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EXISTENCE, UNIQUENESS AND QUALITATIVE PROPERTIES OF GLOBAL SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract We study the existence, uniqueness and qualitative properties of global solutions of abstract differential equations with state-dependent delay. Results on the existence of almost periodic-type solutions (including, periodic, almost periodic, asymptotically almost periodic and almost automorphic solutions) are proved. Some examples of partial differential equations with state-dependent delay arising in population dynamics are presented.

Keywords: state-dependent delay equations; global solutions; strict solutions; analytic semigroup; partial differential equations

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1. Introduction

In this work, we continue our studies in [**16**] on abstract differential equations with state-dependent delay. Specifically, we study the existence, uniqueness and qualitative properties of 'global solutions' for abstract state-dependent delay differential equations of the form

$$u'(t) = Au(t) + F(t, u_{\sigma(t,u_t)}), \quad t \in J, (u_s \in \mathcal{B}_X = C([-p, 0]; X)), \tag{1.1}$$

where $A : D(A) \subset X \to X$ is the generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, $J = [0, \infty)$ or $J = \mathbb{R}$, and $F(\cdot), \sigma(\cdot)$ are continuous functions to be specified later.

The theory of state-dependent delay differential equations is a field of great interest and intense research because of their multiple applications and the fact that the qualitative theory is quite different from the theories of discrete and time-dependent delay. The associated literature is extensive. For ordinary differential equations (ODEs) on finite-dimensional spaces, we cite the early papers by Driver [7, 8], Mackey and Glass [27] and Aiello *et al.* [1]. We also cite the survey by Hartung *et al.* [12], and the papers by Walther [39, 40], Hartung [9, 10], and Hartung and Turi [11] and the references therein. Concerning first-order abstract differential equations with applications to partial differential equations, we refer the reader to Hernandez *et al.* [14], Rezounenko [34], Rezounenko and Wu [35], Kosovalic *et al.* [20] and some recent interesting works by Krisztin and Rezounenko [22], Yunfei *et al.* [26], Kosovalic *et al.* [21] and Hernandez *et al.* [16]. For second-order problems with state-dependent delay, we only cite [4, 5]. Regarding problems defined on unbounded intervals, we cite [19, 23, 29, 38, 40] for equations on finite-dimensional spaces and [2, 6, 24, 32–35] for abstract problems and partial differential equations.

In comparison with the associated literature, in this work we introduce an unified abstract approach motivated by applications and theory; see, for instance, the examples and theoretical developments in [2, 6, 24, 32–35]. In addition, we study the existence of periodic, almost periodic, asymptotically almost periodic and almost automorphic solutions.

It is well known that problems of form (2.1)–(2.2) are (in general) not well posed in the usual space $C([-p,0];X)$, since the map $u \to u_{\sigma(\cdot,u_{(\cdot)})}$ is (in general) not Lipschitz. In order to apply the contraction mapping principle, we use inequalities of the form

$$\|u_{\sigma(\cdot,u_{(\cdot)})} - v_{\sigma(\cdot,v_{(\cdot)})}\|_{C([-p,0];X)} \leq (1 + [v]_{C_{Lip}(J;X)}[\sigma]_{C_{Lip}(J\times\mathcal{B}_X;\mathbb{R})})\|u - v\|_{C(J;X)},$$
$$[u_{\sigma(\cdot,u_{(\cdot)})}]_{C_{Lip}(J;\mathcal{B}_X)} \leq [u_{(\cdot)}]_{C_{Lip}(J;\mathcal{B}_X)}[\sigma]_{C_{Lip}(J\times\mathcal{B}_X;\mathbb{R}^+)}(1 + [u_{(\cdot)}]_{C_{Lip}(J;\mathcal{B}_X)}),$$

and we prove our results working on spaces of Lipschitz functions, a highly non-trivial problem in the semigroup framework. The above inequalities also show that the function $u \to u_{\sigma(\cdot,u_{(\cdot)})}$ introduces a special type of nonlinearity, which has obvious implications concerning the existence of global solutions.

In Theorems 2.1 and 2.2 we establish the existence and uniqueness of solutions for the cases $J = [0,\infty)$ and $J = \mathbb{R}$, respectively. Both theorems are proved assuming that $F(\cdot)$ and $\sigma(\cdot)$ are Lipschitz and that the associated Lipschitz constants are small enough. The cases where $F(\cdot)$ is locally Lipschitz is also considered; see Propositions 2.1 and 2.2. The above results are proved in a very general setting, permitting study of different situations; see, for instance, Corollary 2.1. The existence and uniqueness of periodic, almost periodic, asymptotically almost periodic and almost automorphic solutions are established in Propositions 2.3–2.6. In the last section are presented several applications of partial differential equations arising in population dynamics.

We include now some notation. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this work, $\mathcal{B}_Z = C([-p,0];Z)$, $B_l(z,Z) = \{x \in Z; \|x - z\|_Z \leq r\}$, $\mathcal{L}(Z,W)$ is the space of bounded linear operators from Z into W endowed with the operator norm denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$, and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ if $Z = W$.

Let $I \subset \mathbb{R}$ be an interval. The space $C(I;Z)$ is formed by the continuous bounded functions from I into Z, endowed with a uniform norm denoted by $\|\cdot\|_{C(I;Z)}$. As usual, $C_{Lip}(I;Z)$ is formed by the functions $\xi \in C(I;Z)$ such that $[\zeta]_{C_{Lip}(I;Z)} = \sup_{t,s\in I, t\neq s}((\|\zeta(s) - \zeta(t)\|_Z)/(|\,t - s\,|)) < \infty$, and endowed with the norm $\|\cdot\|_{C_{Lip}(I;Z)} =$

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$\|\cdot\|_{C(I;Z)} + [\cdot]_{Lip(I;Z)}$. The spaces $C(I \times Z; W)$ and $C_{Lip}(I \times Z; W)$ and their norms $\|\cdot\|_{C(I\times Z;W)}$ and $\|\cdot\|_{C_{Lip}(I\times Z;W)}$ are defined in a similar way.

In this work, A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ (not necessarily a C_0-semigroup) on a Banach space $(X, \|\cdot\|)$ and we assume that the general conditions in [**25**, § 2.2.2] are satisfied. In this case, the β-fractional power $(-A)^\beta : D(-A)^\beta \subset X \to X$ ($\beta > 0$) is well defined. For the sake of simplicity, we assume that $0 \in \rho(A)$ and use the notation X_β for the domain of $(-A)^\beta$ endowed with the norm $\|x\|_\beta = \|(-A)^\beta x\|$.

2. Existence of solutions

To begin our studies we discuss the existence of solutions of the problem

$$u'(t) = Au(t) + F(t, u_{\sigma(t,u_t)}), t \geq 0, \tag{2.1}$$

$$u_0 = \varphi \in \mathcal{B}_X = C([-p, 0]; X). \tag{2.2}$$

Next, we adopt the following concepts.

Definition 2.1. A function $u \in C([-p, \infty); X)$ is said to be a mild solution of (2.1)–(2.2) if $u_0 = \varphi$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s, u_{\sigma(s,u_s)})\,\mathrm{d}s, \quad \forall t \in [0, \infty). \tag{2.3}$$

Definition 2.2. A function $u \in C([-p, \infty); X)$ is called a strict solution of (2.1)–(2.2) if $u_{|_{[0,a]}} \in C^1([0, a]; X) \cap C([0, a]; X_1)$ for all $a > 0$, $u_0 = \varphi$ and $u(\cdot)$ satisfies (2.1) on $[0, \infty)$.

Notation 1. In the remainder of this paper, for a Banach space $(V, \|\cdot\|_V)$ and $v \in C([-p, \infty); V)$, we use the notation $v_{(\cdot)}$ for the function $v_{(\cdot)} : [0, \infty) \to \mathcal{B}_V$ given by $v_{(\cdot)}(s) = v_s$. Similarly, for $v \in C(\mathbb{R}; V)$, $v_{(\cdot)} : \mathbb{R} \to \mathcal{B}_V$ is given by $v_{(\cdot)}(s) = v_s$.

The next result follows from [**16**, Lemma 1].

Lemma 2.1 (see [16, Lemma 1]). *Assume $(V, \|\cdot\|_V)$ is a Banach space, $\eta \in C_{Lip}([0, \infty) \times \mathcal{B}_V; \mathbb{R}^+)$, $u, v \in C_{Lip}([-p, \infty); V)$ and $u_0 = v_0 = \varphi$. Then $u_{(\cdot)}, u_{\eta(\cdot,u_{(\cdot)})} \in C_{Lip}([0, \infty); \mathcal{B}_V)$ and*

$$[u_{(\cdot)}]_{C_{Lip}([0,\infty);\mathcal{B}_V)} \leq \max\{[u]_{C_{Lip}([0,\infty);V)}, [\varphi]_{C_{Lip}([-p,0];V)}\}, \tag{2.4}$$

$$[u_{\eta(\cdot,u_{(\cdot)})}]_{C_{Lip}([0,\infty);\mathcal{B}_V)} \leq [u_{(\cdot)}]_{C_{Lip}([0,\infty);\mathcal{B}_V)}[\eta]_{C_{Lip}([0,\infty)\times\mathcal{B}_V;\mathbb{R}^+)}(1 + [u_{(\cdot)}]_{C_{Lip}([0,\infty);\mathcal{B}_V)}), \tag{2.5}$$

$$\|u_{\eta(\cdot,u_{(\cdot)})} - v_{\eta(\cdot,v_{(\cdot)})}\|_{C([0,\infty);\mathcal{B}_V)} \leq (1 + [v_{(\cdot)}]_{C_{Lip}([0,\infty);\mathcal{B}_V)}[\eta]_{C_{Lip}([0,\infty)\times\mathcal{B}_V;\mathbb{R}^+)})\|u - v\|_{C([0,\infty);V)}. \tag{2.6}$$

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Lemma 2.2. *Let $(V, \|\cdot\|_V)$ be a Banach space, $\eta \in C_{Lip}(\mathbb{R} \times \mathcal{B}_V; \mathbb{R}^+)$ and $u, v \in C_{Lip}(\mathbb{R}; V)$. Then $u_{(\cdot)}, u_{\eta(\cdot, u_{(\cdot)})} \in C_{Lip}(\mathbb{R}; \mathcal{B}_V)$, $[u_{(\cdot)}]_{C_{Lip}(\mathbb{R};\mathcal{B}_V)} \le [u]_{C_{Lip}(\mathbb{R};V)}$ and*

$$[u_{\eta(\cdot,u_{(\cdot)})}]_{C_{Lip}(\mathbb{R};\mathcal{B}_V)} \le [u]_{C_{Lip}(\mathbb{R};V)}[\eta]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_V;\mathbb{R}^+)}(1 + [u]_{C_{Lip}(\mathbb{R};V)}), \tag{2.7}$$

$$\|u_{\eta(\cdot,u_{(\cdot)})} - v_{\eta(\cdot,v_{(\cdot)})}\|_{C(\mathbb{R};\mathcal{B}_V)} \le (1 + [v]_{C_{Lip}(\mathbb{R};V)}[\eta]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_V;\mathbb{R}^+)})\|u - v\|_{C(\mathbb{R};V)}. \tag{2.8}$$

To prove our results, we introduce the following condition.

$\mathbf{H^W_{F,Z}}$: $(W, \|\cdot\|_W)$, $(Z, \|\cdot\|_Z)$ are Banach spaces, $(Z, \|\cdot\|_Z) \hookrightarrow (W, \|\cdot\|_W) \hookrightarrow (X, \|\cdot\|)$, $T(\cdot) \in L^1([0,\infty); \mathcal{L}(W, Z))$ and $F \in C_{Lip}([0,\infty) \times \mathcal{B}_Z; W)$. In the following, L_F denotes the Lipschitz constant of F.

Notation 2. For convenience, next we use the notation $\Phi_{Z,W}$ and $\Theta_{Z,W}(\varphi)$ for the constants $\Phi_{Z,W} = \|T(\cdot)\|_{L^1([0,\infty);\mathcal{L}(W,Z))}$ and

$$\Theta_{Z,W}(\varphi) = [T(\cdot)\varphi(0)]_{C_{Lip}([0,\infty);Z)} + \|\varphi\|_{C_{Lip}([-p,0];Z)} + \|T(\cdot)F(0,\varphi)\|_{L^\infty([0,\infty);Z)}.$$

We can prove now our first theorem.

Theorem 2.1. *Assume that the condition $\mathbf{H^W_{F,Z}}$ is satisfied, $\sigma \in C_{Lip}([0,\infty) \times \mathcal{B}_Z; \mathbb{R}^+)$, $\varphi \in C_{Lip}([-p,0]; Z)$, $\sigma(0,\varphi) = 0$, $T(\cdot)\varphi(0) \in C_{Lip}([0,\infty); Z)$, $T(\cdot)F(0,\varphi) \in L^\infty([0,\infty); Z)$, $F([0,\infty) \times K)$ is bounded if $K \subset \mathcal{B}_Z$ is bounded and*

$$1 > 2\Phi_{Z,W} L_F[2[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}(1 + 2(\Theta_{Z,W}(\varphi) + 2\Phi_{Z,W}L_F)) + 1]. \tag{2.9}$$

Then there exists a unique mild solution $u \in C_{Lip}([-p,\infty); Z)$ of (2.1)–(2.2) and $u(\cdot)$ is a strict solution if $\varphi(0) \in X_1$.

Proof. Let $P : \mathbb{R} \to \mathbb{R}$ be the polynomial given by

$$\begin{aligned} P(x) &= \Theta_{Z,W}(\varphi) + 2\Phi_{Z,W}L_F + (\Phi_{Z,W}L_F(2[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)} + 1) - 1)x \\ &\quad + 2\Phi_{Z,W}L_F[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}x^2. \end{aligned} \tag{2.10}$$

From (2.9) we have that $(\Phi_{Z,W}(2[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)} + 1) - 1) < 0$ and

$$\begin{aligned} &(\Phi_{Z,W}(2[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)} + 1) - 1)^2 \\ &\quad - 4(\Theta_{Z,W}(\varphi) + 2\Phi_{Z,W}L_F)2\Phi_{Z,W}L_F[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)} > 0, \end{aligned}$$

which implies that $P(\cdot)$ has a root $R_1 > 0$ and there exists $0 < R < R_1$ such that $P(R) < 0$. From the definition of $\Theta_{Z,W}(\varphi)$ and $P(\cdot)$, it is easy to see that

$$\Theta_{Z,W}(\varphi) + 2\Phi_{Z,W}L_F(1 + R[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}(1 + R)) \le R, \tag{2.11}$$

$$\Phi_{Z,W}L_F(1 + R[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}) < 1. \tag{2.12}$$

Let $\mathcal{S}(R) = \{u \in C([-p,\infty); Z) : u_0 = \varphi,\ [u]_{C_{Lip}([-p,\infty);Z)} \le R\}$, endowed with the metric $d(u,v) = \|u - v\|_{C([0,\infty);Z)}$, and let $\Gamma : \mathcal{S}(R) \to C([-p,\infty); X)$ be the map defined by

$\Gamma u(t) = \varphi(t)$ for $t \in [-p, 0]$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s, u_{\sigma(s,u_s)})\,\mathrm{d}s \quad \text{for } t \in [0, \infty). \tag{2.13}$$

Let $u \in \mathcal{S}(R)$. From the assumptions on $F(\cdot)$ and the inequality

$$\|\Gamma u(t)\|_Z \leq \|T(t)\varphi(0)\|_Z + \|F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})\|_{C([0,\infty);W)}\|T(\cdot)\|_{L^1([0,\infty);\mathcal{L}(W,Z))} < \infty, \tag{2.14}$$

we infer that $\Gamma u(t) \in Z$ and $(\Gamma u)_{|_{[0,\infty)}} \in C([0, \infty); Z)$. To estimate $[\Gamma u]_{C_{Lip}([0,\infty);Z)}$, from Lemma 2.1 we note that $u_{\sigma(\cdot,u_{(\cdot)})} \in C_{Lip}([0, \infty); \mathcal{B}_Z)$, $F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})}) \in C_{Lip}([0, \infty); W)$ and

$$[F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{Lip}([0,\infty);W)} \leq L_F(1 + R[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}(1 + R)).$$

From the above estimates, for non-negative numbers t, h, we have that

$$\begin{aligned}
&\|\Gamma u(t+h) - \Gamma u(t)\|_Z \\
&\quad \leq [T(\cdot)\varphi(0)]_{C_{Lip}([0,\infty);Z)}h + \int_0^h \|T(t+h-s)F(0,\varphi)\|_Z\,\mathrm{d}s \\
&\qquad + \int_0^h \|T(t+h-s)\|_{\mathcal{L}(W,Z)}\|F(s, u_{\sigma(s,u_s)}) - F(0,\varphi)\|_W\,\mathrm{d}s \\
&\qquad + \int_0^t \|T(t-s)\|_{\mathcal{L}(W,Z)}\|F(s+h, u_{\sigma(s+h,u_{s+h})}) - F(s, u_{\sigma(s,u_s)})\|_W\,\mathrm{d}s \\
&\quad \leq [T(\cdot)\varphi(0)]_{C_{Lip}([0,\infty);Z)}h + \|T(\cdot)F(0,\varphi)\|_{L^\infty([0,\infty);Z)}h \\
&\qquad + [F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{Lip}([0,\infty);W)}h\|T(\cdot)\|_{L^1([0,\infty)];\mathcal{L}(W,Z))} \\
&\qquad + [F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{Lip}([0,\infty);W)}\|T(\cdot)\|_{L^1([0,\infty)];\mathcal{L}(W,Z))}h \\
&\quad \leq \Theta_{Z,W}(\varphi)h + 2\Phi_{Z,W}L_F(1 + R[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}(1+R))h,
\end{aligned}$$

which implies that $[\Gamma u]_{C_{Lip}([0,\infty);Z)} \leq R$. Moreover, since $\|\varphi\|_{C_{Lip}([-p,0];Z)} \leq R$, we obtain that $[\Gamma u]_{C_{Lip}([-p,\infty);Z)} \leq R$. Thus, $\Gamma u \in \mathcal{S}(R)$ and $\Gamma(\cdot)$ is a $\mathcal{S}(R)$-valued function. On the other hand, from Lemma 2.1, for $u, v \in \mathcal{S}(R)$ and $t \geq 0$ we get

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\|_Z \\
&\quad \leq \int_0^t \|T(t-s)\|_{\mathcal{L}(W,Z)}L_F\|u_{\sigma(s,u_s)} - v_{\sigma(s,v_s)}\|_{\mathcal{B}_Z}\,\mathrm{d}s \\
&\quad \leq \int_0^t \|T(t-s)\|_{\mathcal{L}(Z,W)}L_F(1 + [v_{(\cdot)}]_{C_{Lip}([0,\infty);\mathcal{B}_Z)}[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)})d(u,v)\,\mathrm{d}s \\
&\quad \leq \Phi_{Z,W}L_F(1 + R[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)})d(u,v),
\end{aligned}$$

which proves that $\Gamma(\cdot)$ is a contraction on $\mathcal{S}(R)$ and there exists a unique mild solution $u \in C_{Lip}([0, \infty); Z)$ of (2.1)–(2.2). In addition, from [**31**, Theorem 4.3.2] we infer that $u(\cdot)$ is a strict solution when $\varphi(0) \in X_1$. □

Remark 2.1. The abstract formulation of Theorem 2.1 permits consideration of different situations and applications. In the next corollary we consider the interesting case in which $W = X_\beta$ and $Z = X_\alpha$ for $1 > \alpha \geq \beta \geq 0$. If $(T(t))_{t\geq 0}$ is exponentially asymptotically stable, then there are $\delta > 0$, $D_{\beta,\alpha} > 0$ (with δ independent of α, β) such that $\|T(s)\|_{\mathcal{L}(X_\beta,X_\alpha)} \leq D_{\beta,\alpha}(\mathrm{e}^{-\gamma s})/(s^{\alpha-\beta})$ for all $s > 0$. Under these conditions, for $\alpha > \beta$ we see that

$$\Phi_{X_\alpha,X_\beta} = \int_0^\infty \|T(s)\|_{\mathcal{L}(X_\beta,X_\alpha)}\,\mathrm{d}\theta \leq D_{\beta,\alpha}\bigg(\int_1^\infty \mathrm{e}^{-\gamma s}\,\mathrm{d}\theta + \int_0^1 \frac{\mathrm{d}\theta}{s^{\alpha-\beta}}\bigg),$$

and hence $\Phi_{X_\alpha,X_\beta} \leq D_{\beta,\alpha}((1/\gamma) + (1/(1+\beta-\alpha)))$. If $\alpha = \beta$, then $\Phi_{X_\alpha,X_\alpha} \leq ((D_{\alpha,\alpha})/\gamma)$.

Corollary 2.1. *Assume $\alpha > \beta \geq 0$, the assumptions in Remark 2.1 and that condition $\mathbf{H}_{\mathbf{F},\mathbf{X}_\alpha}^{\mathbf{X}_\beta}$ is satisfied. Suppose that $\sigma \in C_{Lip}([0,\infty)\times\mathcal{B}_{X_\alpha};\mathbb{R}^+)$, $\sigma(0,\varphi) = 0$, $\varphi(0) \in X_{1+\alpha}$, $\varphi \in C_{Lip}([-p,0];X_\alpha)$, $F(0,\varphi) \in X_\alpha$, $F([0,\infty)\times K)$ is bounded if $K \subset \mathcal{B}_{X_\alpha}$ is bounded and*

$$1 > 2\nu L_F[2[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}(1 + 2(\mu + 2\nu L_F)) + 1] \tag{2.15}$$

where μ, ν are given by $\mu = D_{0,0}((1/\gamma)\|\varphi(0)\|_{X_{1+\alpha}} + \|F(0,\varphi)\|_{X_\alpha}) + \|\varphi\|_{C_{Lip}([-p,0];X_\alpha)}$ and $\nu = D_{\beta,\alpha}((1/\gamma) + (1/(1+\beta-\alpha)))$. Then there exists a unique strict solution $u \in C_{Lip}([-p,\infty);X_\alpha)$ of (2.1)–(2.2). Moreover, $u_{|_{[0,\infty)}} \in C^1([0,\infty);X) \cap C([0,\infty);X_1)$ and $u_{|_{[0,a]}} \in C^\alpha([0,a];X_1) \cap C^{1+\alpha}([0,a];X)$ for all $a > 0$ if $A\varphi(0) + F(0,\varphi) \in X_\alpha$.

Proof. The first assertion follows combining Theorem 2.1 and Remark 2.1 with $W = X_\beta$ and $Z = X_\alpha$. We only note that $\|T(\cdot)F(0,\varphi)\|_{L^\infty([0,\infty);X_\alpha)} \leq D_{0,0}\|F(0,\varphi)\|_{X_\alpha}$ and

$$\|(-A)^\alpha(T(t) - T(s))\varphi(0)\| \leq \int_s^t \|T(\tau)A^{1+\alpha}\varphi(0)\|\,\mathrm{d}\tau \leq \|\varphi(0)\|_{X_{1+\alpha}} D_{0,0}\int_s^t \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau,$$

which implies that $[T(\cdot)\varphi(0)]_{C_{Lip}([0,\infty);X_\alpha)} \leq ((D_{0,0})/\gamma)\|(-A)^{1+\alpha}\varphi(0)\|$.

On the other hand, noting that $F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})}) \in C_{Lip}([0,\infty);X)$, we have that

$$\begin{aligned}
\|Au(t)\| &\leq D_{0,0}\|A\varphi(0)\| + \int_0^t \|AT(t-s)(F(s,u_{\sigma(s,u_s)}) - F(t,u_{\sigma(t,u_t)}))\|\,\mathrm{d}s \\
&\quad + \|(T(t) - I)F(t,u_{\sigma(t,u_t)})\| \\
&\leq D_{0,0}\|A\varphi(0)\| + [F(\cdot,u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{Lip}([0,\infty);X)}\int_0^t D_{0,1}\mathrm{e}^{-\gamma(t-s)}\,\mathrm{d}s \\
&\quad + \|(T(t) - I)F(t,u_{\sigma(t,u_t)})\| \\
&\leq D_{0,0}\|A\varphi(0)\| + \|F(\cdot,u_{\sigma(\cdot,u_{(\cdot)})})\|_{C_{Lip}([0,\infty);X)}\bigg(\frac{D_{0,1}}{\gamma} + 2D_{0,0}\bigg),
\end{aligned}$$

which implies that $\|u\|_{C([0,\infty);X_1)} < \infty$ and $u_{|_{[0,\infty)}} \in C^1([0,\infty);X) \cap C([0,\infty);X_1)$. The last assertion follows from [**25**, Theorem 4.3.1]. □

Introducing some minor modifications to the proof of Theorem 2.1, we can study the case where $F(\cdot)$ is locally Lipschitz. To this end, we introduce the next condition.

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$\mathbf{H^{W}_{F_{loc},Z}}$ $(W, \|\cdot\|_W)$, $(Z, \|\cdot\|_Z)$ are Banach spaces, $(Z, \|\cdot\|_Z) \hookrightarrow (W, \|\cdot\|_W) \hookrightarrow (X, \|\cdot\|)$, $T(\cdot) \in L^1([0,\infty); \mathcal{L}(W,Z))$ and there are functions $N_F, L_F \in C([0,\infty); \mathbb{R})$ such that $\|F(s,\psi)\|_W \leq N_F(r)$ and $\|F(t,\phi) - F(s,\psi)\|_W \leq L_F(r)(| t-s | + \|\psi - \phi\|_{\mathcal{B}_Z})$ for all $r > 0$, $t, s \in \mathbb{R}$ and $\psi, \phi \in B_r(0, \mathcal{B}_Z)$.

Arguing as in the proof of Theorem 2.1, we can prove the next proposition.

Proposition 2.1. *Let condition* $\mathbf{H^{W}_{F_{loc},Z}}$ *hold. Assume that* $\sigma \in C_{Lip}([0,\infty) \times \mathcal{B}_Z; \mathbb{R}^+)$, $\varphi \in C_{Lip}([-p,0]; Z)$, $\sigma(0,\varphi) = 0$, $T(\cdot)\varphi(0) \in C_{Lip}([0,\infty); Z)$, $T(\cdot)F(0,\varphi) \in L^\infty([0,\infty); Z)$, $F([0,\infty) \times K)$ *is bounded if* $K \subset \mathcal{B}_Z$ *is bounded, and there is* $r > \|\varphi\|_{\mathcal{B}_Z}$ *such that (2.9) is valid with* $L_F(r)$ *in place of* L_F *and*

$$\|T(\cdot)\varphi(0)\|_{C([0,\infty);Z)} + N_F(r)\|T(\cdot)\|_{L^1([0,\infty);\mathcal{L}(W,Z))} \leq r. \tag{2.16}$$

Then there exists a unique mild solution $u \in C_{Lip}([-p,\infty); B_r(0,\mathcal{B}_Z))$ *of (2.1)–(2.2). Moreover,* $u(\cdot)$ *is a strict solution if* $\varphi(0) \in X_1$ *and* $u_{|_{[0,a]}} \in C^\alpha([0,a]; X_1) \cap C^{1+\alpha}([0,a]; X)$ *for all* $a > 0$ *if* $A\varphi(0) \in Z$, $A\varphi(0) + F(0,\varphi) \in Z$ *and* $(Z, \|\cdot\|_Z) \hookrightarrow (X_\alpha, \|\cdot\|)$ *for some* $\alpha \in (0,1)$.

Proof. The proof is similar to the proof of Theorem 2.1. Let $P : \mathbb{R} \to \mathbb{R}$ be the polynomial

$$\begin{aligned} P(x) &= \Theta_{Z,W}(\varphi) + 2\Phi_{Z,W}L_F(r) + (\Phi_{Z,W}L_F(r)(2[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)} + 1) - 1)x \\ &\quad + 2\Phi_{Z,W}L_F(r)[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_Z;\mathbb{R}^+)}x^2. \end{aligned} \tag{2.17}$$

From the assumptions, $P(\cdot)$ has a root $R_1(r) > 0$ and we can select $0 < R(r) < R_1(r)$ such that $P(R(r)) < 0$ and the conditions (2.11) and (2.12) are satisfied with $R(r)$ in place of R.

Let $\mathcal{S}(R(r)) = \{u \in C([-p,\infty); Z) : u_0 = \varphi, \ [u]_{C_{Lip}([-p,\infty);Z)} \leq R(r)\}$, endowed with the metric $d(u,v) = \|u - v\|_{C([0,\infty);Z)}$, and $\Gamma : \mathcal{S}(R(r)) \cap B_r(0, C([-p,\infty); Z)) \to C([-p,\infty); Z)$ be defined as in the proof of Theorem 2.1.

Combining (2.14) and (2.16), from the proof of Theorem 2.1 we obtain that $\Gamma(\mathcal{S}(R(r))) \subset B_r(0, C([-p,\infty); Z))$ and that $\Gamma(\cdot)$ is a contraction on $\mathcal{S}(R(r)) \cap B_r(0, C([-p,\infty); Z))$. Thus, there exists a unique mild solution $u \in C_{Lip}([-p,\infty); B_r(0,\mathcal{B}_Z))$ of (2.1)–(2.2). From Theorem 2.1 we have also that $u(\cdot)$ is a strict solution if $\varphi(0) \in X_1$. The other assertions follow, arguing as in the proof of Corollary 2.1. □

2.1. Solutions on $\mathbb{R}$

In this section, we study the existence of solutions for the problem

$$u'(t) = Au(t) + F(t, u_{\sigma(t,u_t)}), t \in \mathbb{R}. \tag{2.18}$$

Next, we assume that $\sigma_-(A) = \{\lambda \in \sigma(A) : Re(\lambda) < 0\}$ and $\sigma_+(A) = \{\lambda \in \sigma(A) : Re(\lambda) > 0\}$ are closed and disjoint, and that $\delta > 0$ is such that $\sup\{Re(\lambda) : \lambda \in \sigma_-(A)\} < -\delta < 0 < \delta < \inf\{Re(\lambda) : \lambda \in \sigma_+(A)\}$. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary such that $\sigma_+(A) \subset \Omega \subseteq \mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$ and let $P : X \to X$ be given by $Px = (1/2\pi i)\int_{\partial\Omega} R(\mu; A)\, x \, d\mu$, with $\partial\Omega$ oriented counterclockwise. Let $X_1 = P(X)$,

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$X_2 = (I - P)(X)$, and let $A_1 : X_1 \to X$, $A_2 : D(A_2) = \{x \in X_1 : x \in X_2, Ax \in X_2\} \to X_2$ be given by $A_1x = Ax$ and $A_2y = Ay$. From [25, Chapter II] we note the following.

(a) P is a projection, $P(X) \subset D(A^n)$ for all $n \in \mathbb{N}$, $T(t)Px = PT(t)x$ for all $x \in X$ and $T(t)X_i \subset X_i$ for $i = 1, 2$ and $t \geq 0$.

(b) For $i \in \mathbb{N}$, $\alpha, \beta \in [0, \infty)$, there are constants $C_{\beta,\alpha}$, C_i such that $\|A^iT(s)P\| \leq C_i e^{\delta s}$ and $\|T(t)(I - P)\|_{\mathcal{L}(X_\beta, X_\alpha)} < C_{\beta,\alpha}(e^{-\gamma t})/(t^{\alpha-\beta})$ for all $s < 0$ and each $t > 0$.

(c) If $f \in L^\infty(\mathbb{R}; X)$ and $u \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; X_1)$ is an X-bounded solution of $x'(t) = Ax(t) + f(t)$, $t \in \mathbb{R}$, then $u(t) = \int_{-\infty}^{t} T(t - \tau)(I - P)f(\tau)\,d\tau - \int_t^\infty T(t - \tau)Pf(\tau)\,d\tau$, for all $t \in \mathbb{R}$.

From the above, we adopt the following concepts.

Definition 2.3. A function $u \in C(\mathbb{R}; X)$ is said to be a mild solution of (2.18) if

$$u(t) = \int_{-\infty}^{t} T(t - s)(I - P)F(s, u_{\sigma(s,u_s)})\,ds - \int_t^\infty T(t - s)PF(s, u_{\sigma(s,u_s)})\,ds, \quad \forall\, t \in \mathbb{R}.$$

Definition 2.4. A function $u \in C(\mathbb{R}; X)$ is said to be a strict solution of (2.18) if $u_{|_{[a,b]}} \in C^1([a, b]; X) \cap C([a, b]; X_1)$ for all $a < b$ and $u(\cdot)$ satisfying (2.18).

To prove our next result, we introduce the following condition.

$\mathcal{H}^{\mathbf{W}}_{\mathbf{F},\mathbf{Z}}$ $(Z, \|\cdot\|_Z) \hookrightarrow (W, \|\cdot\|_W) \hookrightarrow (X, \|\cdot\|)$ are Banach spaces, $T(\cdot)(I - P)$ belongs to $L^1([0, \infty); \mathcal{L}(W, Z))$, $T(\cdot)P \in L^1((-\infty, 0]; \mathcal{L}(W, Z))$ and $F \in C_{Lip}(\mathbb{R} \times \mathcal{B}_Z; W)$. Next, L_F is the Lipschitz constant of F.

We can prove now our next result.

Theorem 2.2. *Assume that the condition $\mathcal{H}^{\mathbf{W}}_{\mathbf{F},\mathbf{Z}}$ is satisfied, $\sigma \in C_{Lip}(\mathbb{R} \times \mathcal{B}_Z; \mathbb{R})$, $F(\mathbb{R} \times K)$ is bounded if $K \subset \mathcal{B}_Z$ is bounded and*

$$1 > 2\Lambda_{Z,W}L_F((2\Lambda_{Z,W}L_F + 1)[\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})} + 1), \tag{2.19}$$

where $\Lambda_{Z,W} = (\|T(\cdot)(I - P)\|_{L^1([0,\infty);\mathcal{L}(W,Z))} + \|T(\cdot)P\|_{L^1((-\infty,0];\mathcal{L}(W,Z))})$. Then there exists a unique strict solution $u \in C_{Lip}(\mathbb{R}; Z)$ of (2.18).

Proof. Let $P : \mathbb{R} \to \mathbb{R}$ be given by

$$\begin{aligned} P(x) = \Lambda_{Z,W}L_F &+ (\Lambda_{Z,W}L_F([\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})} + 1) - 1)x \\ &+ \Lambda_{Z,W}L_F[\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})}x^2. \end{aligned} \tag{2.20}$$

From (2.19) and noting that $\Lambda_{Z,W}([\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})} + 1) - 1 < 0$, we infer that $P(\cdot)$ has a root $R_1 > 0$ and there exists $0 < R < R_1$ such that $P(R) < 0$ and

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$$\Lambda_{Z,W} L_F(1 + [\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})} R(1+R)) \le R,$$
$$\Lambda_{Z,W} L_F(1 + [\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})} R) < 1.$$

Let $\mathfrak{S}(R) = \{u \in C(\mathbb{R}; Z) : u \in C_{Lip}(\mathbb{R}; Z),\ \ [u]_{C_{Lip}(\mathbb{R};Z)} \le R\}$, endowed with the norm $d(u, v) = \|u - v\|_{C(\mathbb{R};Z)}$, and let $\Gamma : \mathfrak{S}(R) \to C(\mathbb{R}, X)$ be the map given by

$$\Gamma u(t) = \int_{-\infty}^{t} T(t-s)(I-P)F(s, u_{\sigma(s,u_s)})\,\mathrm{d}s - \int_{t}^{\infty} T(t-s)PF(s, u_{\sigma(s,u_s)})\,\mathrm{d}s, \quad t \in \mathbb{R}. \tag{2.21}$$

Proceeding as in the proof of Theorem 2.1 and noting that $T(\cdot)(I-P) \in L^1([0,\infty); \mathcal{L}(W, Z))$ and $T(\cdot)P \in L^1((-\infty, 0]; \mathcal{L}(W, Z))$, we can prove that $\Gamma u \in C(\mathbb{R}; Z)$. In addition, from Lemma 2.2, for $t \in \mathbb{R}$ and $h > 0$ we get

$$\begin{aligned}
&\|\Gamma u(t+h) - \Gamma u(t)\|_Z \\
&\quad\le \int_{-\infty}^{t} \|T(t-s)(I-P)\|_{\mathcal{L}(W,Z)} \|F(s+h, u_{\sigma(s+h,u_{s+h})}) - F(s, u_{\sigma(s,u_s)})\|_W\,\mathrm{d}s \\
&\qquad + \int_{t}^{\infty} \|T(t-s)P\|_{\mathcal{L}(W,Z)} \|F(s+h, u_{\sigma(s+h,u_{s+h})}) - F(s, u_{\sigma(s,u_s)})\|_W\,\mathrm{d}s \\
&\quad\le \int_{-\infty}^{t} \|T(t-s)(I-P)\|_{\mathcal{L}(W,Z)} L_F(1 + R[\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})}(1+R))h\,\mathrm{d}s \\
&\qquad + \int_{t}^{\infty} \|T(t-s)P\|_{\mathcal{L}(W,Z)} L_F(1 + R[\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})}(1+R))h\,\mathrm{d}s \\
&\quad\le \Lambda_{Z,W} L_F(1 + R[\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})}(1+R))h,
\end{aligned}$$

and hence $]\Gamma u]_{C_{Lip}(\mathbb{R};Z)} \le R$, which implies that Γ is a $\mathfrak{S}(R)$-valued function. Moreover, for $u, v \in \mathfrak{S}(R)$, we get

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\quad\le \int_{-\infty}^{t} \|T(t-s)(I-P)\|_{\mathcal{L}(W,Z)} L_F(1 + [v]_{C_{Lip}(\mathbb{R};Z)}[\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})}) d(u,v)\,\mathrm{d}s \\
&\qquad + \int_{t}^{\infty} \|T(t-s)P\|_{\mathcal{L}(W,Z)} L_F(1 + [v]_{C_{Lip}(\mathbb{R};Z)}[\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})}) d(u,v)\,\mathrm{d}s \\
&\quad\le \Lambda_{Z,W} L_F(1 + [\sigma]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_Z;\mathbb{R})} R) d(u,v),
\end{aligned}$$

which implies that there exists a unique mild solution $u \in C_{Lip}(\mathbb{R}; Z)$ of (2.18).

We now study the regularity of $u(\cdot)$. For $a < b$, it is easy to see that

$$u(t) = T(t)u(a) + \int_0^t T(t-s)F(s, u_{\sigma(s,u_s)})\,\mathrm{d}s, \quad \forall\, t \in [a, b]. \tag{2.22}$$

Noting that $F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})}) \in C_{Lip}([a, b], X)$, from [**25**, Theorem 4.3.1] it follows that $u \in C^1((a, b]; X) \cap C((a, b]; X_1)$ and $u(\cdot)$ is (in the nomenclature of [**25**]) a classical solution

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of (2.22). Since a is arbitrary, we have that $u(a) \in X_1$, which permits us to conclude that $u(\cdot)$ is a strict solution of (2.22) and in turn prove that $u(\cdot)$ is a strict solution of (2.18). □

Next, we study the existence of solutions for (2.18), assuming that $F(\cdot)$ is locally Lipschitz.

$\mathcal{H}^{\mathbf{W}}_{\mathbf{F_{loc}},\mathbf{Z}}$ $(Z, \|\cdot\|_Z) \hookrightarrow (W, \|\cdot\|_W) \hookrightarrow (X, \|\cdot\|)$ are Banach spaces, $T(\cdot)(I-P) \in L^1([0,\infty); \mathcal{L}(W,Z))$, $T(\cdot)P \in L^1((-\infty,0]; \mathcal{L}(W,Z))$, and $F(\cdot)$ is locally Lipschitz in the sense of condition $\mathbf{H}^{\mathbf{W}}_{\mathbf{F_{loc}},\mathbf{Z}}$, but with $\mathbb{R}$ in place of $[0,\infty)$.

We establish without proof the next proposition.

Proposition 2.2. *Let condition $\mathcal{H}^{\mathbf{W}}_{\mathbf{F_{loc}},\mathbf{Z}}$ hold. Suppose that $\sigma \in C_{Lip}(\mathbb{R} \times \mathcal{B}_Z; \mathbb{R})$, there is $r > 0$ such that the conditions in Theorem 2.2 are valid with $L_F(r)$ in place of L_F, and*

$$N_F(r)(\|T(\cdot)(I-P)\|_{L^1([0,\infty);\mathcal{L}(W,Z))} + \|T(\cdot)P\|_{L^1((-\infty,0];\mathcal{L}(W,Z))}) \leq r. \tag{2.23}$$

Then there exists a unique strict solution $u \in C_{Lip}(\mathbb{R}; B_r(0; Z))$ of (2.18).

In the next sections, we study the existence and uniqueness of almost periodic-type solutions; here, $(V, \|\cdot\|_V)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces and $\omega > 0$.

2.1.1. *Periodic solution*

In what follows, we say that a function $G \in C(\mathbb{R} \times V; Y)$ is ω-periodic if $G(t+\omega, v) = G(t, v)$ for all $(t, v) \in \mathbb{R} \times V$, and we use the notation $C_\omega(\mathbb{R}; V)$ for the space $C_\omega(\mathbb{R}; V) = \{f : \mathbb{R} \to V : f \text{ is } \omega\text{-periodic}\}$ endowed with the norm $d(u, v) = \|u - v\|_{C(\mathbb{R};V)}$.

From Theorem 2.2, we infer the next result.

Proposition 2.3. *Assume that the conditions $\mathcal{H}^{\mathbf{W}}_{\mathbf{F},\mathbf{Z}}$ and (2.19) are satisfied and that $F(\cdot)$, $\sigma(\cdot)$ are ω-periodic. Then there exists a unique ω-periodic strict solution $u(\cdot)$ of the problem (2.18) such that $u \in C_{Lip}(\mathbb{R}; Z) \cap C^\alpha(\mathbb{R}; X_1) \cap C^{1+\alpha}(\mathbb{R}; X)$ for all $\alpha \in (0, 1)$.*

Proof. Let $\mathfrak{S}(R)$ and Γ be defined as in the proof of Theorem 2.2. It is trivial to note that Γu is ω-periodic if $u \in \mathfrak{S}(R) \cap C_\omega(\mathbb{R}; Z)$, which allows us to infer that Γ is a contraction on $\mathfrak{S}(R) \cap C_\omega(\mathbb{R}; Z)$ and there exists a unique mild solution $u \in C_{Lip}(\mathbb{R}; Z) \cap C_\omega(\mathbb{R}; Z)$. Moreover, noting that $F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})}) \in C_\omega(\mathbb{R}, X) \cap C_{Lip}(\mathbb{R}, X)$, we have that $F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})}) \in C^\alpha(\mathbb{R}, X)$ for all $\alpha \in (0, 1)$, which implies (see [**25**, Theorem 4.4.7]) that $u(\cdot)$ is a strict solution and $u \in C^\alpha(\mathbb{R}; X_1) \cap C^{1+\alpha}(\mathbb{R}; X)$. □

If $F(\cdot)$ is locally Lipschitz, we establish without proof the next result.

Corollary 2.2. *Assume that the conditions in Proposition 2.2 are satisfied and that $F(\cdot), \sigma(\cdot)$ are ω-periodic. Then there exists a unique ω-periodic strict solution $u \in C_{Lip}(\mathbb{R}; B_r(0, Z))$ of (2.18) such that $u \in C^\alpha(\mathbb{R}; X_1) \cap C^{1+\alpha}(\mathbb{R}; X)$ for all $\alpha \in (0, 1)$.*

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2.1.2. *Almost periodic and asymptotically almost periodic solutions*

For completeness, we mention some additional concepts and results.

Definition 2.5 (Zaidman [45]). A function $f \in C(\mathbb{R}; V)$ is called almost periodic (a.p.) if for all $\epsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, $\mathcal{H}(\epsilon, f, V)$ such that $\|f(t+\xi) - f(t)\|_V < \varepsilon$ for all $t \in \mathbb{R}$ and $\xi \in \mathcal{H}(\varepsilon, f, V)$.

Definition 2.6 (Zaidman [45]). A function $f \in C([0,\infty); V)$ is said to be asymptotically almost periodic (a.a.p.) if there exists an almost periodic function $g(\cdot)$ and $w \in C([0,\infty); V)$ such that $f(\cdot) = g(\cdot) + w(\cdot)$ and $\lim_{t\to\infty} w(t) = 0$.

Next, we use the notation $AP(V)$ and $AAP(V)$ for the spaces $AP(V) = \{f \in C(\mathbb{R}; V) : f \text{ is a.p.}\}$ and $AAP(V) = \{f \in C([0,\infty); V) : f \text{ is a.a.p.}\}$ endowed with the norms $\|\cdot\|_{C(\mathbb{R};V)}$ and $\|\cdot\|_{C([0,\infty);V)}$. It is well known that $AP(V)$ and $AAP(V)$ are Banach spaces. From [43, 44] we have the next result.

Lemma 2.3. *Assume that $\Omega \subset V$ is open. If $G \in C(\mathbb{R} \times \Omega; Y)$ (respectively $G \in C([0,\infty) \times \Omega; Y)$), $G(\cdot, v) \in AP(Y)$ for all $v \in V$ (respectively $G(\cdot, v) \in AAP(Y)$ for all $v \in V$), $G(\cdot)$ satisfies a local Lipschitz condition at $v \in \Omega$, uniformly at t, and $y \in AP(V)$ (respectively $y \in AAP(V)$) is such that $\overline{\{y(t) : t \in \mathbb{R}\}}^V \subset \Omega$, then $G(\cdot, y(\cdot)) \in AP(Y)$ (respectively $G(\cdot, y(\cdot)) \in AAP(Y)$).*

Concerning the existence of almost periodic solutions for (2.18), we have the next result.

Proposition 2.4. *Assume that the assumptions in Theorem 2.2 are satisfied, $\sigma(\cdot, \psi) \in AP(\mathbb{R})$ and $F(\cdot, \psi) \in AP(W)$ for all $\psi \in \mathcal{B}_Z$. Then there exists a unique strict solution $u \in AP(Z) \cap C_{Lip}(\mathbb{R}; Z)$ of (2.18).*

Proof. Let R, $\mathfrak{S}(R)$ and $\Gamma(\cdot)$ be defined as in the proof of Theorem 2.2. From the proof of Theorem 2.2, it is sufficient to show that $\Gamma(AP(Z) \cap \mathfrak{S}(R)) \subset AP(Z)$.

Let $u \in AP(Z) \cap \mathfrak{S}(R)$. From Lemma 2.3 we have that $\sigma(\cdot, u_{(\cdot)}) \in AP(\mathbb{R})$ and noting that $u(\cdot)$ is uniformly continuous we can prove that $u_{\sigma(\cdot, u_{(\cdot)})} \in AP(\mathcal{B}_Z)$. From the above and Lemma 2.3, $F(\cdot, u_{\sigma(\cdot, u_{(\cdot)})}) \in AP(W)$. Finally, a usual argument allows us to prove that $\Gamma u \in AP(Z)$. □

The next result establishes the existence of a.a.p. solutions for (2.1)–(2.2).

Proposition 2.5. *Let the conditions in Theorem 2.1 hold. Assume that $\|T(t)\|_{\mathcal{L}(Z,Z)} \to 0$ as $t \to \infty$, $\sigma(\cdot, \psi) \in AAP(\mathbb{R})$ and $F(\cdot, \psi) \in AAP(W)$ for all $\psi \in \mathcal{B}_Z$. Then there exists a unique strict solution $u \in AAP(Z) \cap C_{Lip}([0,\infty); Z)$ of (2.1)–(2.2).*

Proof. Let R, $\mathcal{S}(R)$ and $\Gamma(\cdot)$ be defined as in the proof of Theorem 2.1. Arguing as in the proof of Proposition 2.4 it follows that $F(\cdot, u_{\sigma(\cdot, u_{(\cdot)})}) \in AAP(W)$ for all $u \in \mathcal{S}(R) \cap AAP(W)$. Moreover, using the condition on $\|T(t)\|_{\mathcal{L}(Z,Z)}$ and arguing as in the proof of [13, Lemma 2,5], we can prove that $\Gamma u \in AAP(Z)$. □

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2.1.3. *Almost automorphic solutions*

Definition 2.7 (see [17, 45]). A function $f \in C(\mathbb{R}; V)$ is said to be almost automorphic if for all sequences of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\mapsto\infty} f(t+s_n)$ is well defined for each $t \in \mathbb{R}$ and $f(t) = \lim_{n\mapsto\infty} g(t-s_n)$ for all $t \in \mathbb{R}$. In addition, we say that $f(\cdot)$ is compact almost automorphic if the above limits are uniform on compact subsets of $\mathbb{R}$.

Next, the spaces $AA(V) = \{f \in C(\mathbb{R}; V) : f \text{ is almost automorphic}\}$ and $AA_c(V) = \{f \in C(\mathbb{R}; V) : f \text{ is compact almost automorphic}\}$ are endowed with the norm $\|\cdot\|_{C(\mathbb{R};V)}$. We remark that both spaces are Banach spaces.

Definition 2.8. A function $G \in C(\mathbb{R} \times V; Y)$ is said to be compact almost automorphic in $t \in \mathbb{R}$ for each $v \in V$ if for every sequence of real numbers $(s'_n)_{\mathbb{N}}$ there exists a subsequence $(s_n)_{\mathbb{N}}$ of $(s'_n)_{\mathbb{N}}$ such that $H(t,v) := \lim_{n\mapsto\infty} G(t+s_n, v)$ is well defined for all $t \in \mathbb{R}$ and $v \in V$, $G(t,v) = \lim_{n\mapsto\infty} H(t-s_n, v)$ for all $t \in \mathbb{R}$ and $v \in V$, and both limits are uniform on compact subsets of $\mathbb{R}$. The set of such functions will be denoted by $AA_c(V,Y)$.

From [15], we note the next result.

Lemma 2.4. *If* $G \in AA_c(V,Y) \cap C_{Lip}(\mathbb{R} \times V; Y)$ *and* $\xi \in AA_c(V)$, *then* $G(\cdot, \xi(\cdot)) \in AA_c(V)$.

Proposition 2.6. *Assume that the conditions in Theorem 2.2 are satisfied and* $F(\cdot)$ *belongs to* $AA_c(\mathcal{B}_Z, W) \cap C_{Lip}(\mathbb{R} \times \mathcal{B}_Z; W)$. *Then there exists a unique strict solution* $u \in AA_c(Z) \cap C_{Lip}(\mathbb{R}; Z)$ *of (2.18).*

Proof. Let R, $\mathcal{S}(R)$ and $\Gamma(\cdot)$ be defined as in the proof of Theorem 2.2. Let $u \in AA_c(Z) \cap \mathcal{S}(R)$. From Lemma 2.4 it is easy to see that $F(\cdot, u_{\sigma(\cdot, u_{(\cdot)})}) \in AA_c(Z) \cap \mathcal{S}(R)$ and arguing as in the proof of [15, Lemma 2.2] we can show that $\Gamma u \in AA_c(Z)$. □

Remark 2.2. Arguing as above, results on the existence of solutions for the case where $F(\cdot)$ is locally Lipschtz can be proved. We decided not to include additional results.

3. Examples

Next, we present some examples motivated by studies in population dynamics. For the sake of brevity, we assume that $A : D(A) \subset X \to X$ is the generator of an exponentially asymptotically stable analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X with $X = L^2(\Omega; \mathbb{R})$ or $X = C(\Omega; \mathbb{R})$, where $\Omega \subset \mathbb{R}^n$ is an open bounded set with smooth boundary $\partial\Omega$, and we use all the notation and properties in Remark 2.1. Concerning the comments in the introduction of § 2.1, we note that the formula in (**c**) takes the form $u(t) = \int_{-\infty}^{t} T(t-\tau) f(\tau)\, d\tau$. Here, L_S is the Lipschitz constant of a given function $S(\cdot)$.

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Motivated by the problems studied in [**34**], we study the problem

$$u'(t,x) = Au(t)(x) + \int_{\Omega} b(u(\zeta(t,u(t),u_t),y))f(x-y)\,\mathrm{d}y$$
$$+ g(t,u(\zeta(t,u(t),u_t),x)), t \geq 0,\, x \in \Omega, \tag{3.1}$$
$$u(\theta,y) = \varphi(\theta,y), \theta \in [-p,0],\, y \in \Omega, \tag{3.2}$$

where $\varphi \in C_{Lip}([-p,0];X)$, $X = L^2(\Omega;\mathbb{R})$, $f \in C(\mathbb{R}^n;\mathbb{R})$, $g \in C_{Lip}(\mathbb{R}\times\mathbb{R};\mathbb{R})$, $b \in C_{Lip}(\mathbb{R};\mathbb{R})$, the functions $g(\cdot), b(\cdot)$ are bounded and $\zeta \in C_{Lip}(\mathbb{R}\times X\times\mathcal{B}_X;\mathbb{R}^+)$. The study of this class of problems was motivated by the diffusive Nicholson's blowflies equation with state-dependent delay; see [**36**, **37**] for additional details.

In order to use Theorem 2.1, we assume that $(\int_\Omega\int_\Omega f(x-y)^2\,\mathrm{d}y\,\mathrm{d}x)^{1/2} < \infty$ and we define $\sigma : [0,\infty)\times\mathcal{B}_X \to \mathbb{R}$ and $F : \mathcal{B}_X \to X$ by $\sigma(t,\psi) = \zeta(t,\psi(0),\psi)$ and $F(\psi)(x) = \int_\Omega b(\psi(0,y))f(x-y)\,\mathrm{d}y + g(t,\psi(0,x))$. Under these conditions, $\sigma(\cdot)$ and $F(\cdot)$ are Lipschitz, $[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_X;\mathbb{R}^+)} = [\zeta]_{C_{Lip}([0,\infty)\times X\times\mathcal{B}_X;\mathbb{R}^+)}$ and $L_F = L_b(\int_\Omega\int_\Omega f(x-y)^2\,\mathrm{d}y\,\mathrm{d}x)^{1/2} + L_g$. Moreover, if $\varphi(0) \in X_1$, we can assume that the constants $\Theta_{X,X}$ and $\Phi_{X,X}$ in Theorem 2.1 are given by $\Phi_{X,X} = (D_{0,0})/\gamma$ and $\Theta_{X,X}(\varphi) = D_{0,0}(\|A\varphi(0)\| + \|F(0,\varphi)\|) + \|\varphi\|_{C_{Lip}([-p,0];X)}$. In the next result, which follows from Theorem 2.1, we say that $u \in C([-p,\infty);X)$ is a mild or a strict solution of (3.1)–(3.2) if $u(\cdot)$ is a mild or a strict solution of the associated problem (2.1)–(2.2). We adopt a similar nomenclature for the other examples in this section.

Proposition 3.7. *Suppose that* $\varphi(0) \in X_1$, $\zeta(0,\varphi(0),\varphi) = 0$ *and the condition (2.9) is satisfied with* $\Theta_{X,X}(\varphi) = D_{0,0}(\|A\varphi(0)\| + \|F(0,\varphi)\|) + \|\varphi\|_{C_{Lip}([-p,0];X)}$ *and* $\Phi_{X,X} = (D_{0,0})/\gamma$. *Then there exists a unique strict solution* $u \in C_{Lip}([-p,\infty);X)$ *of (3.1)–(3.2).*

We now consider a problem similar to those studied in [**33**]. Consider the problem

$$u'(t,x) = Au(t)(x) + b\bigg(\int_\Omega u(\zeta(t,u(t),u_t),y)f(x-y)l(y)\,\mathrm{d}y\bigg)$$
$$+ g(t,u(\zeta(t,u(t),u_t),x)), \tag{3.3}$$
$$u(\theta,y) = \varphi(\theta,y), \quad \theta \in [-p,0], y \in \Omega, \tag{3.4}$$

for $(t,x) \in J\times\Omega$, where $J = [0,\infty)$ or $J = \mathbb{R}$, $A, X, f(\cdot)$ and $b(\cdot)$ are as in the first example, $g \in C_{Lip}(J\times\mathbb{R};\mathbb{R})$, $f(\cdot), g(\cdot)$ are bounded, $\zeta \in C_{Lip}(J\times X\times\mathcal{B}_X;J)$ and $l \in C_0^\infty(\Omega;\mathbb{R})$.

Let $F : \mathcal{B}_X \to X$ and $\sigma : [0,\infty)\times X\times\mathcal{B}_X \to \mathbb{R}^+$ be defined by $F(\psi)(x) = b(\int_\Omega \psi(0,y)f(x-y)l(y)\,\mathrm{d}y) + g(t,\psi(0,x))$ and $\sigma(t,\psi) = \zeta(t,\psi(0),\psi)$. From Theorems 2.1, 2.2 and Propositions 2.3–2.5 we have the next result. In this result, $\Phi_{X,X} = (D_{0,0})/\gamma$, $\Theta_{X,X}(\varphi) = D_{0,0}(\|A\varphi(0)\| + \|F(0,\varphi)\|) + \|\varphi\|_{C_{Lip}([-p,0];X)}$, $[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_X;\mathbb{R}^+)} = [\zeta]_{C_{Lip}([0,\infty)\times X\times\mathcal{B}_X;\mathbb{R}^+)}$ and $L_F = L_b(\int_\Omega\int_\Omega f^2(x-y)l^2(y)\,\mathrm{d}y\,\mathrm{d}x)^{1/2} + L_g$.

Proposition 3.8. *Suppose* $\varphi(0) \in X_1$ *and* $\varphi \in C_{Lip}([-p,0];X)$.

(a) *Assume that* $J = [0,\infty)$, $\zeta(0,\varphi(0),\varphi) = 0$ *and the inequality (2.9) is satisfied. Then there exists a unique strict solution* $u \in C_{Lip}([0,\infty);X)$ *of the problem (3.3)–(3.4).*

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(b) *If $J = \mathbb{R}$ and (2.19) is verified, then there exists a unique strict solution $u \in C_{Lip}(\mathbb{R}; X)$ of (3.3) on $\mathbb{R}$. If, in addition, $\zeta(\cdot)$ and $g(\cdot)$ are ω-periodic for some $\omega > 0$, then $u(\cdot)$ is ω-periodic.*

(c) *If $J = \mathbb{R}$ and (2.19) is verified, and $\sigma(\cdot, \psi)$ and $F(\cdot, \psi)$ are almost periodic for all $\psi \in \mathcal{B}_X$ (respectively are compact almost automorphic in $t \in \mathbb{R}$), then there exists a unique almost periodic (respectively compact almost automorphic) strict solution of (3.3) on $\mathbb{R}$.*

We now study a problem concerning a diffusive model of haematopoiesis with state-dependent delay of the form

$$u'(t,x) = Au(t)(x) + \frac{\beta u(\zeta(t,u_t),x)}{1 + u^m(\zeta(t,u_t),x)} + g(t, u(\zeta(t,u_t),x)), \quad t \in J, \tag{3.5}$$

$$u(\theta,x) = \varphi(\theta,x), \quad \theta \in [-p,0], \tag{3.6}$$

for $t \in J, x \in \Omega$, where $m \in \mathbb{N}$ is even, $J = \mathbb{R}$ or $J = [0,\infty)$, $g \in C_{Lip}(J \times \mathbb{R}; \mathbb{R})$ is bounded, $\zeta \in C_{Lip}(J \times \mathcal{B}_X; J)$ and $X = L^2(\Omega)$. We cite Mackey and Glass [**27**], Wang and Li [**41**] and Rezounenko [**34**] for additional details on problems of this type.

To study (3.5)–(3.6), we assume that there is $\alpha \in (0,1)$ such that $X_\alpha \hookrightarrow C(\Omega)$. We note, for example, that if A is the realization on X of a strongly elliptic differential operator of order $2m$, then $X_\theta \hookrightarrow C^\nu(\overline{\Omega})$ for $0 \le \nu < 2m\alpha - n/p$; see [**31**, § 8.4] for details.

Denote $\sigma : J \times \mathcal{B}_{X_\alpha} \to J$ and $F : J \times \mathcal{B}_{X_\alpha} \to X$ by $\sigma(t,\psi) = \zeta(t,\psi)$ and $F(t,\psi)(x) = ((\beta\psi(0,x))/(1 + \psi^m(0,x))) + g(t,\psi(0,x))$. From the choice of α, both functions are well defined and are Lipschitz, and $F(\cdot)$ takes bounded sets into bounded sets. In addition, for $r > 0$ and $\psi, \phi \in B_r(0, \mathcal{B}_{X_\alpha})$, we can show that $\|F(\psi)\| \le N_F(r) = \beta(\|i_c\|_{\mathcal{L}(X_\alpha, C(\Omega))} r + \|g\|_{C(\mathbb{R}\times\mathbb{R};\mathbb{R})}) m(\Omega)^{1/2}$ and

$$\begin{aligned} &\|F(\psi) - F(\phi)\| \\ &\quad \le (\beta(1 + (1+2n) r^{2n} \|i_c\|^{2n}_{\mathcal{L}(X_\alpha, C(\Omega))}) + \|i_c\|_{\mathcal{L}(X_\alpha, C(\Omega))} L_g) m(\Omega)^{1/2} \|\psi - \phi\|_{\mathcal{B}_{X_\alpha}}, \end{aligned}$$

where i_c is the inclusion map from X_α into $C(\Omega)$ and $m(\Omega)$ is the Lebesgue measure of Ω. To use Proposition 2.1, we assume $\varphi \in C_{Lip}([-p,0]; X_\alpha)$, $\varphi(0) \in X_{1+\alpha}$ and $F(0,\varphi) \in X_\alpha$. In this case, we can suppose that the numbers $\Theta_{X_\alpha,X}(\varphi)$, $\Phi_{X_\alpha,X}$ and $L_F(r)$ in Proposition 2.1 are given by

$$\Phi_{X_\alpha,X} = D_{0,\alpha}\left(\frac{1}{\gamma} + \frac{1}{1-\alpha}\right) \tag{3.7}$$

$$\Theta_{X_\alpha,X}(\varphi) = D_{0,0}(\|\varphi(0)\|_{X_{1+\alpha}} + \|F(0,\varphi(0))\|_{X_\alpha}) + \|\varphi\|_{C_{Lip}([-p,0];X_\alpha)}, \tag{3.8}$$

$$L_F(r) = (\beta(1 + (1+2n) r^{2n} \|i_c\|^{2n}_{\mathcal{L}(X_\alpha, C(\Omega))}) + L_g) m(\Omega)^{1/2}. \tag{3.9}$$

Moreover, concerning the inequality (2.16), we note that

$$\begin{aligned} &\|T(\cdot)\varphi(0)\|_{C([0,\infty);X_\alpha)} + N_F(r) \|T(\cdot)\|_{L^1([0,\infty);\mathcal{L}(X,X_\alpha))} \\ &\quad \le D_{0,0}\|\varphi(0)\|_{X_\alpha} + (\beta\|i_c\|_{\mathcal{L}(X_\alpha,C(\Omega))} r + \|g\|_{C(\mathbb{R}\times\mathbb{R};\mathbb{R})}) m(\Omega)^{1/2} D_{0,\alpha}\left(\frac{1}{\gamma} + \frac{1}{1-\alpha}\right). \end{aligned}$$

In the next result, which follows from Proposition 2.1, $\Phi_{X_\alpha,X}, \Theta_{X_\alpha,X}(\varphi)$ and $L_F(r)$ are given by (3.7)–(3.9).

Proposition 3.9. *Assume that* $\varphi \in C_{Lip}([-p,0];X_\alpha)$, $\varphi(0) \in X_{1+\alpha}$, $F(0,\varphi) \in X_\alpha$, $\sigma(t,\varphi) = 0$ *and there is* $r > 0$ *with* $\varphi \in B_r(0,\mathcal{B}_{X_\alpha})$ *such that*

$$D_{0,0}\|\varphi(0)\|_{X_\alpha} + (\beta\|i_c\|_{\mathcal{L}(X_\alpha,C(\Omega))}r + \|g\|_{C(\mathbb{R}\times\mathbb{R};\mathbb{R})})m(\Omega)^{1/2}D_{0,\alpha}\left(\frac{1}{\gamma} + \frac{1}{1-\alpha}\right) < r,$$
$$2\Phi_{X_\alpha,X}L_F(r)[[\zeta]_{C_{Lip}([0,\infty)\times\mathcal{B}_X;\mathbb{R}^+)}(1 + 2(\Theta_{X_\alpha,X}(\varphi) + 2\Phi_{X_\alpha,X}L_F(r))) + 1] < 1. \tag{3.10}$$

Then there exists a unique strict solution $u \in C_{Lip}([-p,\infty);X_\alpha)$ *of (3.5)–(3.6) such that* $u_{|_{[0,a]}} \in C^\alpha([0,a];X_1) \cap C^{1+\alpha}([0,a];X)$ *for all* $a > 0$.

Using Corollary 2.2, we can prove the existence of a periodic solution for (3.5) with $J = \mathbb{R}$. In this case, the inequality (2.23) takes the form

$$(\beta\|i_c\|_{\mathcal{L}(X_\alpha,C(\Omega))}r + \|g\|_{C(\mathbb{R}\times\mathbb{R};\mathbb{R})})m(\Omega)^{1/2}D_{0,\alpha}\left(\frac{1}{\gamma} + \frac{1}{1-\alpha}\right) \le r. \tag{3.11}$$

Using the above notation, from Corollary 2.2 we get the next result.

Proposition 3.10. *Assume that* $\zeta \in C_{Lip}(\mathbb{R}\times\mathcal{B}_{X_\alpha};\mathbb{R}) \cap C_\omega(\mathbb{R}\times\mathcal{B}_{X_\alpha};\mathbb{R})$, $g(\cdot)$ *is* ω*-periodic, there is* $r > 0$ *such that (3.11) is valid and* $1 > 2\Lambda_{X_\alpha,X}(r)((2\Lambda_{X_\alpha,X}(r) + 1)[\zeta]_{C_{Lip}(\mathbb{R}\times\mathcal{B}_{X_\alpha};\mathbb{R})} + 1)$. *Then there exists a unique* ω*-periodic strict solution* $u \in C_{Lip}(\mathbb{R};B_r(0;X_\alpha))$ *of (3.5). Moreover,* $u \in C^\beta(\mathbb{R};X_1) \cap C^{1+\beta}(\mathbb{R};X)$ *for all* $\beta \in (0,1)$.

To complete this section, we study the existence of asymptotically almost periodic solutions for a model concerning the Fisher–Kolmogoroff equation and Hutchinson's equation; see [**18**, **30**, **34**, **42**] and the examples in [**23**, Example 1] for details. Consider the diffusive equations with state-dependent delay

$$w'(t,\xi) = Aw(t)(\xi) + \mu(t)w(t,\xi)[1 - w(\zeta(t,w_t),\xi)], \quad t \in \mathbb{R}, \xi \in \Omega, \tag{3.12}$$
$$w(\theta,y) = \varphi(\theta,y), \quad \theta \in [-p,0], y \in \Omega, \tag{3.13}$$

where $\zeta \in C_{Lip}(\mathbb{R}\times\mathcal{B}_X;\mathbb{R})$ and $\mu \in C_{Lip}(\mathbb{R},\mathbb{R})$.

For simplicity, we take $X = C(\Omega)$ and define $F : \mathbb{R}\times\mathcal{B}_X \to X$ and $\sigma : \mathbb{R}\times\mathcal{B}_X \to \mathbb{R}$ by $F(t,\psi)(x) = \mu(t)\psi(0,x)[1 - \psi(0,x)]$ and $\sigma(t,\psi) = \zeta(t,\psi)$. The function $\sigma(\cdot)$ is Lipschitz and for $r > 0$, $t,s \in \mathbb{R}$ and $\psi,\phi \in B_r(0,\mathcal{B}_X)$, $\|F(t,\psi)\| \le \|\mu\|_{C(\mathbb{R};\mathbb{R})}r(1+r)$ and

$$\|F(t,\psi) - F(s,\phi)\| \le [\mu]_{C_{Lip}(\mathbb{R};\mathbb{R})} \mid t-s \mid r(1+r) + \|\mu\|_{C(\mathbb{R};\mathbb{R})}(1+2r)\|\psi-\phi\|_{\mathcal{B}_X},$$

which implies that the condition $\mathbf{H}^{\mathbf{X}}_{\mathbf{F_{loc}},\mathbf{X}}$ is satisfied.

Let $L_F(r) = \|\mu\|_{C_{Lip}(\mathbb{R};\mathbb{R})}(r^2+3r+1)$, $N_F(r) = \|\mu\|_{C(\mathbb{R};\mathbb{R})}r(1+r)$, $\Phi_{X,X} = ((D_{0,0})/(\gamma))$ and

$$\Theta(X,X)(\varphi) = D_{0,0}\left(\|A\varphi(0)\| + \|\mu\|_{C_{Lip}(\mathbb{R};\mathbb{R})}\frac{r}{\gamma}(1+r)\right) + \|\varphi\|_{C_{Lip}([-p,0];X)}. \tag{3.14}$$

Combining the proofs of Propositions 2.1 and 2.5, we can prove the next result.

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Proposition 3.11. *Suppose that* $\varphi \in C_{Lip}([-p,0];X)$, $\varphi(0) \in X_1$, $\zeta(0,\varphi) = 0$ *and there is* $r > 0$ *such that* $\|\varphi\|_{\mathcal{B}_X} < r$, $D_{0,0}\|\varphi(0)\| + \|\mu\|_{C(\mathbb{R};\mathbb{R})} r(1+r)((D_{0,0})/(\gamma)) \leq r$ *and*

$$1 > 2\Phi_{X,X} L_F(r)[2[\sigma]_{C_{Lip}([0,\infty)\times\mathcal{B}_X;\mathbb{R}^+)}(1 + 2(\Theta_{X,X}(\varphi) + 2\Phi_{X,X} L_F(r))) + 1]. \quad (3.15)$$

Then there exists a unique strict solution $u \in C([0,\infty); B_r(0,X)) \cap AAP(X)$ *of (3.12)–(3.13).*

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