \geqslant \frac{\alpha\gamma n}{40} + 1 \geqslant n - |A_j| + 1.$$

Hence there exists $E_s \in \mathcal{E}$ with $V(E_s) \subseteq A_j$. By Proposition 9.3(vi), there exists an edge $X = \{x_1, \ldots, x_k\} \in E(H)$ such that $X \subseteq A_j$ and

$$w_j^*(X) = w_0^*(X) = M_s^{-k}.$$

We would like to use Lemma 9.8 to find copies F of F_s^* with $c(F) = X$, and decrease the weight of X to be able to increase the weight of an appropriate copy of F_s^*. Recall that $S(\omega_j^*)$ is the set of saturated vertices with respect to ω_j^*. We write $S_j = S(\omega_j^*)$ and let $S' = S_j \cup (V(H) \setminus A_j)$. Proposition 9.2(ii) and (9.8) together imply that $|S'| \leqslant (\ell/s + \gamma/40)n$.

For all $1 \leqslant i \leqslant k$, let $N_i = N_H(X \setminus \{x_i\})$. We may assume (by relabelling) that either $|N_1 \cap N_2| < (\ell/s + 2\gamma/3)n$ or $|N_i \cap N_j| \geqslant (\ell/s + 2\gamma/3)n$ for all $1 \leqslant i, j \leqslant k$.

Case 2.1: $(N_1 \cap N_2) \setminus S' \neq \emptyset$. In this case, select $y \in (N_1 \cap N_2) \setminus S'$ and apply Lemma 9.8 with S', y playing the roles of S, y_0. We obtain a copy F_1 of F_s^* such that $c(F_1) = X$ and $p(F_1) \cap S' = \emptyset$. Then $p(F_1) \subseteq A_j \setminus S_j$. Let $P_0 = p(F_1) \setminus U_j$. For $p \in p(F_1) \cap U_j$, by (9.6), $\omega_j^*(p) = 0$. For every $p \in P_0$, by the definitions of A_j and U_j, there exists $J_p \in \mathcal{T}_0^+$ such that $p \in V(J_p)$, and since $p \notin S_j$ we can also choose J_p such that $\alpha_{J_p}(p) \geqslant m_s$. (The J_p might coincide for different $p \in P_0$.) Define ω_{j+1}^* to be such that

$$\partial(J) = \begin{cases} M_s^{-k} & \text{if } J = F_1, \\ -M_s^{-k} & \text{if } J = X, \\ -M_s^{-k}/m_s & \text{if } J = J_p \text{ for some } p \in P_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then ω_{j+1}^* is a weighted fractional $\{F_s^*, E_s^*\}$-tiling. First, note that

$$|A_{j+1}| \geqslant |A_j| - (|V(F_1)| + \sum_{p \in P_0} |V(J_p)|) \geqslant |A_j| - (2k + 2k^2) \geqslant |A_j| - 5k^2.$$

Second, (9.7) implies that $\omega_{j+1}^*(X) = 0$ and $\omega_{j+1}^*(F_1) \geqslant c$, and moreover, for all $p \in P_0$,

$$\omega_{j+1}^*(J_p) \geqslant M_s^{-k}(1 - 1/m_s) \geqslant M_s^{-k-1} \geqslant c.$$

Thus $(\omega_{j+1}^*)_{\min} \geqslant c$. Finally, since $|P_0| \leqslant |p(F_1)| = \ell$, we have

$$\begin{aligned} \phi(\omega_j^*) - \phi(\omega_{j+1}^*) &\geqslant \frac{s}{n}\left(\partial(F_1) + \frac{3}{5}\partial(X) + \sum_{p \in P_0} \partial(J_p)\right) \\ &\geqslant \frac{s}{nM_s^k}\left(\frac{2}{5} - \frac{|P_0|}{m_s}\right) \\ &\geqslant \frac{s}{nM_s^k}\left(\frac{2}{5} - \frac{\ell}{m_s}\right). \end{aligned}$$

By (9.2), $\ell \leqslant k - 1$ and $s \geqslant 5k^2$, we get

$$\frac{\ell}{m_s} \leqslant \frac{k-1}{M_s - 1} \leqslant \frac{k}{M_s} \leqslant \frac{k^2}{s - \ell} \leqslant \frac{k^2}{5k^2 - k + 1} \leqslant \frac{1}{4},$$

where the last inequality holds for every $k \geqslant 3$. Thus

$$\phi\left(\omega_j^*\right) - \phi\left(\omega_{j+1}^*\right) \geqslant 3s/(20nM_s^k) \geqslant \nu_1/n$$

and we are done.

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Case 2.2: $N_1 \cap N_2 \subseteq S'$. Since H is strongly $(1/2+1/(2s)+\gamma,\theta)$-dense and $1/n \ll \gamma, 1/k$, we deduce $|N_1 \cap N_2| \geqslant (1/s+\gamma)n$. Using $N_1 \cap N_2 \subseteq S'$ and (9.8), we have

$$|N_1 \cap N_2 \cap S_j \cap A_j| \geqslant (1/s+\gamma/2)n.$$

By Proposition 9.2(iii), there exists $F_2 \in \mathcal{F}_s^*(H) \cap \mathcal{T}_0^+$ and $|p(F_2) \cap N_1 \cap N_2 \cap S_j \cap A_j| \geqslant 2$. Let y_1', y_1'' be two distinct vertices in $p(F_2) \cap N_1 \cap N_2 \cap S_j \cap A_j$. We claim that

$$\begin{aligned}&\text{there exists } F_2' \in \mathcal{F}_s^*(H) \text{ such that } p(F_2') \setminus p(F_2) \subseteq A_j \setminus (S_j \cup X),\\ &\text{their core vertices satisfy } c(F_2') = c(F_2), \text{ and } \{y_1', y_1''\} \setminus p(F_2') \neq \emptyset.\end{aligned} \tag{9.13}$$

To see where we are heading, if we have found such F_2', then our aim will be to define ω_{j+1}^* by decreasing the weight of F_2 and X, which will then allow us to increase the weight of F_2' and a copy F_1' of F_s^* such that $c(F_1') = X$ and $\{y_1', y_1''\} \cap p(F_1') \neq \emptyset$.

Let us check that (9.13) holds. Let $Z = c(F_2) = \{z_1, \dots, z_k\}$ and for every $1 \leqslant i \leqslant k$ let $Z_i = N_H(Z \setminus \{z_i\})$. Since $y_1' \in p(F_2)$, without loss of generality (by relabelling) we may assume that $y_1' \in Z_1 \cap Z_2$. Suppose first that $(Z_1 \cap Z_2) \setminus (S' \cup X \cup V(F_2))$ is non-empty. Select any $y_1''' \in (Z_1 \cap Z_2) \setminus (S' \cup X \cup V(F_2))$. Thus there exists $F_2' \in \mathcal{F}_s^*(H)$ such that

$$\begin{aligned} c(F_2') &= Z,\\ p(F_2') &= (p(F_2) \setminus \{y_1'\}) \cup \{y_1'''\},\\ p(F_2') \setminus p(F_2) &= \{y_1'''\} \subseteq A_j \setminus (S_j \cup X)\\ \text{and}\quad y_1' &\in \{y_1', y_1''\} \setminus p(F_2'),\end{aligned}$$

as desired. Hence we may assume $Z_1 \cap Z_2 \subseteq S' \cup X \cup V(F_2)$. This implies that

$$|Z_1 \cap Z_2| \leqslant |S' \cup X \cup V(F_2)| \leqslant (\ell/s+\gamma/40)n + |X| + |V(F_2)| < (\ell/s + 2\gamma/3)n.$$

Apply Lemma 9.8 (with $Z, Z_i, S' \cup X \cup V(F_2), y_1'$ playing the roles of X, N_i, S and y_0, respectively) to obtain $F_2' \in \mathcal{F}_s^*$ such that $c(F_2') = Z$ and $p(F_2') \cap (S' \cup X \cup V(F_2) \setminus \{y_1'\}) = \emptyset$. It is easily checked that F_2' satisfies (9.13).

Now take such an F_2' and assume (after relabelling, if necessary) that $y_1' \notin p(F_2')$. Apply Lemma 9.8 (with $X, N_i, S' \cup V(F_2'), y_1'$ playing the roles of X, N_i, S and y_0, respectively) to obtain F_1' such that $c(F_1') = X$ and $p(F_1') \cap (S' \setminus \{y_1'\}) = \emptyset$.

Let

$$P' = (p(F_1') \setminus \{y_1'\}) \cup (p(F_2') \setminus p(F_2))$$

and observe that $P' \subseteq A_j \setminus S_j$. Let $P_0' = P' \setminus U_j$. Arguing as in the previous case, we see that for every $p \in P' \cap U_j$, $\omega_j^*(p) = 0$, and for every $p \in P_0'$ there exists $J_p \in \mathcal{T}_0^+$ such that $p \in V(J_p)$ and $\alpha_{J_p}(p) \geqslant m_s$.

Let ω_{j+1}^* be such that

$$\partial(J) = \begin{cases} M_s^{-k} & \text{if } J = F_1',\\ M_s^{-(k+1)} m_s & \text{if } J = F_2',\\ -M_s^{-k} & \text{if } J \in \{X, F_2\},\\ -M_s^{-k}/m_s & \text{if } J = J_p \text{ for some } p \in P_0',\\ 0 & \text{otherwise.}\end{cases}$$

Since $p(F_1') \cup p(F_2') \subseteq P' \cup p(F_2)$, the decrease of weight in F_2 and the J_p implies that the vertices in $p(F_1') \cup p(F_2')$ get weight at most 1 under ω_{j+1}^*. Using this, it is not difficult to check that ω_{j+1}^* is indeed a weighted fractional $\{F_s^*, E_s^*\}$-tiling.

Note that

$$A_j \setminus A_{j+1} \subseteq V(F_1') \cup V(F_2) \cup V(F_2') \cup \bigcup_{p\in P_0'} V(J_p) \quad \text{and} \quad |P_0'| \leqslant |p(F_1')| + |p(F_2')| = 2\ell.$$

Using that $\ell \leqslant k-1$, we deduce

$$|A_{j+1}| \geqslant |A_j| - 3(k+\ell) - |P_0'|(k+\ell) \geqslant |A_j| - (3+2\ell)(k+\ell) \geqslant |A_j| - 5k^2.$$

Similarly to the previous case, we deduce from (9.7) that $(\omega^*_{j+1})_{\min} \geqslant c$.

Using that $|P_0'| \leqslant 2\ell$, we deduce

$$\begin{aligned}\phi(\omega^*_j) - \phi(\omega^*_{j+1}) &\geqslant \frac{s}{n}\left(\partial(F_1') + \partial(F_2') + \partial(F_2) + \frac{3}{5}\partial(X) + \sum_{p\in P_0'} \partial(J_p)\right)\\ &= \frac{s}{nM_s^k}\left(1 + \frac{m_s}{M_s} - 1 - \frac{3}{5} - \frac{|P_0'|}{m_s}\right)\\ &\geqslant \frac{s}{nM_s^k}\left(\frac{m_s}{M_s} - \frac{3}{5} - \frac{2\ell}{m_s}\right).\end{aligned}$$

From (9.2), $s \geqslant 5k^2$ and $\ell \leqslant k-1$, we deduce

$$\begin{aligned}\frac{m_s}{M_s} - \frac{3}{5} - \frac{2\ell}{m_s} &\geqslant \frac{2}{5} - \frac{1}{M_s} - \frac{2\ell}{m_s}\\ &\geqslant \frac{2}{5} - \frac{1+2\ell}{m_s}\\ &\geqslant \frac{2}{5} - \frac{k(1+2\ell)}{s-\ell-k}\\ &\geqslant \frac{2}{5} - \frac{2k^2-k}{5k^2-2k+1}\\ &= \frac{k+2}{25k^2-10k+5}\\ &\geqslant \frac{k+2}{25k^2}\\ &\geqslant \frac{1}{25k}.\end{aligned}$$

Thus we get

$$\phi(\omega^*_j) - \phi(\omega^*_{j+1}) \geqslant s/(25M_s^k kn) \geqslant \nu_1/n,$$

and we are done. This finishes the proof of Case 2.2 and of Claim 3. □

This concludes the proof of Lemma 9.6. □

10. Remarks and further directions

The following family of examples gives lower bounds for the Turán problems of tight cycles on a number of vertices not divisible by k (and hence for the tiling and covering problem as well). We acknowledge and thank a referee for suggesting this construction. We are not aware of its appearance in the literature before, although it bears some resemblance to examples considered by Mycroft to give lower bounds for tiling problems [24, Section 2].

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Construction 10.1. Let $k \geqslant 2$ and $r > 1$ be a divisor of k. For $n > 0$, we define the k-graph $H^k_{n,r}$ as follows. Given a vertex set V of size n, partition it into r disjoint vertex sets $V_1, \ldots, V_p$ of sizes as equal as possible. Assume that every $x \in V_i$ is labelled with i, for all $1 \leqslant i \leqslant r$. Let $H^k_{n,r}$ be the k-graph on V where the edges are the k-sets such that the sum of the labels of its vertices is congruent to 1 modulo r.

Using this construction, we deduce the following lower bounds for $\mathrm{ex}_{k-1}(n, C^k_s)$ when s is not divisible by k (and therefore also for $c(n, C^k_s)$).

Proposition 10.1. *Let $s > k \geqslant 2$ with s not divisible by k. Let r be a divisor of k which does not divide s. Then $\mathrm{ex}_{k-1}(n, C^k_s) \geqslant \lfloor n/r \rfloor - k + 2$. In particular, $\mathrm{ex}_{k-1}(n, C^k_s) \geqslant \lfloor n/k \rfloor - k + 2$.*

Proof. Given k, r, n, let $H = H^k_{n,r}$ be the k-graph given by Construction 10.1. Since the sets V_i are chosen to have size as equal as possible, we deduce that $|V_i| \geqslant \lfloor n/r \rfloor$ holds for all $1 \leqslant i \leqslant r$. It is easy to check that no edge of H is entirely contained in any set V_i, and that for every $(k-1)$-set S in V, $N(S) = V_j \setminus S$ for some j. Thus $\delta_{k-1}(H) \geqslant \lfloor n/r \rfloor - k + 2$.

We show that H is C^k_s-free. Let C be a tight cycle on t vertices in H. It is enough to show that r divides t (since r does not divide s, it will follow that $t \neq s$). Recall from Construction 10.1 that every $x \in V_i$ is labelled with i. We double-count the sum T of the labels of vertices, over all the edges of C. On one hand, $T \equiv 0 \bmod k$ since each vertex appears in exactly k edges of C and is thus counted k times. Since r divides k, $T \equiv 0 \bmod r$. On the other hand, the sum of the labels of a single edge is congruent to 1 modulo r and there are t of them, thus $T \equiv t \bmod r$. This implies that r divides t. □

Now we discuss covering thresholds. Let $s > k \geqslant 3$. Theorem 1.3 and Proposition 1.2 imply that $c(n, C^k_s) = (1/2 + o(1))n$ for all admissible pairs (k, s) with $s \geqslant 2k^2$. A natural open question is to determine $c(n, C^k_s)$ for the non-admissible pairs (k, s). The smallest case not covered by our constructions is when $(k, s) = (6, 8)$, and Proposition 10.1 implies that $c(n, C^6_8) \geqslant \mathrm{ex}_5(n, C^6_8) \geqslant \lfloor n/3 \rfloor - 4$.

Question 10.2. Is the lower bound for $c(n, C^k_s)$ given by Proposition 10.1 asymptotically tight, for non-admissible pairs (k, s)? In particular, is $c(n, C^6_8) = (1/3 + o(1))n$?

Now we consider the Turán thresholds. Theorem 1.3 and Proposition 1.2 also show that $\mathrm{ex}_{k-1}(n, C^k_s) = (1/2 + o(1))n$ for k even, $s \geqslant 2k^2$ and (k, s) is an admissible pair. We would like to know the asymptotic value of $\mathrm{ex}_{k-1}(n, C^k_s)$ in the cases not covered by our constructions. Proposition 10.1 implies that $\mathrm{ex}_{k-1}(n, C^k_s) \geqslant \lfloor n/k \rfloor - k + 2$ for s not divisible by k; but on the other hand, if $s \equiv 0 \bmod k$ then $\mathrm{ex}_{k-1}(n, C^k_s) = o(n)$, which follows easily from Theorem 1.2.

The simplest open case is when $k = 3$ and s is not divisible by 3. Note that $C^3_4 = K^3_4$, and the lower bound $\mathrm{ex}_2(n, C^3_4) \geqslant (1/2 + o(1))n$ holds in this case [7]. We conjecture that in the case $k = 3$, for $s > 4$ and not divisible by three, the lower bound given by Proposition 10.1 describes the correct asymptotic behaviour of $\mathrm{ex}_{k-1}(n, C^k_s)$.

Conjecture 10.3. $\mathrm{ex}_2(n, C^3_s) = (1/3 + o(1))n$ *for every $s > 4$ with $s \not\equiv 0 \bmod 3$.*

Finally, we discuss tiling thresholds. Let (k, s) be an admissible pair such that $s \geqslant 5k^2$. If k is even, then Theorem 1.4 and Proposition 1.3 imply that $t(n, C^k_s) = (1/2 + 1/(2s) + o(1))n$. We conjecture that for k odd, the bound given by Proposition 1.3 is asymptotically tight.

Conjecture 10.4. *Let (k, s) be an admissible pair such that $k \geqslant 3$ is odd and $s \geqslant 5k^2$. Then $t(n, C^k_s) = (1/2 + k/(4s(k-1) + 2k) + o(1))n$.*

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Note that, for k odd, the conjectured extremal example given by Proposition 1.3 is an example of the so-called space barrier construction. However, it is different from the common construction which is obtained by attaching a new vertex set W to an F-free k-graph and adding all possible edges incident with W. On the other hand, for k even, it is indeed the common construction of a space barrier.

It would also be interesting to find bounds on the Turán, covering and tiling thresholds that hold whenever $k < s \leqslant 5k^2$. The known thresholds for these kind of k-graphs do not necessarily follow the pattern of the bounds we have found for longer cycles. For example, note that C^k_{k+1} is a complete k-graph on $k+1$ vertices, which suggests that for lower values of s the problem behaves in a different way. Concretely, when $(k, s) = (3, 4)$, it is known that $t(n, C^3_4) = (3/4 + o(1))n$ [17, 23].

Question 10.5. Given $k \geqslant 3$, what is the minimum s such that $t(n, C^k_s) \leqslant (1/2 + 1/(2s) + o(1))n$ holds?

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Appendix A. Hypergraph regularity

In Section 8 we stated modified versions of some regularity statements which follow from easy modifications of the original statements or proofs. In this appendix we sketch how to guarantee that these properties hold.

A.1 Avoiding fixed $(k-1)$-graphs

Our version of the regular slice lemma (Theorem 8.1) includes an additional property (that of 'avoiding' a fixed $(k-1)$-graph $\mathcal{S}$ on the same vertex set as G) which is not present in the original statement [2, Lemma 10]. We claim that extra property follows already from their proof by doing one simple extra step.

Their proof of the regular slice lemma can be summarized as follows (we refer the reader to [2] for the precise definitions). First they obtain an 'equitable family of partitions' $\mathcal{P}^*$ from (a strengthened version of) the hypergraph regularity lemma. This can be used to find suitable complexes in the following way: first, for each pair of clusters of $\mathcal{P}^*$, select a 2-cell uniformly at random; then, for each triple of clusters of $\mathcal{P}^*$, select a 3-cell uniformly at random which is supported on the corresponding previously selected 2-cells; and so on, until we select $(k-1)$-cells. This will always output a (t_0, t_1, ε)-equitable $(k-1)$-complex $\mathcal{J}$, and the task is to check that, with positive probability, $\mathcal{J}$ is actually a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice satisfying the 'desired properties' with respect to the reduced k-graph.

Having selected $\mathcal{J}$ at random as before, the most technical part of the proof is to show that the 'desired properties' of the reduced k-graph (labelled (a), (b) and (c) in [2, Lemma 10]) hold with probability tending to 1 whenever n goes to infinity. Thankfully that part of the proof does not require any modification for our purposes. Moreover, the selected $\mathcal{J}$ will be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice with probability at least $1/2$. This is shown by upper-bounding the expected number of k-sets of clusters of $\mathcal{J}$ for which G is not (ε_k, r)-regular, and an application of Markov's inequality (see [2, pp. 65–66]). It is a natural adaptation of this method that will show that $\mathcal{J}$ is also $(3\theta^{1/2}, \mathcal{S})$-avoiding with probability at least $2/3$.

Let $\mathcal{S}$ be a $(k-1)$-graph on $V(G)$ of size at most $\theta\binom{n}{k-1}$. We only need to consider the edges of $\mathcal{S}$ which are $\mathcal{P}$-partite. Every $\mathcal{P}$-partite edge of $\mathcal{S}$ is supported in exactly one $(k-1)$-cell of the family of partitions $\mathcal{P}^*$, which by [2, Claim 32] is present in $\mathcal{J}$ with probability

$$p = \prod_{i=2}^{k-1} d_i^{\binom{k-1}{j}}.$$

Thus the expected size of $|E(\mathcal{S}) \cap E(\mathcal{J}_{k-1})|$ is at most $|E(\mathcal{S})|p \leqslant \theta p\binom{n}{k-1}$. By Markov's inequality, with probability at least $2/3$ we have

$$|E(H) \cap E(\mathcal{J}_{k-1})| \leqslant 3\theta p\binom{n}{k-1}.$$

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By the above discussion, with positive probability $\mathcal{J}$ satisfies all of the properties of [2, Lemma 10] and also that

$$|E(\mathcal{S}) \cap E(\mathcal{J}_{k-1})| \leqslant 3\theta p \binom{n}{k-1}.$$

Thus we may assume $\mathcal{J}$ satisfies all of the previous properties simultaneously, and it is only necessary to check that $\mathcal{J}$ is $(3\theta^{1/2}, \mathcal{S})$-avoiding.

Let t be the number of clusters of $\mathcal{P}$ and m the size of a cluster in $\mathcal{P}$. For each $(k-1)$-set of clusters Y, $\mathcal{J}_Y$ has $(1 \pm \varepsilon_k/10)pm^{k-1}$ edges (see [2, Fact 7]). We say a $(k-1)$-set of clusters Y is *bad* if $|\mathcal{J}_Y \cap E(\mathcal{S})| > \sqrt{6\theta}|\mathcal{J}_Y|$ and let $\mathcal{Y}$ be the set of bad $(k-1)$-sets. Then

$$3\theta p \binom{n}{k-1} \geqslant \sum_Y |\mathcal{J}_Y \cap E(\mathcal{S})| \geqslant |\mathcal{Y}|\sqrt{6\theta}(1-\varepsilon_k/10)pm^{k-1},$$

which implies $|\mathcal{Y}| \leqslant 3\theta^{1/2}\binom{t}{k-1}$. It follows that $\mathcal{J}$ is $(3\theta^{1/2}, \mathcal{S})$-avoiding, as desired.

A.2 Embedding lemma

Note that [4, Theorem 2] is stronger than Lemma 8.5 in the sense that it allows embeddings of k-graphs with bounded maximum degree whose number of vertices is linear in m, but we do not require that property here.

The main technical difference between Lemma 8.5 and Theorem 2 in [4] is that their lemma asks for the stronger condition that for all $e \in E(H)$ intersecting the vertex classes $\{X_{i_j} : 1 \leqslant j \leqslant k\}$, the k-graph G should be (d, ε_k, r)-regular with respect to the k-set of clusters $\{V_{i_j} : 1 \leqslant j \leqslant k\}$, such that the value d does not depend on e, and $1/d \in \mathbb{N}$, whereas we allow G to be (d_e, ε_k, r)-regular for some $d_e \geqslant d$ depending on e and not necessarily satisfying $1/d_e \in \mathbb{N}$. By the discussion after Lemma 4.6 in [21], we can reduce to that case by working with a sub-k-complex of $\mathcal{J} \cup G$ which is $(d, d_{k-1}, d_{k-2}, \ldots, d_2, \varepsilon_k, \varepsilon, r)$-regular, whose existence is guaranteed by an application of the 'slicing lemma' [4, Lemma 8].

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