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SKEW POLYNOMIALS AND ALGEBRAIC REFLEXIVITY

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Abstract. For an arbitrary K-algebra R, an R, K-bimodule M is algebraically reflexive if the only K-endomorphisms of M leaving invariant every R-submodule of M are the scalar multiplications by elements of R. Hadwin has shown for an infinite field K and $R = K[x]$ that R is reflexive as an R, K-bimodule. This paper provides a generalisation by giving a skew polynomial version of his result.

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1. Introduction. If V is a vector space over a field K and $\mathcal{L}$ is a lattice of subspaces then alg $\mathcal{L}$ is defined to be the algebra of all K-endomorphisms of V which leave $\mathcal{L}$ point-wise fixed; dually if R is a subalgebra of End ${}_KV$, then lat R is defined to be the lattice of all subspaces of V which are left invariant by every element of R. Combining these two functors produces alg (lat R), an algebra containing R, and R is called reflexive when these two algebras are equal. Thus, an algebra of operators on a vector space is called *reflexive* when no larger algebra of operators has the same lattice of invariant subspaces. This terminology originated with Halmos [**3**] although earlier work also explored some of this area; for example, the lattice of invariant subspaces of a single linear transformation was studied in [**1**].

In [**2**] a start was made on studying an extension of the concept of alg lat where the focus moved to classes of representations, that is R, K-bimodules, of a K-algebra R. For an arbitrary K-algebra R a bimodule ${}_RM_K$ is said to be (algebraically) reflexive if, whenever $\alpha \in$ End ${}_KM$ is such that $\alpha m \in Rm$ for all $m \in M$, then there is an $r \in R$ such that α is simply left multiplication by r.

More generally, define for rings R, A and bimodule ${}_R\mathrm{M}_A$ [**2**]

$$\text{alglat } M := \{\alpha \in \text{End } M_{\mathrm{A}} \mid \alpha m \in Rm \text{ for all } m \in M\},$$

and let $\lambda(r) \in \mathrm{End}_K M$ send m to rm for all $m \in M$, setting

$$\lambda(R) := \{\lambda(r) : r \in R\},$$

then $\lambda(R) \subseteq$ alglat M and M is reflexive when this is equality. In Halmos' notation alglat M becomes alg (lat $\lambda(R)$) and ${}_RM_K$ is a reflexive bimodule exactly when $\lambda(R)$ is a reflexive algebra of operators on ${}_RM_K$. It is of interest to note that Leptin [**5**] considered the concept of alglat, although not in this notation, for topological modules in a study of completeness and linear compactness.

A central lemma in the paper by Hadwin [**4**] states that if K is an infinite field and $R = K[x]$, the polynomial ring, then the algebra ${}_RR_K$ is reflexive as an R, K-bimodule. It is the aim of this paper to establish a skew polynomial version of this result.

2. Results.

LEMMA 1. *Let $M = Rm$ be an R, A-bimodule with A a subring of R such that $km = mk$ for all $k \in A$. Suppose there is a set $\mathcal{E}$ of additive group homomorphisms from Rm into Am such that*

(1) $\epsilon(rv) = \epsilon(r\epsilon(v))$ *for all* $r \in R$, $v \in M$;

(2) $\epsilon(u) = 0$ *for all* $\epsilon \in \mathcal{E}$ *implies* $u = 0$.

Then ${}_R M_A$ *is reflexive.*

Proof. Let $\alpha \in$ alglat M. Then $\alpha m = am$ for some $a \in R$, and $\alpha - \lambda(a) = \beta \in$ alglat M with $\beta m = 0$. We show that $\beta = 0$, whence $\alpha = \lambda(a)$.

Let $u \in M$. Then $\beta u = ru$ for some $r \in R$. Also, for all $\epsilon \in \mathcal{E}$, there is $s \in R$ with $\beta(u - \epsilon(u)) = s(u - \epsilon(u))$. Hence, $ru - \beta(\epsilon(u)) = su - s\epsilon(u)$. Now $u \in M$ so $\epsilon(u) = km$ for some $k \in A$, so $\beta(\epsilon(u)) = \beta(km) = \beta(mk) = \beta(m)k = 0$. Thus $ru = su - s\epsilon(u)$. Therefore $\epsilon(ru) = \epsilon(su - s\epsilon(u)) = \epsilon(su) - \epsilon(s\epsilon(u)) = \epsilon(su) - \epsilon(su)$ for all $\epsilon \in \mathcal{E}$. By (2) $ru = 0$, so $\beta u = 0$ for all $u \in U$ and so $\beta = 0$. □

THEOREM 1. *Let $R = A[x; \sigma, \delta]$ be a skew polynomial ring over a domain A with $\sigma : A \to A$ a monomorphism and δ a σ-derivation on A. Let $K = \{a \in Z(A) \mid \sigma a = a, \delta a = 0\}$. If K is infinite, then ${}_R R_A$ is reflexive.*

Proof. We apply Lemma 1 to the R, A-bimodule R, so $m = 1$. For each $k \in K$ define the evaluation map $\epsilon_k : R \to A$ by

$$\epsilon_k : \sum_{i=0}^{n} r_i x^i \mapsto \sum_{i=0}^{n} r_i k^i, \quad r_i \in A.$$

It is clear that ϵ_k is an additive group homomorphism from R to A. We now verify that the conditions of Lemma 1 are satisfied by ϵ_k.

(1) Let $r = \sum_{i=0}^{m} r_i x^i$, $v = \sum_{j=0}^{n} v_j x^j$. Then $\epsilon_k(rv) = \sum_{i,j} \epsilon_k(r_i x^i v_j x^j)$ so it suffices to show (1) for the monomial cases.

Consider $\epsilon_k(ax^i bx^j)$ with $a, b \in A$. Inductively, $x^i b = \sum_{l=0}^{i} t_l(b) x^l$, where $t_l = \sum_{w \in W_l} w$ and W_l is the set of all words in σ, δ with $l \sqcup \sigma$'s and $i - l \sqcup \delta$'s. Now

$$x^{i+1} b = \sum x t_l(b) x^l = \sum_l \sum_w x w(b) x^l.$$

Since $xw(b) = (\sigma w)(b)x + (\delta w)(b)$, σw is a word with $l + 1 \sqcup \sigma$'s and $i - l \sqcup \delta$'s and δw is a word with $l \sqcup \sigma$'s and $i - l + 1 \sqcup \delta$'s; thus the general word with $l + 1 \sqcup \sigma$'s and $i - l \sqcup \delta$'s beginning with a δ comes from a word with $l + 1 \sqcup \sigma$'s and $i - l - 1 \sqcup \delta$'s (in W_{l+1}).

Thus

$$\epsilon_k(ax^i bx^j) = \epsilon_k\left(\sum_l \sum_w aw(b) x^l x^j\right) = \sum_l \sum_w aw(b) k^l k^j.$$

Also, $\epsilon_k(ax^i \epsilon_k(bx^j)) = \epsilon_k(ax^i bk^j) = \epsilon_k(a(\sum_l \sum_w w(b)x^l)k^j)$. Now $xk = \sigma(k)x + \delta(k) = kx$, by the definition of K, so $\epsilon_k(ax^i \epsilon_k(bx^j)) = \sum_{l,w} aw(b)k^j k^l$. Thus $\epsilon_k(ax^i \epsilon_k(bx^j)) = \epsilon_k(ax^i bx^j)$ and $\epsilon_k(r\epsilon_k(v)) = \epsilon_k(rv)$.

(2) Suppose $\epsilon_k(u) = 0$ for all $k \in K$. Let $u = \sum_{i=0}^{n} u_i x^i$, $u_i \in A$. Then $\sum_{i=0}^{n} u_i k^i = 0$ for all $k \in K$.

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Let $k_0, k_1, \ldots, k_n$ be distinct elements of K. Then set

$$V = \begin{bmatrix} 1 & \vdots & \vdots & 1 \\ k_0 & \vdots & \vdots & k_n \\ k_0^2 & \vdots & \vdots & k_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ k_0^n & \vdots & \vdots & k_n^n \end{bmatrix},$$

the $(n+1)\times(n+1)$ Vandermonde matrix. Now $(u_0, \ldots, u_n)V = 0$ and V is invertible over the field of fractions of K, so $u_0 = u_1 = \cdots = u_n = 0$. Hence $u = 0$.

By Lemma 1 ${}_RR_A$ is reflexive. □

Corollary 1 [**4**, Lemma 0]. *If K is an infinite field and $R = K[x]$, then ${}_RR_K$ is reflexive.*

The next result gives sufficient conditions for every non-torsion module to be reflexive and generalises Hadwin's result concerning non-torsion modules over $K[x]$.

Theorem 2. *Let R be a K-algebra domain and suppose that R has finite Goldie dimension. If ${}_RR_K$ is reflexive, then every non-torsion R-module is reflexive.*

Proof. Let M be a non-torsion R-module. Let $m_0 \in M$ be such that $rm_0 \neq 0$ for all $0 \neq r \in R$, $\alpha \in$ alglat ${}_RM_K$, and $m \in M$. Note that Rm is a homomorphic image of R and $Rm_0 \cong R$. Thus if $Rm_0 \cap Rm = 0$, then $Rm_0 \oplus Rm$ is reflexive as an R, K-bimodule by [**2**, Corollary 1.9]. Therefore if $\alpha m_0 = r_0 m_0$, with r_0 uniquely determined by the torsion-freeness of m_0, then $\alpha m = r_0 m$ also.

Now suppose $Rm_0 \cap Rm \neq 0$. We show first that m is torsion-free. Let $0 \neq x = a_0 m_0 = am$ for some $a_0, a \in R$. If $b \in R$ and $bx = 0$ then $ba_0 m_0 = 0$, so $ba_0 = 0$ by the choice of m_0, and finally $b = 0$ from the domain property. Thus $L = \{r \in R \mid rm = 0\}$ is such that $Ra \cap L = 0$. Now $\dim R \geq \dim(Ra \oplus L) = \dim Ra + \dim L = \dim R + \dim L$, hence $\dim L = 0$. Therefore $L = 0$ and m is torsion-free.

We now have unique $r_0, r \in R$ such that $\alpha m_0 = r_0 m_0$ and $\alpha m = rm$. Moreover, from the reflexivity of $Rm_0 \cong Rm \cong R$ we have $\alpha x = r_0 x = rx$ and so $r_0 = r$ since x is also torsion-free. Thus $\alpha m = r_0 m$ and M is reflexive. □

Corollary 2 [**4**, Theorem 1]. *If K is an infinite field and $R = K[x]$, then every non-torsion R-module is reflexive.*

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