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On the Singular Sheaves in the Fine Simpson Moduli Spaces of 1-dimensional Sheaves

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Abstract. In the Simpson moduli space M of semi-stable sheaves with Hilbert polynomial $dm - 1$ on a projective plane we study the closed subvariety M' of sheaves that are not locally free on their support. We show that for $d \geqslant 4$, it is a singular subvariety of codimension 2 in M. The blow up of M along M' is interpreted as a (partial) modification of $M \smallsetminus M'$ by line bundles (on support).

1 Introduction

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, let V be a vector space over $\mathbb{k}$ of dimension 3, and let $\mathbb{P}_2 = \mathbb{P}V$ be the corresponding projective plane. Consider a linear polynomial with integer coefficients $P(m) = dm + c \in \mathbb{Z}[m]$, $d \in \mathbb{Z}_{>0}$, $c \in \mathbb{Z}$. Let $M := M_P(\mathbb{P}_2)$ be the Simpson coarse moduli space [10] of semi-stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial P. As shown in [7], M is an irreducible locally factorial variety of dimension $d^2 + 1$, smooth if $\gcd(d, c) = 1$.

Singular Sheaves

The sheaves from M are torsion sheaves on $\mathbb{P}_2$; they are supported on curves of degree d. Restricted to their (Fitting) support, most of the sheaves in M are line bundles with Euler–Poincaré characteristic c. The fibres of the morphism $M \to \mathbb{P}S^d V^*$, $[\mathcal{F}] \mapsto \operatorname{Supp}\mathcal{F}$ that sends a class of a sheaf to its support are Jacobians over smooth curves. Over singular curves the fibres can be seen as compactified Jacobians. Sheaves that are line bundles on their support constitute an open subvariety M_B of M. Its complement M', the closed subvariety of sheaves that are not locally free on their support, is in general non-empty. This way one can consider M as a compactification of M_B. We call the sheaves from the boundary $M' = M \smallsetminus M_B$ *singular*.

The boundary M' does not have the minimal codimension in general. Loosely speaking, one glues together too many different directions at infinity. For example, for $c \in \mathbb{Z}$ with $\gcd(3, c) = 1$, all moduli spaces M_{3m+c} are isomorphic to the universal plane cubic curve and M'_{3m+c} is a smooth subvariety of codimension 2 isomorphic to the universal singular locus of a cubic curve [6].

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The Main Result

As demonstrated in [5], for $P(m) = 4m+c$ with $\gcd(4,c) = 1$, the subvariety M' is singular of codimension 2. The main result of this paper is the following generalization of [5].

Theorem 1.1 *For an integer $d \geqslant 4$, let $M = M_{dm-1}(\mathbb{P}_2)$ be the Simpson moduli space of (semi-)stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm-1$. Let $M' \subseteq M$ be the subvariety of singular sheaves. Then M' is singular of codimension 2.*

Using our understanding of M' and the construction from [6], we obtain, as a consequence, Theorem 4.7 which allows interpreting the blow up of M along M' as a modification of the boundary M' by a divisor consisting generically of line bundles.

Structure of the Paper

In Section 2 we recall the results from [9] and identify the open subvariety M_0 in M of sheaves without global sections with an open subvariety of a projective bundle $\mathbb{B}$ over a variety of Kronecker modules N. In Section 3, using a convenient characterization of free ideals of fat curvilinear points on planar curves from Appendix A, we show that a generic fibre of $M_0' = M_0 \cap M'$ over N is a union of projective subspaces of codimension 2 in the fibre of $\mathbb{B}$. Thus it is singular of codimension 2, which allows us to prove the main result. In Section 4 we briefly discuss how our analysis can be used to modify the boundary M' by line bundles (on support).

Some Notations and Conventions

Dealing with homomorphisms between direct sums of line bundles and identifying them with matrices, we consider the matrices acting on elements from the right, *i.e.*, the composition $X \xrightarrow{A} Y \xrightarrow{B} Z$ is given by the matrix $A \cdot B$. In particular, a section of a direct sum of line bundles $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$ is identified with the row-vector of sections of $\mathcal{E}_i$, $i = 1, \dots, m$.

2 Basic Constructions

2.1 Kronecker Modules

Let $\mathbb{V}$ be the affine space of Kronecker modules $(n-1)\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\Phi} n\mathcal{O}_{\mathbb{P}_2}$. There is a natural group action of $G = (\mathrm{GL}_{n-1}(\mathbb{k}) \times \mathrm{GL}_n(\mathbb{k}))/\mathbb{k}^*$ on $\mathbb{V}$. Since $\gcd(n-1,n) = 1$, all semistable points of this action are stable and G acts freely on the open subset of stable points $\mathbb{V}^s$. Then $\Phi \in \mathbb{V}^s$ if and only if Φ does not lie in the same orbit with a Kronecker module with a zero block of size $j \times (n-j)$, $1 \leqslant j \leqslant n-1$ [2, Proposition 15], [4, 6.2]. In particular, Kronecker modules with linearly independent maximal minors are stable. There exists a geometric quotient $N = N(3; n-1, n) = \mathbb{V}^s/G$, which is a smooth projective variety of dimension $(n-1)n$. For more details, consult [4, §6] and [2, §III].

The cokernel of a stable Kronecker module $\Phi \in \mathbb{V}^s$ is an ideal of a zero-dimensional scheme Z of length $(n-1)n/2$ provided that the maximal minors of Φ are coprime. In this case the maximal minors $d_0, \dots, d_{n-1}$ are linearly independent and there is a resolution

$$0 \to (n-1)\mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{\Phi} n\mathcal{O}_{\mathbb{P}_2}(-n+1) \xrightarrow{\begin{pmatrix} d_0 \\ \vdots \\ d_{n-1} \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_Z \to 0. \tag{2.1}$$

Moreover, Z does not lie on a curve of degree $n-2$. Let $\mathbb{V}_0$ denote the open subvariety of $\Phi \in \mathbb{V}^s$ of Kronecker modules with coprime maximal minors. Let $N_0 \subseteq N$ be the corresponding open subvariety in the quotient space.

This way one obtains a morphism from $N_0 \subseteq N$ to the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}_2$ of length $l = (n-1)n/2$ that sends a class of $\Phi \in \mathbb{V}^s$ to the zero scheme of its maximal minors. Since every zero-dimensional scheme of length l that does not lie on a curve of degree $n-2$ has a minimal resolution of type (2.1), this gives an isomorphism of N_0 and the open subvariety $H_0 \subseteq H = \mathbb{P}_2^{[l]}$ consisting of Z that do not lie on a curve of degree $n-2$. The complement of H_0 is an irreducible hypersurface [3, p. 119].

2.2 Projective Bundles Over N

Let $\mathbb{U}_1 = (n-1)\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ and let $\mathbb{U}_2 = n\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$. Consider the trivial vector bundles $\mathbb{V} \times \mathbb{U}_1$ and $\mathbb{V} \times \mathbb{U}_2$ over $\mathbb{V}$ and the following morphism between them: $\mathbb{V} \times \mathbb{U}_1 \xrightarrow{F} \mathbb{V} \times \mathbb{U}_2$, $(\Phi, L) \mapsto (\Phi, L \cdot \Phi)$.

Lemma 2.1 *The morphism F is injective over $\mathbb{V}^s$.*

Proof Let $\Phi \in \mathbb{V}^s$ and assume $L \cdot \Phi = 0$ for some non-zero $0 \neq L = (l_1, \dots, l_{n-1}) \in \mathbb{U}_1$. Since $l_i \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$, the dimension of the vector space generated by $\{l_i\}_i$ is at most 3. So for some $B \in \mathrm{GL}_{n-1}(\mathbb{k})$ we get $L \cdot B = L' = (l_1', l_2', l_3', 0, \dots, 0)$ such that the first non-zero entries are linearly independent. Then since Φ is semistable if and only if $B^{-1}\Phi$ is semistable and since $L\Phi = (LB) \cdot (B^{-1}\Phi) = 0$, we may assume without loss of generality that $L = (l_1, l_2, l_3, 0, \dots, 0)$ and the first non-zero entries of L are linearly independent.

If $l_1 \neq 0$, $l_2 = l_3 = 0$, then $L\Phi = 0$ implies that the first row of Φ is zero, which contradicts the stability of Φ.

If $l_1 \neq 0$, $l_2 \neq 0$, $l_3 = 0$, then the syzygy module of $(l_1, l_2)^{\mathrm{T}}$ is generated by $(l_2, -l_1)$. So the columns of the first two rows of Φ are scalar multiples of $\begin{pmatrix} l_2 \\ -l_1 \end{pmatrix}$ and hence, after performing elementary transformations on the columns of Φ, Φ is equivalent to a matrix with a zero block of size $2 \times (n-1)$, which contradicts the stability of Φ.

If $l_i \neq 0$, $i = 1, 2, 3$, then the syzygy module of $(l_1, l_2, l_3)^{\mathrm{T}}$ is generated by three linearly independent generators $(0, l_3, -l_2)$, $(-l_3, 0, l_1)$, and $(l_2, -l_1, 0)$, which implies that Φ is equivalent to a matrix with a zero block of size $3 \times (n-3)$, and this contradicts the stability of Φ. ∎

Therefore, $\mathbb{V}^s \times \mathbb{U}_1 \xrightarrow{F} \mathbb{V}^s \times \mathbb{U}_2$ is a vector subbundle, and hence the cokernel of F is a vector bundle of rank $6 \cdot n - 3 \cdot (n-1) = 3n + 3 = 3d$, denoted by E.

The group action of $\mathrm{GL}_{n-1}(\mathbb{k})\times\mathrm{GL}_n(\mathbb{k})$ on $\mathbb{V}^s\times\mathbb{U}_1$ and $\mathbb{V}^s\times\mathbb{U}_2$ induces a group action of $\mathrm{GL}_{n-1}(\mathbb{k})\times\mathrm{GL}_n(\mathbb{k})$ on E, and hence an action of $G = (\mathrm{GL}_{n-1}(\mathbb{k})\times\mathrm{GL}_n(\mathbb{k}))/\mathbb{k}^*$ on $\mathbb{P}E$. Finally, since the stabilizer of $\Phi \in \mathbb{V}^s$ under the action of G is trivial, it acts trivially on the fibres of $\mathbb{P}E$, and thus $\mathbb{P}E$ descends to a projective $\mathbb{P}_{3n+2}$-bundle

$$\mathbb{B} \xrightarrow{\nu} N = N(3; n-1, n) = \mathbb{V}^s/G.$$

Let $\mathbb{W}$ be the affine variety of morphisms

$$\mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\mathcal{O}_{\mathbb{P}_2}(-1). \tag{2.2}$$

Notice that $\mathbb{W}$ can be identified with $\mathbb{V} \times \mathbb{U}_2$ by the isomorphism

$$\mathbb{V} \times \mathbb{U}_2 \to \mathbb{W}, \quad (\Phi, Q) \mapsto \begin{pmatrix} Q \\ \Phi \end{pmatrix}.$$

The group $G' = (\mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\mathcal{O}_{\mathbb{P}_2}(-2)) \times \mathrm{Aut}(n\mathcal{O}_{\mathbb{P}_2}(-1))/\mathbb{k}^*$ acts on $\mathbb{W} \cong \mathbb{V} \times \mathbb{U}_2$. As shown in [9, Proposition 7.7], $\mathbb{B}$ is a geometric quotient of $\mathbb{V}^s \times \mathbb{U}_2 \setminus \mathrm{Im}\, F$ with respect to G'.

2.3 Moduli Space $M_{dm-1}(\mathbb{P}_2)$

Let $d \geqslant 4$, $d = n+1$, be an integer. Let $M = M_{dm-1}(\mathbb{P}_2)$ be the Simpson moduli space of (semi-)stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm-1$. In [9] it was shown that M contains an open dense subvariety M_0 of isomorphism classes of sheaves $\mathcal{F}$ with $h^0(\mathcal{F}) = 0$.

2.3.1 Sheaves in M_0

By [9, Claim 4.2] the sheaves in M without global sections are exactly the cokernels of the injective morphisms (2.2) with $A = \left(\begin{smallmatrix} Q \\ \Phi \end{smallmatrix}\right)$, $\det A \not\equiv 0$, $\Phi \in \mathbb{V}^s$. This allows one to describe M_0 as an open subvariety in $\mathbb{B}$.

Let $\mathbb{W}_0$ be the open subvariety of injective morphisms in $\mathbb{W}$ parameterizing the points from M_0. Since the determinant of a matrix from the image of F is zero, one sees that $\mathbb{W}_0 \subseteq \mathbb{V}^s \times \mathbb{U}_2 \setminus \mathrm{Im}\, F$, which allows one to conclude that $M_0 = \mathbb{W}_0/G'$ is an open subvariety of $\mathbb{B}$.

Let $\mathbb{B}_0 = \mathbb{B}|_{N_0}$ be the restriction of $\mathbb{B}$ to the open subscheme $N_0 \subseteq N$. From the exact sequence (2.1) it follows that a matrix $A = \left(\begin{smallmatrix} Q \\ \Phi \end{smallmatrix}\right) \in \mathbb{W}$, $\Phi \in \mathbb{V}_0$, has zero determinant if and only if A lies in the image of F. Therefore, the fibres of $\mathbb{B}$ over N_0 are contained in M_0 and thus $\mathbb{B}_0 \subseteq M_0$. As shown in [11], the codimension of the complement of $\mathbb{B}_0$ in M is at least 2.

2.3.2 Sheaves in $\mathbb{B}_0$

Sheaves in $\mathbb{B}_0$ are exactly the twisted ideal sheaves $\mathcal{I}_{Z\subseteq C}(d-3)$ of zero-dimensional schemes Z of length l lying on a curve C of degree d such that Z is not contained in a curve of degree $d-3$. In other words the sheaves $\mathcal{F}$ in $\mathbb{B}_0$ are given by the short exact sequences $0 \to \mathcal{F} \to \mathcal{O}_C(d-3) \to \mathcal{O}_Z \to 0$ with $Z \subseteq C$ as described.

A fibre over a point $[\Phi]$ from N_0 can be seen as the space of plane curves of degree d through the corresponding subscheme of l points. The identification is given by the map $v^{-1}([\Phi]) \ni [A] \mapsto \langle \det A \rangle \in \mathbb{P}(S^d V^*)$. Indeed, if two matrices over $[\Phi]$ are equivalent, then their determinants are equal up to a non-zero constant multiple, and hence the map above is well defined. On the other hand, if two matrices $A = \left(\begin{smallmatrix} Q \\ \Phi \end{smallmatrix}\right)$ and $A' = \left(\begin{smallmatrix} Q' \\ \Phi \end{smallmatrix}\right)$ have equal determinants, then $Q - Q'$ lies in the syzygy module of the maximal minors of Φ and hence (*cf.* (2.1)) is a linear combination of the rows of Φ, which means that A and A' are equivalent.

3 Singular Sheaves

Let M' be the subvariety of singular sheaves in M and let $M'_0 = M' \cap M_0$. Let us consider the restriction of v to M'_0 and describe some of its fibres.

3.1 Generic Fibres

Let N_c be the open subset of N_0 that corresponds to l different points. Under the isomorphism $N_0 \cong H_0$, it corresponds to the open subvariety $H_c \subseteq H_0$ of the configurations of l points on $\mathbb{P}_2$ that do not lie on a curve of degree $d-3$.

Let $[\Phi] \in N_c$ and let $Z = \{\mathrm{pt}_1, \dots, \mathrm{pt}_l\}$ be the corresponding zero-dimensional scheme. Let $m_1, \dots, m_l$ be the monomials of degree $d-3$ in variables x_0, x_1, x_2 ordered, say, in lexicographical order:

$$m_1 = x_0^{d-3}, \quad m_2 = x_0^{d-4}x_1, \quad m_3 = x_0^{d-4}x_2, \quad \dots .$$

Since Z does not lie on a curve of degree $d-3$, the matrix

$$\begin{pmatrix} m_1(\mathrm{pt}_1) & \dots & m_l(\mathrm{pt}_1) \\ \vdots & \ddots & \vdots \\ m_1(\mathrm{pt}_l) & \dots & m_l(\mathrm{pt}_l) \end{pmatrix}$$

has full rank. Assume without loss of generality that $\mathrm{pt}_1 = \langle 1, 0, 0 \rangle$. Then the matrix is

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ m_1(\mathrm{pt}_2) & m_2(\mathrm{pt}_2) & \dots & m_l(\mathrm{pt}_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(\mathrm{pt}_l) & m_2(\mathrm{pt}_l) & \dots & m_l(\mathrm{pt}_l) \end{pmatrix}$$

and therefore the matrix

$$\begin{pmatrix} m_1(\mathrm{pt}_2) & m_2(\mathrm{pt}_2) & \dots & m_l(\mathrm{pt}_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(\mathrm{pt}_l) & m_2(\mathrm{pt}_l) & \dots & m_l(\mathrm{pt}_l) \end{pmatrix}$$

has full rank and thus there exists a homogeneous polynomial q of degree $d-3$ vanishing at the points $\mathrm{pt}_2, \dots, \mathrm{pt}_l$, with the coefficient 1 in front of the monomial $m_1 = x_0^{d-3}$. Therefore, the forms $x_0^2 x_1 q$ and $x_0^2 x_2 q$ vanish at Z. Notice that $x_0^2 x_1 q$ has the monomial $x_0^{d-1}x_1$ but does not have $x_0^{d-1}x_2$, and $x_0^2 x_2 q$ has $x_0^{d-1}x_2$ but does not have $x_0^{d-1}x_1$.

Let $\mathcal{E}$ be a sheaf over $[\Phi]$. By Lemma A.1 it is singular at pt_1 if and only if pt_1 is a singular point of the support C of $\mathcal{E}$. The latter holds if and only if in the homogeneous polynomial of degree d defining C, the coefficients of the monomials $x_0^{d-1}x_1$ and $x_0^{d-1}x_2$ vanish. Therefore, taking into account the considerations above, sheaves

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over $[\Phi]$ singular at pt_1 constitute a projective subspace of codimension 2 in the fibre $F := \nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$. Since our argument can be repeated for each point pt_i, $i = 1, \dots, l$, we conclude that the sheaves over $[\Phi]$ singular at pt_i constitute a projective subspace F_i of codimension 2 in the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$.

Now let us clarify how the linear subspaces in the fibre corresponding to different points pt_i intersect with each other. First of all notice that Z must contain a triple of non-collinear points because otherwise Z must lie on a line. Therefore, in addition to the assumption $\mathrm{pt}_1 = \langle 1, 0, 0\rangle$, we can assume without loss of generality that $\mathrm{pt}_2 = \langle 0, 1, 0\rangle$, $\mathrm{pt}_3 = \langle 0, 0, 1\rangle$. Then the conditions for being singular at these three different points read as the absence of the following monomials in the equation of C:

$$\begin{aligned} &x_0^{d-1}x_1, x_0^{d-1}x_2 \quad \text{for the point } \mathrm{pt}_1;\\ &x_1^{d-1}x_0, x_1^{d-1}x_2 \quad \text{for the point } \mathrm{pt}_2;\\ &x_2^{d-1}x_0, x_2^{d-1}x_1 \quad \text{for the point } \mathrm{pt}_3. \end{aligned} \tag{3.1}$$

These conditions are clearly independent of each other. Moreover, these conditions are independent from the conditions imposed on the curves of degree d by the requirement $Z \subseteq C$. This is the case since the forms $x_0^2x_1q$ and $x_0^2x_2q$ constructed above clearly do not have monomials $x_1^{d-1}x_0$, $x_1^{d-1}x_2$ and $x_2^{d-1}x_0$, $x_2^{d-1}x_1$. This produces a way to construct the curves through Z with only one of the monomials from (3.1).

We conclude that for every pair of different indices $\operatorname{codim}_F F_i \cap F_j = 4$. Moreover, for every triple of different indices i, j, ν corresponding to three non-collinear points $\mathrm{pt}_i, \mathrm{pt}_j, \mathrm{pt}_\nu$ from Z, we have $\operatorname{codim}_F F_i \cap F_j \cap F_\nu = 6$. Finally we obtain the following.

Lemma 3.1 *The fibres of M_0' over N_c are unions of l different linear subspaces of $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$ of codimension 2 such that each pair intersects in codimension 4 and each triple corresponding to three non-collinear points intersects in codimension 6. In particular the fibres are singular.*

Remark 3.2 One can show that $\operatorname{codim}_F F_i \cap F_j \cap F_\nu = 6$ for every triple of different indices i, j, ν corresponding to three different points $\mathrm{pt}_i, \mathrm{pt}_j, \mathrm{pt}_\nu$ from Z.

Remark 3.3 In general it is not true that F_i intersect transversally. For example, for $d = 6$ and $Z = \{\mathrm{pt}_1, \dots, \mathrm{pt}_{10}\}$ with

$$\begin{array}{llll} \mathrm{pt}_1 = \langle 1,0,0\rangle, & \mathrm{pt}_2 = \langle 0,1,0\rangle, & \mathrm{pt}_3 = \langle 0,0,1\rangle, & \mathrm{pt}_4 = \langle 0,1,1\rangle,\\ \mathrm{pt}_5 = \langle 0,1,-1\rangle, & \mathrm{pt}_6 = \langle 1,-2,0\rangle, & \mathrm{pt}_7 = \langle 1,2,-1\rangle, & \mathrm{pt}_8 = \langle 1,1,-2\rangle,\\ \mathrm{pt}_9 = \langle 1,-1,1\rangle, & \mathrm{pt}_{10} = \langle 1,1,-1\rangle, & & \end{array}$$

we have $\operatorname{codim}_F F_1 \cap F_2 \cap F_3 \cap F_4 = 8$, but $\operatorname{codim}_F F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5 = 9$.

3.2 Fibres Over N_1

Let N_1 be the open subset of $N_0 \smallsetminus N_c$ that corresponds to $l-2$ different simple points and one double point. Let $[\Phi] \in N_1$ and let $Z = \{\mathrm{pt}_1\} \cup \{\mathrm{pt}_2, \dots, \mathrm{pt}_{l-1}\}$ be the corresponding zero-dimensional scheme, where pt_1 is a double point. Without loss of

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generality, applying if necessary a coordinate change, we may assume that pt_1 is given by the ideal (x_1^2, x_2).

3.2.1 Sheaves Singular at the Double Point

Since Z does not lie on a curve of degree $d-3$, the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ m_1(\mathrm{pt}_2) & m_2(\mathrm{pt}_2) & m_3(\mathrm{pt}_2) & \dots & m_l(\mathrm{pt}_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1(\mathrm{pt}_{l-1}) & m_2(\mathrm{pt}_{l-1}) & m_3(\mathrm{pt}_{l-1}) & \dots & m_l(\mathrm{pt}_{l-1}) \end{pmatrix}$$

has full rank. Therefore, there exist homogeneous polynomials q and q' of degree $d-3$ vanishing at the points $\mathrm{pt}_2, \dots, \mathrm{pt}_{l-1}$ such that q does not have monomial $x_0^{d-4}x_1$ but has term x_0^{d-3}, and q' does not have monomial x_0^{d-3} but has term $x_0^{d-4}x_1$. This implies that the forms $x_0^2x_2q$ and $x_0^2x_1q'$ vanish at Z.

Let $\mathcal{E}$ be a sheaf over $[\Phi]$. Let C be its support. By Lemma A.2 one concludes that $\mathcal{E}$ is singular at pt_1 if and only if in the homogeneous polynomial of degree d defining C, the coefficients of the monomials $x_0^{d-2}x_1^2$ and $x_0^{d-1}x_2$ vanish. Therefore, taking into account the considerations above, sheaves over $[\Phi]$ singular at pt_1 constitute a projective subspace of codimension 2 in the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$.

3.2.2 Sheaves Singular at Simple Points

Assume in addition that $\mathrm{pt}_2 = \langle 0, 0, 1\rangle$. We can still do this without loss of generality because Z cannot lie on a line. Then the matrix above is

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ m_1(\mathrm{pt}_3) & m_2(\mathrm{pt}_3) & m_3(\mathrm{pt}_3) & \dots & m_l(\mathrm{pt}_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1(\mathrm{pt}_{l-1}) & m_2(\mathrm{pt}_{l-1}) & m_3(\mathrm{pt}_{l-1}) & \dots & m_l(\mathrm{pt}_{l-1}) \end{pmatrix}.$$

Then, as in 3.1, we obtain a homogeneous polynomial q'' vanishing at $Z \smallsetminus \{\mathrm{pt}_2\}$ with coefficient 1 in front of the monomial $m_l = x_2^{d-3}$. Then the polynomials $x_1x_2^2q''$ and $x_0x_2^2q''$ vanish at Z. The former has $x_1x_2^{d-1}$ but does not have $x_0x_2^{d-1}$, and the latter has $x_0x_2^{d-1}$ but does not have $x_1x_2^{d-1}$. This means that the sheaves over $[\Phi]$ singular at pt_2 constitute a projective subspace of codimension 2 in the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$. This also shows that the sheaves over $[\Phi]$ singular at pt_j such that pt_1 and pt_j do not lie on a line constitute a projective subspace of codimension 2 in the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$.

Suppose there exists a point pt_j such that pt_1 and pt_j lie on a line. Without loss of generality we can assume that in this case $\mathrm{pt}_j = \langle 0,1,0\rangle$. Then we can construct a homogeneous polynomial q''' of degree $d-3$ through $Z \setminus \{\mathrm{pt}_j\}$ with coefficient 1 in front of the monomial x_1^{d-3}. Then the polynomials $x_0 x_1^2 q'''$ and $x_1^2 x_2 q'''$ vanish at Z. The former has $x_0 x_1^{d-1}$ but does not have $x_1^{d-1}x_2$, and the latter has $x_1^{d-1}x_2$ but does not have $x_0 x_1^{d-1}$. This means that the sheaves over $[\Phi]$ singular at pt_j such that pt_1 and pt_j lie on a line constitute a projective subspace of codimension 2 in the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$. This concludes the proof of the following lemma.

Lemma 3.4 *The fibres of M_0' over N_1 are unions of $l-1$ different linear subspaces of $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$ of codimension 2. In particular the fibres are singular.*

3.3 Main Result

Now we are able to prove Theorem 1.1.

Singularities

Notice that a generic fibre of a surjective morphism of smooth varieties must be smooth. Therefore, since a generic fibre of M_0' over N is singular as demonstrated in 3.1, we conclude that M_0' is singular. Therefore, M' is a singular subvariety in M.

Dimension

Since, as shown in [11], the codimension of the complement of $\mathbb{B}_0$ in M is at least 2, in order to demonstrate that $\operatorname{codim}_M M' = 2$, it is enough to show that $\operatorname{codim}_{\mathbb{B}_0} \mathbb{B}_0 \cap M' = 2$. Denote $\mathbb{B}_c = \mathbb{B}|_{N_c}$, $\mathbb{B}_1 = \mathbb{B}|_{N_1}$. Since the complement of $N_c \sqcup N_1$ in N_0 has codimension 2, it is enough to show $\operatorname{codim}_{\mathbb{B}_c \sqcup \mathbb{B}_1} M' \cap (\mathbb{B}_c \sqcup \mathbb{B}_1) = 2$. The latter follows immediately since the codimension of fibres of M_0' over $N_c \sqcup N_1$ is 2. This concludes the proof.

Smooth Locus of M'

Proposition 3.5 *The smooth locus of M' over N_c coincides with the locus of sheaves corresponding to $Z \subseteq C$ such that only one of the points in Z is a singular point of C.*

Proof Notice that $H_c \cong N_c$ can be seen as an open subscheme in $S^l\mathbb{P}_2$. Taking the composition of a local section (in analytic or étale topology) of the quotient $\prod_1^l \mathbb{P}_2 \to S^l\mathbb{P}_2$ with the projection $\prod_1^l \mathbb{P}_2 \to \mathbb{P}_2$ to the j-th factor, we get, locally around a given $Z_0 \in H_c$, l different local choices $N_c \cong H_c \supseteq U \xrightarrow{p_j} \mathbb{P}_2$, $j = 1,\dots,l$ of one point in $Z \in U \subseteq H_c$. Shrinking U if necessary, we can assume that $\mathbb{B} \to N_c$ is trivial over U. Then by 3.1 the subvariety $S_j \subseteq \mathbb{B}|_U$ of those sheaves given by $Z \subseteq C$ that are singular at point $p_j(Z) \in \mathbb{P}_2$ is isomorphic to a product of U with a linear subspace of $\mathbb{P}_{3d-1}$ of codimension 2, *i.e.*, with $\mathbb{P}_{3d-3}$. Therefore, S_j is smooth. Notice that $M' \cap \mathbb{B}|_U$ is isomorphic to the union of S_j, $j = 1,\dots,l$. Therefore, $\bigcup_j S_j \setminus \bigcup_{j\neq i} S_j \cap S_i$ is smooth, which proves the required statement. ∎

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4 Modifying the Boundary by Line Bundles

4.1 Normal Spaces at $M' \cap \mathbb{B}_c$

Let $[\mathcal{F}] \in \mathbb{B}_c \cap M'$ be the isomorphism class of a singular sheaf represented by a curve C of degree d and a configuration of l points $Z \subseteq C$, $Z = \{\mathrm{pt}_1, \dots, \mathrm{pt}_l\}$. Assume without loss of generality that $Z \cap \operatorname{Sing} C = \{\mathrm{pt}_1, \dots, \mathrm{pt}_r\} := Z'$, $0 < r \leqslant l$.

Using, for every $j = 1, \dots, r$, the local choice of a point p_j and the morphism $M \to \mathbb{P}S^d V^*$, $[\mathcal{G}] \mapsto \operatorname{Supp} \mathcal{G}$, we obtain locally around $[\mathcal{F}]$ a morphism $\mathbb{B}_c \supseteq U_{[\mathcal{F}]} \xrightarrow{\rho_j} C_d$ from a neighbourhood of $[\mathcal{F}]$ to the universal planar curve C_d of degree d, *i.e.*, to the variety of pairs (C', pt'), where pt' is a point of a curve C' of degree d. For a fixed $1 \leqslant j \leqslant r$, ρ_j sends an isomorphism class of a sheaf given by a pair consisting of a curve C' of degree d and a configuration $\{\mathrm{pt}'_1, \dots, \mathrm{pt}'_l\} \subseteq C'$ to the pair (C', pt'_j).

This induces a linear map of the tangent spaces $\mathrm{T}_{[\mathcal{F}]} \mathbb{B}_c \to \mathrm{T}_{(C, \mathrm{pt}_j)} C_d$. Let C'_d denote the universal singular locus of C_d. Since ρ_j maps S_j to C'_d, we also obtain the induced linear map on the normal spaces

$$N^{(j)}_{[\mathcal{F}]} := \mathrm{T}_{[\mathcal{F}]} \mathbb{B}_c / \mathrm{T}_{[\mathcal{F}]} S_j \xrightarrow{N_{[\mathcal{F}]}(\rho_j)} \mathrm{T}_{(C, \mathrm{pt}_j)} C_d / \mathrm{T}_{(C, \mathrm{pt}_j)} C'_d =: N_{(C, \mathrm{pt}_j)}.$$

Lemma 4.1 *The linear map $N^{(j)}_{[\mathcal{F}]} \xrightarrow{N_{[\mathcal{F}]}(\rho_j)} N_{(C, \mathrm{pt}_j)}$ constructed above is an isomorphism of 2-dimensional vector spaces.*

Proof Let $F = \nu^{-1}(\nu([\mathcal{F}]))$ be the fibre of $[\mathcal{F}]$. As already noticed, F can be seen as the space of curves of degree d through Z. For a fixed $j \in \{1, \dots, r\}$ let F'_j be the fibre of S_j over $\nu([\mathcal{F}])$, which can be seen as the subspace of F of those curves through Z singular at pt_j. Let F_{pt_j} be the space of curves of degree d through pt_j and let F'_{pt_j} be its subspace of curves singular at pt_j.

Our analysis in 3.1 implies that ρ_j induces an isomorphism

$$\mathrm{T}_C F / \mathrm{T}_C F'_j \cong \mathrm{T}_C F_{\mathrm{pt}_j} / \mathrm{T}_C F'_{\mathrm{pt}_j},$$

which concludes the proof, because both S_j and C'_d are locally trivial over N and $\mathbb{P}_2$, respectively, and hence $N^{(j)}_{[\mathcal{F}]} \cong \mathrm{T}_C F / \mathrm{T}_C F'_j$, $N_{(C, \mathrm{pt}_j)} \cong \mathrm{T}_C F_{\mathrm{pt}_j} / \mathrm{T}_C F'_{\mathrm{pt}_j}$. ∎

4.2 R-bundles

Let $\mathfrak{U}$ denote the universal family on $M \times \mathbb{P}_2$. Consider a germ of a morphism γ of a smooth curve T to $\mathbb{B}_c$ mapping $0 \in T$ to $\gamma(0) = [\mathcal{F}]$. Let $\boldsymbol{\mathcal{F}}$ be the pullback of $\mathfrak{U}$ along $\gamma \times \mathrm{id}_{\mathbb{P}_2}$.

If γ is not tangent to S_j at $[\mathcal{F}]$, then $\boldsymbol{\mathcal{F}}$ represents $[\mathcal{F}]$ as a flat 1-parameter degeneration of sheaves $[\boldsymbol{\mathcal{F}}_t] = \gamma(t)$ non-singular at point $p_j(\nu \circ \gamma(t))$, where $\boldsymbol{\mathcal{F}}_t = \boldsymbol{\mathcal{F}}|_{\{t\} \times \mathbb{P}_2}$. If γ is not tangent to S_j at $[\mathcal{F}]$ for all $j = 1, \dots, r$, then $\boldsymbol{\mathcal{F}}$ is a degeneration of non-singular sheaves to $\mathcal{F} = \boldsymbol{\mathcal{F}}_0$.

Let $\widetilde{T \times \mathbb{P}_2} \xrightarrow{\sigma} T \times \mathbb{P}_2$ be the blow-up $\mathrm{Bl}_{\{0\} \times Z'}(T \times \mathbb{P}_2)$ and let $D_1 = D_1(Z')$ be its exceptional divisor, which is a disjoint union of projective planes $D_1(p_j) \cong \mathbb{P}_2$, $j = 1, \dots, r$.

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The fibre of the flat morphism $\widetilde{T\times\mathbb{P}_2}\xrightarrow{\sigma}T\times\mathbb{P}_2\xrightarrow{pr_1}T$ over 0 is a reduced surface $D(Z')=D(p_1,\dots,p_r)$ obtained by blowing-up $\mathbb{P}_2$ at $\{\mathrm{pt}_1,\dots,\mathrm{pt}_r\}$ and attaching the surfaces $D_1(\mathrm{pt}_j)\cong\mathbb{P}_2$, $j=1,\dots,r$, to $D_0(Z')=D_0(\mathrm{pt}_1,\dots,\mathrm{pt}_r)=\mathrm{Bl}_{Z'}\mathbb{P}_2$ along the exceptional lines $L_1,\dots,L_r$.

Let $\mathcal{E}$ be the sheaf on $\widetilde{T\times\mathbb{P}_2}$ obtained as the quotient of the pullback $\sigma^*(\mathcal{F})$ by the subsheaf $\mathcal{T}or_{\mathcal{I}_{D_1}}(\sigma^*(\mathcal{F}))$ generated by the sections annihilated by the ideal sheaf $\mathcal{I}_{D_1}$ of the exceptional divisor D_1.

Lemma 4.2 *Assume that for all $j=1,\dots,r$, γ is not tangent to S_j at $[\mathcal{F}]$. Then the sheaf $\mathcal{E}$ is a flat family of 1-dimensional sheaves. Its fibres $\mathcal{E}_t$, $t\neq 0$, are non-singular sheaves on $\mathbb{P}_2$, the fibre $\mathcal{E}=\mathcal{E}_0$ is a 1-dimensional non-singular sheaf on $D(Z')$.*

Proof We shall show that our definition of $\mathcal{E}$ locally coincides with the construction from [6].

Fix some local coordinates x_i,y_i at pt_i. Then in some neighbourhood U_i of pt_i the sheaf $\mathcal{F}$ is just an ideal sheaf of pt_i on C and hence can be given as the cokernel of a morphism

$$2\mathcal{O}_{U_i}\xrightarrow{A}2\mathcal{O}_{U_i},\quad A=\begin{pmatrix}x_i&y_i\\a_i&b_i\end{pmatrix},$$

for some polynomials a_i,b_i in x_i,y_i. As $\mathrm{pt}_i\in\operatorname{Sing}C$ for $i=1,\dots,r$, a_i and b_i do not have constant terms.

Let $\mathcal{U}_d$ denote the universal family on $C_d\times\mathbb{P}_2$. Then for a fixed $j\in\{1,\dots,r\}$, locally around the point $[\mathcal{F}]\times\mathrm{pt}_j\in\mathbb{B}_c\times\mathbb{P}_2$, $\mathcal{U}$ is isomorphic to the pullback of $\mathcal{U}_d$ along $\rho_j\times\mathrm{id}_{\mathbb{P}_2}$. Then $\mathcal{F}$ is isomorphic locally around $0\times\mathrm{pt}_j$ to the pullback of $\mathcal{U}_d$ along $\gamma_j\times\mathrm{id}_{\mathbb{P}_2}$, where $\gamma_j=\rho_j\circ\gamma$.

The latter means that the family $\mathcal{F}$ is given around $0\times\mathrm{pt}_j$ as a cokernel of the morphism

$$2\mathcal{O}_{T\times U_j}\xrightarrow{\begin{pmatrix}x_j&y_j\\v_j&w_j\end{pmatrix}+t\cdot B(t)}2\mathcal{O}_{T\times U_j},\quad B(t)=\begin{pmatrix}b_{11}(t)&b_{12}(t)\\b_{21}(t)&b_{22}(t)\end{pmatrix},$$

where $b_{11}(t),b_{12}(t)$ take values in $\Bbbk$ and $b_{21}(t),b_{22}(t)$ are polynomials of degree not bigger than d.

The blow-up $\widetilde{T\times\mathbb{P}_2}$ over $0\times\mathrm{pt}_j\in T\times\mathbb{P}_2$ is locally just the blow-up $\mathrm{Bl}_{0\times\mathrm{pt}_j}T\times U_j=:\widetilde{T\times U_j}$. It can be seen as a subvariety in $T\times U_j\times\mathbb{P}_2$ given by the (2×2)-minors of the matrix $\begin{pmatrix}t&x_j&y_j\\u_0&u_1&u_2\end{pmatrix}$, where u_0,u_1,u_2 are some homogeneous coordinates of the last factor $\mathbb{P}_2$. The canonical section s of $\mathcal{O}_{\widetilde{T\times\mathbb{P}_2}}(D_1)$ is locally given by t/u_0, x_j/u_1, and y_j/u_2.

As in [6] one "divides" the pullback $\sigma^*(A+tB(t))$ by s and obtains a family $\mathcal{E}'$ of one-dimensional sheaves given by a locally free resolution

$$0\to 2\mathcal{O}_{\widetilde{T\times U_j}}(D_1)\xrightarrow{\phi(A,B)}2\mathcal{O}_{\widetilde{T\times U_j}}\to\mathcal{E}'\to 0.$$

We claim that this construction coincides with taking the quotient by the subsheaf annihilated by $\mathcal{I}_{D_1}$. This follows from a diagram chase on the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathcal{K} \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_{\widetilde{T\times U_j}} \arrow[r, "\sigma^*(A+tB(t))"] \arrow[d, "\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)"'] & 2\mathcal{O}_{\widetilde{T\times U_j}} \arrow[r] \arrow[d, equal] & \sigma^*\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{\widetilde{T\times U_j}}(D_1) \arrow[r, "\phi(A,B)"] \arrow[d] & 2\mathcal{O}_{\widetilde{T\times U_j}} \arrow[r] & \mathcal{E}' \arrow[r] \arrow[d] & 0 \\
 & \mathcal{K} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

More precisely, one shows that $\mathcal{K}$ is exactly the subsheaf of $\sigma^*\mathcal{F}$ annihilated by s.

Now the flatness of $\mathcal{E}$ and the local freeness of $\mathcal{E}$ on its support follow from [6, Lemma 5.1 and its preceding discussion] . ■

Remark 4.3 Notice that the fibre of $\sigma^*\mathcal{F}$ over $0 \in T$ is not a 1-dimensional sheaf. Its support contains D_1. The subsheaf $\mathcal{T}or_{\mathcal{I}_{D_1}}(\sigma^*(\mathcal{F}))$ of $\sigma^*(\mathcal{F})$ is the maximal subsheaf supported completely in D_1.

Remark 4.4 The sheaf $\mathcal{E}_0$ depends only on the derivative of γ at 0, *i.e.*, only on the induced map $\mathrm{T}_0\, T \to \mathrm{T}_{[\mathcal{F}]}\, \mathbb{B}_0$ of tangent spaces.

Definition 4.5 The sheaf $\mathcal{E}_0$ on $D(Z')$, as above, is called an R-bundle associated with the pair $Z \subseteq C$, $Z' = Z \cap \operatorname{Sing} C$.

The following generalizes [6, Definition 5.5].

Definition 4.6 Two R-bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ associated with $Z \subseteq C$ on $D(\mathrm{pt}_1, \dots, \mathrm{pt}_r)$ are called *equivalent* if there exists an automorphism ϕ of $D(\mathrm{pt}_1, \dots, \mathrm{pt}_r)$ that acts identically on the surface $D_0(\mathrm{pt}_1, \dots, \mathrm{pt}_r)$ such that $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$.

Theorem 4.7 *Assume that the components $F_1, \dots, F_r$ intersect transversally at $[\mathcal{F}]$. Then the equivalence classes of R-bundles associated with $Z \subseteq C$ are in one-to-one correspondence with the points of the product of projective lines $\prod_{j=1}^r \mathbb{P}N_{[\mathcal{F}]}^{(j)}$.*

Proof The classes of R-bundles around $D_1(\mathrm{pt}_j)$ are parameterized by $\mathbb{P}N_{(C,\mathrm{pt}_j)} \cong \mathbb{P}N_{[\mathcal{F}]}^{(j)}$ by [6, Proposition 5.6] and Lemma 4.1. Since the components $F_1, \dots, F_r$ intersect transversally, the map $\mathrm{T}_{[\mathcal{F}]}\, \mathbb{B}_c \smallsetminus \bigcup_{j=1}^r \mathrm{T}_{[\mathcal{F}]}\, S_j \to \prod_{j=1}^r \mathbb{P}N_{[\mathcal{F}]}^{(j)}$ is surjective, which means that the class of an R-bundle around a given exceptional plane $D_1(\mathrm{pt}_j)$ is independent of the class of the R-bundle around other planes $D_1(\mathrm{pt}_i)$, $i \neq j$. This concludes the proof. ■

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Remark 4.8 The assumption of Theorem 4.7 is always satisfied at least for $r \leqslant 3$ by Remark 3.2. In particular, it is the case for $d = 4$. Remark 3.2 also implies that the locus of $[\mathcal{F}] \in \mathbb{B}_c$ that do not satisfy the assumption on transversality lies at least in codimension 7.

Remark 4.9 All possible R-bundles can be produced simultaneously as fibres of a family of sheaves over (an open subset of) $\widetilde{M} := \mathrm{Bl}_{M'} M$. Indeed, pull back the universal family over M to a family $\mathfrak{U}_{\widetilde{M}}$ over $\widetilde{M}$. Let

$$\widetilde{\widetilde{M} \times \mathbb{P}_2} \to \widetilde{M} \times \mathbb{P}_2$$

be the blow-up along the subvariety in $\widetilde{M} \times \mathbb{P}_2$ where $\mathfrak{U}_{\widetilde{M}}$ is singular, pull $\mathfrak{U}_{\widetilde{M}}$ back and consider the quotient $\widetilde{\mathfrak{U}}$ by the subsheaf annihilated by the ideal sheaf of the exceptional divisor of this blow-up.

Let $\mathbb{B}_{\mathrm{gen}}$ denote the open locus of those $[\mathcal{F}] \in \mathbb{B}_0$ satisfying the conditions of Theorem 4.7. Then $\widetilde{\mathbb{B}}_{\mathrm{gen}} := \mathrm{Bl}_{M' \cap \mathbb{B}_{\mathrm{gen}}} \mathbb{B}_{\mathrm{gen}}$ is smooth by [8, Theorem 1.2 and Theorem 1.3]. The fibre of the exceptional divisor over $[\mathcal{F}] \in \mathbb{B}_{\mathrm{gen}}$ coincides with the product $\prod_{j=1}^{r} \mathbb{P}N_{[\mathcal{F}]}^{(j)}$. The restriction of $\widetilde{\mathfrak{U}}$ to $\widetilde{\mathbb{B}}_{\mathrm{gen}}$ is a flat family of 1-dimensional non-singular sheaves: non-singular $(dm-1)$-sheaves together with R-bundles associated with $Z \subseteq C$, $Z \cap \operatorname{Sing} C \neq \varnothing$.

This allows us to see the blow-up $\widetilde{\mathbb{B}}_{\mathrm{gen}}$ as a process that substitutes the boundary $M' \cap \mathbb{B}_{\mathrm{gen}}$ by a divisor consisting of non-singular 1-dimensional sheaves, which generalizes the construction from [6].

4.3 Further Work

A more detailed discussion of the relation of the blow-up $\mathrm{Bl}_{M'} M$ with modifying the boundary of M by line bundles should follow in a separate paper.

A On the Ideals of Points on Planar Curves

Let $R = \mathcal{O}_{C,p}$ be a local $\Bbbk$-algebra of a curve C at point $p \in C$. Let $I \subseteq R$ be an ideal of R. As a submodule of a free module, I is a torsion-free R-module. If R is regular, *i.e.*, if p is a smooth point of C, then I is free. Therefore, the non-regularity of R is a necessary condition for the non-freeness of I.

A.1 Ideals of Simple Points on a Curve

Let $\mathfrak{m} = \mathfrak{m}_{C,p}$ be the maximal ideal of R and let $\Bbbk_p = R/\mathfrak{m}$ be the local ring of the structure sheaf of the one point subscheme $\{p\} \subseteq C$. The following is a slight reformulation of the well-know fact about local 1-dimensional rings [1, Proposition 9.2].

Lemma A.1 *The maximal ideal $\mathfrak{m}$ is a free R-module if and only if R is regular.*

Proof If $\mathfrak{m}$ is free, then $\mathfrak{m}$ is a principal ideal and therefore R is regular. If R is regular, then $\mathfrak{m}$ is principal and thus free, since R is a domain in this case. ■

A.2 Ideals of Fat Curvilinear Points on a Planar Curve

Assume that C is a planar curve locally defined as the zero locus of $f \in \Bbbk[x,y]$. Assume $p = 0$, consider a fat curvilinear point Z at p given by the ideal

$$(x - h(y), y^n) \subseteq \Bbbk[x,y], \quad h(y \in \Bbbk[y]), h(0) = 0, \deg h < n,$$

and assume that Z is a subscheme of C. Let $I \subseteq R$ be its ideal.

Since we assumed $Z \subseteq C$, one can write

$$f = \det \begin{pmatrix} x - h(y) & y^n \\ u(y) & v(x,y) \end{pmatrix}, \quad u(y) \in \Bbbk[y], v(x,y) \in \Bbbk[x,y].$$

Lemma A.2 *Keeping the notations as above, let R be a non-regular ring,* i.e., *let p be a singular point of C. Then I is non-free if and only if $u(0) = 0$.*

If I is free, then it is generated by $x - h(y)$ and there is an isomorphism

$$R \cong I, \quad r \mapsto r \cdot (x - h(y)).$$

Proof First we will show that I is generated by one element if and only if $u(0) \neq 0$.

Since $R = \Bbbk[x,y]_{(x,y)}/(f)$, it is enough to answer the question when the ideal

$$(x - h(y), y^n, f) \subseteq \Bbbk[x,y]_{(x,y)}$$

equals (ξ, f) for some $\xi \in \Bbbk[x,y]_{(x,y)}$.

"$\Leftarrow$": If $u(0) \neq 0$, then $(x - h(y), y^n, f) = (x - h(y), f)$.

"$\Rightarrow$": Let $(x - h(y), y^n, f) = (\xi, f)$. Without loss of generality we can assume that

$$\xi \in (x - h(y), y^n) \quad \text{and} \quad \xi = a \cdot (x - h(y)) + b \cdot y^n$$

for some $a, b \in \Bbbk[x,y]_{(x,y)}$. In order to show that $u(0) \neq 0$, we suppose that the contrary holds true, *i.e.*, $u(0) = 0$.

Since one can embed $\Bbbk[x,y]_{(x,y)}$ into the ring of formal power series $\Bbbk[[x,y]]$, we are going to consider the elements of $\Bbbk[x,y]_{(x,y)}$ as power series in x, y.

Since $x - h(y) \in (\xi, f)$, then $x - h(y) = c \cdot \xi + d \cdot f = ca(x - h(y)) + cby^n + df$. As the orders of cby^n and df are at least 2, we conclude that a and c are units. We can assume without loss of generality that $\xi = x - h(y) + \eta(x,y)$, $\operatorname{ord} \eta(h(y), y) \geqslant n$. As $y^n \in (\xi, f)$, it must hold $y^n = C\xi + Df$ for some C and D. Since by our assumption $u(0) = 0$, evaluating this equality at $x = h(y)$, we conclude that

$$y^n = C(h(y), y) \cdot \eta(h(y), y) + D(h(y), y) \cdot (-u(y) \cdot y^n).$$

Therefore, C must be a unit and $\operatorname{ord} \xi(h(y), y) = \operatorname{ord} \eta(h(y), y) = n$.

Substituting x by $h(y)$ in the equality $x - h(y) = c \cdot \xi + d \cdot f$, we get

$$0 = c(h(y), y) \cdot \eta(h(y), y) + d(h(y), y) \cdot (-u(y) \cdot y^n).$$

Since c is a unit, it contains a non-zero constant term, and hence the product

$$c(h(y), y) \cdot \eta(h(y), y)$$

has order n. On the other hand, since $u(0) = 0$ by our assumption, the order of $d(h(y), y) \cdot u(y) y^n$ is at least $n + 1$. We obtain a contradiction, which shows that our assumption was wrong.

Notice that if I is free, then it must be one-generated. On the other hand, we see that if I is one-generated, then it is generated by $x - h(y)$. In this case $u(0) \neq 0$ and therefore f is not divisible by $x - h(y)$. Thus $x - h(y)$ is not a zero divisor in R and there is an isomorphism $R \cong I$, $r \mapsto r(x - h(y))$. The latter means that I is one-generated if and only if it is free, which concludes the proof. ■

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