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ON DEL PEZZO FIBRATIONS IN POSITIVE CHARACTERISTIC

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Abstract We establish two results on three-dimensional del Pezzo fibrations in positive characteristic. First, we give an explicit bound for torsion index of relatively torsion line bundles. Second, we show the existence of purely inseparable sections with explicit bounded degree. To prove these results, we study log del Pezzo surfaces defined over imperfect fields.

Keywords: minimal model program; generic fibres; del Pezzo surfaces; positive characteristic

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Contents

1. Introduction

The minimal model conjecture predicts that an arbitrary algebraic variety is birational to either a minimal model or a Mori fibre space $\pi: V \to B$. A distinguished property of Mori fibre spaces in characteristic zero is that any relative numerically trivial line bundle is automatically trivial (cf. [18, Lemma 3.2.5]). In [41, Theorem 1.4], the second author constructs counterexamples to the same statement in positive characteristic. More specifically, if the characteristic is two or three, then there exists a Mori fibre space $\pi: V \to B$ and a line bundle L on V such that $\dim V = 3$, $\dim B = 1$, $L \equiv_\pi 0$, and $L \not\sim_\pi 0$. Then it is tempting to ask how bad the torsion indices can be. One of the main results of this paper is to give such an explicit upper bound of torsion indices for three-dimensional del Pezzo fibrations.

Theorem 1.1 (Theorem 8.2). *Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi: V \to B$ be a projective k-morphism such that $\pi_*\mathcal{O}_V = \mathcal{O}_B$, where V is a three-dimensional $\mathbb{Q}$-factorial normal quasi-projective variety over k and B is a smooth curve over k. Assume that there exists an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is klt and $\pi: V \to B$ is a $(K_V + \Delta)$-Mori fibre space. Let L be a π-numerically trivial Cartier divisor on V. Then the following hold:*

(1) *If $p \geqslant 7$, then $L \sim_\pi 0$.*
(2) *If $p \in \{3, 5\}$, then $p^2 L \sim_\pi 0$.*
(3) *If $p = 2$, then $16L \sim_\pi 0$.*

We also prove a theorem of Graber–Harris–Starr type for del Pezzo fibrations in positive characteristic.

Theorem 1.2 (Theorem 8.1). *Let k be an algebraically closed field of characteristic $p > 0$.*

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Let $\pi : V \to B$ be a projective k-morphism such that $\pi_* \mathcal{O}_V = \mathcal{O}_B$, V is a normal three-dimensional variety over k, and B is a smooth curve over k. Assume that there exists an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is klt and $-(K_V + \Delta)$ is π-nef and π-big. Then the following hold:

(1) *There exists a curve C on V such that $C \to B$ is surjective and the following properties hold:*
   (a) *If $p \geqslant 7$, then $C \to B$ is an isomorphism.*
   (b) *If $p \in \{3, 5\}$, then $K(C)/K(B)$ is a purely inseparable extension of degree $\leqslant p$.*
   (c) *If $p = 2$, then $K(C)/K(B)$ is a purely inseparable extension of degree $\leqslant 4$.*

(2) *If B is a rational curve, then V is rationally chain connected.*

Theorem 1.2 can be considered as a generalisation of classical Tsen's theorem, i.e. the existence of sections on ruled surfaces. Tsen's theorem was used to establish the log minimal model program in characteristic $p > 5$ [4, § 3.4]. Also, Tsen's theorem was used to show that $H^i(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for threefolds X of Fano type in characteristic $p > 5$ when $i > 0$ (cf. [15, Theorem 1.3]).

The proofs of Theorems 1.1 and 1.2 are carried out by studying the generic fibre $X := V \times_B \operatorname{Spec} K(B)$ of π, which is a surface of del Pezzo type defined over an imperfect field. Roughly speaking, Theorems 1.1 and 1.2 hold by the following two theorems.

Theorem 1.3 (Theorem 4.10). *Let k be a field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type. Let L be a numerically trivial Cartier divisor on X. Then the following hold:*

(1) *If $p \geqslant 7$, then $L \sim 0$.*
(2) *If $p \in \{3, 5\}$, then $pL \sim 0$.*
(3) *If $p = 2$, then $4L \sim 0$.*

Theorem 1.4 (Theorem 6.12). *Let k be a C_1-field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then*

(1) *If $p \geqslant 7$, then $X(k) \neq \emptyset$;*
(2) *If $p \in \{3, 5\}$, then $X(k^{1/p}) \neq \emptyset$;*
(3) *If $p = 2$, then $X(k^{1/4}) \neq \emptyset$.*

1.1. Sketch of the proof of Theorem 1.3

Let us overview some of the ideas used in the proof of Theorem 1.3. By considering the minimal resolution and running a minimal model program, the problem is reduced to the case when X is a regular surface of del Pezzo type which has a K_X-Mori fibre space structure $X \to B$. In particular, it holds that $\dim B = 0$ or $\dim B = 1$.

1.1.1. The case when $\dim B = 0$. Assume that $\dim B = 0$. In this case, X is a regular del Pezzo surface. We first classify $Y := (X \times_k \overline{k})_{\mathrm{red}}^N$ (Theorem 1.5). We then compare $X \times_k \overline{k}$ with $Y = (X \times_k \overline{k})_{\mathrm{red}}^N$ (Theorem 1.6).

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Theorem 1.5 (Theorem 3.3). *Let k be a field of characteristic $p > 0$. Let X be a projective normal surface over k with canonical singularities such that $k = H^0(X, \mathcal{O}_X)$ and $-K_X$ is ample. Then the normalisation Y of $(X \times_k \overline{k})_{\mathrm{red}}$ satisfies one of the following properties:*

(1) *$X \times_k \overline{k}$ is normal. Moreover, $X \times_k \overline{k}$ has at worst canonical singularities. In particular, $Y \simeq X \times_k \overline{k}$ and $-K_Y$ is ample.*
(2) *Y is isomorphic to a Hirzebruch surface, i.e. a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.*
(3) *Y is isomorphic to a weighted projective surface $\mathbb{P}(1, 1, m)$ for some positive integer m.*

Theorem 1.6 (cf. Theorem 3.7). *Let k be a field of characteristic $p > 0$. Let X be a projective normal surface over k with canonical singularities such that $k = H^0(X, \mathcal{O}_X)$ and $-K_X$ is ample. Let Y be the normalisation of $(X \times_k \overline{k})_{\mathrm{red}}$ and let*

$$\mu : Y \to X \times_k \overline{k}$$

be the induced morphism.

(1) *If $p \geqslant 5$, then μ is an isomorphism and Y has at worst canonical singularities.*
(2) *If $p = 3$, then the absolute Frobenius morphism $F_{X\times_k\overline{k}}$ of $X \times_k \overline{k}$ factors through μ:*
$$F_{X\times_k\overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$
(3) *If $p = 2$, then the second iterated absolute Frobenius morphism $F^2_{X\times_k\overline{k}}$ of $X \times_k \overline{k}$ factors through μ:*
$$F^2_{X\times_k\overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$

Note that Theorem 1.5 shows that $Y = (X \times_k \overline{k})^N_{\mathrm{red}}$ is a rational surface. In particular, any numerically trivial line bundle on Y is trivial. By Theorem 1.6, if L' denotes the pullback of L to $X \times_k \overline{k}$, then it holds that $L'^4 \simeq \mathcal{O}_{X\times_k\overline{k}}$ in the case (3). Then the flat base change theorem implies that also L^4 is trivial.

We now discuss the proofs of Theorems 1.5 and 1.6. Roughly speaking, we apply Reid's idea [32, cf. the proof of Theorem 1.1] to prove Theorem 1.5 by combining with a rationality criterion (Lemma 3.2). As for Theorem 1.6, we use the notion of Frobenius length of geometric normality $\ell_F(X/k)$ introduced in [42] (cf. Definition 3.4, Remark 3.5). Roughly speaking, if $p = 2$, then we can prove that $\ell_F(X/k) \leqslant 2$ by computing certain intersection numbers (cf. the proof of Proposition 3.6). Then a general result on $\ell_F(X/k)$ (Remark 3.5) implies (3) of Theorem 1.6.

1.1.2. The case when $\dim B = 1$. Assume that $\dim B = 1$, i.e. $\pi : X \to B$ is a K_X-Mori fibre space to a curve B. Since X is of del Pezzo type, we have that the extremal ray R of $\overline{\mathrm{NE}}(X)$ that is not corresponding to $\pi : X \to B$ is spanned by an integral curve Γ, i.e. $R = \mathbb{R}_{\geqslant 0}[\Gamma]$. In particular, $\Gamma \to B$ is a finite surjective morphism of curves. If $K_X \cdot \Gamma < 0$, then the problem is reduced to the above case (1.1.1) by contracting Γ. Even if $K_X \cdot \Gamma = 0$, then we may contract Γ and apply the same strategy. Hence, it is enough to treat the case when $K_X \cdot \Gamma > 0$. Note that the numerically trivial Cartier

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divisor L on X descends to B, i.e. we have $L \sim \pi^* L_B$ for some Cartier divisor L_B on B. Then, a key observation is that the extension degree $[K(\Gamma) : K(B)]$ is at most five (Proposition 4.7). For example, if $p > 5$, then $\Gamma \to B$ is separable. Then the Hurwitz formula implies that $-K_B$ is ample; hence, $L_B \sim 0$. If $K(\Gamma)/K(B)$ is purely inseparable of degree p^e, then it holds that $L_B^{p^e} \sim 0$ since $-K_{\Gamma^N}$ is ample. For the remaining case, i.e. $p = 2$, $[K(\Gamma) : K(B)] = 4$, and $K(\Gamma)/K(B)$ is inseparable but not purely inseparable, we prove that $H^0(B, L_B^4) \neq 0$ by applying Galois descent for the separable closure of $K(\Gamma)/K(B)$ (cf. the proof of Proposition 4.9).

1.2. Sketch of the proof of Theorem 1.4

Let us overview some of the ideas used in the proof of Theorem 1.4. The first step is the same as § 1.1, i.e. considering the minimal resolution and running a minimal model program, we reduce the problem to the case when X is a regular surface of del Pezzo type which has a K_X-Mori fibre space structure $X \to B$.

1.2.1. The case when $\dim B = 0$. Assume that $\dim B = 0$. In this case, X is a regular del Pezzo surface with $\rho(X) = 1$. Since the p-degree of a C_1-field is at most one (Lemma 6.1), it follows from [11, Theorem 14.1] that X is geometrically normal. Then Theorem 1.5 implies that the base change $X \times_k \overline{k}$ is a canonical del Pezzo surface, i.e. $X \times_k \overline{k}$ has at worst canonical singularities and $-K_{X \times_k \overline{k}}$ is ample. In particular, we have that $1 \leqslant K_X^2 \leqslant 9$. Note that if X is smooth, then it is known that X has a k-rational point (cf. [19, Theorem IV.6.8]). Following the same strategy as in [19, Theorem IV.6.8], we can show that $X(k) \neq \emptyset$ if $K_X^2 \leqslant 4$ (Lemma 6.3). For the remaining cases $5 \leqslant K_X^2 \leqslant 9$, we use results established in [34], which restrict the possibilities for the type of singularities on $X \times_k \overline{k}$. For instance, if $p \geqslant 11$, then [34, Theorem 6.1] shows that the singularities on $X \times_k \overline{k}$ are of type $A_{p^e - 1}$. However, such singularities cannot appear because the minimal resolution V of $X \times_k \overline{k}$ satisfies $\rho(V) \leqslant 9$. Hence, X is actually smooth if $p \geqslant 11$ (Proposition 5.2). For the remaining cases $p \leqslant 7$, we study the possibilities one by one so that we are able to deduce what we desire. For more details, see § 6.1.

1.2.2. The case when $\dim B = 1$. Assume that $\dim B = 1$, i.e. $\pi : X \to B$ is a K_X-Mori fibre space to a curve B. Then the outline is similar to the one in (1.1.2). Let us use the same notation as in (1.1.2). The typical case is that $-K_B$ is ample. In this case, B has a rational point. Then also the fibre of π over a rational point, which is a conic curve, has a rational point. Although we need to overcome some technical difficulties, we may apply this strategy up to suitable purely inseparable covers for almost all the cases (cf. the proof of Proposition 6.10). There is one case where we cannot apply this strategy: $p = 2$, $K_X \cdot \Gamma > 0$, and $K(\Gamma)/K(B)$ is inseparable and not purely inseparable. In this case, we can prove that $-K_B$ is actually ample (Proposition 6.9).

1.3. Large characteristic

Using the techniques developed in this paper, we also prove the following theorem, which shows that some a priori possible pathologies of log del Pezzo surfaces over imperfect fields can appear exclusively in small characteristic.

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Theorem 1.7 (cf. Corollary 5.5 and Theorem 5.7). *Let k be a field of characteristic $p \geqslant 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then X is geometrically integral over k and $H^i(X, \mathcal{O}_X) = 0$ for any $i > 0$.*

As a consequence, we deduce the following result on del Pezzo fibrations in large characteristic.

Corollary 1.8. *Let k be an algebraically closed field of characteristic $p \geqslant 7$. Let $\pi : V \to B$ be a projective k-morphism of normal k-varieties such that $\pi_*\mathcal{O}_V = \mathcal{O}_B$ and $\dim V - \dim B = 2$. Assume that there exists an effective $\mathbb{Q}$-divisor Δ on V such that (V, Δ) is klt and $-(K_V + \Delta)$ is π-nef and π-big. Then general fibres of π are integral schemes and there is a non-empty open subset B' of B such that the equation $(R^i\pi_*\mathcal{O}_V)|_{B'} = 0$ holds for any $i > 0$.*

The authors do not know whether surfaces of del Pezzo type are geometrically normal if the characteristic is sufficiently large. On the other hand, even if p is sufficiently large, regular surfaces of del Pezzo type can be non-smooth. More specifically, for an arbitrary imperfect field k of characteristic $p > 0$, we construct a regular surface of del Pezzo type which is not smooth (Proposition 7.2).

1.4. Related results

In this subsection, we summarise known results on log del Pezzo surfaces mainly over imperfect fields.

1.4.1. Vanishing theorems. We first summarise results over algebraically closed fields of characteristic $p > 0$. It is well known that smooth rational surfaces satisfy the Kodaira vanishing theorem (cf. [28, Proposition 3.2]). However, the Kawamata–Viehweg vanishing theorem fails even for smooth rational surfaces (cf. [5, Theorem 3.1]). Moreover, the surface used in [5, Theorem 3.1] is a weak del Pezzo surface if the base field is of characteristic two [5, Lemma 2.4]. Also in characteristic three, there exists a surface of del Pezzo type which violates the Kawamata–Viehweg vanishing [3, Theorem 1.1]. On the other hand, if the characteristic is sufficiently large, it is known that surfaces of del Pezzo type satisfy the Kawamata–Viehweg vanishing by [7, Theorem 1.2].

We now overview known results over imperfect fields. If the characteristic is two or three, there exists a surface X of del Pezzo type such that $H^1(X, \mathcal{O}_X) \neq 0$ (cf. § 7.1). On the other hand, regular del Pezzo surfaces of characteristic $p \geqslant 5$ satisfy the Kawamata–Viehweg vanishing theorem as shown in [8, Theorem 1.1].

1.4.2. Geometric properties. In characteristics two and three, there exist regular del Pezzo surfaces which are not geometrically reduced (cf. § 7.1). On the other hand, Patakfalvi and Waldron prove that regular del Pezzo surfaces are geometrically normal if the base field is of characteristic $p \geqslant 5$ (cf. [30, Theorem 1.5]). Furthermore, Fanelli and Schröer show that a regular del Pezzo surface X is geometrically normal in every characteristic p if $[k : k^p] \leqslant p$ and $\rho(X) = 1$ (cf. [11, Theorem 14.1]).

2. Preliminaries

2.1. Notation

In this subsection, we summarise notations that we will use in this paper.

(1) We will freely use the notation and terminology in [16] and [20].

(2) We say that a noetherian scheme X is *excellent* (respectively *regular*) if the local ring $\mathcal{O}_{X,x}$ at any point $x \in X$ is excellent (respectively regular). For the definition of excellent local rings, we refer to [27, § 32].

(3) For a scheme X, its *reduced structure* X_{red} is the reduced closed subscheme of X such that the induced morphism $X_{\text{red}} \to X$ is surjective.

(4) For an integral scheme X, we define the *function field* $K(X)$ of X to be $\mathcal{O}_{X,\xi}$ for the generic point ξ of X.

(5) For a field k, we say that X is a *variety over* k or a k-*variety* if X is an integral scheme that is separated and of finite type over k. We say that X is a *curve* over k or a k-*curve* (respectively a *surface* over k or a k-*surface*, respectively a *threefold* over k) if X is a k-variety of dimension one (respectively two, respectively three).

(6) For a field k, we denote by $\overline{k}$ (respectively k^{sep}) an algebraic closure (respectively a separable closure) of k. If k is of characteristic $p > 0$, then we set $k^{1/p^\infty} := \bigcup_{e=0}^{\infty} k^{1/p^e} = \bigcup_{e=0}^{\infty}\{x \in \overline{k} \mid x^{p^e} \in k\}$.

(7) For an $\mathbb{F}_p$-scheme X, we denote by $F_X \colon X \to X$ the *absolute Frobenius morphism*. For a positive integer e, we denote by $F_X^e \colon X \to X$ the eth iterated absolute Frobenius morphism.

(8) If k is a field of characteristic $p > 0$ such that $[k : k^p] < \infty$, we define its p-*degree* $\text{p-deg}(k)$ as the non-negative integer n such that $[k : k^p] = p^n$. The p-degree $\text{p-deg}(k)$ is also called the degree of imperfection in some literature.

(9) If $k \subset k'$ is a field extension and X is a k-scheme, we denote $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ by $X \times_k k'$ or $X_{k'}$.

(10) Let k be a field, let X be a scheme over k and let $k \subset k'$ be a field extension. We denote by $X(k')$ the set of the k-morphisms $\operatorname{Hom}_k(\operatorname{Spec} k', X)$. Note that if X is a scheme of finite type over k and $k \subset k'$ is a purely inseparable extension, then the induced map $\theta : X(k') \to X$ is injective and its image $\theta(X(k'))$ consists of closed points of X.

(11) Let L be a Cartier divisor on a variety X over k. We define the *base locus* $\text{Bs}(L)$ of L by

$$\text{Bs}(L) := \bigcap_{s \in H^0(X,L)} \{x \in X \mid s(x) = 0\}.$$

In particular, $\text{Bs}(L)$ is a closed subset of X.

(12) Let k be an algebraically closed field. For a normal surface X over k and a canonical singularity $x \in X$ (i.e. a rational double point), we refer to the table at [1, pages 15–17] for the list of equations of types A_n, D_n^m, and E_n^m. For example, we say that x is a canonical singularity of type A_n if the henselisation of $\mathcal{O}_{X,x}$ is

isomorphic to $k\{x, y, z\}/(z^{n+1} + xy)$, where $k\{x, y, z\}$ denotes the henselisation of the local ring of $k[x, y, z]$ at the maximal ideal (x, y, z).

2.2. Geometrically klt singularities

The purpose of this subsection is to introduce the notion of geometrically klt singularities and its variants.

Definition 2.1. Let (X, Δ) be a log pair over a field k such that k is algebraically closed in $K(X)$. We say that (X, Δ) is *geometrically klt* (respectively terminal, canonical, lc) if $(X \times_k \overline{k}, \Delta \times_k \overline{k})$ is klt (respectively terminal, canonical, lc).

Lemma 2.2. *Let k be a field. Let X and Y be varieties over k which are birational to each other. Then X is geometrically reduced over k if and only if Y is geometrically reduced over k.*

Proof. Recall that for a k-scheme, being geometrically reduced is equivalent to being S_1 and geometrically R_0. Since both X and Y are S_1, the assertion follows from the fact that being geometrically R_0 is a condition on the generic point. □

We prove a descent result for such singularities.

Proposition 2.3. *Let (X, Δ) be a geometrically klt (respectively terminal, canonical, lc) pair such that k is algebraically closed in $K(X)$. Then (X, Δ) is klt (respectively terminal, canonical, lc).*

Proof. We only treat the klt case, as the others are analogous. Let $\pi\colon Y \to X$ be a birational k-morphism, where Y is a normal variety and we write $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. It suffices to prove that $\lfloor \Delta_Y \rfloor \leqslant 0$. Thanks to Lemma 2.2, Y is geometrically integral. Let $\nu\colon W \to Y \times_k \overline{k}$ be the normalisation morphism and let us consider the following commutative diagram:

$$\begin{array}{ccc} W & & \\ \downarrow{\scriptstyle \nu} & & \\ Y \times_k \overline{k} & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \pi_{\overline{k}}} & & \downarrow{\scriptstyle \pi} \\ X \times_k \overline{k} & \xrightarrow{f} & X. \end{array}$$

Denote by $\psi := \pi_{\overline{k}} \circ \nu$ and $h := g \circ \nu$ the composite morphisms. We have

$$K_W + \Delta_W := \psi^*(K_{X_{\overline{k}}} + \Delta_{\overline{k}}) = h^*\pi^*(K_X + \Delta) = h^*(K_Y + \Delta_Y).$$

By [40, Theorem 4.2], there exists an effective $\mathbb{Z}$-divisor D such that

$$h^*(K_Y + \Delta_Y) = K_Y + D + h^*\Delta_Y,$$

and, thus, $\Delta_W = D + h^*\Delta_Y \geqslant h^*\Delta_Y$. Since $(X_{\overline{k}}, \Delta_{\overline{k}})$ is klt, any coefficient of Δ_W is < 1. Then any coefficient of Δ_Y is < 1, and, thus, (X, Δ) is klt. □

Remark 2.4. If k is a perfect field, being klt is equivalent to being geometrically klt by [20, Proposition 2.15]. However, over imperfect fields, being geometrically klt is a strictly stronger condition. As an example, let k be an imperfect field of characteristic $p > 0$ and consider the log pair $(\mathbb{A}^1_k, \frac{2}{3}P)$, where P is a closed point whose residue field $\kappa(P)$ is a purely inseparable extension of k of degree p. This pair is klt over k, but it is not geometrically lc.

2.3. Surfaces of del Pezzo type

In this subsection, we summarise some basic properties of surfaces of del Pezzo type over arbitrary fields. For later use, we introduce some terminology. Note that del Pezzo surfaces in our notation allow singularities.

Definition 2.5. Let k be a field. A k-surface X is *del Pezzo* if X is a projective normal surface such that $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. A k-surface X is *weak del Pezzo* if X is a projective normal surface such that $-K_X$ is a nef and big $\mathbb{Q}$-Cartier divisor.

Definition 2.6. Let k be a field. A k-surface X is *of del Pezzo type* if X is a projective normal surface over k and there exists an effective $\mathbb{Q}$-divisor $\Delta \geqslant 0$ such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. In this case, we say that (X, Δ) is a log del Pezzo pair.

We study how the property of being of del Pezzo type behaves under birational transformations.

Lemma 2.7. *Let k be a field. Let X be a k-surface of del Pezzo type. Let $f : Y \to X$ be the minimal resolution of X. Then Y is a k-surface of del Pezzo type.*

Proof. Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a log del Pezzo pair. We define a $\mathbb{Q}$-divisor Δ_Y by $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Since $f : Y \to X$ is the minimal resolution of X, we have that Δ_Y is an effective $\mathbb{Q}$-divisor. The pair (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is nef and big. By perturbing the coefficients of Δ_Y, we can find an effective $\mathbb{Q}$-divisor Γ such that (Y, Γ) is klt and $-(K_Y + \Gamma)$ is ample. □

Lemma 2.8. *Let k be a field. Let (X, Δ) be a two-dimensional projective klt pair over k. Let H be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then there exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A such that $A \sim_{\mathbb{Q}} H$ and $(X, \Delta + A)$ is klt.*

Proof. Thanks to the existence of log resolutions for excellent surfaces [24], the same proof of [15, Lemma 2.8] works in our setting. □

Lemma 2.9. *Let k be a field. Let X be a k-surface of del Pezzo type. Let $f : X \to Y$ be a birational k-morphism to a projective normal k-surface Y. Then Y is a k-surface of del Pezzo type.*

Proof. Let Δ be an effective $\mathbb{Q}$-divisor such that (X, Δ) is a log del Pezzo pair. Set $H := -(K_X + \Delta)$, which is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. By Lemma 2.8, there exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A such that $A \sim_{\mathbb{Q}} H$ and $(X, \Delta + A)$ is klt. Then the pair

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$(Y, f_*\Delta + f_*A)$ is klt and $K_X + \Delta + A \sim_{\mathbb{Q}} f^*(K_Y + f_*\Delta + f_*A) \sim_{\mathbb{Q}} 0$. It follows from [39, Corollary 4.11] that Y is $\mathbb{Q}$-factorial. By Nakai's criterion, the $\mathbb{Q}$-divisor f_*A is ample. In particular, $(Y, f_*\Delta)$ is a log del Pezzo pair. □

2.4. Geometrically canonical del Pezzo surfaces

In this subsection, we collect results on the anti-canonical systems of geometrically canonical del Pezzo surfaces we will need later.

2.4.1. Canonical del Pezzo surfaces over algebraically closed fields. We verify that the results in [19, Chapter III, § 3] hold for del Pezzo surfaces with canonical singularities over algebraically closed fields. Recall that we say that X is a canonical (weak) del Pezzo surface over a field k if X is a surface over k, X is (weak) del Pezzo in the sense of Definition 2.5, and $(X, 0)$ is canonical in the sense of [20, Definition 2.8].

Proposition 2.10. *Let X be a canonical weak del Pezzo surface over an algebraically closed field k. Then the following hold:*

(1) $H^2(X, \mathcal{O}_X(-mK_X)) = 0$ *for any non-negative integer m.*
(2) $H^i(X, \mathcal{O}_X) = 0$ *for any $i > 0$.*
(3) $H^0(X, \mathcal{O}_X(-K_X)) \neq 0$.
(4) $H^1(X, \mathcal{O}_X(mK_X)) = 0$ *for any integer m.*
(5) $h^0(X, \mathcal{O}_X(-mK_X)) = 1 + \frac{m(m+1)}{2}K_X^2$ *for any non-negative integer m.*

Proof. The assertion (1) follows from Serre duality. We now show (2). It follows from [37, Theorem 5.4 and Remark 5.5] that X has at worst rational singularities. Then the assertion (2) follows from the fact that X is a rational surface [38, Theorem 3.5].

We now show (3). By $H^2(X, \mathcal{O}_X(-K_X)) = 0$ and the Riemann–Roch theorem, we have $h^0(X, \mathcal{O}_X(-K_X)) \geqslant 1 + K_X^2 > 0$. Thus, (3) holds.

We now show (4). By (3), there exists an effective Cartier divisor D such that $D \sim -K_X$. In particular, D is effective, nef, and big. It follows from [6, Proposition 3.3] that

$$H^1(X, \mathcal{O}_X(-nD)) = H^1(X, \mathcal{O}_X(K_X + nD)) = 0$$

for any $n \in \mathbb{Z}_{>0}$. Replacing D by $-K_X$, the assertion (4) holds. Thanks to (1) and (4), assertion (5) follows from the Riemann–Roch theorem. □

Lemma 2.11. *Let Y be a canonical weak del Pezzo surface over an algebraically closed field k. If a divisor $\sum_{i=1}^r a_iC_i \in |-K_Y|$ is not irreducible or not reduced, then every C_i is a smooth rational curve.*

Proof. Taking the minimal resolution of Y, we may assume that Y is smooth. Fix an index $1 \leqslant i_0 \leqslant r$. By adjunction, we have

$$2p_a(C_{i_0}) - 2 = -C_{i_0} \cdot \left(\sum_{i \neq i_0} \frac{a_i}{a_{i_0}} C_i\right) - \frac{a_{i_0} - 1}{a_{i_0}} C_{i_0} \cdot (-K_Y). \qquad (2.11.1)$$

Note that both the terms on the right-hand side are non-positive.

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Since Y is smooth and $\sum_i a_i C_i$ is nef and big, it follows from [38, Theorem 2.6] that $H^1(X, -n\sum_i a_i C_i) = 0$ for $n \gg 0$. Hence, $\sum_i a_i C_i$ is connected. Therefore, if $\sum_i a_i C_i$ is reducible, the first term in the right-hand side of (2.11.1) is strictly negative; hence $p_a(C_{i_0}) < 0$.

If $a_{i_0} \geqslant 2$ and $C_{i_0} \cdot K_Y < 0$, then the second term in the right-hand side of (2.11.1) is strictly negative; hence $p_a(C_{i_0}) < 0$. If $C_{i_0} \cdot K_Y = 0$, then C_i is a smooth rational curve with $C_i^2 = -2$. □

Proposition 2.12. *Let Y be a canonical weak del Pezzo surface over an algebraically closed field k. Let $\mathrm{Bs}(-K_Y)$ be the base locus of $-K_Y$, which is a closed subset of Y. Then the following hold:*

(1) $\mathrm{Bs}(-K_Y)$ *is empty or* $\dim(\mathrm{Bs}(-K_Y)) = 0$.

(2) *A general member of the linear system $|-K_Y|$ is irreducible and reduced.*

Proof. Taking the minimal resolution of Y, we may assume that Y is smooth. Using Proposition 2.10, the same proof of [9, Theorem 8.3.2.i] works in our setting so that (1) holds and general members of $|-K_Y|$ are irreducible.

It is enough to show that a general member of $|-K_Y|$ is reduced. Suppose it is not. Then there exists $a > 1$ such that a general member is of the form $aC \in |-K_Y|$ for some curve C. In particular, C is a smooth rational curve by Lemma 2.11. Recall that we have the short exact sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y(C)) \to H^0(C, \mathcal{O}_C(C)) \to 0.$$

Since $H^1(Y, \mathcal{O}_Y) = 0$ (Proposition 2.10), we have that $h^0(Y, \mathcal{O}_Y(C)) = 1 + h^0(C, \mathcal{O}_C(C))$. As C is a smooth rational curve, we conclude by the Riemann–Roch theorem that $h^0(Y, \mathcal{O}_Y(C)) = 2 + C^2$.

We now consider the induced map

$$\begin{aligned} \theta : H^0(Y, \mathcal{O}_Y(C)) &\to H^0(Y, \mathcal{O}_Y(aC)) \simeq H^0(Y, \mathcal{O}_Y(-K_Y)) \\ \varphi &\mapsto \varphi^a. \end{aligned}$$

Since a general member of $|-K_Y|$ is of the form aD for some $D \geqslant 0$, θ is a dominant morphism if we consider θ as a morphism of affine spaces. Therefore, it holds that

$$h^0(Y, \mathcal{O}_Y(-K_Y)) \leqslant h^0(Y, \mathcal{O}_Y(C)) = 2 + C^2 = -K_Y \cdot C \leqslant K_Y^2,$$

which contradicts Proposition 2.10. □

2.4.2. Anti-canonical systems on geometrically canonical del Pezzo surfaces.

In this section, we study anti-canonical systems on geometrically canonical del Pezzo surfaces over an arbitrary field k and we describe their anti-canonical model when the anti-canonical degree is small.

We need the following results on geometrically integral curves of genus one.

Lemma 2.13. *Let k be a field. Let C be a geometrically integral Gorenstein projective curve over k of arithmetic genus one with $k = H^0(C, \mathcal{O}_C)$. Let L be a Cartier divisor on C and let $R(C, L) := \bigoplus_{m \geqslant 0} H^0(C, mL)$ be the graded k-algebra. Then the following hold:*

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(i) *If* $\deg_k(L) = 1$, *then* $\mathrm{Bs}(L) = \{P\}$ *for some* k*-rational point* P *and* $R(C, L)$ *is generated by* $\bigoplus_{1 \leqslant j \leqslant 3} H^0(C, jL)$ *as a* k*-algebra.*

(ii) *If* $\deg_k(L) \geqslant 2$, *then* L *is globally generated and* $R(C, L)$ *is generated by* $H^0(C, L) \oplus H^0(C, 2L)$ *as a* k*-algebra.*

(iii) *If* $\deg_k L \geqslant 3$, *then* L *is very ample and* $R(C, L)$ *is generated by* $H^0(C, L)$ *as a* k*-algebra.*

Proof. See [42, Lemma 11.10 and Proposition 11.11]. □

Proposition 2.14. *Let* k *be a field. Let* X *be a geometrically canonical weak del Pezzo surface over* k *such that* $k = H^0(X, \mathcal{O}_X)$. *Let* $R(X, -K_X) = \bigoplus_{m \geqslant 0} H^0(X, \mathcal{O}_X(-mK_X))$ *be the graded* k*-algebra. Then the following hold.*

(1) *If* m *is a positive integer such that* $mK_X^2 \geqslant 2$, *then* $|-mK_X|$ *is base point free.*

(2) *If* $K_X^2 = 1$, *then* $\mathrm{Bs}(-K_X) = \{P\}$ *for some* k*-rational point* P.

(3) *If* $K_X^2 = 1$, *then* $R(X, -K_X)$ *is generated by* $\bigoplus_{1 \leqslant j \leqslant 3} H^0(X, -jK_X)$ *as a* k*-algebra.*

(4) *If* $K_X^2 = 2$, *then* $R(X, -K_X)$ *is generated by* $H^0(X, -K_X) \oplus H^0(X, -2K_X)$ *as a* k*-algebra.*

(5) *If* $K_X^2 \geqslant 3$, *then* $R(X, -K_X)$ *is generated by* $H^0(X, -K_X)$ *as a* k*-algebra.*

In particular, if $-K_X$ *is ample, then* $|-6K_X|$ *is very ample.*

Proof. Consider the following condition.

(2)' If $K_X^2 = 1$, then $\mathrm{Bs}(-K_X)$ is not empty and of dimension zero.

Since $K_X^2 = 1$, (2) and (2)' are equivalent. Note that to show (1), (2)', and (3)–(5), we may assume that k is algebraically closed.

From now on, let us prove (1)–(5) under the condition that k is algebraically closed. It follows from Proposition 2.12 that a general member C of $|-K_X|$ is a prime divisor.

Since C is a Cartier divisor and X is Gorenstein, then C is a Gorenstein curve. By adjunction, C is a Gorenstein curve of arithmetic genus $p_a(C) = 1$. By Proposition 2.10, we have the following exact sequence for every integer m:

$$0 \to H^0(X, -(m-1)K_X) \to H^0(X, -mK_X) \to H^0(C, -mK_X|_C) \to 0.$$

By the above exact sequence, the assertions (1) and (2) follow from (3) and (2) of Lemma 2.13, respectively.

We prove the assertions (3), (4), and (5). By the above short exact sequence, it is sufficient to prove the same statement for the k-algebra $R(C, \mathcal{O}_C(-K_X))$, which is the content of Lemma 2.13. □

Theorem 2.15. *Let* k *be a field. Let* X *be a geometrically canonical del Pezzo surface over* k *such that* $H^0(X, \mathcal{O}_X) = k$. *Then the following hold:*

(1) *If* $K_X^2 = 1$, *then* X *is isomorphic to a weighted hypersurface in* $\mathbb{P}_k(1, 1, 2, 3)$ *of degree six.*

(2) *If* $K_X^2 = 2$, *then* X *is isomorphic to a weighted hypersurface in* $\mathbb{P}_k(1, 1, 1, 2)$ *of degree four.*

(3) *If* $K_X^2 = 3$, *then* X *is isomorphic to a hypersurface in* $\mathbb{P}_k^3$ *of degree three.*

(4) *If* $K_X^2 = 4$, *then* X *is isomorphic to a complete intersection of two quadric hypersurfaces in* $\mathbb{P}_k^4$.

Proof. Using Proposition 2.14, the proof is the same as in [19, Theorem III.3.5]. □

2.5. Mori fibre spaces to curves

In this subsection, we summarise properties of regular curves with anti-ample canonical divisor and of Mori fibre space of dimension two over arbitrary fields.

Lemma 2.16. *Let* k *be a field. Let* C *be a projective Gorenstein integral curve over* k. *Then the following are equivalent:*

(1) ω_C^{-1} *is ample.*

(2) $H^1(C, \mathcal{O}_C) = 0$.

(3) C *is a conic curve of* $\mathbb{P}_K^2$, *where* $K := H^0(C, \mathcal{O}_C)$.

(4) $\deg_k \omega_C = -2 \dim_k(H^0(C, \mathcal{O}_C))$.

Proof. It follows from [39, Corollary 2.8] that (1), (2), and (4) are equivalent. Clearly, (3) implies (1). By [20, Lemma 10.6], (1) implies (3). □

Lemma 2.17. *Let* k *be a field and let* C *be a projective Gorenstein integral curve over* k *such that* $k = H^0(C, \mathcal{O}_C)$ *and* ω_C^{-1} *is ample. Then the following hold:*

(1) *If* C *is geometrically integral over* k, *then* C *is smooth over* k.

(2) *If the characteristic of* k *is not two, then* C *is geometrically reduced over* k.

(3) *If the characteristic of* k *is not two and* C *is regular, then* C *is smooth over* k.

Proof. By Lemma 2.16, C is a conic curve in $\mathbb{P}_k^2$. Thus, the assertion (1) follows from the fact that an integral conic curve over an algebraically closed field is smooth.

Let us show (2) and (3). Since the characteristic of k is not two and C is a conic curve in $\mathbb{P}_k^2$, we can write

$$C = \operatorname{Proj} k[x, y, z]/(ax^2 + by^2 + cz^2)$$

for some $a, b, c \in k$. Since C is an integral scheme, two of a, b, c are not zero. Hence, C is reduced. Thus, (2) holds. If C is regular, then each of a, b, c is nonzero; hence, C is smooth over k. □

Proposition 2.18. *Let* k *be a field. Let* $\pi : X \to B$ *be a* K_X*-Mori fibre space from a projective regular* k*-surface* X *to a projective regular* k*-curve with* $k = H^0(B, \mathcal{O}_B)$. *Let* b *be a (not necessarily closed) point. Then the following hold:*

(1) *The fibre* X_b *is irreducible.*

(2) *The equation* $\kappa(b) = H^0(X_b, \mathcal{O}_{X_b})$ *holds.*

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(3) *The fibre X_b is reduced.*

(4) *The fibre X_b is a conic in $\mathbb{P}^2_{\kappa(b)}$.*

(5) *If* $\mathrm{char}\, k \neq 2$*, then any fibre of* π *is geometrically reduced.*

(6) *If* $\mathrm{char}\, k \neq 2$ *and* k *is separably closed, then* π *is a smooth morphism.*

Proof. If X_b is not irreducible, it contradicts the hypothesis $\rho(X/B) = 1$. Thus, (1) holds.

Let us show (2). Since π is flat, the integer

$$\chi := \dim_{\kappa(b)} H^0(X_b, \mathcal{O}_{X_b}) - \dim_{\kappa(b)} H^1(X_b, \mathcal{O}_{X_b}) \in \mathbb{Z}$$

is independent of $b \in B$. Since $H^1(X_b, \mathcal{O}_{X_b}) = 0$ for any $b \in B$, it suffices to show that $\dim_{\kappa(b)} H^0(X_b, \mathcal{O}_{X_b}) = 1$ for some $b \in B$. This holds for the case when b is the generic point of B. Hence, (2) holds.

Let us prove (3). It is clear that the generic fibre is reduced. We may assume that $b \in B$ is a closed point. Assume that X_b is not reduced. By (1), we have $X_b = mC$ for some prime divisor C and $m \in \mathbb{Z}_{\geqslant 2}$. Since $-K_X \cdot_{\kappa(b)} X_b = 2$, we have that $m = 2$. Then we obtain an exact sequence:

$$0 \to \mathcal{O}_X(-C)|_C \to \mathcal{O}_{X_b} \to \mathcal{O}_C \to 0.$$

Since $C^2 = 0$ and ω_C^{-1} is ample, we have that $\mathcal{O}_X(-C)|_C \simeq \mathcal{O}_C$. Since $H^1(C, \mathcal{O}_C) = 0$, we get an exact sequence:

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(X_b, \mathcal{O}_{X_b}) \to H^0(C, \mathcal{O}_C) \to 0.$$

Then we obtain $\dim_{\kappa(b)} H^0(X_b, \mathcal{O}_{X_b}) \geqslant 2$, which contradicts (2). Hence, (3) holds.

We now show (4). By [39, Corollary 2.9], $\deg_{\kappa(b)} \omega_{X_b} = (K_X + X_b) \cdot_{\kappa(b)} X_b < 0$. Hence, (4) follows from (2) and Lemma 2.16.

The assertions (5) and (6) follow from Proposition 2.17. □

2.6. Twisted forms of canonical singularities

The aim of this subsection is to prove Proposition 2.27. The main idea is to bound the purely inseparable degree of regular non-smooth points on geometrically normal surfaces according to the type of singularities. For this, the notion of Jacobian number plays a crucial role.

Definition 2.19. Let k be a field of characteristic $p > 0$. Let R be an equi-dimensional k-algebra essentially of finite type over k. Let $J_{R/k}$ be its Jacobian ideal of R over k (cf. [17, Definition 4.4.1 and Proposition 4.4.4]). We define the *Jacobian number* of R/k as $\nu(R) := \nu(R/k) := \dim_k(R/J_{R/k})$. Note that $\nu(R/k) < \infty$ if $R/J_{R/k}$ is an artinian ring and its residue fields are finite extensions of k.

Remark 2.20. Let $k \subset k'$ be a field extension of characteristic $p > 0$ and let R be an equi-dimensional k-algebra essentially of finite type over k. Then the following hold:

(1) By [17, Definition 4.4.1], we get

$$J_{R/k} \cdot (R \otimes_k k') = J_{R \otimes_k k'/k'}.$$

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In particular, if $R/J_{R/k}$ is an artinian ring and its residue fields are finite extensions of k, then we have $\nu(R/k) = \nu(R \otimes_k k'/k')$.

(2) Assume that k is a perfect field. By [17, Definition 4.4.9], $\operatorname{Spec}(R/J_{R/k})$ set-theoretically coincides with the non-regular locus of $\operatorname{Spec} R$.

(3) Assume that R is of finite type over k. Then (1) and (2) imply that $\operatorname{Spec}(R/J_{R/k})$ set-theoretically coincides with the non-smooth locus of $\operatorname{Spec} R \to \operatorname{Spec} k$.

Remark 2.21. In our application, R will be assumed to be a local ring $\mathcal{O}_{X,x}$ at a closed point x of a geometrically normal surface X over k. In this case, (3) of Remark 2.20 implies that $R/J_{R/k}$ is an artinian local ring whose residue field is a finite extension of k. Hence, $\nu(R/k) = \dim_k(R/J_{R/k})$ is well defined as in Definition 2.19.

To treat local situations, let us recall the notion of essentially étale ring homomorphisms. For its fundamental properties, we refer to [13, § 2.8].

Definition 2.22. Let $f: R \to S$ be a local homomorphism of local rings. We say that f is *essentially étale* if there exists an étale R-algebra $\overline{S}$ and a prime ideal $\mathfrak{p}$ of $\overline{S}$ such that $\mathfrak{p}$ lies over the maximal ideal of R and S is R-isomorphic to $\overline{S}_{\mathfrak{p}}$.

Lemma 2.23. *Let k be a field. Let $f: R \to S$ be an essentially étale local k-algebra homomorphism of local rings which are essentially of finite type over k. Let $\mathfrak{m}_R$ and $\mathfrak{m}_S$ be the maximal ideals of R and S, respectively. Set $\kappa(R) := R/\mathfrak{m}_R$ and $\kappa(S) := S/\mathfrak{m}_S$. Then the following hold:*

(1) *If M is an R-module of finite length whose support is contained in the maximal ideal $\mathfrak{m}_R$, then the equation*

$$\dim_k(M \otimes_R S) = [\kappa(S) : \kappa(R)] \dim_k M$$

holds.

(2) *Suppose that R is an integral domain, $R/J_{R/k}$ is an artinian ring, and $\kappa(R)$ is a finite extension of k. Then the equation*

$$\nu(S/k) = [\kappa(R) : \kappa(S)]\nu(R/k)$$

holds.

Proof. Let us show (1). Since M is a finitely generated R-module, there exists a sequence of R-submodules $M =: M_0 \supset M_1 \supset \cdots \supset M_n = 0$ such that $M_i/M_{i+1} \simeq R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ by [27, Theorem 6.4]. Since the support of M is $\mathfrak{m}_R$, we have $\mathfrak{p} = \mathfrak{m}_R$. As $R \to S$ is flat, the problem is reduced to the case when $M = R/\mathfrak{m}_R = \kappa(R)$. In this case, we have

$$\kappa(R) \otimes_R S = (R/\mathfrak{m}_R) \otimes_R S \simeq S/\mathfrak{m}_R S = S/\mathfrak{m}_S = \kappa(S),$$

where the equality $S/m_R S = S/m_S$ follows from the assumption that f is a localisation of an unramified homomorphism. Hence, (1) holds.

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Let us show (2). Set $n := \dim R$. We use the description of the Jacobian of R via Fitting ideals (cf. [17, Discussion 4.4.7]): $J_{R/k} = \mathrm{Fit}_n(\Omega^1_{R/k})$ and $J_{S/k} = \mathrm{Fit}_n(\Omega^1_{S/k})$. We have

$$J_{S/k} = \mathrm{Fit}_n(\Omega^1_{S/k}) = \mathrm{Fit}_n(\Omega^1_{R/k} \otimes_R S) = \mathrm{Fit}_n(\Omega^1_{R/k})S = J_{R/k}S,$$

where the third equality follows from (3) of [36, Tag 07ZA]. As $f : R \to S$ is flat, we obtain $S/J_{S/k} \simeq (R/J_{R/k}) \otimes_R S$. By (1) and Definition 2.19, the assertion (2) holds. □

Example 2.24. Let k be a field of characteristic $p > 0$. Let $X = \mathrm{Spec}\, R$ be a surface over k such that

(i) $X \times_k \overline{k} = \mathrm{Spec}\,(R \otimes_k \overline{k})$ is a normal surface;

(ii) $X \times_k \overline{k}$ has a unique singular point x, and x is a canonical singularity of type A_{p^n-1}.

We prove that $\nu(R/k) = p^n$. By Remark 2.20, we have $\nu(R/k) = \nu(R \otimes_k \overline{k}/\overline{k})$. In order to compute $\nu(R \otimes_k \overline{k}/\overline{k})$, it is sufficient to localise at the singular point by [17, Corollary 4.4.5]. Thus, we can suppose that k is algebraically closed and R is a local k-algebra.

By [1, pages 16–17] (cf. (12) of § 2.1), the henselisation R^h of R is isomorphic to

$$k\{x, y, z\}/(z^{p^n} + xy).$$

In particular, there exist essentially étale local k-algebra homomorphisms $R \to S$ and $k[x, y, z]/(z^{p^n} - xy) \to S$. A direct computation shows $\nu(k[x, y, z]/(z^{p^n} - xy)) = p^n$. Thus, by Lemma 2.23, we have

$$\nu(R) = \nu(S) = \nu(k[x, y, z]/(z^{p^n} - xy)) = p^n.$$

The following is a generalisation of [11, Lemma 14.2].

Lemma 2.25. *Let k be a field of characteristic $p > 0$. Let $X = \mathrm{Spec}\, R$, where R is an equi-dimensional local k-algebra of essentially finite type over k. Let x be the closed point of X. Suppose that $R/J_{R/k}$ is a local artinian ring and its residue field $\kappa(x)$ is a finite extension of k. Then $[\kappa(x) : k]$ is a divisor of $\nu(R/k)$.*

Proof. Let $R/J_{R/k} =: M_0 \supset M_1 \supset \cdots \supset M_n = 0$ be a composition sequence of $R/J_{R/k}$-submodules (cf. [27, Theorem 6.4]). Since $R/J_{R/k}$ is an artinian local ring, it holds that $M_i/M_{i+1} \simeq \kappa(x)$ for any i. We have

$$\nu(R/k) = \dim_k(R/J_{R/k}) = \sum_{i=0}^{n-1} \dim_k(M_i/M_{i+1}) = n \dim_k \kappa(x) = n[\kappa(x) : k].$$

We thus conclude that $[\kappa(x) : k]$ is a divisor of $\nu(R/k)$. □

Lemma 2.26. *Let X be a regular variety over a separably closed field k. Suppose that $X_{\overline{k}} = X \times_k \overline{k}$ is a normal variety with a unique singular point y. Let x be the image of y by the induced morphism $X_{\overline{k}} \to X$. Then the following hold:*

(1) *$[\kappa(x) : k]$ is a divisor of $\nu(\mathcal{O}_{X,x})$.*

(2) *$X \times_k \kappa(x)$ is not regular.*

Proof. Since k is separably closed, the induced morphism $X_{\overline{k}} \to X$ is a universal homeomorphism. Note that the local ring $\mathcal{O}_{X,x}$ is not geometrically regular over k. Applying Lemma 2.25 to the local ring $\mathcal{O}_{X,x}$, we deduce that $[\kappa(x) : k]$ is a divisor of $\nu(\mathcal{O}_{X,x})$. Thus, (1) holds. Consider the base change $\pi : X \times_k \kappa(x) \to X$. Let x' be the point on $X \times_k \kappa(x)$ lying over x. Note that x' is a $\kappa(x)$-rational point of $X \times_k \kappa(x)$ whose base change by $(-) \times_{\kappa(x)} \overline{k}$ is not regular. By [11, Corollary 2.6], we conclude that $X \times_k \kappa(x)$ is not regular at x'. □

We now explain how the previous results can be used to construct closed points with purely inseparable residue field on a regular surface. This will be used in § 6 to find purely inseparable points on regular del Pezzo surfaces.

Proposition 2.27. *Let X be a regular surface over k. Suppose that $X_{\overline{k}} = X \times_k \overline{k}$ is a normal surface over $\overline{k}$ with a unique singular point y. Assume that y is a canonical singularity of type A_{p^n-1}. Let z be the image of y by the induced morphism $X_{\overline{k}} \to X_{k^{1/p^n}} = X \times_k k^{1/p^n}$. Then z is a k^{1/p^n}-rational point on $X_{k^{1/p^n}}$.*

Proof. Set $R := \mathcal{O}_{X,x}$, where x is the unique closed point along which X is not smooth. Let k^{sep} be the separable closure of k. For $R_{k^{\mathrm{sep}}} := R \otimes_k k^{\mathrm{sep}}$, it follows from Example 2.24 that $\nu(R_{k^{\mathrm{sep}}}) = p^n$. Lemma 2.26 implies that $k^{\mathrm{sep}} \subset \kappa(z)$ is purely inseparable and $[\kappa(z) : k]$ is a divisor of p^n. In particular, $\kappa(z) \subset (k^{\mathrm{sep}})^{1/p^n}$.

Consider the Galois extension $k^{1/p^n} \subset (k^{\mathrm{sep}})^{1/p^n}$ and denote by G its Galois group. For $X_{(k^{\mathrm{sep}})^{1/p^n}} := X \times_k (k^{\mathrm{sep}})^{1/p^n}$, G acts on the set $X_{(k^{\mathrm{sep}})^{1/p^n}}((k^{\mathrm{sep}})^{1/p^n})$. The unique singular $(k^{\mathrm{sep}})^{1/p^n}$-rational point on $X_{(k^{\mathrm{sep}})^{1/p^n}}$ is fixed under the G-action. Thus, it descends to a k^{1/p^n}-rational point on $X_{k^{1/p^n}}$. □

3. Behaviour of del Pezzo surfaces under base changes

In this section, we study the behaviour of canonical del Pezzo surfaces over an imperfect field k under the base changes to the algebraic closure $\overline{k}$.

3.1. Classification of base changes of del Pezzo surfaces

In this subsection, we give classification of base changes of del Pezzo surfaces with canonical singularities over imperfect fields (Theorem 3.3). To this end, we need two auxiliary lemmas: Lemmas 3.1 and 3.2. The former one classifies $\mathbb{Q}$-factorial surfaces over algebraically closed fields whose anti-canonical bundles are sufficiently positive. Its proof is based on a simple but smart idea by Reid (cf. the proof of [32, Theorem 1.1]). The latter one, i.e. Lemma 3.2, gives a rationality criterion for the base changes of log del Pezzo surfaces.

Lemma 3.1. *Let k be an algebraically closed field. Let Y be a projective normal $\mathbb{Q}$-factorial surface over k such that $-K_Y \equiv A + D$ for an ample Cartier divisor A and a pseudo-effective $\mathbb{Q}$-divisor D. Let $\mu : Z \to Y$ be the minimal resolution of Y. Then one of the following assertions holds:*

(1) *$D \equiv 0$ and Y has at worst canonical singularities.*

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(2) *Z is isomorphic to a $\mathbb{P}^1$-bundle over a smooth projective curve.*

(3) *$Z \simeq \mathbb{P}^2$.*

Proof. Assuming that (1) does not hold, let us prove that either (2) or (3) holds. We have

$$K_Z + E = \mu^* K_Y$$

for some effective μ-exceptional $\mathbb{Q}$-divisor E on Z. In particular, it holds that

$$K_Z + E + \mu^*(D) = \mu^*(K_Y + D) \equiv -\mu^* A.$$

Since (1) does not hold, we have that $D \not\equiv 0$ or $E \neq 0$. Then we get

$$K_Z + \mu^* A \equiv -E - \mu^*(D) \not\equiv 0;$$

hence, $K_Z + \mu^* A$ is not nef. By the cone theorem for a smooth projective surface [21, Theorem 1.24], there is a curve C that spans a $(K_Z + \mu^* A)$-negative extremal ray R of $\overline{\mathrm{NE}}(Z)$. Note that C is not a (-1)-curve. Indeed, otherwise $\mu(C)$ is a curve and we obtain $\mu^* A \cdot C > 0$, which induces a contradiction:

$$(K_Z + \mu^* A) \cdot C \geqslant -1 + 1 = 0.$$

It follows from the classification of the K_Z-negative extremal rays [21, Theorem 1.28] that either $Z \simeq \mathbb{P}^2$ or Z is a $\mathbb{P}^1$-bundle over a smooth projective curve. In any case, one of (2) and (3) holds. □

Lemma 3.2. *Let (X, Δ) be a projective two-dimensional klt pair over a field of characteristic $p > 0$ such that $-(K_X + \Delta)$ is nef and big. Assume that $k = H^0(X, \mathcal{O}_X)$. Then $(X \times_k \overline{k})_{\mathrm{red}}$ is a rational surface.*

Proof. See [29, Proposition 2.20]. □

We now give a classification of the base changes of del Pezzo surfaces with canonical singularities.

Theorem 3.3. *Let k be a field of characteristic $p > 0$. Let X be a canonical del Pezzo surface over k with $k = H^0(X, \mathcal{O}_X)$. Then the normalisation Y of $(X \times_k \overline{k})_{\mathrm{red}}$ satisfies one of the following properties:*

(1) *X is geometrically canonical over k. In particular, $Y \simeq X \times_k \overline{k}$ and $-K_Y$ is ample.*

(2) *X is not geometrically normal over k and Y is isomorphic to a Hirzebruch surface, i.e. a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.*

(3) *X is not geometrically normal over k and Y is isomorphic to a weighted projective surface $\mathbb{P}(1, 1, m)$ for some positive integer m.*

Proof. Replacing k by its separable closure, we may assume that k is separably closed. Let $f : Y \to X$ be the induced morphism and let $\mu : Z \to Y$ be the minimal resolution of Y. By [40, Theorem 4.2], there is an effective $\mathbb{Z}$-divisor D on Y such that

- $K_Y + D = f^* K_X$ and
- if $X \times_k \overline{k}$ is not normal, then $D \neq 0$.

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Since $-K_X$ is an ample Cartier divisor, so is $-f^*K_X$. Moreover, it follows from [40, Lemmas 2.2 and 2.5] that Y is $\mathbb{Q}$-factorial. Hence, we may apply Lemma 3.1 to $-K_Y = -f^*K_X + D$.

By Lemma 3.2, Y is a rational surface. Thus, if (2) or (3) of Lemma 3.1 holds, then one of (1)–(3) of Theorem 3.3 holds, as desired. Therefore, let us treat the case when (1) of Lemma 3.1 holds. Then it holds that $D = 0$ and Y has at worst canonical singularities. In this case, we have that $Y = X \times_k \overline{k}$ and X is geometrically canonical. Hence, (1) of Theorem 3.3 holds, as desired. □

3.2. Bounds on Frobenius length of geometric non-normality

In this subsection, we give an upper bound for the Frobenius length of geometric non-normality for canonical del Pezzo surfaces (Proposition 3.6). We start by recalling its definition (Definition 3.4) and fundamental properties (Remark 3.5).

Definition 3.4. Let k be a field of characteristic $p > 0$. Let X be a proper normal variety over k such that $k = H^0(X, \mathcal{O}_X)$. The *Frobenius length of geometric non-normality* $\ell_F(X/k)$ of X/k is defined by

$$\ell_F(X/k) := \min\{\ell \in \mathbb{Z}_{\geqslant 0} \mid (X \times_k k^{1/p^\ell})^N_{\mathrm{red}} \text{ is geometrically normal over } k^{1/p^\ell}\}.$$

Remark 3.5. Let k and X be as in Definition 3.4. Set $\ell := \ell_F(X/k)$. Let (k', Y) be one of $(k^{1/p^\infty}, (X \times_k k^{1/p^\infty})^N_{\mathrm{red}})$ and $(\overline{k}, (X \times_k \overline{k})^N_{\mathrm{red}})$. We summarise some results from [42, § 5].

(1) The existence of the right-hand side of Definition 3.4 is assured by [42, Remark 5.2].

(2) If X is not geometrically normal, then ℓ is a positive integer [42, Remark 5.3] and there exist nonzero effective Weil divisors $D_1, \ldots, D_\ell$ such that

$$K_Y + (p-1)\sum_{i=1}^{\ell} D_i \sim f^*K_X,$$

where $f : Y \to X$ denotes the induced morphism [42, Proposition 5.11].

(3) The ℓth iterated absolute Frobenius morphism $F^\ell_{X\times_k k'}$ factors through the induced morphism $Y \to X \times_k k'$ [42, Proposition 5.4 and Theorem 5.9]:

$$F^\ell_{X\times_k k'} : X \times_k k' \to Y \to X \times_k k'.$$

Proposition 3.6. *Let k be a field of characteristic $p > 0$. Let X be a canonical del Pezzo surface over k with $k = H^0(X, \mathcal{O}_X)$. Let Y be the normalisation of $(X \times_k \overline{k})_{\mathrm{red}}$ and let $f : Y \to X$ be the induced morphism. Assume that the linear equivalence*

$$K_Y + \sum_{i=1}^{r} C_i \sim f^*K_X$$

holds for some prime divisors $C_1, \ldots, C_r$ (not necessarily $C_i \neq C_j$ for $i \neq j$). Then it holds that $r \leqslant 2$.

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Proof. Set $C := \sum_{i=1}^{r} C_i$. We have $K_Y + C \sim f^* K_X$. If $C = 0$, then there is nothing to show. Hence, we may assume that $C \neq 0$. In particular, X is not geometrically normal. In this case, it follows from Theorem 3.3 that Y is isomorphic to either a Hirzebruch surface or $\mathbb{P}(1, 1, m)$ for some $m > 0$.

We first treat the case when $Y \simeq \mathbb{P}(1, 1, m)$. If $m = 1$, then the assertion is obvious. Hence, we may assume that $m \geqslant 2$. In this case, for the minimal resolution $g : Z \to Y$, we have that

$$K_Z + \frac{m-2}{m}\Gamma = g^* K_Y,$$

where Γ is the negative section of the fibration $Z \to \mathbb{P}^1$ such that $\Gamma^2 = -m$. Note that m is the $\mathbb{Q}$-factorial index of Y, i.e. mD is Cartier for any $\mathbb{Z}$-divisor D on Y. We have that

$$-K_Z = \frac{m-2}{m}\Gamma - g^* K_Y \equiv \frac{m-2}{m}\Gamma + g^* C - g^* f^* K_X.$$

Consider the intersection number with a fibre F_Z of $Z \to \mathbb{P}^1$:

$$2 = \left(\frac{m-2}{m}\Gamma + g^* C - g^* f^* K_X\right) \cdot F_Z \geqslant \frac{m-2}{m} + C \cdot g_*(F_Z) + 1.$$

Thus, we obtain

$$2 \geqslant C \cdot (m g_*(F_Z)) \geqslant r,$$

where the last inequality holds since $m g_*(F_Z)$ is an ample Cartier divisor. Therefore, we obtain $r \leqslant 2$, as desired.

It is enough to treat the case when Y is a Hirzebruch surface. For a fibre F of $\pi : Y \to \mathbb{P}^1$, we have that

$$-2 + C \cdot F = (K_Y + C) \cdot F = f^* K_X \cdot F \leqslant -1;$$

hence, $C \cdot F \leqslant 1$. There are two possibilities: $C \cdot F = 1$ or $C \cdot F = 0$.

Assume that $C \cdot F = 1$. Then there is a section Γ of π and a π-vertical $\mathbb{Z}$-divisor C' such that $C = \Gamma + C'$. Consider the intersection number with Γ:

$$-2 + \Gamma \cdot C' = (K_Y + \Gamma + C') \cdot \Gamma = (K_Y + C) \cdot \Gamma = f^* K_X \cdot \Gamma \leqslant -1.$$

Therefore, we have $\Gamma \cdot C' \leqslant 1$. This implies that either $C' = 0$ or C' is a prime divisor. In any case, we get $r \leqslant 2$, as desired.

We may assume that $C \cdot F = 0$, i.e. C is a π-vertical divisor. Let Γ be a section of π such that $\Gamma^2 \leqslant 0$. We have that

$$-2 + C \cdot \Gamma = (K_Y + \Gamma + C) \cdot \Gamma \leqslant (K_Y + C) \cdot \Gamma = f^* K_X \cdot \Gamma \leqslant -1.$$

Hence, we obtain $C \cdot \Gamma \leqslant 1$, which implies $r \leqslant 1$. □

Theorem 3.7. *Let k be a field of characteristic $p > 0$. Let X be a canonical del Pezzo surface over k such that $k = H^0(X, \mathcal{O}_X)$. Let Y be the normalisation of $(X \times_k \overline{k})_{\mathrm{red}}$ and let*

$$\mu : Y \to X \times_k \overline{k}$$

be the induced morphism.

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(1) *If* $p \geqslant 5$, *then* X *is geometrically canonical, i.e.* μ *is an isomorphism and* Y *has at worst canonical singularities.*

(2) *If* $p = 3$, *then* $\ell_F(X/k) \leqslant 1$ *and the absolute Frobenius morphism* $F_{X\times_k \overline{k}}$ *of* $X \times_k \overline{k}$ *factors through* μ*:*

$$F_{X\times_k \overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$

(3) *If* $p = 2$, *then* $\ell_F(X/k) \leqslant 2$ *and the second iterated absolute Frobenius morphism* $F^2_{X\times_k \overline{k}}$ *of* $X \times_k \overline{k}$ *factors through* μ*:*

$$F^2_{X\times_k \overline{k}} : X \times_k \overline{k} \to Y \xrightarrow{\mu} X \times_k \overline{k}.$$

Proof. The assertion follows from Remark 3.5 and Proposition 3.6. □

4. Numerically trivial line bundles on log del Pezzo surfaces

The purpose of this section is to give an explicit upper bound on the torsion index of numerically trivial line bundles on log del Pezzo surfaces over imperfect fields (Theorem 4.10). To achieve this result, we use the minimal model program to reduce the problem to the case when our log del Pezzo surface admits a Mori fibre space structure $\pi : X \to B$. The cases $\dim B = 0$ and $\dim B = 1$ will be settled in Theorem 4.1 and Proposition 4.9, respectively.

4.1. Canonical case

In this subsection, we study numerically trivial Cartier divisor on del Pezzo surfaces with canonical singularities.

Theorem 4.1. *Let* k *be a field of characteristic* $p > 0$. *Let* X *be a canonical weak del Pezzo surface over* k *such that* $k = H^0(X, \mathcal{O}_X)$. *Let* L *be a numerically trivial Cartier divisor on* X. *Then the following hold:*

(1) *If* $p \geqslant 5$, *then* $L \sim 0$.

(2) *If* $p = 3$, *then* $3L \sim 0$.

(3) *If* $p = 2$, *then* $4L \sim 0$.

Proof. We first reduce the problem to the case when $-K_X$ is ample. It follows from [39, Theorem 4.2] that $-K_X$ is semi-ample. As $-K_X$ is also big, $|-mK_X|$ induces a birational morphism $f : X \to Y$ to a projective normal surface Y. Then it holds that K_Y is $\mathbb{Q}$-Cartier and $K_X = f^*K_Y$. In particular, Y has at worst canonical singularities. Then [39, Theorem 4.4] enables us to find a numerically trivial Cartier divisor L_Y on Y such that $f^*L_Y \sim L$. Hence, the problem is reduced to the case when $-K_X$ is ample.

We only treat the case when $p = 2$, as the other cases are easier. By Theorem 3.7, the second iterated absolute Frobenius morphism

$$F^2_{X\times_k \overline{k}} : X \times_k \overline{k} \to X \times_k \overline{k}$$

factors through the normalisation $(X \times_k \overline{k})^N_{\mathrm{red}}$ of $(X \times_k \overline{k})_{\mathrm{red}}$:

$$F^2_{X\times_k \overline{k}} : X \times_k \overline{k} \to (X \times_k \overline{k})^N_{\mathrm{red}} \xrightarrow{\mu} X \times_k \overline{k},$$

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where μ denotes the induced morphism. Set $\mathcal{L} := \mathcal{O}_X(L)$ and let $\mathcal{L}_{\overline{k}}$ be the pullback of $\mathcal{L}$ to $X \times_k \overline{k}$. Since $(X \times_k \overline{k})^N_{\mathrm{red}}$ is a normal rational surface by Lemma 3.2, any numerically trivial invertible sheaf is trivial: $\mu^*\mathcal{L}_{\overline{k}} \simeq \mathcal{O}_{(X\times_k\overline{k})^N_{\mathrm{red}}}$. As $F^2_{X\times_k\overline{k}}$ factors through μ, we have that

$$\mathcal{L}^4_{\overline{k}} = (F^2_{X\times_k\overline{k}})^*\mathcal{L}_{\overline{k}} \simeq \mathcal{O}_{X\times_k\overline{k}}.$$

Then it holds that

$$H^0(X, \mathcal{L}^4) \otimes_k \overline{k} \simeq H^0(X \times_k \overline{k}, \mathcal{L}^4_{\overline{k}}) \simeq H^0(X \times_k \overline{k}, \mathcal{O}_{X\times_k\overline{k}}) \neq 0.$$

Hence, we obtain $H^0(X, \mathcal{L}^4) \neq 0$, i.e. $4L \sim 0$. □

4.2. Essential step for the log case

In this subsection, we study the torsion index of numerically trivial line bundles on log del Pezzo surfaces admitting the following special Mori fibre space structure onto a curve.

Notation 4.2. We use the following notations:

(1) k is a field of characteristic $p > 0$.

(2) X is a regular k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$ and $\rho(X) = 2$.

(3) B is a regular projective curve over k such that $k = H^0(B, \mathcal{O}_B)$.

(4) $\pi : X \to B$ is a K_X-Mori fibre space.

(5) Let $R = \mathbb{R}_{\geqslant 0}[\Gamma]$ be the extremal ray which does not correspond to π, where Γ denotes a curve on X. Note that $\pi(\Gamma) = B$. Set $d_\Gamma := \dim_k H^0(\Gamma, \mathcal{O}_\Gamma) \in \mathbb{Z}_{>0}$ and $m_\Gamma := [K(\Gamma) : K(B)] \in \mathbb{Z}_{>0}$. We denote by $\pi_\Gamma : \Gamma \to B$ the induced morphism.

(6) Assume that $K_X \cdot \Gamma > 0$.

Lemma 4.3. *We use Notation 4.2. Then the following hold:*

(7) $\Gamma^2 \leqslant 0$.

(8) *There exists a rational number α such that $0 \leqslant \alpha < 1$ and $(X, \alpha\Gamma)$ is a log del Pezzo pair.*

Proof. The assertion (7) follows from Lemma 4.4. Let us prove (8). By Notation 4.2(2), there is an effective $\mathbb{Q}$-divisor Δ such that (X, Δ) is a log del Pezzo pair. We write $\Delta = \alpha\Gamma + \Delta'$ for some rational number $0 \leqslant \alpha < 1$ and an effective $\mathbb{Q}$-divisor Δ' with $\Gamma \not\subset \mathrm{Supp}(\Delta')$. Since $\overline{\mathrm{NE}}(X)$ is generated by Γ and a fibre F of the morphism $\pi : X \to B$, we conclude that any prime divisor C such that $C \neq \Gamma$ is nef. In particular, Δ' is nef. Hence, $(X, \alpha\Gamma)$ is a log del Pezzo pair. Thus, (8) holds. □

Lemma 4.4. *Let k be a field. Let X be a projective $\mathbb{Q}$-factorial normal surface over k Let $R = \mathbb{R}_{\geqslant 0}[\Gamma]$ is an extremal ray of $\overline{\mathrm{NE}}(X)$, where Γ is a curve on X. If $\Gamma^2 > 0$, then $\rho(X) = 1$.*

Proof. We may apply the same argument as in [37, Theorem 3.21, Proof of the case where $C^2 > 0$ in page 20]. □

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The first step is to prove that $m_\Gamma \leqslant 5$ (Proposition 4.7). To this end, we find an upper bound and a lower bound for α (Lemmas 4.5 and 4.6).

Lemma 4.5. *We use Notation 4.2. Take a closed point b of B and set $F_b := \pi^*(b)$. Let $\kappa(b)$ be the residue field at b and set $d(b) := [\kappa(b) : k]$. Then the following hold:*

(1) $K_X \cdot_k F_b = -2d(b)$.

(2) $\Gamma \cdot_k F_b = m_\Gamma d(b)$.

(3) *If α is a rational number such that $-(K_X + \alpha\Gamma)$ is ample, then $\alpha m_\Gamma < 2$.*

Proof. Let us show (1). We have that

$$\deg_k \omega_{F_b} = (K_X + F_b) \cdot_k F_b = K_X \cdot_k F_b < 0.$$

Hence, Lemma 2.16 implies that

$$K_X \cdot_k F_b = \deg_k \omega_{F_b} = -2d(b).$$

Thus, (1) holds. Clearly, (2) holds.

Let us show (3). Since $-(K_X + \alpha\Gamma)$ is ample, (1) and (2) imply that

$$0 > (K_X + \alpha\Gamma) \cdot_k F_b = -2d(b) + \alpha m_\Gamma d(b).$$

Thus, (3) holds. □

Lemma 4.6. *We use Notation 4.2. Then the following hold:*

(1) $(K_X + \Gamma) \cdot_k \Gamma = -2d_\Gamma < 0$.

(2) *For a rational number β with $0 \leqslant \beta \leqslant 1$, it holds that*

$$(K_X + \beta\Gamma) \cdot_k \Gamma \geqslant d_\Gamma(1 - 3\beta).$$

(2) *If α is a rational number such that $0 \leqslant \alpha < 1$ and $-(K_X + \alpha\Gamma)$ is ample, then it holds that $1/3 < \alpha$.*

Proof. We fix a rational number α such that $0 \leqslant \alpha < 1$ and $-(K_X + \alpha\Gamma)$ is ample, whose existence is guaranteed by Lemma 4.3.

Let us show (1). It holds that

$$(K_X + \Gamma) \cdot_k \Gamma \leqslant (K_X + \alpha\Gamma) \cdot_k \Gamma < 0,$$

where the first inequality follows from $\Gamma^2 \leqslant 0$ and $0 \leqslant \alpha < 1$, whilst the second one holds since $-(K_X + \alpha\Gamma)$ is ample. Therefore, by adjunction and Lemma 2.16, we deduce $(K_X + \Gamma) \cdot_k \Gamma = \deg_k \omega_\Gamma = -2d_\Gamma$. Thus, (1) holds.

Let us show (2). For $k_\Gamma := H^0(\Gamma, \mathcal{O}_\Gamma)$, the equation $d_\Gamma = [k_\Gamma : k]$ (Notation 4.2(5)) implies that

$$K_X \cdot_k \Gamma = \deg_k(\omega_X|_\Gamma) = d_\Gamma \cdot \deg_{k_\Gamma}(\omega_X|_\Gamma) \in d_\Gamma\mathbb{Z}.$$

Combining with $K_X \cdot_k \Gamma > 0$ (Notation 4.2(6)), we obtain $K_X \cdot_k \Gamma \geqslant d_\Gamma$. Hence, it holds that

$$\begin{aligned}(K_X+\beta\Gamma)\cdot_k\Gamma &= (1-\beta)K_X\cdot_k\Gamma+\beta(K_X+\Gamma)\cdot_k\Gamma\\ &= (1-\beta)K_X\cdot_k\Gamma+\beta(-2d_\Gamma)\geqslant(1-\beta)d_\Gamma+\beta(-2d_\Gamma)=d_\Gamma(1-3\beta).\end{aligned}$$

Thus, (2) holds. The assertion (3) follows from (2). □

Proposition 4.7. *We use Notation 4.2. It holds that* $m_\Gamma \leqslant 5$.

Proof. We fix a rational number α such that $0 \leqslant \alpha < 1$ and $-(K_X+\alpha\Gamma)$ is ample, whose existence is guaranteed by Lemma 4.3. Then the inequality $m_\Gamma < 6$ holds by

$$\frac{2}{m_\Gamma} > \alpha > \frac{1}{3},$$

where the first and second inequalities follow from Lemmas 4.5 and 4.6, respectively. □

To prove the main result of this subsection (Proposition 4.9), we first treat the case when $K(\Gamma)/K(B)$ is separable or purely inseparable.

Lemma 4.8. *We use Notation 4.2. Let* L_B *be a numerically trivial Cartier divisor on* B*. Then the following hold:*

(1) *If* $K(\Gamma)/K(B)$ *is a separable extension, then* ω_B^{-1} *is ample and* $L_B \sim 0$.

(2) *If* $K(\Gamma)/K(B)$ *is a purely inseparable morphism of degree* p^e *for some* $e \in \mathbb{Z}_{>0}$*, then* $p^e L_B \sim 0$.

Proof. We first prove (1). Assume that $K(\Gamma)/K(B)$ is a separable extension. Let $\Gamma^N \to \Gamma$ be the normalisation of Γ. Set $\pi_{\Gamma^N} : \Gamma^N \to B$ to be the induced morphism. Since ω_Γ^{-1} is ample, so is $\omega_{\Gamma^N}^{-1}$. Hence, we obtain $H^1(\Gamma^N, \mathcal{O}_{\Gamma^N}) = 0$ (Lemma 2.16). Thanks to the Hurwitz formula (cf. [25, Theorem 4.16 in § 7]), we have that $H^1(B, \mathcal{O}_B) = 0$; thus, ω_B^{-1} is ample (Lemma 2.16). In particular, the numerically trivial Cartier divisor L_B is trivial, i.e. $L_B \sim 0$. Thus, (1) holds.

We now show (2). Since $K(\Gamma)/K(B)$ is a purely inseparable morphism of degree p^e, the eth iterated absolute Frobenius morphism $F_B^e : B \to B$ factors through the induced morphism $\pi_{\Gamma^N} : \Gamma^N \to B$:

$$F_B^e : B \to \Gamma^N \xrightarrow{\pi_{\Gamma^N}} B.$$

It holds that $\pi_{\Gamma^N}^* L_B \sim 0$; hence, $p^e L_B = (F_B^e)^* L_B \sim 0$. Thus, (2) holds. □

Proposition 4.9. *We use Notation 4.2. Let* L *be a numerically trivial Cartier divisor on* X*. Then the following hold:*

(1) *If* $p \geqslant 7$*, then* $L \sim 0$.

(2) *If* $p \in \{3, 5\}$*, then* $pL \sim 0$.

(3) *If* $p = 2$*, then* $4L \sim 0$.

Proof. By [39, Theorem 4.4], there exists a numerically trivial Cartier divisor L_B on B such that $\pi^* L_B \sim L$. If $K(\Gamma)/K(B)$ is separable, then Lemma 4.8(1) implies that $L \sim 0$. Therefore, we may assume that $K(\Gamma)/K(B)$ is not a separable extension. Thanks to Proposition 4.7, we have

$$[K(\Gamma) : K(B)] = m_\Gamma \leqslant 5.$$

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Let us show (1). Assume $p \geqslant 7$. In this case, there does not exist an inseparable extension $K(\Gamma)/K(B)$ with $[K(\Gamma) : K(B)] \leqslant 5$. Thus, (1) holds.

Let us show (2). Assume $p \in \{3, 5\}$. Since $K(\Gamma)/K(B)$ is not a separable extension and $[K(\Gamma) : K(B)] \leqslant 5$, it holds that $K(\Gamma)/K(B)$ is a purely inseparable extension of degree p. Hence, Lemma 4.8(2) implies that $pL \sim 0$. Thus, (2) holds.

Let us show (3). Assume $p = 2$. Since $K(\Gamma)/K(B)$ is not a separable extension and $[K(\Gamma) : K(B)] \leqslant 5$, there are the following three possibilities:

(i) $K(\Gamma)/K(B)$ is a purely inseparable extension of degree 2.

(ii) $K(\Gamma)/K(B)$ is a purely inseparable extension of degree 4.

(iii) $K(\Gamma)/K(B)$ is an inseparable extension of degree 4 which is not purely inseparable.

If (i) or (ii) holds, then Lemma 4.8(2) implies that $4L \sim 0$. Hence, we may assume that (iii) holds. Let $\Gamma^N \to \Gamma$ be the normalisation of Γ. Corresponding to the separable closure of $K(B)$ in $K(\Gamma) = K(\Gamma^N)$, we obtain the following factorisation:

$$\Gamma^N \to B_1 \to B,$$

where $K(\Gamma^N)/K(B_1)$ is a purely inseparable extension of degree two and $K(B_1)/K(B)$ is a separable extension of degree two. In particular, $K(B_1)/K(B)$ is a Galois extension. Set $G := \mathrm{Gal}(K(B_1)/K(B)) = \{\mathrm{id}, \sigma\}$. Since $L_B|_{\Gamma^N} \sim L|_{\Gamma^N} \sim 0$ and the absolute Frobenius morphism $F_{B_1} : B_1 \to B_1$ factors through $\Gamma^N \to B_1$, it holds that $2L_B|_{B_1} \sim 0$. In particular, we have that $H^0(B_1, 2L_B|_{B_1}) \neq 0$. Fix $0 \neq s \in H^0(B_1, 2L_B|_{B_1})$. We obtain

$$0 \neq s\sigma(s) \in H^0(B_1, 4L_B|_{B_1})^G.$$

As $s\sigma(s)$ is G-invariant, $s\sigma(s)$ descends to B, i.e. there is an element

$$t \in H^0(B, 4L_B)$$

such that $t|_{B_1} = s\sigma(s)$. In particular, we obtain $t \neq 0$; hence, $4L_B \sim 0$. Therefore, we have $4L \sim 0$. □

4.3. General case

We are ready to prove the main theorem of this section.

Theorem 4.10. *Let k be a field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type. Let L be a numerically trivial Cartier divisor on X. Then the following hold:*

(1) *If $p \geqslant 7$, then $L \sim 0$.*

(2) *If $p \in \{3, 5\}$, then $pL \sim 0$.*

(3) *If $p = 2$, then $4L \sim 0$.*

Proof. Replacing k by $H^0(X, \mathcal{O}_X)$, we may assume that $k = H^0(X, \mathcal{O}_X)$. Furthermore, replacing X by its minimal resolution, we may assume that X is regular by Lemma 2.7. We run a K_X-MMP:

$$\varphi : X =: X_0 \to X_1 \to \cdots \to X_n.$$

Since $-K_X$ is big, the end result X_n is a K_{X_n}-Mori fibre space. It follows from [39, Theorem 4.4(3)] that there exists a Cartier divisor L_n with $\varphi^* L_n \sim L$. Since also X_n is of del Pezzo type by Lemma 2.9, we may replace X by X_n. Let $\pi : X \to B$ be the induced K_X-Mori fibre space.

If $\dim B = 0$, then we conclude by Theorem 4.1. Hence, we may assume that $\dim B = 1$. Since X is a surface of del Pezzo type, there is an effective $\mathbb{Q}$-divisor such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample. Hence, any extremal ray of $\overline{\mathrm{NE}}(X)$ is spanned by a curve. Note that $\rho(X) = 2$ and a fibre of $\pi : X \to B$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$. Let $R = \mathbb{R}_{\geqslant 0}[\Gamma]$ be the other extremal ray, where Γ is a curve on X. To summarise, (1)–(5) of Notation 4.2 hold. There are the following three possibilities:

(i) $\Gamma^2 \geqslant 0$.

(ii) $\Gamma^2 < 0$ and $K_X \cdot \Gamma \leqslant 0$.

(iii) $\Gamma^2 < 0$ and $K_X \cdot \Gamma > 0$.

Assume (i). In this case, any curve C on X is nef. Since $-(K_X + \Delta)$ is ample, also $-K_X$ is ample. Therefore, we conclude by Theorem 4.1.

Assume (ii). In this case, $-K_X$ is nef and big. Again, Theorem 4.1 implies the assertion of Theorem 4.10.

Assume (iii). In this case, all the conditions (1)–(6) of Notation 4.2 hold. Hence, the assertion of Theorem 4.10 follows from Proposition 4.9. □

5. Results in large characteristic

In this section, we prove the existence of geometrically normal birational models of log del Pezzo surfaces over imperfect fields of characteristic at least seven (Theorem 5.4). As consequences, we prove geometric integrality (Corollary 5.5) and vanishing of irregularity for such surfaces (Theorem 5.7).

5.1. Analysis up to birational modification

The purpose of this subsection is to prove Theorem 5.4. To this end, we establish auxiliary results on Mori fibre spaces (Propositions 5.2 and 5.3). We start by recalling the following well-known relation between the Picard rank and the anti-canonical volume of del Pezzo surfaces.

Lemma 5.1. *Let Y be a smooth weak del Pezzo surface over an algebraically closed field k. Then $\rho(Y) = 10 - K_Y^2$. In particular, it holds that $\rho(Y) \leqslant 9$.*

Proof. Let $Y =: Y_1 \to Y_2 \to \cdots \to Y_n = Z$ be a K_Y-MMP, where Z is a weak del Pezzo surface endowed with a K_Z-Mori fibre space $Z \to B$. It is sufficient to prove the relation $\rho(Z) = 10 - K_Z^2$, which is well known (cf. [21, Theorem 1.28]). □

Proposition 5.2. *Let k be a field of characteristic $p \geqslant 11$. Let X be a regular del Pezzo k-surface such that $k = H^0(X, \mathcal{O}_X)$. Then X is smooth over k.*

Proof. By Theorem 3.7, $X \times_k \overline{k}$ has at most canonical singularities. By [34, Theorem 6.1],

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such singularities are of type A_{p^e-1}. Since $X \times_k \overline{k}$ is a canonical del Pezzo surface, its minimal resolution $\pi : Y \to X \times_k \overline{k}$ is a smooth weak del Pezzo surface and we have

$$9 \geqslant \rho(Y) \geqslant \rho(X \times_k \overline{k}) + \sum_{x \in \mathrm{Sing}(X \times_k \overline{k})} (p-1) \geqslant \sum_{x \in \mathrm{Sing}(X \times_k \overline{k})} 10,$$

where the first inequality follows from Lemma 5.1 and the last inequality holds by $p \geqslant 11$. Thus, we obtain $\mathrm{Sing}(X \times_k \overline{k}) = \emptyset$, as desired. □

Proposition 5.3. *Let k be a field of characteristic $p > 0$. Let X be a regular k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Assume that there is a K_X-Mori fibre space $\pi : X \to B$ to a projective regular k-curve B. Let Γ be a curve which spans the extremal ray of $\overline{\mathrm{NE}}(X)$ not corresponding to π. Then the following hold:*

(1) *If $K_X \cdot \Gamma < 0$ (respectively $\leqslant 0$), then $-K_X$ is ample (respectively nef and big). If $p \geqslant 5$, then ω_B^{-1} is ample and B is smooth over k.*

(2) *If $K_X \cdot \Gamma > 0$ and $p \geqslant 7$, then ω_B^{-1} is ample and B is smooth over k.*

(3) *If $K_X \cdot \Gamma > 0$, $p \geqslant 7$, and k is separably closed, then Γ is a section of π and π is smooth. In particular, X is smooth over k.*

Proof. The first part of assertion (1) follows immediately from Kleimann's criterion for ampleness (respectively [23, Theorem 2.2.16]). Assume $p \geqslant 5$. The anti-canonical model Z of X is geometrically normal by Theorem 3.7, and, thus, $H^1(Z, \mathcal{O}_Z) = 0$. This implies that $H^1(X, \mathcal{O}_X) = 0$ and $H^1(B, \mathcal{O}_B) = 0$. Hence, the assertion (1) holds by Lemmas 2.16 and 2.17.

Let us show (2). The field extension $K(\Gamma)/K(B)$ corresponding to the induced morphism $\pi_\Gamma : \Gamma \to B$ is separable (Proposition 4.7). Thus, B is a curve such that ω_B^{-1} is ample (Lemma 4.8). Since $p > 2$, B is a k-smooth curve by Lemma 2.17. Thus, (2) holds.

Let us show (3). It follows from Proposition 2.18(6) that π is a smooth morphism. Hence, it suffices to show that $\pi_\Gamma : \Gamma \to B$ is a section of π. Since $K(\Gamma)$ is separable over $K(B)$ and B is smooth over k, $K(\Gamma)$ is separable over k, i.e. $K(\Gamma)$ is geometrically reduced over k. Hence, also Γ is geometrically reduced over k. Since $X_{\overline{k}}$ is a smooth projective rational surface with $\rho(X_{\overline{k}}) = 2$, $X_{\overline{k}}$ is a Hirzebruch surface, and $\pi_{\overline{k}} : X_{\overline{k}} \to B_{\overline{k}}$ is a projection. Since the pullback $\Gamma_{\overline{k}}$ of Γ is a curve with $\Gamma_{\overline{k}}^2 < 0$ by Lemma 4.4, $\Gamma_{\overline{k}}$ is a section of $\pi_{\overline{k}} : X_{\overline{k}} \to B_{\overline{k}}$. The base change $\Gamma_{\overline{k}} \to B_{\overline{k}}$ is an isomorphism, hence so is the original one $\pi_\Gamma : \Gamma \to B$. Thus, (3) holds. □

Theorem 5.4. *Let k be a separably closed field of characteristic $p \geqslant 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then there exists a birational map $X \dashrightarrow Y$ to a projective normal k-surface Y such that one of the following properties holds:*

(1) *Y is a regular del Pezzo surface such that $k = H^0(Y, \mathcal{O}_Y)$ and $\rho(Y) = 1$. In particular, Y is geometrically canonical over k. Moreover, if $p \geqslant 11$, then Y is smooth over k.*

(2) *There is a smooth projective morphism $\pi : Y \to B$ such that $B \simeq \mathbb{P}^1_k$ and the fibre $\pi^{-1}(b)$ is isomorphic to $\mathbb{P}^1_{k(b)}$ for any closed point b of B, where $k(b)$ denotes the*

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residue field of b. In particular, Y is smooth over k and $Y \times_k \overline{k}$ is a Hirzebruch surface.

Proof. Let $f : Z \to X$ be the minimal resolution of X. By Lemma 2.7, Z is a k-surface of del Pezzo type. We run a K_Z-MMP:

$$Z =: Z_0 \to Z_1 \to \cdots \to Z_n =: Y.$$

By Lemma 2.9, the surfaces Z_i are of del Pezzo type. The end result Y is a K_Y-Mori fibre space $\pi : Y \to B$. If $\dim B = 0$, then Y is a regular del Pezzo surface; hence, (1) holds by Theorem 3.7 and Proposition 5.2. If $\dim B = 1$, then Proposition 5.3 implies that (2) holds. □

Corollary 5.5. *Let k be a field of characteristic $p \geqslant 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then X is geometrically integral over k.*

Proof. We may assume that k is separably closed. It is enough to show that X is geometrically reduced [40, Lemma 2.2]. By Lemma 2.2, we may replace X by a surface birational to X. Then the assertion follows from Theorem 5.4. □

5.2. Vanishing of $H^1(X, \mathcal{O}_X)$

In this subsection, we prove that surfaces of del Pezzo type over an imperfect field of characteristic $p \geqslant 7$ have vanishing irregularity.

Lemma 5.6. *Let k be a field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. If X is geometrically normal over k, then it holds that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

Proof. The assertion immediately follows from Lemma 3.2. □

Theorem 5.7. *Let k be a field of characteristic $p \geqslant 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.*

Proof. We may assume that k is separably closed. Let $X \dashrightarrow Y$ be the birational morphism as in the statement of Theorem 5.4. Lemma 5.6 implies that $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.

Let $\varphi : W \to X$ and $\psi : W \to Y$ be birational morphisms from a regular projective surface W. Since both Y and W are regular, we have that $H^i(W, \mathcal{O}_W) = 0$ for $i > 0$. Then the Leray spectral sequence implies that $H^1(X, \mathcal{O}_X) = 0$. It is clear that $H^j(X, \mathcal{O}_X) = 0$ for $j \geqslant 2$. □

Remark 5.8. We now give an alternative proof of Theorem 5.7. We use the same notation as in [12, Chapter 9]. Assume that $H^1(X, \mathcal{O}_X) \neq 0$ and let us derive a contradiction. We may assume that k is separably closed. Since X is geometrically integral over k (Corollary 5.5), X has a k-rational point, i.e. $X(k) \neq \emptyset$. By [12, Theorem 9.2.5 and Corollary 9.4.18.3], there exists a scheme $\mathbf{Pic}_{X/k}$ that represents any of the functors $\mathrm{Pic}_{X/k}$, $\mathrm{Pic}_{X/k,(\text{ét})}$, and $\mathrm{Pic}_{X/k,(\text{fppf})}$. Then $\mathbf{Pic}_{X/k}$ is a group k-scheme which is locally of

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finite type over k [12, Proposition 9.4.17] and its connected component $\mathbf{Pic}^0_{X/k}$ containing the identity is an open and closed group subscheme of finite type over k [12, Proposition 9.5.3]. By $H^1(X, \mathcal{O}_X) \neq 0$ and $H^2(X, \mathcal{O}_X) = 0$, $\mathbf{Pic}^0_{X/k}$ is smooth and $\dim \mathbf{Pic}^0_{X/k} > 0$ [12, Remark 9.5.15 and Theorem 9.5.11]. Since k is separably closed, $\mathbf{Pic}^0_{X/k}(k)$ is an infinite set. In particular, there exists a numerically trivial Cartier divisor L on X with $L \not\sim 0$. This contradicts Theorem 4.10.

In characteristic zero, it is known that the image of a variety of Fano type under a surjective morphism remains of Fano type (cf. [14, Theorem 5.12]). The same result is false over imperfect fields of low characteristic as shown in [41, Theorem 1.4]. We now prove that this phenomenon can appear exclusively in low characteristic.

Corollary 5.9. *Let k be a field of characteristic $p \geqslant 7$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$ and let $\pi : X \to Y$ be a projective k-morphism such that $\pi_* \mathcal{O}_X = \mathcal{O}_Y$. Then Y is a k-variety of Fano type. Furthermore, if $\dim Y = 1$, then Y is smooth over k.*

Proof. We distinguish two cases according to $\dim Y$. If $\dim Y = 2$, then π is birational and we conclude by Lemma 2.9. If $\dim Y = 1$, then thanks to the Leray spectral sequence, we have an injection:

$$H^1(Y, \mathcal{O}_Y) \hookrightarrow H^1(X, \mathcal{O}_X),$$

where $H^1(X, \mathcal{O}_X) = 0$ by Theorem 5.7. Therefore, ω_Y^{-1} is ample by Lemma 2.16 and Y is smooth over k by Lemma 2.17. □

6. Purely inseparable points on log del Pezzo surfaces

The aim of this section is to construct purely inseparable points of bounded degree on log del Pezzo surfaces X over C_1-fields of positive characteristic (Theorem 6.12). Since we may take birational model changes, the problem is reduced to the case when X has a Mori fibre space structure $X \to B$. The case when $\dim B = 0$ and $\dim B = 1$ are treated in §§ 6.1 and 6.2, respectively. In § 6.3, we prove the main result of this section (Theorem 6.12).

6.1. Purely inseparable points on regular del Pezzo surfaces

In this subsection, we prove the existence of purely inseparable points with bounded degree on geometrically normal regular del Pezzo surfaces over C_1-fields. If $K_X^2 \leqslant 4$, then we apply the strategy as in [19, Theorem IV.6.8] (Lemma 6.3). We analyse the remaining cases by using a classification result given by [34, § 6] and Proposition 2.27. We first relate the C_r-condition (for definition of C_r-field, see [19, Definition IV.6.4.1]) for a field of positive characteristic to its p-degree.

Lemma 6.1. *Let k be a field of characteristic $p > 0$. If r is a positive integer and k is a C_r-field, then $\text{p-deg}(k) \leqslant r$, where $\text{p-deg}(k) := \log_p [k : k^p]$. In particular, if k is a C_1-field, then $\text{p-deg}(k) \leqslant 1$.*

Proof. Suppose by contradiction that $[k : k^p] \geqslant p^{r+1}$. Let $s_1, \ldots, s_{p^r+1}$ be elements of

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k which are linearly independent over k^p. Let us consider the following homogeneous polynomial of degree p:

$$P := \sum_{k=1}^{p^r+1} s_i x_i^p = s_1 x_1^p + \cdots + s_{p^r+1} x_{p^r+1}^p \in k[x_1, \ldots, x_{p^r+1}].$$

Since $s_1, \ldots, s_{p^r+1}$ are linearly independent over k^p, the polynomial P has only the trivial solution in k. In particular, k is not a C_r-field. □

We then study rational points on geometrically normal del Pezzo surfaces of degree $\leqslant 4$ (compare with [19, Exercise IV.6.8.3]). We need the following result.

Lemma 6.2 (cf. Exercise IV.6.8.3.2 of [19]). *Let k be a C_1-field. Let S be a weighted hypersurface of degree 4 in $\mathbb{P}_k(1, 1, 1, 2)$. Then $S(k) \neq \emptyset$.*

Proof. Let us recall the definition of normic forms [19, Definition IV.6.4.2]. A homogeneous polynomial $h \in k[y_1, \ldots, y_m]$ of degree m is called a normic form if $h = 0$ has only the trivial solution in k. If k has a normic form of degree two, then the same argument as in the proof of [19, Theorem IV.6.7] works.

Suppose now that k does not have a normic form of degree two. We can write $\mathbb{P}_k(1, 1, 1, 2) = \operatorname{Proj} k[x_0, x_1, x_2, x_3]$, where $\deg x_0 = \deg x_1 = \deg x_2 = 1$ and $\deg x_3 = 2$. Let

$$F(x_0, x_1, x_2, x_3) := cx_3^2 + f(x_0, x_1, x_2)x_3 + g(x_0, x_1, x_2) \in k[x_0, x_1, x_2, x_3]$$

be the defining polynomial of S, where $c \in k$ and $f(x_0, x_1, x_2), g(x_0, x_1, x_2) \in k[x_1, x_2, x_3]$. If $c = 0$, then $F(0, 0, 0, 1) = 0$. Thus, we may assume that $c \neq 0$. Fix $(a_0, a_1, a_2) \in k^3 \setminus \{(0, 0, 0)\}$. Set $\alpha := f(a_0, a_1, a_2) \in k$ and $\beta := g(a_0, a_1, a_2) \in k$. Since $h(X, Y) := cX^2 + \alpha XY + \beta Y^2$ is not a normic form, there is $(u, v) \in k^2 \setminus \{(0, 0)\}$ such that $h(u, v) = cu^2 + \alpha uv + \beta v^2 = 0$. Since $c \neq 0$, we obtain $v \neq 0$. Therefore, it holds that $F(a_0, a_1, a_2, u/v) = c(u/v)^2 + \alpha(u/v) + \beta = 0$, as desired. □

Lemma 6.3. *Let X be a geometrically normal regular del Pezzo surface over a C_1-field k of characteristic $p > 0$ such that $k = H^0(X, \mathcal{O}_X)$. If $K_X^2 \leqslant 4$, then $X(k) \neq \emptyset$.*

Proof. Since X is geometrically normal, then it is geometrically canonical by Theorem 3.3. Thus, we can apply Theorem 2.15 and we distinguish the cases according to the degree of K_X.

If $K_X^2 = 1$, then X has a k-rational point by Proposition 2.14(2). If $K_X^2 = 2$, then X can be embedded as a weighted hypersurface of degree 4 in $\mathbb{P}_k(1, 1, 1, 2)$ and we apply Lemma 6.2 to conclude that it has a k-rational point. If $K_X^2 = 3$, then X is a cubic hypersurface in $\mathbb{P}_k^3$, and, thus, it has a k-rational point by definition of C_1-field. If $K_X^2 = 4$, then X is a complete intersection of two quadrics in $\mathbb{P}^4$ and thus it has a k-rational point by [22, Corollary in page 376]. □

We now discuss the existence of purely inseparable points on geometrically normal regular del Pezzo surfaces over C_1-fields.

Proposition 6.4. *Let X be a regular del Pezzo surface over a C_1-field k of characteristic $p \geqslant 7$ such that $k = H^0(X, \mathcal{O}_X)$. Then $X(k) \neq \emptyset$.*

Proof. If X is a smooth del Pezzo surface, we conclude that there exists a k-rational point by [19, Theorem IV.6.8]. If $p \geqslant 11$, then X is smooth by Proposition 5.2 and we conclude.

It suffices to treat the case when $p = 7$ and X is not smooth. By Theorem 3.7(2), X is geometrically canonical. By [34, Theorem 6.1], any singular point of the base change $X_{\overline{k}} = X \times_k \overline{k}$ is of type A_{p^n-1}. It follows from Lemma 5.1 that $X_{\overline{k}}$ has a unique A_6 singular point. Thus, by Lemma 5.1, we have $K_X^2 \leqslant 3$; hence, Lemma 6.3 implies $X(k) \neq \emptyset$. □

Proposition 6.5. *Let X be a regular del Pezzo surface over a C_1-field k of characteristic $p \in \{3, 5\}$ such that $k = H^0(X, \mathcal{O}_X)$. If X is geometrically normal over k, then $X(k^{1/p}) \neq \emptyset$.*

Proof. It is sufficient to consider the case when X is not smooth by [19, Theorem IV.6.8]. By Theorem 3.3, $X_{\overline{k}}$ has canonical singularities.

If $p = 5$ and X is not smooth, then the singularities of $X_{\overline{k}}$ must be of type A_4 or E_8^0 according to [34, Theorem 6.1 and Theorem 6.4]. If $X_{\overline{k}}$ has one singular point of type E_8^0 or two singular points of type A_4, then $K_X^2 = 1$ by Lemma 5.1. Thus, we conclude that X has a k-rational point by Lemma 6.3. If $X_{\overline{k}}$ has a unique singular point of type A_4, it follows from Proposition 2.27 that $X(k^{1/p}) \neq \emptyset$.

If $p = 3$ and X is not smooth, then the singularities of $X_{\overline{k}}$ must be of type A_2, A_8, E_6^0, or E_8^0 according to [34, Theorem 6.1 and Theorem 6.4]. If one of the singular points is of the type A_8, E_6^0, and E_8^0, then $K_X^2 \leqslant 3$ by Lemma 5.1, and we conclude $X(k) \neq \emptyset$ by Lemma 6.3. Thus, we may assume that all the singularities of $X_{\overline{k}}$ are of type A_2. If there is a unique singularity of type A_2 on $X_{\overline{k}}$, then it follows from Proposition 2.27 that $X(k^{1/3}) \neq \emptyset$. Therefore, we may assume that there are at least two singularities of type A_2 on $X_{\overline{k}}$. Then it holds that $K_X^2 \leqslant 5$. By [9, Table 8.5 in page 431], we have that $K_X^2 \neq 5$; hence, $K_X^2 \leqslant 4$. Thus, Lemma 6.3 implies $X(k) \neq \emptyset$. □

Proposition 6.6. *Let X be a regular del Pezzo surface over a C_1-field k of characteristic $p = 2$ such that $k = H^0(X, \mathcal{O}_X)$. If X is geometrically normal, then $X(k^{1/4}) \neq \emptyset$.*

Proof. It is sufficient to consider the case when X is not smooth by [19, Theorem IV.6.8]. The singularities of $X_{\overline{k}}$ are canonical by Theorem 3.3. Hence, by [34, Theorem in page 57], they must be of type A_1, A_3 A_7, D_n^0 with $4 \leqslant n \leqslant 8$ or E_n^0 for $n = 6, 7, 8$. We distinguish five cases for the singularities appearing on $X_{\overline{k}}$.

(1) There exists at least a singular point of type A_7, D_n^0 with $n \geqslant 5$ or E_n^0 for $n = 6, 7, 8$.

(2) There are at least two singular points with one being of type A_3.

(3) There exists at least one singular point of type D_4^0.

(4) There is a unique singular point of type A_3.

(5) All the singular points are of type A_1.

In case (1), it holds that $K_X^2 \leqslant 4$. Hence, we obtain $X(k) \neq \emptyset$ by Lemma 6.3. In case (2), if $K_X^2 \leqslant 4$, then Lemma 6.3 again implies $X(k) \neq \emptyset$. Hence, we may assume that $K_X^2 = 5$. Then there exist exactly two singular points P and Q on $X_{\overline{k}}$ such that P is of type A_3 and Q is of type A_1. However, this cannot occur by [9, Table 8.5 at page 431].

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In case (3), we have that $K_X^2 \leqslant 5$. However, a D_4^0 singularity cannot appear on a del Pezzo of degree five according to [9, Table 8.5 at page 431]. Thus, $K_X^2 \leqslant 4$ and Lemma 6.3 implies $X(k) \neq \emptyset$. In case (4), we apply Proposition 2.27 to conclude that $X(k^{1/4}) \neq \emptyset$.

In case (5), consider $X_{(k^{\text{sep}})^{1/2}}$. By Proposition 2.27, on $X_{(k^{\text{sep}})^{1/2}}$ there are singular points $\{P_i\}_{i=1}^m$ of type A_1 such that $\kappa(P_i) = (k^{\text{sep}})^{1/2}$ and their union $\coprod_i P_i$ is the non-smooth locus of $X_{(k^{\text{sep}})^{1/2}}$. Let $Y = \text{Bl}_{\coprod_i P_i} X_{(k^{\text{sep}})^{1/2}}$ be the blowup of $X_{(k^{\text{sep}})^{1/2}}$ along $\coprod_i P_i$. Since each P_i is a $(k^{\text{sep}})^{1/2}$-rational point whose base change to the algebraic closure is a canonical singularity of type A_1, the surface Y is smooth. Since the closed subscheme $\coprod_i P_i$ is invariant under the action of the Galois group $\text{Gal}((k^{\text{sep}})^{1/2}/k^{1/2})$, the birational $(k^{\text{sep}})^{1/2}$-morphism $Y \to X_{(k^{\text{sep}})^{1/2}}$ descends to a birational $k^{1/2}$-morphism $Z \to X_{k^{1/2}}$, where Z is a smooth projective surface over $k^{1/2}$ whose base change to the algebraic closure is a rational surface. It holds that $Z(k^{1/2}) \neq \emptyset$ by [19, Theorem IV.6.8], which implies $X(k^{1/2}) \neq \emptyset$. □

6.2. Purely inseparable points on Mori fibre spaces

In this subsection, we discuss the existence of purely inseparable points on log del Pezzo surfaces over C_1-fields admitting Mori fibre space structures onto curves. We start by recalling auxiliary results.

Lemma 6.7. *Let k be a C_1-field and let C be a regular projective curve such that $k = H^0(C, \mathcal{O}_C)$ and $-K_C$ is ample. Then it holds that $C \simeq \mathbb{P}_k^1$. In particular, $C(k) \neq \emptyset$.*

Proof. Since C is a geometrically integral conic curve in $\mathbb{P}_k^2$ (Lemma 2.16), the assertion follows from the definition of C_1-field. □

Lemma 6.8. *Let X be a regular projective surface over a C_1-field k of characteristic $p > 0$ such that $k = H^0(X, \mathcal{O}_X)$. Let $\pi : X \to B$ be a K_X-Mori fibre space to a regular projective curve B. Then the following hold:*

(1) *Let $k \subset k'$ be an algebraic field extension. If $B(k') \neq \emptyset$, then $X(k') \neq \emptyset$.*
(2) *If $-K_B$ is ample, then $X(k) \neq \emptyset$.*

Proof. Let us show (1). Let b be a closed point in B such that $k \subset \kappa(b) \subset k'$. By Proposition 2.18, the fibre X_b is a conic in $\mathbb{P}^2_{\kappa(b)}$. By [22, Corollary in page 377], $\kappa(b)$ is a C_1-field; hence, we deduce $X_{\kappa(b)}(\kappa(b)) \neq \emptyset$. Thus, (1) holds. The assertion (2) follows from Lemma 6.7 and (1) for the case when $k' = k$. □

To discuss the case when $p = 2$, we first handle a complicated case in characteristic two.

Proposition 6.9. *Let k be a field of characteristic two such that $[k : k^2] \leqslant 2$. Let X be a regular k-surface of del Pezzo type and let $\pi : X \to B$ be a K_X-Mori fibre space to a curve B. Let Γ be a curve which spans the K_X-negative extremal ray which is not corresponding to π. Assume that*

(1) *$K_X \cdot \Gamma > 0$ and*

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(2) *$K(\Gamma)/K(B)$ is an inseparable extension of degree four which is not purely inseparable.*

Then $-K_B$ is ample.

Proof. We divide the proof into several steps.

Step 1. In order to show the assertion of Proposition 6.9, we may assume that

(3) B is not smooth over k;

(4) p-deg$(k) = 1$, i.e. $[k : k^2] = 2$; and

(5) the generic fibre of π is not geometrically reduced.

Proof. If (3) does not hold, then B is a smooth curve over k. Since $(X_{\overline{k}})_{\mathrm{red}}$ is a rational surface by Lemma 3.2, $B_{\overline{k}}$ is a smooth rational curve. Then $-K_B$ is ample, as desired. Thus, we may assume (3). From now on, we assume (3).

If (4) does not hold, then k is a perfect field. In this case, B is smooth over k, which contradicts (3). Thus, we may assume (4).

Let us prove the assertion of Proposition 6.9 if (5) does not hold. In this case, the generic fibre $X_{K(B)}$ of $\pi : X \to B$ is a geometrically integral regular conic over $K(B)$. Thus, it is smooth over $K(B)$ by Lemma 2.17. We use notation as in Notation 4.2. Lemma 4.3(8) enables us to find a rational number α such that $0 \leqslant \alpha < 1$ and $(X, \alpha\Gamma)$ is a log del Pezzo pair. Then Lemma 4.5(3) implies that $\alpha m_\Gamma < 2$. Since our assumption (2) implies $m_\Gamma = [K(\Gamma) : K(B)] = 4$, we have that $\alpha < 1/2$. By the assumption (2) and $\alpha < 1/2$, the induced pair $(X_{\overline{K(B)}}, \alpha\Gamma|_{X_{\overline{K(B)}}})$ on the geometric generic fibre is F-pure. It follows from [10, Corollary 4.10] that $-K_B$ is ample. Hence, we may assume that (5) holds. This completes the proof of Step 1. □

From now on, we assume that (3)–(5) of Step 1 hold.

Step 2. X and B are geometrically integral over k. X is not geometrically normal over k.

Proof. Since $[k : k^2] = 2$, it follows from [35, Theorem 2.3] that X and B are geometrically integral over k (note that $\log_2[k : k^2]$ is called the degree of imperfection for k in [35, Theorem 2.3]). If X is geometrically normal over k, then also B is geometrically normal over k, i.e. B is smooth over k. This contradicts (3) of Step 1. This completes the proof of Step 2. □

We now introduce some notation. Set $k_1 := k^{1/2}$. By Step 2, $X \times_k k_1$ is integral and non-normal (cf. [42, Proposition 2.10(3)]). Let $\nu : X_1 := (X \times_k k_1)^N \to X \times_k k_1$ be its normalisation. Let $X_1 \to B_1$ be the Stein factorisation of the induced morphism $X_1 \to X \to B$. To summarise, we have a commutative diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{\ \nu\ } & X \times_k k_1 & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B \times_k k_1 & \longrightarrow & B. \end{array}$$

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Let $C \subset X \times_k k_1$ and $D \subset X_1$ be the closed subschemes defined by the conductors for ν. For $K := K(B)$, we apply the base change $(-) \times_B \operatorname{Spec} K$ to the above diagram:

$$\begin{array}{ccccc} V_1 & \longrightarrow & V \times_K L & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K_1 & \longrightarrow & \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K, \end{array}$$

where $V := X \times_B K$, $L := K(B \times_k k_1) = K(B) \otimes_k k_1$, and $K_1 = K(B_1)$. Since taking Stein factorisations commute with flat base changes, the morphism $V_1 \to \operatorname{Spec} K_1$ coincides with the Stein factorisation of the induced morphism $V_1 \to \operatorname{Spec} K$.

Step 3. C dominates B.

Proof. Assuming that C does not dominate B, let us derive a contradiction. Since B is geometrically integral over k (Step 2), we can find a non-empty open subset B' of B such that B' is smooth over k and the image of C on B is disjoint from B'. Let B'_1, X', and X'_1 be the inverse images of B' to B_1, X, and X_1, respectively. Then the resulting diagram is as follows:

$$\begin{array}{ccccc} X'_1 & \xrightarrow{\simeq} & X' \times_k k_1 & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi'} \\ B'_1 & \xrightarrow{\simeq} & B' \times_k k_1 & \longrightarrow & B'. \end{array}$$

Since $X'_1 \simeq X' \times_k k_1 = X' \times_k k^{1/2}$ is normal, it holds that X' is geometrically normal over k.

Let $\pi'_{\overline{k}} : X'_{\overline{k}} \to B'_{\overline{k}}$ be the base change of π' to the algebraic closure $\overline{k}$. Since X' is geometrically normal over k, $X'_{\overline{k}}$ is a normal surface. Note that $B'_{\overline{k}}$ is a smooth curve. Since general fibres of $\pi'_{\overline{k}} : X'_{\overline{k}} \to B'_{\overline{k}}$ are $K_{X'_{\overline{k}}}$-negative and $(\pi'_{\overline{k}})_*\mathcal{O}_{X'_{\overline{k}}} = \mathcal{O}_{B'_{\overline{k}}}$, general fibres of $\pi'_{\overline{k}}$ are isomorphic to $\mathbb{P}^1_{\overline{k}}$. Then the generic fibre of $\pi'_{\overline{k}} : X'_{\overline{k}} \to B'_{\overline{k}}$ is smooth, hence so is the generic fibre of $\pi : X \to B$. This contradicts (5) of Step 1. This completes the proof of Step 3. □

Step 4. The following hold:

(i) L/K is a purely inseparable extension of degree two.

(ii) V is a regular conic curve on $\mathbb{P}^2_K$ which is not geometrically reduced over K.

(iii) $V_1 \to V \times_K L$ is the normalisation of $V \times_K L$.

(iv) $V \times_K L$ is an integral scheme which is not regular.

(v) The restriction $D|_{V_1}$ of the conductor D to V_1 satisfies $D_{V_1} = Q$, where Q is a K_1-rational point.

(vi) V_1 is isomorphic to $\mathbb{P}^1_{K_1}$.

(vii) $[K_1 : L]$ is a purely inseparable extension of degree two, and $K_1 = K^{1/2}$.

Proof. The assertions (i)–(iii) follow from the construction. Step 3 implies (iv). Let us show (v). For the induced morphism $\varphi : V_1 \to V$, we have that

$$K_{V_1} + D|_{V_1} \sim \varphi^* K_V.$$

Since $-K_V$ is ample, it holds that

$$0 > \deg_{K_1}(K_{V_1} + D|_{V_1}) \geqslant -2 + \deg_{K_1}(D|_{V_1}),$$

which implies $\deg_{K_1}(D|_{V_1}) \leqslant 1$. Step 3 implies that $D|_{V_1} \neq 0$; hence, $D|_{V_1}$ consists of a single rational point. Thus, (v) holds.

Let us show (vi). Since V_1 has a K_1-rational point around which V_1 is regular, V_1 is smooth around this point. In particular, Lemma 2.2 implies that V_1 is geometrically reduced. Then V_1 is a geometrically integral conic curve in $\mathbb{P}^2_{K_1}$. Therefore, V_1 is smooth over K_1. Since V_1 has a K_1-rational point, V_1 is isomorphic to $\mathbb{P}^1_{K_1}$. Thus, (vi) holds.

Let us show (vii). The inclusion $K_1 \subset K^{1/2}$, which is equivalent to $K_1^2 \subset K$, follows from the fact that K is algebraically closed in $K(V)$ and the following:

$$K_1^2 \subset K(V_1)^2 = K(V \times_K L)^2 = (K(V) \otimes_K L)^2 \subset K(V).$$

It follows from [2, Theorem 3] that the p-degree p-deg(K) is two, i.e. $[K^{1/2} : K] = 4$ (note that the p-degree is called the degree of imperfection in [2]). Hence, it is enough to show that $K_1 \neq L$. Assume that $K_1 = L$. Then V_1 is smooth over L by (vi). Hence, $V \times_K L$ is geometrically integral over L. Therefore, V is geometrically integral over K, which contradicts (5) of Step 1. This completes the proof of Step 4. □

Step 5. Set-theoretically, C does not contain $\Gamma \times_k k_1$.

Proof. Assuming that C contains $\Gamma \times_k k_1$, let us derive a contradiction. In this case, the set-theoretic inclusion

$$f^{-1}(\Gamma) \subset \nu^{-1}(C) = D$$

holds, where $f : X_1 \to X$ is the induced morphism. Since $B_1 \to B$ is a universal homeomorphism and the geometric generic fibre $\Gamma \times_B \operatorname{Spec} \overline{K}$ of $\Gamma \to B$ consists of two points, the geometric generic fibre of $D \to B_1$ contains two distinct points. In particular, it holds that $\deg_{K_1}(D|_{V_1}) \geqslant 2$. However, this contradicts (v) of Step 4. This completes the proof of Step 5. □

Step 6. $-K_{B_1}$ is ample.

Proof. It follows from Lemma 4.3(8) that there is a rational number α such that $0 \leqslant \alpha < 1$ and $(X, \alpha\Gamma)$ is a log del Pezzo pair. Consider the pullback:

$$K_{X_1} + D + \alpha f^*\Gamma = f^*(K_X + \alpha\Gamma).$$

Take the geometric generic fibre W of $\pi_1 : X_1 \to B_1$, i.e. $W = V_1 \times_{K_1} \operatorname{Spec} \overline{K}_1 \simeq \mathbb{P}^1_{\overline{K}_1}$ (Step 4(vi)). It is clear that $-(K_W + (D + \alpha f^*\Gamma)|_W)$ is ample. Since $D|_{V_1} = Q$ is a rational point (Step 4(v)), its pullback $D|_W =: Q_W$ to W is a closed point on W. As

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$-(K_W + (D+\alpha f^*\Gamma)|_W)$ is ample, all the coefficients of $B := (\alpha f^*\Gamma)|_W$ must be less than one. Therefore, Step 5 implies that $(W, (D+\alpha f^*\Gamma)|_W)$ is F-pure. It follows from [10, Corollary 4.10] that $-K_{B_1}$ is ample. This completes the proof of Step 6. □

Step 7. $-K_B$ is ample.

Proof. As $-K_{B_1}$ is ample (Step 6), Lemma 2.16 implies that $H^1(B_1, \mathcal{O}_{B_1}) = 0$. Since $K(B_1) = K(B)^{1/2}$ (Step 4(vii)), the morphism $B_1 \to B$ coincides with the absolute Frobenius morphism of B. Hence, B_1 and B are isomorphic as schemes. Thus, the vanishing $H^1(B_1, \mathcal{O}_{B_1}) = 0$ implies $H^1(B, \mathcal{O}_B) = 0$. Then $-K_B$ is ample by Lemma 2.16. This completes the proof of Step 7. □

Step 7 completes the proof of Proposition 6.9. □

Proposition 6.10. *Let X be a regular k-surface of del Pezzo type over a C_1-field k of characteristic $p > 0$ such that $k = H^0(X, \mathcal{O}_X)$. Let $\pi : X \to B$ be a K_X-Mori fibre space to a regular projective curve. Then the following hold:*

(1) *If $p \geqslant 7$, then $X(k) \neq \emptyset$.*

(2) *If $p = \{3, 5\}$, then $X(k^{1/p}) \neq \emptyset$.*

(3) *If $p = 2$, then $X(k^{1/4}) \neq \emptyset$.*

Proof. Let $R = \mathbb{R}_{\geqslant 0}[\Gamma]$ be the extremal ray of $\overline{\mathrm{NE}}(X)$ not corresponding to $\pi : X \to B$. In particular, we have $\pi(\Gamma) = B$. We distinguish two cases:

(I) $K_X \cdot \Gamma \leqslant 0$;

(II) $K_X \cdot \Gamma > 0$.

Suppose that (I) holds. In this case, $-K_X$ is nef and big. If $p > 2$, then the generic fibre $X_{K(B)}$ is a smooth conic. In particular, the base change $X_{\overline{K(B)}}$ is strongly F-regular. By [10, Corollary 4.10], $-K_B$ is ample. Hence, Proposition 6.10 implies $X(k) \neq \emptyset$.

We now treat the case when (I) holds and $p = 2$. Then $-K_X$ is semi-ample and big. Let Z be its anti-canonical model. In particular, Z is a canonical del Pezzo surface. By Theorem 3.7, we have $\ell_F(Z/k) \leqslant 2$. Therefore, for $k_W := k^{1/4}$ and $W := (Z \times_k k_W)^N_{\mathrm{red}}$, W is geometrically normal over k_W. In particular, $H^0(W, \mathcal{O}_W) = k_W = k^{1/4}$. We have the following commutative diagram:

$$\begin{array}{ccccc}
Y & \xrightarrow{\nu} & X \times_k k^{1/4} & \longrightarrow & X \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\
W & \xrightarrow{\mu} & Z \times_k k^{1/4} & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} k^{1/4} & = & \operatorname{Spec} k^{1/4} & \longrightarrow & \operatorname{Spec} k,
\end{array}$$

where μ and ν are the normalisations. It follows from Theorem 3.3 that W is geometrically klt and $H^1(W, \mathcal{O}_W) = 0$. Since the morphism $Y \to W$ is birational and W is klt by Proposition 2.3, it holds that $H^1(Y, \mathcal{O}_Y) = 0$.

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Consider the Stein factorisation $\pi_1\colon Y \to B_1$ of the induced morphism $Y \to X \xrightarrow{\pi} B$. Since $H^1(Y, \mathcal{O}_Y) = 0$, we conclude that $H^1(B_1, \mathcal{O}_{B_1}) = 0$. In particular, since k_W is a C_1-field, it holds that $B_1 \simeq \mathbb{P}^1_{k_W}$ (Lemma 6.7). Thanks to [40, Theorem 4.2], we can find an effective divisor D on Y such that $K_Y + D = f^*K_X$. Since $-K_X$ is big, also $-K_Y$ is big. Fix a general k_W-rational point $c \in B_1$ and let F_c be its π_1-fibre. Since we take c to be general, F_c avoids the non-regular points of Y. By adjunction, $\omega_{F_c}^{-1}$ is ample. This implies that F is a conic on $\mathbb{P}^2_{k_W}$. Hence, $Y(k^{1/4}) = Y(k_W) \neq \emptyset$. Therefore, we deduce $X(k^{1/4}) \neq \emptyset$.

We suppose (II) holds. We have $[K(\Gamma) : K(B)] \leqslant 5$ by Proposition 4.7. If $K(\Gamma)/K(B)$ is separable, then $-K_B$ is ample (Lemma 4.8). Then Proposition 6.10 implies $X(k) \neq \emptyset$. Hence, we may assume that $K(\Gamma)/K(B)$ is inseparable. If $K(\Gamma)/K(B)$ is not purely inseparable, then $-K_B$ is ample by Proposition 6.9. Again, Proposition 6.10 implies $X(k) \neq \emptyset$. Hence, it is enough to treat the case when $K(\Gamma)/K(B)$ is purely inseparable. Since $[K(\Gamma) : K(B)] \leqslant 5$, it suffices to prove that $X(k^{1/p^e}) \neq \emptyset$ for the positive integer e defined by $[K(\Gamma) : K(B)] = p^e$. Set $C := \Gamma^N$. Since ω_Γ^{-1} is ample, also $-K_C$ is ample. Hence, Proposition 6.10 implies $C(k') \neq \emptyset$, where $k' := H^0(C, \mathcal{O}_C)$. Since

$$k'^{p^e} \subset K(\Gamma)^{p^e} \subset K(B),$$

it holds that $k'^{p^e} \subset k$. Therefore, we obtain $X(k^{1/p^e}) \neq \emptyset$, as desired. □

6.3. General case

In this subsection, using the results proven above, we prove the main result in this section (Theorem 6.12). We present a generalisation of the Lang–Nishimura theorem on rational points. Although the argument is similar to the one in [31, Proposition A.6], we include the proof for the sake of completeness.

Lemma 6.11 (Lang–Nishimura). *Let k be a field. Let $f : X \dashrightarrow Y$ be a rational map between k-varieties. Suppose that X is regular and Y is proper over k. Fix a closed point P on X. Then there exists a closed point Q on Y such that $k \subset \kappa(Q) \subset \kappa(P)$, where $\kappa(P)$ and $\kappa(Q)$ denote the residue fields.*

Proof. The proof is by induction on $n := \dim X$. If $n = 0$, then there is nothing to show. Suppose $n > 0$. Consider the blowup $\pi : \mathrm{Bl}_P X \to X$ at the closed point P. Since X is regular, the π-exceptional divisor E is isomorphic to $\mathbb{P}^{n-1}_{\kappa(P)}$ by [25, § 8, Theorem 1.19]. Consider now the induced map $f : \mathrm{Bl}_P X \dashrightarrow Y$. By the valuative criterion of properness, the map f induces a rational map $E = \mathbb{P}^{n-1}_{\kappa(P)} \dashrightarrow Y$ from the π-exceptional divisor E. Then by the induction hypothesis, Y has a closed point Q whose residue field is contained in $\kappa(P)$. □

Theorem 6.12. *Let k be a C_1-field of characteristic $p > 0$. Let X be a k-surface of del Pezzo type such that $k = H^0(X, \mathcal{O}_X)$. Then the following hold:*

(1) *If $p \geqslant 7$, then $X(k) \neq \emptyset$;*

(2) *if $p \in \{3, 5\}$, then $X(k^{1/p}) \neq \emptyset$;*

(3) *if $p = 2$, then $X(k^{1/4}) \neq \emptyset$.*

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Proof. Let $Y \to X$ be the minimal resolution of X. We run a K_Y-MMP $Y =: Y_0 \to Y_1 \to \cdots \to Y_n =: Z$. Note that the end result is a Mori fibre space. Thanks to Lemma 6.11, we may replace X by Z. Hence, it is enough to treat the following two cases:

(i) X is a regular del Pezzo surface with $\rho(X) = 1$.

(ii) There exists a Mori fibre space structure $\pi : X \to B$ to a curve B.

Assume (i). By Lemma 6.1, we have $\text{p-deg}(k) \leqslant 1$. Therefore, X is geometrically normal by [11, Theorem 14.1]. Thus, we conclude by Propositions 6.4, 6.5, and 6.6. If (ii) holds, then the assertion follows from Proposition 6.10. □

7. Pathological examples

In this section, we collect pathological features appearing on surfaces of del Pezzo type over imperfect fields.

7.1. Summary of known results

We first summarise previously known examples of pathologies appearing on del Pezzo surfaces over imperfect fields.

7.1.1. Geometric properties. We have shown that if $p \geqslant 7$ and X is a surface of del Pezzo type, then X is geometrically integral (Corollary 5.5). We have established a partial result on geometric normality (Theorem 5.4). Let us summarise known examples in small characteristic related to these properties.

(1) Let $\mathbb{F}$ be a perfect field of characteristic $p > 0$ and let $k := \mathbb{F}(t_1, t_2, t_3)$. Then

$$X := \operatorname{Proj} k[x_0, x_1, x_2, x_3]/(x_0^p + t_1 x_1^p + t_2 x_2^p + t_3 x_3^p)$$

is a regular projective surface which is not geometrically reduced over k. It is easy to show that $H^0(X, \mathcal{O}_X) = k$. If the characteristic of k is two or three, then $-K_X$ is ample; hence, X is a regular del Pezzo surface.

(2) There exist a field of characteristic $p = 2$ and a regular del Pezzo surface X over k such that $H^0(X, \mathcal{O}_X) = k$, X is geometrically reduced over k, and X is not geometrically normal over k (see [26, Main Theorem]).

(3) If k is an imperfect field of characteristic $p = 2, 3$ there exists a geometrically normal regular del Pezzo surface X of Picard rank one which is not smooth (see [11, § 14, Equation (27)]). In [11, Theorem 14.8], an example of a regular geometrically integral but geometrically non-normal del Pezzo surface of Picard rank two is constructed when $p = 2$.

(4) If k is an imperfect field of characteristic $p \in \{2, 3\}$, then there exists a k-surface X of del Pezzo type such that $H^0(X, \mathcal{O}_X) = k$, X is geometrically reduced over k, and X is not geometrically normal over k [41].

7.1.2. Vanishing of $H^1(X, \mathcal{O}_X)$. We have shown that if X is a surface of del Pezzo type over a field of characteristic $p \geqslant 7$, then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Let us summarise known examples in small characteristic which violate the vanishing of $H^1(X, \mathcal{O}_X)$.

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(1) If k is an imperfect field of characteristic $p = 2$, then there exists a regular weak del Pezzo surface X such that $H^1(X, \mathcal{O}_X) \neq 0$ (see [33]).

(2) There exist an imperfect field of characteristic $p = 2$ and a regular del Pezzo surface X such that $H^1(X, \mathcal{O}_X) \neq 0$ (see [26, Main theorem]).

(3) If k is an imperfect field of characteristic $p \in \{2, 3\}$, then there exists a surface X of del Pezzo type such that $H^1(X, \mathcal{O}_X) \neq 0$ (see [41]).

Remark 7.1. Since $h^1(X, \mathcal{O}_X)$ is a birational invariant for surfaces with klt singularities, the previous examples do not admit regular k-birational models which are geometrically normal. This shows that Theorem 5.4 cannot be extended to characteristics two and three.

7.2. Non-smooth regular log del Pezzo surfaces

In this subsection, we construct examples of regular k-surfaces of del Pezzo type which are not smooth (cf. Theorem 5.4).

Proposition 7.2. *Let k be an imperfect field of characteristic $p > 0$. Then there exists a k-regular surface X of del Pezzo type which is not smooth over k.*

Proof. Fix a k-line L on $\mathbb{P}^2_k$. Let $Q \in L$ be a closed point such that $k(Q)/k$ is a purely inseparable extension of degree p whose existence is guaranteed by the assumption that k is imperfect. Consider the blowup $\pi : X \to \mathbb{P}^2_k$ at the point Q. We have

$$K_X = \pi^* K_{\mathbb{P}^2_k} + E \quad \text{and} \quad \widetilde{L} + E = \pi^* L,$$

where E denotes the π-exceptional divisor and $\widetilde{L}$ is the proper transform of L. Since $\widetilde{L} \cup E$ is simple normal crossing and the $\mathbb{Q}$-divisor

$$-(K_X + \widetilde{L} + \epsilon E) = \pi^*(K_X + L) - \epsilon E$$

is ample for any $0 < \epsilon \ll 1$, the pair $(X, (1-\delta)\widetilde{L} + \epsilon E)$ is log del Pezzo for $0\delta \ll 1$. Hence, X is of del Pezzo type.

It is enough to show that X is not smooth. There exists an affine open subset $\operatorname{Spec} k[x, y] = \mathbb{A}^2_k$ of $\mathbb{P}^2_k$ such that $Q \in \operatorname{Spec} k[x, y]$ and the maximal ideal corresponding to Q can be written as $(x^p - \alpha, y)$ for some $\alpha \in k \setminus k^p$. Let X' be the inverse image of $\operatorname{Spec} k[x, y]$ by π. Since blowups commute with flat base changes, the base change $X'_{\overline{k}}$ is isomorphic to the blowup of $\operatorname{Spec} \overline{k}[x, y]$ along the non-reduced ideal $((x-\beta)^p, y)$, where $\beta \in \overline{k}$ with $\beta^p = \alpha$.

After choosing appropriate coordinate, $X'_{\overline{k}}$ is isomorphic to the blowup of $\mathbb{A}^2_{\overline{k}} = \operatorname{Spec} \overline{k}[x', y']$ along (x'^p, y'). We can directly check that $X'_{\overline{k}}$ contains an affine open subset of the form $\operatorname{Spec} k[s, y, u]/(st - u^p)$, which is not smooth. □

Remark 7.3. The surface X constructed in Proposition 7.2 is del Pezzo (respectively weak del Pezzo) if and only if $p = 2$ (respectively $p \leqslant 3$). Indeed, $-E^2 = [k(Q) : k] = p$ implies $K_X \cdot_k E = (K_X + E) \cdot_k E - E^2 = -2p + p = -p$. Thus, the desired conclusion

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follows from

$$K_X \cdot_k \widetilde{L} = K_X \cdot_k \pi^* L - K_X \cdot_k E = -3 + p.$$

8. Applications to del Pezzo fibrations

In this section, we give applications of Theorems 4.10 and 6.12 on log del Pezzo surfaces over imperfect fields to the birational geometry of threefold fibrations. The first application is to rational chain connectedness.

Theorem 8.1. *Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi : V \to B$ be a projective k-morphism such that $\pi_* \mathcal{O}_V = \mathcal{O}_B$, V is a normal threefold over k, and B is a smooth curve over k. Assume that there exists an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is klt and $-(K_V + \Delta)$ is π-nef and π-big. Then the following hold:*

(1) *There exists a curve C on V such that $C \to B$ is surjective and the following properties hold:*
   (a) *If $p \geqslant 7$, then $C \to B$ is an isomorphism.*
   (b) *If $p \in \{3, 5\}$, then $K(C)/K(B)$ is a purely inseparable extension of degree $\leqslant p$.*
   (c) *If $p = 2$, then $K(C)/K(B)$ is a purely inseparable extension of degree $\leqslant 4$.*

(2) *If B is a rational curve, then V is rationally chain connected.*

Proof. Let us show (1). Thanks to [19, Chapter IV, Theorem 6.5], $K(B)$ is a C_1-field. Then Theorem 6.12 implies the assertion (1). The assertion (2) follows from (1) and the fact that general fibres are rationally connected (see Lemma 3.2). □

The second application is to Cartier divisors on Mori fibre spaces which are numerically trivial over the bases.

Theorem 8.2. *Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi : V \to B$ be a projective k-morphism such that $\pi_* \mathcal{O}_V = \mathcal{O}_B$, where X is a $\mathbb{Q}$-factorial normal quasi-projective threefold and B is a smooth curve. Assume that there exists an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is klt and $\pi : V \to B$ is a $(K_V + \Delta)$-Mori fibre space. Let L be a π-numerically trivial Cartier divisor on V. Then the following hold:*

(1) *If $p \geqslant 7$, then $L \sim_\pi 0$.*
(2) *If $p \in \{3, 5\}$, then $p^2 L \sim_\pi 0$.*
(3) *If $p = 2$, then $16L \sim_\pi 0$.*

Proof. We only prove the theorem in the case when $p = 2$ since the other cases are similar and easier. Since the generic fibre $V_{K(B)}$ is a $K(B)$-surface of del Pezzo type, we have by Theorem 4.10 that $4L|_{V_{K(B)}} \sim 0$. Therefore, $4L$ is linearly equivalent to a vertical divisor, i.e. we have

$$4L \sim \sum_{i=1}^{r} \ell_i D_i,$$

where $\ell_i \in \mathbb{Z}$ and D_i is a prime divisor such that $\pi(D_i)$ is a closed point b_i.

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Since $\rho(V/B) = 1$ and V is $\mathbb{Q}$-factorial, all the fibres of π are irreducible. Hence, we can write $\pi^*(b_i) = n_i D_i$ for some $n_i \in \mathbb{Z}_{>0}$. Let m_i be the Cartier index of D_i, i.e. the minimum positive integer m such that mD_i is Cartier. Since the divisor $\pi^*(b_i) = n_i D_i$ is Cartier, then there exists $r_i \in \mathbb{Z}_{>0}$ such that $n_i = r_i m_i$.

We now prove that r_i is a divisor of 4. Since $K(B)$ is a C_1-field and the generic fibre is a surface of del Pezzo type, we conclude by Theorem 6.12 that there exists a curve Γ on V such that the degree d of the morphism $\Gamma \to B$ is a divisor of 4. By the equation

$$r_i \cdot (m_i D_i) \cdot \Gamma = n_i D_i \cdot \Gamma = \pi^*(b_i) \cdot \Gamma = d,$$

r_i is a divisor of 4.

Therefore, it holds that $4m_i D_i \sim_\pi 0$. On the other hand, the divisor $4L = \sum_{i=1}^r \ell_i D_i$ is Cartier; hence, we have that $\ell_i = s_i m_i$ for some $s_i \in \mathbb{Z}$. Therefore, it holds that

$$16L \sim \sum_{i=1}^r 4\ell_i D_i \sim \sum_{i=1}^r s_i (4m_i D_i) \sim_\pi 0,$$

as desired. □

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