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Free and projective Banach lattices

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We define and prove the existence of free Banach lattices in the category of Banach lattices and contractive lattice homomorphisms, and establish some of their fundamental properties. We give much more detailed results about their structure in the case when there are only a finite number of generators, and give several Banach lattice characterizations of the number of generators being, respectively, one, finite or countable. We define a Banach lattice P to be *projective* if, whenever X is a Banach lattice, J is a closed ideal in X, $Q\colon X \to X/J$ is the quotient map, $T\colon P \to X/J$ is a linear lattice homomorphism and $\varepsilon > 0$, there exists a linear lattice homomorphism $\hat{T}\colon P \to X$ such that $T = Q \circ \hat{T}$ and $\|\hat{T}\| \leqslant (1+\varepsilon)\|T\|$. We establish the connection between projective Banach lattices and free Banach lattices, describe several families of Banach lattices that are projective and prove that some are not.

1. Introduction

Free and projective objects have not played anywhere near as important a role in analysis as in algebra; nevertheless, there has been some work done on these objects, mainly with the results that one would expect. For example, the existence of free and projective Banach spaces is virtually folklore, but is uninteresting as both are of the form $\ell_1(I)$ for an arbitrary index set I. The existence of free vector lattices over an arbitrary number of generators is also long established and holds no real surprises; see [1] or [3] for details. In this paper we investigate free and projective Banach lattices. Some of our results are rather surprising and although we are able to answer many questions we are forced to leave several unanswered.

It is almost obvious that, if it exists, the free Banach lattice over $\mathfrak{a}$ generators must be the completion of the free vector lattice over $\mathfrak{a}$ generators for some lattice norm. That the required norm actually exists is easily proved, but describing it in concrete and readily identifiable terms is not so easy. Indeed, except in the case when $\mathfrak{a} = 1$, it is not a classical Banach lattice norm at all. In fact it is only in the case when $\mathfrak{a}$ is finite that the free Banach lattice over $\mathfrak{a}$ generators is even isomorphic to an AM-space.

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We primarily devote § 2 to establishing the notation, while § 3 recapitulates the existing theory of free vector lattices. We then prove the existence of free Banach lattices in § 4 and give a representation on a compact Hausdorff space in § 5. We establish some of the basic properties of free Banach lattices in § 6. The finitely generated free Banach lattices are by far the easiest ones to understand, and we investigate their structure in § 7. In § 8 we give some characterizations of free Banach lattices over, respectively, one, a finite number or a countable number of generators, amongst all free Banach lattices. In preparation for looking at projective Banach lattices, in § 9 we investigate when disjoint families in quotient Banach lattices X/J can be lifted to disjoint families in X, giving a positive result for countable families and a negative result for larger ones. We prove the connection between free and projective Banach lattices in § 10, and in § 11 find some classes of Banach lattices that are, or are not, projective. Finally, § 12 contains some open problems.

Let us emphasize at this point that this paper is set in the category of Banach lattices and linear lattice homomorphisms. There is a substantial theory of *injective* Banach lattices (and indeed we refer to them later) but this is set in the context of Banach lattices and positive (or regular) operators.[1] Thus, there is no reason to expect any kind of duality between the two notions.

2. Notation

In this short section we establish the notation that we use concerning functions and function spaces. If A and X are non-empty sets, then, as usual, X^A denotes the set of all maps from A into X. If $\emptyset \neq B \subseteq A$, then we let $r_B\colon X^A \to X^B$ denote the restriction map with $r_B\xi = \xi|_B$ for $\xi \in X^A$. Clearly, r_B is surjective. On occasion we will also write ξ_B in place of $r_B(\xi)$.

The space of all real-valued functions on X^A, $\mathbb{R}^{X^A}$, is a vector lattice under the pointwise operations. Again, we consider the setting where B is a non-empty subset of A and define $j_B\colon \mathbb{R}^{X^B} \to \mathbb{R}^{X^A}$ by $(j_B f)(\xi) = f(\xi_B)$ for $\xi \in X^A$ and $f \in \mathbb{R}^{X^B}$. This makes j_B an injective lattice homomorphism. The following description of the image of j_B is easily verified.

Lemma 2.1. *If A, B and X are non-empty sets with $B \subseteq A$ and $f \in \mathbb{R}^{X^A}$, then the following are equivalent.*

(1) *$f \in j_B(\mathbb{R}^{X^B})$.*

(2) *If $\xi, \eta \in X^A$ with $\xi_B = \eta_B$, then $f(\xi) = f(\eta)$.*

[1] In fact, although we can find no explicit proof in the literature, there is no non-zero injective in the category of Banach lattices and linear lattice homomorphisms. Indeed, suppose that F were a non-zero injective. Let $\mathfrak{a}$ be strictly greater than the cardinality of F^* and let μ be the product of $\mathfrak{a}$ many copies of the measure that assigns mass $\frac{1}{2}$ to each of 0 and 1 in $\{0, 1\}$. This is a homogenous measure space and each order interval in $\mathcal{L}_1(\mu)$ has the property that the least cardinality of a dense subset is precisely $\mathfrak{a}$; see [19, § 26] for details. In particular, every order interval has cardinality at least $\mathfrak{a}$. As μ is finite, the same is true of $\mathcal{L}_\infty(\mu)$. Pick any non-zero $y \in F_+$. As F is alleged to be injective, there is a linear lattice homomorphism T extending the map that takes the constantly-one function in $\mathcal{L}_1(\mu)$, $\mathbf{1}$, to y. The adjoint of this maps F^* into $\mathcal{L}_1(\mu)^* = \mathcal{L}_\infty(\mu)$ and is interval preserving; see [13, theorem 1.4.19]. In particular, if $f \in F^*_+$ with $f(y) > 0$, then $T^* f(\mathbf{1}) = f(T\mathbf{1}) = f(y) > 0$, so the image of the order interval $[0, f]$ will be a non-zero order interval in $\mathcal{L}_\infty(\mu)$ that has cardinality at least $\mathfrak{a}$. This contradicts the fact that $[0, f]$ has cardinality strictly less than $\mathfrak{a}$.

We now specialize somewhat by assuming that $X \subseteq \mathbb{R}$ and that $0 \in X$. This means that if $\xi \in X^A$, $\emptyset \neq B \subseteq A$ and χ_B is the characteristic function of B, then the pointwise product $\xi\chi_B \in X^A$.

Lemma 2.2. *If $\emptyset \neq B \subseteq A$ and $0 \in X \subseteq \mathbb{R}$, then the map $P_B\colon \mathbb{R}^{X^A} \to j_B(\mathbb{R}^{X^B})$ defined by*

$$(P_B f)(\xi) = f(\xi\chi_B) \quad (\xi \in X^A,\ f \in \mathbb{R}^{X^A})$$

is a linear lattice homomorphism and a projection onto $j_B(\mathbb{R}^{X^B})$. Furthermore, if $B_1, B_2 \subseteq A$ are non-empty sets with non-empty intersection, then $P_{B_1}P_{B_2} = P_{B_2}P_{B_1} = P_{B_1\cap B_2}$.

Proof. It is clear that P_B is a well-defined vector lattice homomorphism of X^A into itself. If $\xi, \eta \in X^A$ are such that $\xi_B = \eta_B$, then $(P_B f)(\xi) = f(\xi\chi_B) = f(\eta\chi_B) = (P_B f)(\eta)$, so, by lemma 2.1, $P_B f \in j_B(\mathbb{R}^{X^B})$ for all $f \in \mathbb{R}^{X^A}$. If $f \in \mathbb{R}^{X^B}$, then for any $\xi \in X^A$ we have $P_B(j_B f)(\xi) = (j_B f)(\xi\chi_B) = (j_B f)(\xi)$, as ξ and $\xi\chi_B$ coincide on B and using lemma 2.1 again. Thus, P_B is indeed a projection.

Finally, if $f \in \mathbb{R}^{X^A}$ and $\xi \in X^A$, then

$$\begin{aligned} P_{B_1}P_{B_2}f(\xi) &= (P_{B_2})(f\chi_{B_1}) \\ &= f(\xi\chi_{B_1}\chi_{B_2}) \\ &= f(\xi\chi_{B_1\cap B_2}) \\ &= (P_{B_1\cap B_2}f)(\xi), \end{aligned}$$

which shows that $P_{B_1}P_{B_2} = P_{B_1\cap B_2}$. Similarly, $P_{B_2}P_{B_2} = P_{B_2\cap B_1} = P_{B_1\cap B_2}$, and the proof is complete. □

In future, we identify $\mathbb{R}^{X^B}$ with the vector sublattice $j_B(\mathbb{R}^{X^B})$ of $\mathbb{R}^{X^A}$.

If L is any vector lattice and D is a non-empty subset of L, then $\langle D\rangle$ denotes the vector sublattice of L generated by D. All elements of $\langle D\rangle$ can be obtained from those of D by the application of a finite number of multiplications, additions, suprema and infima. The following simple consequence of this observation may also be proved directly.

Lemma 2.3. *If L and M are vector lattices, $T\colon L \to M$ is a vector lattice homomorphism and $\emptyset \neq D \subseteq L$, then $\langle T(D)\rangle = T(\langle D\rangle)$.*

We specialize further now to the case when $X = \mathbb{R}$. On the space $\mathbb{R}^A$ we can consider the product topology, which is the topology of pointwise convergence on A. By definition, this is the weakest topology such that all the functions $\delta_a\colon \xi \mapsto \xi(a)$ are continuous on $\mathbb{R}^A$ for each $a \in A$. As a consequence, we certainly have $\langle\{\delta_a\colon a \in A\}\rangle \subset C(\mathbb{R}^A)$. In fact, we can do rather better than this. A function $f\colon \mathbb{R}^A \to \mathbb{R}$ is *homogeneous* if $f(t\xi) = tf(\xi)$ for $\xi \in \mathbb{R}^A$ and $t \in [0,\infty)$. The space $H(\mathbb{R}^A)$ of continuous homogeneous real-valued functions on $\mathbb{R}^A$ is a vector sublattice of $C(\mathbb{R}^A)$ and, clearly, $\langle\{\delta_a\colon a \in A\}\rangle \subset H(\mathbb{R}^A)$.

3. Free vector lattices

In this section we recapitulate much of the theory of free vector lattices, to make this work as self-contained as possible and in order to both establish our notation

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(which may not coincide with that used in other papers on free vector lattices) and point out some properties that we will use later.

Definition 3.1. If A is a non-empty set, then a *free vector lattice* over A is a pair (F, ι), where F is a vector lattice and $\iota\colon A \to F$ is a map with the property that for any vector lattice E and any map $\phi\colon A \to E$ there exists a unique vector lattice homomorphism $T\colon F \to E$ such that $\phi = T \circ \iota$.

It follows immediately from this definition that the map ι must be injective, as we can certainly choose E and ϕ to make ϕ injective. Many of the results that follow are almost obvious, but we prefer to make them explicit.

Proposition 3.2. *If (F, ι) is a free vector lattice over A, then F is generated, as a vector lattice, by $\iota(A)$.*

Proof. Let G be the vector sublattice of F generated by $\iota(A)$. Define $\phi\colon A \to G$ by $\phi(a) = \iota(a)$; it then follows from the definition that there exists a unique vector lattice homomorphism $T\colon F \to G$ with $T(\iota(a)) = \phi(a) = \iota(a)$ for $a \in A$. If $j\colon G \to F$ is the inclusion map, then $j \circ T\colon F \to F$ is a vector lattice homomorphism with $(j \circ T)(\iota(a)) = j(\iota(a)) = \iota(a)$ for $a \in A$. The identity on F, I_F, is also a vector lattice homomorphism from F into itself with $I_F(\iota(a)) = \iota(a)$. The uniqueness part of the definition of a free vector lattice applied to the map $a \mapsto \iota(a)$, of A into F, tells us that these two maps are equal, so $j \circ T = I_F$, from which we see that $F \subseteq G$, and therefore $F = G$ as claimed. □

The definition of a free vector lattice makes the following result easy to prove.

Proposition 3.3. *If (F, ι) and (G, κ) are free vector lattices over a non-empty set A, then there exists a (unique) vector lattice isomorphism $T\colon F \to G$ such that $T(\iota(a)) = \kappa(a)$ for $a \in A$.*

In view of this we will just refer to a free vector lattice (F, ι) over a set A as the *free vector lattice over A* (or sometimes as the *free vector lattice generated by A* when we identify A with a subset of that free vector lattice). We denote it by $\mathrm{FVL}(A)$. It is clear that if A and B are sets of equal cardinality, then $\mathrm{FVL}(A)$ and $\mathrm{FVL}(B)$ are isomorphic vector lattices, so $\mathrm{FVL}(A)$ depends only on the cardinality of the set A. Thus, we also use the notation $\mathrm{FVL}(\mathfrak{a})$ for $\mathrm{FVL}(A)$ when $\mathfrak{a}$ is the cardinality of A. This is the notation that can be found elsewhere in the literature. We retain both versions so that we can handle proper inclusions of $\mathrm{FVL}(B)$ into $\mathrm{FVL}(A)$ when $B \subset A$ even when A and B have the same cardinality.

If $\iota\colon A \to \mathrm{FVL}(A)$ is the embedding of A into $\mathrm{FVL}(A)$ specified in the definition, then we often write δ_a for $\iota(a)$ and refer to the set $\{\delta_a\colon a \in A\}$ as the *free generators* of $\mathrm{FVL}(A)$.

A slight rewording of the definition of a free vector lattice is sometimes useful, trading off uniqueness of the lattice homomorphism for specifying that $\iota(A)$ is a generating set. The proof of this follows immediately from results above.

Proposition 3.4. *If A is a non-empty set, then the vector lattice F is the free vector lattice over A if and only if the following hold.*

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(1) *There exists a subset* $\{\delta_a\colon a \in A\} \subset F$, *with* $\delta_a \neq \delta_b$ *if* $a \neq b$, *that generates* F *as a vector lattice.*

(2) *For every vector lattice* E *and any family* $\{x_a\colon a \in A\} \subset E$ *there exists a vector lattice homomorphism* $T\colon F \to E$ *such that* $T(\delta_a) = x_a$ *for* $a \in A$.

We will find the next simple result useful later.

Proposition 3.5. *Let* A *be a non-empty set and let* $\{\delta_a\colon a \in A\}$ *be the free generators of* $\mathrm{FVL}(A)$. *Let* B *and* C *be non-empty subsets of* A *with* $B \cap C \neq \emptyset$.

(1) *The vector sublattice of* $\mathrm{FVL}(A)$ *generated by* $\{\delta_b\colon b \in B\}$ *is (isomorphic to) the free vector lattice* $\mathrm{FVL}(B)$.

(2) *There is a lattice homomorphism projection* P_B *from* $\mathrm{FVL}(A)$ *onto* $\mathrm{FVL}(B)$.

(3) $P_C P_B = P_B P_C = P_{B\cap C}$.

Proof. (1) Let F denote the vector sublattice of $\mathrm{FVL}(A)$ generated by $\{\delta_b\colon b \in B\}$. Suppose that E is a vector lattice and $\pi\colon B \to E$ is any map. There exists a unique vector lattice homomorphism $T\colon \mathrm{FVL}(A) \to E$ with $T(\delta_b) = \pi(b)$ for $b \in B$ and $T(\delta_a) = 0$ for $a \in A \setminus B$. The restriction S of T to F gives us a vector lattice homomorphism $S\colon F \to E$ with $S(\delta_b) = \pi(b)$. It follows from proposition 3.4 that $F = \mathrm{FVL}(B)$.

(2) The free property of $\mathrm{FVL}(A)$ gives a (unique) lattice homomorphism

$$P_B\colon\ \mathrm{FVL}(A) \to \mathrm{FVL}(A)$$

with $P_B(\delta_b) = \delta_b$ if $b \in B$ and $P_B(\delta_a) = 0$ if $a \in A \setminus B$. As P_B maps the generators of $\mathrm{FVL}(A)$ into $\mathrm{FVL}(B)$, we certainly have $P_B(\mathrm{FVL}(A)) \subseteq \mathrm{FVL}(B)$. Also, P_B is the identity on the generators of $\mathrm{FVL}(B)$, and so is the identity linear operator on $\mathrm{FVL}(B)$, so P_B is indeed a projection.

(3) If $a \in B\cap C$, then $P_C P_B \delta_a = P_B P_C \delta_a = P_{B\cap C}\delta_a = \delta_a$, while if $a \notin B\cap C$, then $P_C P_B \delta_a = P_B P_C \delta_a = P_{B\cap C}\delta_a = 0$. Thus, the three vector lattice homomorphisms $P_B P_C$, $P_C P_B$ and $P_{B\cap C}$ coincide on a set of generators of $\mathrm{FVL}(A)$, and are therefore equal. □

So far all our discussions of free vector lattices have been rather academic, as we have not shown that they exist. However, it was shown in [1] (see also [3]) that they do exist. In essence we have the following.

Theorem 3.6. *For any non-empty set* A, $\mathrm{FVL}(A)$ *exists and is the vector sublattice of* $\mathbb{R}^{\mathbb{R}^A}$ *generated by* δ_a $(a \in A)$ *where* $\delta_a(\xi) = \xi(a)$ *for* $\xi \in \mathbb{R}^A$.

It is reasonable to ask how this representation of $\mathrm{FVL}(A)$ interacts with the properties of free vector lattices noted above. Under the notation of § 2, if $\emptyset \neq B \subseteq A$, then the map $j_B\colon \mathbb{R}^{\mathbb{R}^B} \to \mathbb{R}^{\mathbb{R}^A}$ is a vector lattice embedding of $\mathbb{R}^{\mathbb{R}^B}$ into $\mathbb{R}^{\mathbb{R}^A}$. This corresponds precisely to the embedding of $\mathrm{FVL}(B)$ into $\mathrm{FVL}(A)$ as indicated in proposition 3.5. If we use δ_a to denote the map $\xi \mapsto \xi(a)$ on $\mathbb{R}^A$ and η_b for the map $\xi \mapsto \xi(b)$ on $\mathbb{R}^B$, then we have, for $b \in B$ and $\xi \in \mathbb{R}^A$, that

$$(j_B\eta_b)(\xi) = \eta_b(\xi_B) = \xi(b) = \delta_b(\xi),$$

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so $j_B\eta_b = \delta_b$. We know from § 2 that j_B is a vector lattice homomorphism so that $j_B(\mathrm{FVL}(B))$ is the vector sublattice of $\mathrm{FVL}(A)$ generated by $\{\delta_b\colon b \in B\}$, which is precisely what was described in proposition 3.5.

Also, if $B \subseteq A$, then we may consider $\mathrm{FVL}(B) \subseteq \mathrm{FVL}(A) \subseteq \mathbb{R}^{\mathbb{R}^A}$. The projection map $P_B\colon \mathrm{FVL}(A) \to \mathrm{FVL}(B)$ defined in proposition 3.5(2) is then precisely the restriction to $\mathrm{FVL}(A)$ of the projection $P_B\colon \mathbb{R}^{\mathbb{R}^A} \to \mathbb{R}^{\mathbb{R}^B}$ described in lemma 2.2. We temporarily denote this projection by $\tilde{P}_B$ to distinguish it from the abstract projection. Once we establish equality, that distinction will not be required and we will omit the tilde. As P_B and $\tilde{P}_B$ are both vector lattice homomorphisms, it suffices to prove this equality for the generators of $\mathrm{FVL}(A)$. If $b \in B$, then

$$(\tilde{P}_B\delta_b)(\xi) = \delta_b(\xi\chi_B) = (\xi\chi_B)(b) = \xi(b) = \delta_b(\xi)$$

for $\xi \in \mathbb{R}^A$, so $\tilde{P}_B\delta_b = \delta_b = P_B\delta_b$. If, on the other hand, $a \in A \setminus B$, then

$$(\tilde{P}_B\delta_a)(\xi) = \delta_a(\xi\chi_B) = 0$$

for $\xi \in \mathbb{R}^A$, so $\tilde{P}_B\delta_a = 0 = P_B\delta_a$.

A few more observations will be of use later.

Proposition 3.7. *If A is a non-empty set and $\mathcal{F}(A)$ denotes the collection of all non-empty finite subsets of A, then*

$$\mathrm{FVL}(A) = \bigcup_{B \in \mathcal{F}(A)} \mathrm{FVL}(B).$$

Proof. Any element of $\mathrm{FVL}(A)$ is in the vector sublattice of $\mathrm{FVL}(A)$ generated by a finite number of generators $\{\delta_{a_1}, \delta_{a_2}, \dots, \delta_{a_n}\}$, and so lies in $\mathrm{FVL}(\{a_1, a_2, \dots, a_n\})$. □

Proposition 3.8. *If A is a finite set, then $\sum_{a\in A} |\delta_a|$ is a strong order unit for* $\mathrm{FVL}(A)$.

Proof. The proof is obvious, as $\mathrm{FVL}(A)$ is generated by the set $\{\delta_a\colon a \in A\}$. □

Lemma 3.9. *The real-valued vector lattice homomorphisms on* $\mathrm{FVL}(A)$ *are precisely the evaluations at points of $\mathbb{R}^A$.*

Proof. It is clear that if $\xi \in \mathbb{R}^A$, then the map $\omega_\xi\colon f \mapsto f(\xi)$ is a real-valued vector lattice homomorphism on $\mathbb{R}^{\mathbb{R}^A}$, and therefore on $\mathrm{FVL}(A)$. Note, in particular, that $\omega_\xi(\delta_a) = \delta_a(\xi) = \xi(a)$. Conversely, if ω is a real-valued vector lattice homomorphism on $\mathrm{FVL}(A)$, then we may define $\xi \in \mathbb{R}^A$ by $\xi(a) = \omega(\delta_a)$ for $a \in A$. We now see that, for this ξ, ω_ξ is a real-valued vector lattice homomorphism on $\mathrm{FVL}(A)$ with $\omega_\xi(\delta_a) = \xi(a) = \omega(\delta_a)$. The two maps ω and ω_ξ coincide on a set of generators of $\mathrm{FVL}(A)$, and so, being vector lattice homomorphisms, are equal. □

4. Free Banach lattices

Definition 4.1. If A is a non-empty set, then a *free Banach lattice* over A is a pair (X, ι), where X is a Banach lattice and $\iota\colon A \to X$ is a bounded map with the property that for any Banach lattice Y and any bounded map $\kappa\colon A \to Y$ there

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exists a unique vector lattice homomorphism $T\colon X \to Y$ such that $\kappa = T \circ \iota$ and $\|T\| = \sup\{\|\kappa(a)\|\colon a \in A\}$.

It is clear that the set $\{\iota(a)\colon a \in A\}$ generates X as a Banach lattice (see proposition 3.2).

REMARK 4.2. The definition forces each $\iota(a)$ to have norm precisely 1. This is because, if $\kappa(a) = 1 \in \mathbb{R}$ for each $a \in A$, the map T that is guaranteed to exist has norm 1, so $1 = \|T(\iota(a))\| \leqslant \|\iota(a)\|$. On the other hand, if we take $\kappa = \iota$, then T is the identity operator, with norm 1, so $\sup\{\|\iota(a)\|\colon a \in A\} = 1$.

PROPOSITION 4.3. *If (X, ι) and (Y, κ) are free Banach lattices over a non-empty set A, then there exists a (unique) isometric order isomorphism $T\colon X \to Y$ such that $T(\iota(a)) = \kappa(a)$ for $a \in A$.*

Proof. As (X, ι) is free, there exists a vector lattice homomorphism $T\colon X \to Y$ with $T(\iota(a)) = \kappa(a)$ for $a \in A$ with $\|T\| = \sup\{\|\kappa(a)\|\colon a \in A\} = 1$, by the preceding proposition. There similarly exists a contractive vector lattice homomorphism $S\colon Y \to X$ with $S(\kappa(a)) = \iota(a)$. By uniqueness, the compositions $S \circ T$ and $T \circ S$ must be the identity operators. This suffices to prove our claim. □

In a similar way as for the free vector lattice case, we use the notation $\mathrm{FBL}(A)$ for the free Banach lattice over A if it exists (which we will shortly show is the case). Since we know that if A and B have the same cardinality, then $\mathrm{FBL}(A)$ and $\mathrm{FBL}(B)$ are isometrically order isomorphic, we also use the notation $\mathrm{FBL}(\mathfrak{a})$ to denote a free Banach lattice on a set of cardinality $\mathfrak{a}$. Again, we also use the notation δ_a for $\iota(a)$ and refer to $\{\delta_a\colon a \in A\}$ as the *free generators* of $\mathrm{FBL}(A)$.

Our first task is to show that free Banach lattices do indeed exist.

DEFINITION 4.4. If A is a non-empty set, then we define a mapping from $\mathrm{FVL}(A)^\sim$ into the extended non-negative reals by

$$\|\phi\|^\dagger = \sup\{|\phi|(|\delta_a|)\colon a \in A\}.$$

We also define

$$\mathrm{FVL}(A)^\dagger = \{\phi \in \mathrm{FVL}(A)^\sim\colon \|\phi\|^\dagger < \infty\},$$

which is clearly a vector lattice ideal in the Dedekind complete vector lattice $\mathrm{FVL}(A)^\sim$.

Suppose that a positive functional ϕ vanishes on each $|\delta_a|$. Each element x of $\mathrm{FVL}(A)$ lies in the sublattice of $\mathrm{FVL}(A)$ generated by a finite set of generators $\{a_k\colon 1 \leqslant k \leqslant n\}$. By proposition 3.8, $e = \sum_{k=1}^n |\delta_{a_k}|$ is a strong order unit for that sublattice. Thus, there exists $\lambda \in \mathbb{R}$ with $|x| \leqslant \lambda e$ such that $|\phi(x)| \leqslant \phi(|x|) \leqslant \phi(\lambda e) = \lambda \sum_{k=1}^n \phi(|\delta_{a_k}|) = 0$, and thus $\phi = 0$. It is now clear that $\|\cdot\|^\dagger$ is a lattice norm on $\mathrm{FVL}(A)^\dagger$. Given the embedding of $\mathrm{FVL}(A)$ in $\mathbb{R}^{\mathbb{R}^A}$ given in theorem 3.6, if $\xi \in \mathbb{R}^A$, then $\omega_\xi \in \mathrm{FVL}(A)^\dagger$ if and only if the map $\xi\colon A \to \mathbb{R}$ is bounded, and then $\|\omega_\xi\|^\dagger = \sup_{a \in A} |\xi(a)|$. By lemma 3.9, these maps are lattice homomorphisms. Note that if A is an infinite set, then there exists an unbounded $\xi \in \mathbb{R}^A$ that induces $\omega_\xi \in \mathrm{FVL}(A)^\sim \setminus \mathrm{FVL}(A)^\dagger$.

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Definition 4.5. For $f \in \mathrm{FVL}(A)$, where A is a non-empty set, define

$$\|f\|_F = \sup\{\phi(|f|) \colon \phi \in \mathrm{FVL}(A)^\dagger_+,\ \|\phi\|^\dagger \leqslant 1\}.$$

Proposition 4.6. *For any non-empty set A, $\|\cdot\|_F$ is a lattice norm on* $\mathrm{FVL}(A)$.

Proof. Our first step is to show that $\|\cdot\|_F$ is real valued. By proposition 3.7, any $f \in \mathrm{FVL}(A)$ actually lies in $\mathrm{FVL}(B)$ for some finite subset $B \subseteq A$. By proposition 3.8, $\mathrm{FVL}(B)$ has a strong order unit $\sum_{b\in B} |\delta_b|$, so there exists λ with $|f| \leqslant \lambda \sum_{b\in B} |\delta_b|$. If $\phi \in \mathrm{FVL}(A)^\dagger_+$ with $\|\phi\|^\dagger \leqslant 1$, then

$$\phi(|f|) \leqslant \phi\biggl(\lambda \sum_{b\in B} |\delta_b|\biggr) = \lambda \sum_{b\in B} \phi(|\delta_b|) \leqslant \lambda \sum_{b\in B} 1,$$

so $\|f\|_F$ is certainly finite.

If $\|f\|_F = 0$, then $\phi(|f|) = 0$ for all $\phi \in \mathrm{FVL}(A)^\dagger_+$. Using the observation above, $f(\xi) = \omega_\xi(f) = 0$ for any bounded function $\xi \colon A \to \mathbb{R}$. But there exists a finite set $B \subset A$ such that $f \in \mathrm{FVL}(B)$, so that $f(\xi) = f(\xi\chi_B)$ for all $\xi \in \mathbb{R}^A$. As each $\xi\chi_B$ is bounded, $f(\xi) = 0$ for all $\xi \in \mathbb{R}^A$, and therefore $f = 0$.

That $\|\cdot\|_F$ is sublinear and positively homogeneous is obvious, so $\|\cdot\|_F$ is a norm on $\mathrm{FVL}(A)$, which is clearly a lattice norm. □

Note, in particular, that we certainly have $\|\delta_a\|_F = 1$ for all $a \in A$. In fact, this construction gives us our desired free Banach lattices.

Theorem 4.7. *For any non-empty set A, the pair consisting of the completion of* $\mathrm{FVL}(A)$*, under the norm $\|\cdot\|_F$, and the map $\iota \colon a \to \delta_a$ is the free Banach lattice over A.*

Proof. Suppose that Y is any Banach lattice and that $\kappa \colon A \to Y_1$, the unit ball of Y. There exists a vector lattice homomorphism $T \colon \mathrm{FVL}(A) \to Y$ with $T(\iota(a)) = \kappa(a)$ for all $a \in A$, as $\mathrm{FVL}(A)$ is free. We claim that if $f \in \mathrm{FVL}(A)$ with $\|f\|_F \leqslant 1$, then $\|Tf\| = \||Tf|\| = \|T(|f|)\| \leqslant 1$ in Y, where we have the used that fact that the norm in Y is a lattice norm and that T is a lattice homomorphism. If this were not the case, then we could find $\psi \in Y^*_{1+}$, a positive linear functional on Y with norm at most 1, with $\psi(T(|f|)) > 1$. As $\|T(\iota(a))\| = \|\kappa(a)\| \leqslant 1$ for all $a \in A$, we have $\||T(\iota(a))|\| = \|T(|\iota(a)|)\| \leqslant 1$, using again the fact that T is a lattice homomorphism. Thus, $|\psi(T(|\iota(a)|))| \leqslant 1$ for all $a \in A$. Using the functional $\psi \circ T$ in the definition of $\|f\|_F$, we see that $\|f\|_F \geqslant \psi(T(|f|)) > 1$, contradicting our assumption that $\|f\|_F \leqslant 1$.

The completion of $\mathrm{FVL}(A)$ is a Banach lattice, and T extends by continuity to it while still taking values in Y, as Y is complete. □

We will eventually need to know the relationship between different free Banach lattices, so we record now the following result.

Proposition 4.8. *If B is a non-empty subset of A, then* $\mathrm{FBL}(B)$ *is isometrically order isomorphic to the closed sublattice of* $\mathrm{FBL}(A)$ *generated by $\{\delta_b \colon b \in B\}$. Furthermore, there exists a contractive lattice homomorphic projection P_B of* $\mathrm{FBL}(A)$ *onto* $\mathrm{FBL}(B)$.

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Proof. Recall from proposition 3.5 that FVL(B) is isomorphic to the sublattice of FVL(A) generated by $\{\delta_b : b \in B\}$, and that there exists a lattice homomorphism projection P_B of FVL(A) onto FVL(B) with $P_B(\delta_b) = \delta_b$ if $b \in B$ and $P_B(\delta_a) = 0$ if $a \in A \backslash B$. As $\|\delta_b\|_F = 1$ in both FBL(A) and FBL(B), there exist contractive lattice homomorphisms of FBL(B) into FBL(A) and of FBL(B) onto FBL(A) that act in the same way on the generators so extend the homomorphisms between FVL(A) and FVL(B). The conclusion is now clear. □

There is also a simple relationship between their duals. This is a consequence of the following result, which is surely well known but for which we can find no convenient reference; see [21, IV.12, problem 6] and [7, lemma VI.3.3, pp. xiv, 858] for similar results.

PROPOSITION 4.9. *If P is a contractive lattice homomorphism projection from a Banach lattice X onto a closed sublattice Y, then P^*Y^* is a weak*-closed band in X^*, which is isometrically order isomorphic to Y^*.*

Proof. Write ker(P) for the kernel of P, which is a lattice ideal in X, and define

$$Z = \{\phi \in X^* : \phi|_{\ker(P)} \equiv 0\},$$

which is a weak*-closed band in X^*. It is clear that $P^*X^* = Z$.

Define $J : Y^* \to X^*$ by $J\phi = \phi \circ P$ and note that $J : Y^* \to Z$ with $\|J\| \leqslant \|P\|$. If $\phi \in Z$, then $J(\phi|_Y) = \phi$, so J is actually an isometry of Y^* onto Z. It is clear that both J and J^{-1} are positive. Thus, $J : Y^* \to P^*X^*$ is actually an isometric order isomorphism. □

COROLLARY 4.10. *If B is a non-empty subset of A, then* FBL(B)* *is isometrically order isomorphic to a weak*-closed band in* FBL(A)*.

As in the algebraic case, if B and C are two subsets of A with $B \cap C \neq \emptyset$ of A, then $P_B P_C = P_C P_B = P_{B \cap C}$.

In particular, the embedding of the finitely generated free closed sublattices are important.

PROPOSITION 4.11. *Let $\mathcal{F}(A)$ be the collection of all non-empty finite subsets of A, ordered by inclusion. The net of projections $\{P_B : B \in \mathcal{F}(A)\}$ in* FBL(A) *converges strongly to the identity in* FBL(A).

Proof. If $f \in$ FVL(A), then there actually exists $B_0 \in \mathcal{F}(A)$ with $P_B(f) = f$ whenever $B_0 \subset B$. Recall that each P_B is a contraction. If $\varepsilon > 0$ and $f \in$ FBL(A), choose $f' \in$ FVL(A) with $\|f - f'\|_F < \varepsilon/2$, and then $B_0 \in \mathcal{F}(A)$ with $P_B(f') = f'$ for $B_0 \subset B$. Then, if $B_0 \subset B$,

$$\|P_B f - f\|_F \leqslant \|P_B f - P_B f'\| + \|P_B f' - f'\|_F + \|f' - f\|_F < \varepsilon,$$

which completes the proof. □

Before looking at some properties of FBL(A) in detail, we ask about its normed dual.

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Proposition 4.12. *If A is any non-empty set, then the three normed spaces*

$$(\mathrm{FVL}(A)^\dagger, \|\cdot\|^\dagger), \qquad (\mathrm{FVL}(A), \|\cdot\|_F)^* \quad \textit{and} \quad \mathrm{FBL}(A)^*$$

are isometrically order isomorphic.

Proof. If $\phi \in \mathrm{FBL}(A)^*$, then the restriction map $\phi \mapsto \phi|_{\mathrm{FVL}(A)}$ is an order isomorphism, by continuity, and as $\|\delta_a\| = 1$ we have that $|\phi|(\delta_a)\| \leqslant \|\phi\|$, so $\|\phi|_{\mathrm{FVL}(A)}\|^\dagger \leqslant \|\phi\|$. On the other hand, as each $\|\delta_a\| = 1$ we see that

$$\begin{aligned}\|\phi\| &= \||\phi|\| \\ &= \sup\{|\phi|(f) \colon \|f\|_F \leqslant 1\} \\ &\leqslant \sup\{|\phi|(|\delta_a|) \colon a \in A\} \\ &= \|\phi_{|\,\mathrm{FVL}(A)|}\|^\dagger,\end{aligned}$$

so the isometric order isomorphism of the first and third spaces is proved. The identification of the second and third follows from the density of $\mathrm{FVL}(A)$ in $\mathrm{FBL}(A)$. □

As we noted above, if A is infinite, then $\mathrm{FVL}(A)^\dagger \neq \mathrm{FVL}(A)^\sim$. On the other hand, we have the following.

Proposition 4.13. *If $n \in \mathbb{N}$, then $\mathrm{FBL}(n)^*$ is isometrically order isomorphic to the whole of $\mathrm{FVL}(n)^\sim$ under the norm $\|\cdot\|^\dagger$.*

Proof. All that remains to establish is that $\|\phi\|^\dagger$ is finite for all $\phi \in \mathrm{FVL}(n)$. Given that $\|\phi\|^\dagger$ is, in this case, a finite supremum of real values $|\phi|(|\delta_a|)$, this is clear. □

5. A smaller representation space

The set $\Delta_A = [-1,1]^A$ is a compact subset of $\mathbb{R}^A$. We call a function $f \colon \Delta_A \to \mathbb{R}$ *homogeneous* if $f(t\xi) = tf(\xi)$ for $\xi \in \Delta_A$ and $t \in [0,1]$ (this is consistent with the definition for functions on $\mathbb{R}^A$). The space of continuous homogeneous real-valued functions on Δ_A is denoted by $H(\Delta_A)$. If we equip $C(\Delta_A)$ with the supremum norm $\|\cdot\|_\infty$, then $H(\Delta_A)$ is a closed vector sublattice of $C(\Delta_A)$ (and hence $H(\Delta_A)$ is itself a Banach lattice with respect to this norm).

Lemma 5.1. *The restriction map $R \colon H(\mathbb{R}^A) \to H(\Delta_A)$ is an injective vector lattice homomorphism.*

Proof. The only part of the proof that is not completely trivial is that the map R is injective. Suppose that $f \in H(\mathbb{R}^A)$ and $Rf = 0$. If $\xi \in \mathbb{R}^A$, consider the net $\{\xi\chi_B \colon B \in \mathcal{F}(A)\}$, where $\mathcal{F}(A)$ is the collection of all non-empty finite subsets of A ordered by inclusion; we then have $\xi\chi_B \to_{\mathcal{F}(A)} \xi$ in $\mathbb{R}^A$. For any $B \in \mathcal{F}(A)$, there exists $t > 0$ such that $t\xi\chi_B \in [-1,1]^A$, so $tf(\xi\chi_B) = f(t\xi\chi_B) = 0$ by homogeneity. Hence, $f(\xi\chi_B) = 0$, and thus $f(\xi) = 0$ by the continuity of f, so $f = 0$. □

It should be noted that the restriction map is not surjective unless A is a finite set.

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Example 5.2. It suffices to prove the non-surjectiveness in the case when $A = \mathbb{N}$. Define $g \in H(\Delta_{\mathbb{N}})$ by $g(\xi) = \sum_{k=1}^{\infty} 2^{-k}\xi(k)$ for $\xi \in \Delta_{\mathbb{N}}$. Suppose that there exists $f \in H(\mathbb{R}^{\mathbb{N}})$ with $Rf = g$. Define $\eta \in \mathbb{R}^{\mathbb{N}}$ by $\eta(k) = 2^k$ and let $\eta_n = \eta\chi_{\{1,\dots,n\}}$, for $n \in \mathbb{N}$, so that $\eta_n \to \eta$ in $\mathbb{R}^{\mathbb{N}}$. But, for each $n \in \mathbb{N}$ we have

$$f(\eta_n) = 2^n f(2^{-n}\eta_n) = 2^n g(2^{-n}\eta_n) = n.$$

As f is supposed to be continuous, this is impossible.

Note that this example also shows that the space $H(\mathbb{R}^A)$, equipped with the sup-norm over Δ_A, is not complete if A is infinite. This is one of the reasons that we use the space $H(\Delta_A)$.

In general, $\mathrm{FVL}(A)$ may be identified with a vector sublattice of $H(\mathbb{R}^A)$ (see theorem 3.6), which in turn, courtesy of lemma 5.1, may be identified with a vector sublattice of $H(\Delta_A)$ via the restriction map R. This identification extends to $\mathrm{FBL}(A)$. The proof of this turns out to be slightly more tricky than might have been anticipated.

For the sake of convenience, we denote by $J = J_A$ the restriction to $\mathrm{FVL}(A)$ of the restriction map $R\colon H(\mathbb{R}^A) \to H(\Delta_A)$. Since $\|J\delta_a\|_\infty = 1$ for all $a \in A$, it is clear that $\|J\| = 1$, and so $\|Jf\|_\infty \leqslant \|f\|_F$ for all $f \in \mathrm{FVL}(A)$. Since $H(\Delta_A)$ is a Banach lattice with respect to $\|\cdot\|_\infty$, J extends by continuity to a lattice homomorphism $J\colon \mathrm{FBL}(A) \to H(\Delta_A)$ with $\|J\| = 1$. Note that, by the universal property of $\mathrm{FBL}(A)$, J is the unique lattice homomorphism from $\mathrm{FBL}(A)$ into $H(\Delta_A)$ satisfying $J\delta_a = \delta|_{a\Delta_A}$, $a \in A$. This implies, in particular, that if B is a non-empty subset of A, then J_B is the restriction of J_A to $\mathrm{FBL}(B)$ (see proposition 4.8). The problem is to show that this extension J is injective.

First we consider the situation when A is finite, in which case everything is very nice indeed.

Proposition 5.3. *For any non-empty finite set A, the map $J\colon \mathrm{FBL}(A) \to H(\Delta_A)$ is a surjective norm and lattice isomorphism.*

Proof. We claim that $\|f\|_F \leqslant n\|Jf\|_\infty$, $f \in \mathrm{FVL}(A)$, where n is the cardinality of A. Indeed, if $f \in \mathrm{FVL}(A)$, then

$$|Jf| \leqslant \|Jf\|_\infty \bigvee_{a\in A} |J(\delta_a)|,$$

so

$$|f| \leqslant \|Jf\|_\infty \bigvee_{a\in A} |\delta_a|,$$

and hence

$$\|f\|_F \leqslant \|Jf\|_\infty \Big\| \bigvee_{a\in A} |\delta_a| \Big\|_F \leqslant \|Jf\|_\infty \sum_{a\in A} \|\delta_a\|_F = n\|Jf\|_\infty.$$

This proves the claim. Consequently, $\|Jf\|_\infty \leqslant \|f\|_F \leqslant n\|Jf\|_\infty$, $f \in \mathrm{FVL}(A)$, which implies that $J\colon \mathrm{FBL}(A) \to H(\Delta_A)$ is a norm and lattice isomorphism. It remains to be shown that J is surjective. For this purpose, denote by S_A the compact subset of Δ_A given by $S_A = \{\xi \in \Delta_A\colon \|\xi\|_A = 1\}$. Since A is finite,

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the restriction map $r\colon H(\Delta_A) \to C(S_A)$ is a surjective norm and lattice isomorphism. Since the functions $\{\delta_{a|S_A}\colon a \in A\}$ separate the points of S_A, it follows via the Stone–Weierstrass theorem that $(r \circ J)(\mathrm{FBL}(A)) = C(S_A)$, and hence $J(\mathrm{FBL}(A)) = H(\Delta_A)$. The proof is complete. □

This norm isomorphism is not an isometry unless $n = 1$. In fact, if $a_1, \dots, a_n \in A$ are distinct, then $\| \bigvee_{j=1}^{n} |\delta_{a_j}| \|_F = n$ (indeed, consider the lattice homomorphism $T\colon \mathrm{FBL}(A) \to \ell_1^n$ satisfying $T(\delta_{a_j}) = e_j$, $1 \leqslant j \leqslant n$, where e_j denotes the jth unit vector in ℓ_1^n).

Sometimes it is convenient to use the following, slightly weaker, description.

Corollary 5.4. *For any non-empty finite set A,* FBL(A) *is linearly order isomorphic to $H(\mathbb{R}^A)$.*

Proof. We need only observe that the restriction map $R\colon H(\mathbb{R}^A) \to H(\Delta_A)$ is onto whenever A is finite. □

To show that the lattice homomorphism $J\colon \mathrm{FBL}(A) \to H(\Delta_A)$ is injective in general, we make use of real-valued linear lattice homomorphisms on FBL(A), which will later allow us to characterize these in general, something that is worth knowing anyway!

Theorem 5.5. *If A is a non-empty set, then $\omega\colon \mathrm{FBL}(A) \to \mathbb{R}$ is a lattice homomorphism if and only if there exists $\xi \in \Delta_A$ and $0 \leqslant \lambda \in \mathbb{R}$ such that $\omega(f) = \lambda Jf(\xi)$ for all $f \in \mathrm{FBL}(A)$.*

Proof. If ω is a real-valued lattice homomorphism on FBL(A), then it follows from lemma 3.9 that there exists $\eta \in \mathbb{R}^A$ such that $\omega(f) = f(\eta)$, $f \in \mathrm{FVL}(A)$. As FBL(A) is a Banach lattice, ω is $\|\cdot\|_F$-bounded, and so

$$\sup_{a\in A} |\eta(a)| = \sup_{a\in A} |\omega(\delta_a)| = \|\omega\| < \infty.$$

Hence, there exists a $\lambda = \|\omega\| > 0$ such that $\xi = \lambda^{-1}\eta \in \Delta_A$. If $f \in \mathrm{FVL}(A)$, then

$$\omega(f) = f(\eta) = \lambda f(\lambda^{-1}\eta) = \lambda Jf(\xi).$$

Given $f \in \mathrm{FBL}(A)$, choose a sequence (g_n) in FVL(A) with $\|f - g_n\|_F \to 0$, so that $\|Jf - Jg_n\|_\infty \to 0$, and hence $Jg_n(\xi) \to Jf(\xi)$. Thus,

$$\omega(f) = \lim_{n\to\infty} \omega(g_n) = \lambda \lim_{n\to\infty} Jg_n(\xi) = \lambda Jf(\xi).$$

The converse is clear, as, if $\xi \in \Delta_A$ and $0 \leqslant \lambda \in \mathbb{R}$, the formula $\omega(f) = \lambda Jf(\xi)$, $f \in \mathrm{FBL}(A)$, defines a lattice homomorphism on FBL(A). □

It is clear already that, for $f \in \mathrm{FVL}(A)$, $f = 0$ if and only if $Jf = 0$, if and only if $\omega(f) = 0$ for every $\|\cdot\|_F$-bounded real-valued lattice homomorphism on FVL(A). We need this equivalence for $f \in \mathrm{FBL}(A)$.

Corollary 5.6. *For any non-empty set A and $f \in \mathrm{FBL}(A)$ the following are equivalent:*

(i) $f = 0$,

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(ii) $\omega(f) = 0$ *for all real-valued lattice homomorphisms on* $\mathrm{FBL}(A)$,

(iii) $Jf = 0$.

Proof. Clearly, (i) implies (iii), and that (iii) implies (ii) follows directly from theorem 5.5.

Now assume that (ii) holds. Note first that it follows from proposition 5.3 that for any non-empty finite subset $B \subseteq A$ the restriction of J to $\mathrm{FBL}(B)$ is injective. For such a set B, the map $f \mapsto (JP_B f)(\xi)$, $f \in \mathrm{FBL}(A)$, is a real-valued lattice homomorphism on $\mathrm{FBL}(A)$ for each $\xi \in \Delta_A$, so $JP_B f = 0$. As J is injective on $\mathrm{FBL}(B)$, this shows that $P_B f = 0$. It follows from proposition 4.11 that $P_B f \to f$ for $\|\cdot\|_F$, so $f = 0$. This suffices to complete the proof. □

Corollary 5.7. *If A is any non-empty set, then the lattice homomorphism*

$$J \colon \mathrm{FBL}(A) \to H(\Delta_A)$$

is injective, so $\mathrm{FBL}(A)$ *is linearly order isomorphic to a vector sublattice of* $H(\Delta_A)$.

V. Troitsky (personal communication) pointed out to the authors that there is no similar embedding of $\mathrm{FBL}(\mathbb{N})$ into $H(\mathbb{R}^{\mathbb{N}})$. Note also that, although we have no need of the fact, the image of $\mathrm{FBL}(A)$ is actually a lattice ideal in $H(\Delta_A)$.

In the following, we identify $\mathrm{FBL}(A)$ with the vector sublattice $J(\mathrm{FBL}(A))$ of $H(\Delta_A)$.

As we have seen in proposition 4.8, if B is a non-empty subset of A, then $\mathrm{FBL}(B)$ may be identified isometrically with the closed vector sublattice of $\mathrm{FBL}(A)$ generated by $\{\delta_b \colon b \in B\}$, and there exists a canonical contractive lattice homomorphic projection P_B in $\mathrm{FBL}(A)$ onto $\mathrm{FBL}(B)$. It should be noted that we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{FBL}(A) & \xrightarrow{J_A} & H(\Delta_A) \\ {\scriptstyle k_B}\uparrow & & \uparrow{\scriptstyle j_B} \\ \mathrm{FBL}(B) & \xrightarrow[J_B]{} & H(\Delta_B) \end{array}$$

where j_B is the restriction to $H(\Delta_B)$ of the injective lattice homomorphism j_B introduced in § 2, and k_B is the isometric lattice embedding of $\mathrm{FBL}(B)$ into $\mathrm{FBL}(A)$ guaranteed by proposition 4.8. Note also that j_B is an isometry. The commutativity of the diagram follows by considering the action of the maps on the free generators of $\mathrm{FBL}(B)$. Consequently, the canonical embedding of $\mathrm{FBL}(B)$ into $\mathrm{FBL}(A)$ is compatible with the canonical embedding of $H(\Delta_B)$ into $H(\Delta_A)$. It can similarly be seen that the following diagram also commutes:

$$\begin{array}{ccc} \mathrm{FBL}(A) & \xrightarrow{J_A} & H(\Delta_A) \\ {\scriptstyle P_B}\downarrow & & \downarrow{\scriptstyle (j_B)^{-1}\circ P_B} \\ \mathrm{FBL}(B) & \xrightarrow[J_B]{} & H(\Delta_B) \end{array}$$

The next proposition describes this in terms of FBL(A) considered as a vector sublattice of $H(\Delta_A)$. We consider $\mathbb{R}^{\Delta_B}$ as a subspace of $\mathbb{R}^{\Delta_A}$ as explained in § 2.

Recall that if B is a non-empty subset of A, then for any $\xi \in \Delta_A$ we denote by ξ_B the restriction of ξ to B, so $\xi_B \in \Delta_B$.

Proposition 5.8. *Suppose that B is a non-empty subset of A. If we consider* FBL(A) *as a vector sublattice of $H(\Delta_A)$, we have the following.*

(i) *The canonical projection P_B of* FBL(A) *onto* FBL(B) *is given by $P_B f(\xi) = f(\xi\chi_B)$, $\xi \in \Delta_A$, for all $f \in$* FBL(A).

(ii) *If $f \in$* FBL(A)*, then a necessary and sufficient condition for f to belong to* FBL(B) *is that $f(\xi) = f(\eta)$ whenever $\xi, \eta \in \Delta_A$ with $\xi_B = \eta_B$.*

Proof. (i) Let P_B be the canonical projection in FBL(A) onto FBL(B) (see proposition 4.8), so that $P_B\delta_a = \delta_a$ if $a \in B$ and $P_B\delta_a = 0$ if $a \in A \setminus B$. If $f \in$ FVL(A), then it follows from the observations preceding proposition 3.7 that $P_B f(\xi) = f(\xi\chi_B)$, $\xi \in \Delta_A$. Given $f \in$ FBL(A), let (f_n) be a sequence in FVL(A) such that $\|f - f_n\|_F \to 0$, which implies that $\|f - f_n\|_\infty \to 0$, and so $f_n(\xi) \to f(\xi)$, $\xi \in \Delta_A$. Furthermore, $\|P_B f - P_B f_n\|_F \to 0$, and hence $P_B f_n(\xi) \to P_B f(\xi)$, $\xi \in \Delta_A$. Since $P_B f_n(\xi) = f_n(\xi\chi_B) \to f(\xi\chi_B)$, we may conclude that $P_B f(\xi) = f(\xi\chi_B)$, $\xi \in \Delta_A$.

(ii) *Necessity.* If $f \in$ FBL(B) and $\xi, \eta \in \Delta_A$ are such that $\xi_B = \eta_B$, then $\xi\chi_B = \eta\chi_B$, and hence it follows from (i) that

$$f(\xi) = P_B f(\xi) = f(\xi\chi_B) = f(\eta\chi_B) = P_B f(\eta) = f(\eta).$$

Sufficiency. If $f \in$ FBL(A) is such that $f(\xi) = f(\eta)$ whenever $\xi, \eta \in \Delta_A$ with $\xi_B = \eta_B$, then $P_B f(\xi) = f(\xi\chi_B) = f(\xi)$, as $(\xi\chi_B)_B = \xi_B$, for all $\xi \in \Delta_A$, and hence $f = P_B f \in$ FBL(B). □

Recall that a sublattice H of a lattice L is said to be *regularly embedded* if every subset of H with a supremum (respectively, infimum) in H has the same supremum (respectively, infimum) in L. If we are dealing with vector lattices, it suffices to consider the case of a subset of H that is downward directed in H to 0 and check that it also has infimum 0 in L.

Proposition 5.9. *If A is any non-empty set and B is a non-empty subset of A, then* FBL(B) *is regularly embedded in* FBL(A).

Proof. Suppose that $(f_\gamma)_{\gamma \in \Gamma}$ is a downward directed net in FBL(B) such that $f_\gamma \downarrow_\gamma 0$ in FBL(B), and suppose that $g \in$ FBL(A) satisfies $0 < g \leqslant f_\gamma$ for all $\gamma \in \Gamma$. Let $\xi_0 \in \Delta_A$ be such that $g(\xi_0) > 0$. We claim that we may assume that $\xi_0\chi_B \neq 0$. If our chosen ξ_0 is such that $\xi_0\chi_B = 0$, i.e. $\xi_0 = \xi_0\chi_{A\setminus B}$, then consider $\xi_\varepsilon = \xi_0 + \varepsilon\xi_B$. Since $\xi_\varepsilon \to \xi_0$ in Δ_A as $\varepsilon \downarrow 0$ and g is continuous, we may choose $\varepsilon \in (0, 1]$ with $g(\xi_\varepsilon) > 0$ and then replace ξ_0 by this ξ_ε. Given $b \in B$, define $h \in H(\Delta_A)$ by setting

$$h(\xi) = g\left(\xi\chi_B + \frac{|\xi(b)|}{\|\xi_0\chi_B\|_\infty}\xi_0\chi_{A\setminus B}\right), \quad \xi \in \Delta_A.$$

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We claim that $h \in \mathrm{FBL}(A)$. Indeed, define the lattice homomorphism $T\colon H(\Delta_A) \to H(\Delta_A)$ by setting

$$Tf(\xi) = f\left(\xi\chi_B + \frac{|\xi(b)|}{\|\xi_0\chi_B\|_\infty}\xi_0\chi_{A\setminus B}\right), \quad \xi \in \Delta_A,$$

for all $f \in H(\Delta_A)$. Observing that

$$T\delta_a = \delta_a\chi_B(a) + \frac{|\delta_b|}{\|\xi_0\chi_B\|_\infty}\delta_a(\xi_0)\chi_{A\setminus B}(a),$$

it follows that $T\delta_a \in \mathrm{FVL}(A)$ for all $a \in A$, and that $\sup_{a\in A}\|T\delta_a\|_F < \infty$. Consequently, there exists a unique lattice homomorphism $S\colon \mathrm{FBL}(A) \to \mathrm{FBL}(A)$ such that $S\delta_a = T\delta_a$ for all $a \in A$. Evidently, $Tf = Sf$ for all $f \in \mathrm{FVL}(A)$. Given $f \in \mathrm{FBL}(A)$, we may approximate f with a sequence (f_n) with respect to $\|\cdot\|_F$. Since convergence with respect to $\|\cdot\|_F$ implies pointwise convergence on Δ_A, it follows that $Sf = Tf$ (see the proof of proposition 5.8). This implies, in particular, that $h = Tg = Sg \in \mathrm{FBL}(A)$, which proves our claim.

If $\xi, \eta \in \Delta_A$ are such that $\xi_B = \eta_B$, then $h(\xi) = h(\eta)$, and so, by proposition 5.8 and lemma 2.1, it follows that $h \in \mathrm{FBL}(B)$. If $\xi \in \Delta_A$, then

$$\xi_B = \left(\xi\chi_B + \frac{|\xi(b)|}{\|\xi_0\chi_B\|_\infty}\xi_0\chi_{A\setminus B}\right)_B$$

(recall that the subscript B indicates taking the restriction to the subset B), and hence

$$\begin{aligned} f_\gamma(\xi) &= f_\gamma\left(\xi\chi_B + \frac{|\xi(b)|}{\|\xi_0\chi_B\|_\infty}\xi_0\chi_{A\setminus B}\right) \\ &\geqslant g\left(\xi\chi_B + \frac{|\xi(b)|}{\|\xi_0\chi_B\|_\infty}\xi_0\chi_{A\setminus B}\right) \\ &= h(\xi), \quad \xi \in \Delta_A, \end{aligned}$$

that is, $f_\gamma \geqslant h \geqslant 0$ for all $\gamma \in \Gamma$. We may conclude that $h = 0$.

It follows, in particular, that

$$g\left(\xi_0\chi_B + \frac{|\xi_0(b)|}{\|\xi_0\chi_B\|_\infty}\xi_0\chi_{A\setminus B}\right) = 0, \quad b \in B.$$

Applying this to $b = b_n$, where (b_n) is a sequence in B satisfying $|\xi_0(b_n)| \to \|\xi_0\chi_B\|_\infty$, the continuity of g implies that

$$g(\xi_0) = g(\xi_0\chi_B + \xi_0\chi_{A\setminus B}) = 0,$$

which is a contradiction. The proof is complete. □

6. Some properties of free Banach lattices

If X is a non-empty set and $f\colon X \to \mathbb{R}$, then we let $O_f = \{x \in X\colon f(x) \neq 0\}$, and if W is a non-empty subset of $\mathbb{R}^X$, then we define $O_W = \bigcup\{O_f\colon f \in W\}$. Although probably well known, we know of no convenient reference for the following result.

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Proposition 6.1. *If X is a Hausdorff topological space, L is a vector sublattice of $C(X)$ and the open set O_L is connected, then the only projection bands in L are $\{0\}$ and L.*

Proof. Suppose that B is a projection band in L, so $L = B \oplus B^d$. If $f \in B$ and $g \in B^d$, then $f \perp g$; hence $O_f \cap O_g = \emptyset$, and therefore $O_B \cap O_{B^d} = \emptyset$. Given $x \in O_L$ there exists $0 \neq f \in L_+$ with $f(x) > 0$. We may write $f = f_1 \oplus f_2$ with $0 \leqslant f_1 \in B$ and $0 \leqslant f_2 \in B^d$. Clearly, either $f_1(x) > 0$ or $f_2(x) > 0$. I.e. $x \in O_{f_1} \cup O_{f_2} \subset O_B \cup O_{B^d}$. Hence, $O_L \subset O_B \cup O_{B^d}$, and therefore $O_L = O_B \cup O_{B^d}$. The sets O_B and O_{B^d} are both open and disjoint and O_L is, by hypothesis, connected. This is only possible if either O_B or O_{B^d} is empty, which says that either $L = B^d$ or $L = B$. □

Corollary 6.2. *If $|A| \geqslant 2$, then the only projection bands in* FBL(A) *are* $\{0\}$ *and* FBL(A).

Proof. By corollary 5.7 we are able to identify FBL(A) with a vector sublattice of $H(\Delta_A) \subset C(\Delta_A)$. Observe that

$$O_{\mathrm{FBL}(A)} \supset \bigcup_{a\in A} O_{\delta_a} = \bigcup_{a\in A} \{\xi \in \Delta_A : \xi(a) \neq 0\} = \Delta_A \setminus \{0\}.$$

Clearly, $O_{\mathrm{FBL}(A)} \subset \Delta_A \setminus \{0\}$ so that $O_{\mathrm{FBL}(A)} = \Delta_A \setminus \{0\}$, which, provided that $|A| \geqslant 2$, is (pathwise) connected. □

Corollary 6.3. *If $|A| \geqslant 2$, then* FBL(A) *is not Dedekind σ-complete.*

Corollary 6.4. *If $|A| \geqslant 2$, then* FBL(A) *has no atoms.*

Proof. The linear span of an atom is always a projection band. □

Corollary 6.5. *If $a \in A$, then $|\delta_a|$ is a weak order unit for* FBL(A).

Proof. If $f \in \mathrm{FBL}(A)$ and $f \perp |\delta_a|$, then $O_f \subset \{\xi \in \Delta_A : \xi(a) = 0\}$, and the latter set has an empty interior, so $O_f = \emptyset$, and hence $f = 0$. □

Corollary 6.6. *Every disjoint system in* FBL(A) *is at most countable.*

Proof. If $\{u_i : i \in I\}$ is a disjoint family of strictly positive elements of FBL(A), then the corresponding sets O_{u_i} are non-empty disjoint open subsets of Δ_A. As $\Delta_A = [-1,1]^A$ is a product of separable spaces, [18, theorem 2] tells us that Δ_A can contain only countably many disjoint non-empty open sets, so the families of all O_{u_i} and of all u_i are indeed countable. □

The same result is true for FVL(A), first proved by Weinberg in [24]. It can also be found, with essentially the current proof, in [1].

Recall that an Archimedean vector lattice is *order separable* if every subset $D \subset L$ contains an at most countable subset with the same upper bounds in L as D has. This is equivalent to every order bounded disjoint family of non-zero elements being at most countable [12, theorem 29.3]. Corollary 6.6 thus actually tells us that the universal completion of FBL(A) [12, definition 50.4] is always order separable.

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Every Banach lattice is a quotient of a free Banach lattice. We can actually make this statement quite precise. The following lemma is well known, dating back, in the case when $\mathfrak{a} = \aleph_0$, to a result of Banach and Mazur [2]. A more accessible proof, again in the case when $\mathfrak{a} = \aleph_0$ (although the modifications needed for the general case are minor), are given as part of the proof of [6, ch. VII, theorem 5].

LEMMA 6.7. *Let X be a Banach space and let D be a dense subset of the unit ball of X. If $x \in X$ and $\|x\| < 1$, then there exist sequences (x_n) in D and (α_n) in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |\alpha_n| < 1$ and $x = \sum_{n=1}^{\infty} \alpha_n x_n$.*

PROPOSITION 6.8. *Let X be a Banach lattice. If D is a dense subset of the unit ball of X of cardinality $\mathfrak{a}$, then there exists a closed ideal J in $\mathrm{FBL}(\mathfrak{a})$ such that X is isometrically order isomorphic to $\mathrm{FBL}(\mathfrak{a})/J$.*

Proof. Let $D = \{x_a : a \in \mathfrak{a}\}$. By the definition of a free Banach lattice, there exists a unique contractive lattice homomorphism $T\colon \mathrm{FBL}(\mathfrak{a}) \to X$ with $T(\delta_a) = x_a$ for each $a \in \mathfrak{a}$. If $x \in X$ with $\|x\| < 1$, then lemma 6.7 gives us the sequences (x_{a_n}) in D and (α_n) in $\mathbb{R}$ with $\sum_{n=1}^{\infty} |\alpha_n| < 1$ and $x = \sum_{n=1}^{\infty} \alpha_n x_{a_n}$. If we define $f \in \mathrm{FBL}(\mathfrak{a})$ by $f = \sum_{n=1}^{\infty} \alpha_n \delta_{a_n}$, noting that this series converges absolutely, then $\|f\|_F < 1$ and $Tf = x$. This shows that T maps the open unit ball in $\mathrm{FBL}(\mathfrak{a})$ onto the open unit ball in X. In particular, T is surjective.

Take J to be the kernel of T and let $Q\colon \mathrm{FBL}(\mathfrak{a}) \to \mathrm{FBL}(\mathfrak{a})/J$ be the quotient map. Let $U\colon \mathrm{FBL}(\mathfrak{a})/J \to X$ be defined by $U(Qf) = Tf$ for $f \in \mathrm{FBL}(\mathfrak{a})$, which is clearly well defined. It is also clear that U is a contractive lattice isomorphism. As T maps the open unit ball of $\mathrm{FBL}(\mathfrak{a})$ onto the open unit ball of X, and Q maps the open unit ball of $\mathrm{FBL}(\mathfrak{a})$ onto the open unit ball of $\mathrm{FBL}(\mathfrak{a})/J$, it follows that U maps the open unit ball of $\mathrm{FBL}(\mathfrak{a})/J$ onto the open unit ball of X, so U is an isometry. □

COROLLARY 6.9. *Let X be a Banach lattice. If D is a dense subset of the unit ball of X of cardinality $\mathfrak{a}$, then $\mathrm{FBL}(\mathfrak{a})^*$ contains a weak*-closed band that is isometrically order isomorphic to X^*.*

Proof. If $T\colon \mathrm{FBL}(\mathfrak{a}) \to X$ is the quotient map from proposition 6.8, then $T^*\colon X^* \to \mathrm{FBL}(\mathfrak{a})^*$ is an isometry and its range, which is $\ker(T)^{\perp}$, is a weak*-closed band. As T is a surjective lattice homomorphism, T^* is actually a lattice isomorphism. □

In particular note the following.

COROLLARY 6.10. *If $\mathfrak{a}$ is any cardinal, then there exists a weak*-closed band in $\mathrm{FBL}(\mathfrak{a})^*$ that is isometrically order isomorphic to $\ell_\infty(\mathfrak{a})$.*

Proof. If $\mathfrak{a}$ is infinite, then we need merely note that the unit ball of $\ell_1(\mathfrak{a})$ has a dense subset of cardinality $\mathfrak{a}$ and that $\ell_\infty(\mathfrak{a})$ may be identified with $\ell_1(\mathfrak{a})^*$.

Suppose that $\mathrm{card}(A) = \mathfrak{a}$ is finite. For $a \in A$ we write ξ_a for that element of $\Delta_A = [-1, 1]^A$ with $\xi_a(a) = 1$ and $\xi_a(b) = 0$ if $a \neq b$. If $b \in A$, then $|\delta_b|(\xi_a) = |\delta_b(\xi_a)| = 1$ if $a = b$ and $|\delta_b|(\xi_a) = |\delta_b(\xi_a)| = 0$ if $a \neq b$. It follows from theorem 5.5 that the functional $f \mapsto f(\xi_a)$ is a lattice homomorphism on $\mathrm{FBL}(A)$, and therefore an atom of $\mathrm{FBL}(A)^*$, of norm 1. Finite sums of such maps also have norm 1. This embeds a copy of $\ell_\infty(A)$ isometrically onto an order ideal in $\mathrm{FBL}(A)^*$, which, as it is finite dimensional, is certainly a weak*-closed band. □

Corollary 6.11. *If X is a separable Banach lattice, then X is isometrically order isomorphic to a Banach lattice quotient of* $\mathrm{FBL}(\aleph_0)$ *and* X^* *is isometrically order isomorphic to a weak***-closed band in* $\mathrm{FBL}(\aleph_0)^*$.

This illustrates quite effectively what rich structure free Banach lattices and their duals have. For example, if X and Y are separable Banach lattices such that no two non-zero bands in X^* and Y^* are isometrically isomorphic, then the isometrically order isomorphic bands in $\mathrm{FBL}(\aleph_0)^*$ must be disjoint in the lattice theoretical sense. So, for example, we have the following.

Corollary 6.12. *In* $\mathrm{FBL}(\aleph_0)^*$ *there exist mutually disjoint weak***-closed bands* A *and* B_p $(p \in (1,\infty])$ *with* B_p *isometrically order isomorphic to* $L_p([0,1])$ *and* A *isometrically order isomorphic to* ℓ_∞.

This gives continuum many disjoint non-zero elements in $\mathrm{FBL}(\aleph_0)^*$, which should be contrasted with corollary 6.6.

7. The structure of finitely generated free Banach lattices

We will see shortly that $\mathrm{FBL}(n)$ is not an AM-space unless $n = 1$, but it does have a lot of AM-structure provided that n is finite.

If we have only a finite number of generators, n say, then we may identify $\mathrm{FBL}(n)$ with $H(\Delta_n)$, where Δ_n is now a product of n copies of $[-1,1]$. In this setting, it might be more useful to consider the restriction of these homogeneous functions to the union of all the proper faces of Δ_n, which we denote by F_n. An alternative description of this set is that it is the points in $\mathbb{R}^n$ with supremum norm equal to 1. Each of the generators δ_k $(1 \leqslant k \leqslant n)$ takes the value $+1$ on one maximal proper face of F_n of dimension $n-1$, and the value -1 on the complementary face. These faces are precisely the maximal proper faces of Δ_n. The restriction map from $H(\Delta_n)$ to $C(F_n)$ is a surjective vector lattice isomorphism and an isometry from the supremum norm over Δ_n to the supremum norm over F_n. We also know that these norms are equivalent to the free norm. Thus, when we identify $\mathrm{FBL}(n)$ with $C(F_n)$, even though the norms are not the same, the closed ideals, band, quotients, etc. remain the same, so we can read off many structural results from those for $C(K)$ spaces. Whenever we refer to the free norm on $C(F_n)$, we refer to the free norm generated using the generators that take value ± 1 on the maximal proper faces.

In particular, we may identify the dual of $\mathrm{FBL}(n)$ with the space of regular Borel measures on F_n, $\mathcal{M}(F_n)$. We see in theorem 8.1 that unless $n = 1$ the dual of the free norm $\|\cdot\|^\dagger$ is definitely not the usual norm $\|\cdot\|_1$, under which $\mathcal{M}(F_n)$ is an AL-space. However, there remains a lot of AL-structure in this dual.

Proposition 7.1. *If* $\mu \in \mathcal{M}(F_n)$ *is supported by a maximal proper face of* Δ_n*, then* $\|\mu\|^\dagger = \|\mu\|_1$.

Proof. Suppose first that $\mu \geqslant 0$. Let the free generators be denoted by $\delta_1, \delta_2, \ldots, \delta_n$. If G is the maximal proper face in question, we may suppose that $G \subset \delta_1^{-1}(1)$. As $|\delta_k| \leqslant \mathbf{1}$ on F_n, for $1 \leqslant k \leqslant n$, we have

$$\int |\delta_k| \, \mathrm{d}\mu \leqslant \int \mathbf{1} \, \mathrm{d}\mu = \|\mu\|_1,$$

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and on taking the maximum we have $\|\mu\|^\dagger \leqslant \|\mu\|_1$. On the other hand, $|\delta_1| \equiv 1$ on G, so

$$\|\mu\|^\dagger \geqslant \int |\delta_1| \, \mathrm{d}\mu = \|\mu\|_1,$$

and we have equality. Both $\|\cdot\|_1$ and $\|\cdot\|^\dagger$ are lattice norms, so in the general case we have that

$$\|\mu\|^\dagger = \||\mu|\|^\dagger = \||\mu|\|_1 = \|\mu\|_1,$$

and the proof is complete. □

Corollary 7.2. *If $f \in C(F_n)$ and there is a maximal proper face G such that f vanishes off G, then $\|f\|_F = \|f\|_\infty$.*

Proof. If $\mu \in \mathcal{M}(F_n)$, then we may write $\mu = \mu_G + \mu_{F_n \setminus G}$, where $\mu_A(X) = \mu(A \cap X)$, and note that $\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu_G$. If $\|\mu\|^\dagger \leqslant 1$, then $\|\mu_G\|^\dagger = \|\mu_G\|_1 \leqslant 1$ as $|\mu_G| \leqslant |\mu|$. Thus,

$$\begin{aligned} \|f\|_F &= \sup\left\{ \int |f| \, \mathrm{d}|\mu| \colon \|\mu\|^\dagger \leqslant 1 \right\} \\ &\leqslant \sup\left\{ \int |f| \, \mathrm{d}|\mu| \colon \|\mu\|_1 \leqslant 1 \right\} \\ &= \|f\|_\infty, \end{aligned}$$

and the embedding of $\mathrm{FBL}(n)$ into $H(\Delta_n)$ is a contraction, so $\|f\|_F \geqslant \|f\|_\infty$. □

This means that certain closed ideals in $\mathrm{FBL}(n)$ are actually AM-spaces, namely, those that may be identified with functions on F_n that vanish on a closed set A whose complement is contained in a single proper face of F_n. Rather more interesting is an analogous result for quotients.

In general, if J is a closed ideal in a Banach lattice X, then $(X/J)^*$ may be identified, both in terms of order and norm, with the ideal $J^\circ = \{f \in X^* \colon f|_J \equiv 0\}$. We know that if A is a closed subset of a compact Hausdorff space K and J^A denotes the closed ideal $J^A = \{f \in C(K) \colon f|_A \equiv 0\}$, then when $C(K)$ is given the supremum norm the normed quotient $C(K)/J^A$ is isometrically order isomorphic to $C(A)$ under its supremum norm, and its dual is isometrically order isomorphic to the space of measures on K that are supported by A. In the particular case when $K = F_n$, we may still identify quotients algebraically in the same way, but the description of the quotient norm has to be modified slightly. That means that the quotient norm may be described in a similar manner to our original description of the free norm, as follows.

Proposition 7.3. *If A is a closed subset of F_n and $C(F_n)$ is normed by its canonical free norm, then $C(F_n)/J^A$ is isometrically order isomorphic to $C(A)$, where $C(A)$ is normed by*

$$\|f\|_A = \sup\left\{ \int |f| \, \mathrm{d}|\mu_A| \colon \|\mu\|^\dagger \leqslant 1 \right\}.$$

In this supremum we may restrict to measures μ supported by A.

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In particular, we have the following, using proposition 7.1.

Corollary 7.4. *If A is a closed subset of a proper face of F_n and $C(F_n)$ is normed by its canonical free norm, then $C(F_n)/J^A$ is isometrically order isomorphic to $C(A)$ under its supremum norm.*

The free vector lattices over a finite number of generators exhibit a lot of symmetry. For example, it is not difficult to see that $\mathrm{FVL}(n)$ is invariant under rotations. In studying symmetry of $\mathrm{FBL}(n)$ it makes things clearer to identify $\mathrm{FBL}(n)$ with the space $C(S^{n-1})$ rather than with $C(F_n)$, where S^{n-1} is the Euclidean unit sphere in $\mathbb{R}^n$, even though the description of the free norm is made slightly more difficult. In the case $n = 2$, we are looking at continuous functions on the unit circle, and the dual free norm is given by

$$\|\mu\|^\dagger = \int_{S^1} |\sin(t)| \, \mathrm{d}|\mu|(t) \vee \int_{S^1} |\cos(t)| \, \mathrm{d}|\mu|(t).$$

In particular, if η_x denotes the unit measure concentrated at x, then

$$\|\eta_x\|^\dagger = |\sin(x)| \vee |\cos(x)|,$$

which is certainly not rotation invariant. Note also that

$$\|\eta_x + \eta_{x+\pi/2}\|^\dagger = (|\sin(x)| + |\sin(x+\pi/2)|) \vee (|\cos(x)| + |\cos(x+\pi/2)|).$$

In fact, only rotations through multiples of $\pi/2$ are isometries on $C(S^1)$ for the free norm. Of course, all rotations of $\mathrm{FBL}(n)$ will be isomorphisms.

There is an obvious procedure for obtaining a rotation invariant norm from the free norm, namely, to take the average, with respect to Haar measure on the group of rotations, of the free norms of rotations of a given element. Although this will certainly not be the free norm, given that it is derived in a canonical manner from the free norm we might expect that either it is a familiar norm or else it is of some independent interest. It turns out to not be familiar. This is again easiest to see in the dual.

If we denote this symmetric free norm by $\|\cdot\|_S$ and its dual norm by $\|\cdot\|^S$, then we have

$$\|\eta_x\|^S = \|\eta_{x+\pi/2}\|^S = \frac{1}{2\pi}\int_0^{2\pi} |\sin(t)| \vee |\cos(t)| \, \mathrm{d}t = \frac{2\sqrt{2}}{\pi}$$

and

$$\begin{aligned}\|\eta_x + \eta_{x+\pi/2}\|^S &= \frac{1}{2\pi}\int_0^{2\pi} (|\sin(x)| + |\sin(x+\pi/2)|) \vee (|\cos(x)| + |\cos(x_\pi/2)|) \, \mathrm{d}t \\ &= \frac{4}{\pi},\end{aligned}$$

so the symmetric free norm is not an AL-norm, which is the natural symmetric norm on $C(S^1)^*$, nor an AM-norm. In fact, $\|\eta_x + \eta_{x+t}\|^S$ can take any value between $4/\pi$ and $4\sqrt{2}/\pi$, so the symmetric free norm cannot be any L^p norm either, implausible though that would be anyway.

8. Characterizing the number of generators

Apart from wanting to understand how the number of generators affects the Banach lattice structure of $\mathrm{FBL}(A)$, we would like to know when $\mathrm{FBL}(A)$ is a classical Banach lattice or has various properties generally considered desirable. The answer to this is 'not very often'! It turns out that such properties can be used to characterize the number of generators, at least in a rather coarse manner.

In fact several properties that are normally considered 'good' are only possessed by a free Banach lattice if it has only one generator. We gather several of these into our first result. We know that, in the finitely generated case, both $\mathrm{FBL}(n)$ and its dual have a certain amount of AM-structure. There is another area of Banach lattice theory where the same is true, namely, for injective Banach lattices in the category of Banach lattices and contractive positive operators (see [10]). As injective Banach lattices are certainly Dedekind complete, we cannot have $\mathrm{FBL}(n)$ injective if $n > 1$. It might be thought possible that $\mathrm{FBL}(A)^*$ is injective, but that also turns out to be false unless $|A| = 1$.

THEOREM 8.1. *If A is a non-empty set, then the following are equivalent.*

(1) $|A| = 1$.

(2) $\mathrm{FBL}(A)$ *is isometrically an AM-space.*

(3) $\mathrm{FBL}(A)$ *is isomorphic to an AL-space.*

(4) *Every bounded linear functional on* $\mathrm{FBL}(A)$ *is order continuous.*

(5) *There is a non-zero order continuous linear functional on* $\mathrm{FBL}(A)$.

(6) $\mathrm{FBL}(A)^*$ *is an injective Banach lattice.*

Proof. If A is a singleton, then $\Delta_A = [-1, 1]$ and $\mathrm{FBL}(A)$ may be identified with $H(\Delta_A)$, which in turn may be identified with $\mathbb{R}^2$. The generator is the pair $g = (-1, 1)$. The positive linear functionals ϕ such that $\phi(|g|) \leqslant 1$ are those described by pairs of reals (ϕ_1, ϕ_2) with $|\phi_1| + |\phi_2| \leqslant 1$. The free norm that they induce on $\mathbb{R}^2$ is precisely the supremum norm.

If $|A| > 1$, then, by corollary 6.10, $\mathrm{FBL}(A)^*$ contains an order isometric copy of $\ell_\infty(A)$, so is not an AL-space, and therefore $\mathrm{FBL}(A)$ is not an AM-space. This establishes that (1) $\Leftrightarrow$ (2).

It is clear that (1) $\Rightarrow$ (3), although even in this case it is clear that $\mathrm{FBL}(1)$ is not isometrically an AL-space. $\mathrm{FBL}(2)$, on the other hand, is isomorphic to continuous functions on a square, so is certainly not isomorphic to an AL-space. In view of proposition 4.8 and the fact that every closed sublattice of an AL-space is itself an AL-space, we see that (3) $\Rightarrow$ (1).

It is clear that (1) $\Rightarrow$ (4) $\Rightarrow$ (5). To show that (5) $\Rightarrow$ (1), suppose that $|A| > 1$ and that ϕ is a non-zero order continuous linear functional on $\mathrm{FBL}(A)$. By continuity of ϕ and density of $\mathrm{FVL}(A)$ in $\mathrm{FBL}(A)$, $\phi|_{\mathrm{FVL}(A)} \neq 0$. Similarly, as

$$\mathrm{FVL}(A) = \bigcup \{\mathrm{FVL}(F) \colon F \subseteq A,\ |F| < \infty\},$$

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we may choose a finite subset $F \subseteq A$ with $\phi|_{\mathrm{FVL}(F)} \neq 0$, so certainly $\phi|_{\mathrm{FBL}(F)} \neq 0$. Without loss of generality, as long as $|A| > 1$ we may assume that $|F| > 1$. As $\mathrm{FBL}(F)$ is regularly embedded in $\mathrm{FBL}(A)$, by proposition 5.9, $\phi|_{\mathrm{FBL}(F)}$ is order continuous. As a vector lattice, we may identify $\mathrm{FBL}(F)$ with $C(S_F)$, where S_F is the ℓ_∞ unit sphere in Δ_F. Certainly, S_F is a dense in itself metrizable (and hence separable) compact Hausdorff space, so it follows from [19, proposition 19.9.4] that $\phi|_{\mathrm{FBL}(F)} = 0$, contradicting our original claim.

Certainly, $\mathrm{FBL}(1)^*$, being an AL-space, is injective (see [11, proposition 3.2]). We know from corollary 4.10 that if $|A| > 1$, then $\mathrm{FBL}(2)^*$ is isometrically order isomorphic to a (projection) band in $\mathrm{FBL}(A)^*$. If $\mathrm{FBL}(A)^*$ were injective, then certainly $\mathrm{FBL}(2)^*$ would also be injective. Recall that [10, proposition 3G] tells us that an injective Banach lattice either contains a sublattice isometric to ℓ_∞, or else is isometrically isomorphic to a finite AM-direct sum of AL-spaces. We know that $\mathrm{FBL}(2)$ is order and norm isomorphic to continuous functions on the square F_2, so $\mathrm{FBL}(2)^*$ is norm and order isomorphic to the space of measures on F_2, and thus certainly has an order continuous norm. Thus it does not contain even an isomorphic copy of ℓ_∞ by [13, corollary 2.4.3], so it certainly suffices to show that $\mathrm{FBL}(2)^*$ cannot be decomposed into a non-trivial finite AM-direct sum of bands of any nature.

The dual of $\mathrm{FBL}(2)$ can be identified, as a vector lattice, with the regular Borel measures on F_2. The dual free norm amounts to

$$\|\mu\| = \max\left\{ \int |\delta_1| \,\mathrm{d}|\mu|, \int |\delta_2| \,\mathrm{d}|\mu| \right\},$$

where δ_i is the projection onto the ith coordinate. It is clear that $\int |\delta_1| \,\mathrm{d}|\mu| = 0$ if and only if μ is supported by $S_1 = \{\langle 0, -1\rangle, \langle 0, 1\rangle\}$, while $\int |\delta_2| \,\mathrm{d}|\mu| = 0$ if and only if μ is supported by $S_2 = \{\langle -1, 0\rangle, \langle 1, 0\rangle\}$. If any non-trivial AM-decomposition of $\mathrm{FBL}(2)^*$ were possible, into $J \oplus K$, say, then we can choose $0 \neq \mu \in J_+$ and $0 \neq \nu \in K_+$. We may assume that $\|\mu\| = \|\nu\| = 1$, and therefore $\|\mu + \nu\| = 1$. The fact that $\|\mu\| = \|\nu\| = 1$ means that

$$\int |\delta_1| \,\mathrm{d}\mu \vee \int |\delta_2| \,\mathrm{d}\mu = \int |\delta_1| \,\mathrm{d}\nu \vee \int |\delta_2| \,\mathrm{d}\nu = 1.$$

Suppose that $\int |\delta_1| \,\mathrm{d}\mu = \int |\delta_1| \,\mathrm{d}\nu = 1$; we then have

$$1 = \|\mu + \nu\| \geqslant \int |\delta_1| \,\mathrm{d}(\mu + \nu) = \int |\delta_1| \,\mathrm{d}\mu + \int |\delta_1| \,\mathrm{d}\nu = 2,$$

which is impossible. Similarly, we cannot have $\int |\delta_2| \,\mathrm{d}\mu = \int |\delta_2| \,\mathrm{d}\nu = 1$. If $\int |\delta_1| \,\mathrm{d}\mu = \int |\delta_2| \,\mathrm{d}\nu = 1$, then the fact that $1 = \|\mu + \nu\| \leqslant \int |\delta_1| \,\mathrm{d}(\mu + \nu)$ tells us that $\int |\delta_1| \,\mathrm{d}\nu = 0$, so ν is supported by S_1. Similarly, we see that $\int |\delta_2| \,\mathrm{d}\mu = 0$, so μ is supported by S_2. This implies that $\mathrm{FBL}(2)^*$ is supported by $S_1 \cup S_2$, which is impossible. A similar contradiction arises if $\int |\delta_2| \,\mathrm{d}\mu = \int |\delta_1| \,\mathrm{d}\nu = 1$. □

It is already clear that free Banach lattices on more than one generator are not going to be amongst the classical Banach lattices. Isomorphism with AM-spaces is still possible and turns out to determine whether or not the number of generators is finite.

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Theorem 8.2. *If A is any non-empty set, then the following are equivalent.*

(1) *A is finite.*

(2) $\mathrm{FBL}(A)$ *is isomorphic to $H(\Delta_A)$ under the supremum norm.*

(3) $\mathrm{FBL}(A)$ *has a strong order unit.*

(4) $\mathrm{FBL}(A)$ *is isomorphic to an AM-space.*

(5) $\mathrm{FBL}(A)^*$ *has an order continuous norm.*

Proof. We have already seen that (1) ⇒ (2) ⇒ (3). It is well known and simple to prove that (3) ⇒ (4). That (4) ⇒ (5) is because the dual of an AM-space is an AL-space that has an order continuous norm, and order continuity of the norm is preserved under (not necessarily isometric) isomorphisms. In order to complete the proof we need only prove that (5) ⇒ (1).

If A is infinite, then $\mathrm{FBL}(A)^*$ contains a weak*-closed band that is isometrically order isomorphic to ℓ_∞, by corollary 6.10. By [13, theorem 2.4.14], this is equivalent to $\mathrm{FBL}(\mathfrak{a})^*$ *not* having an order continuous norm (and to many other conditions as well). □

In a similar vein, we can characterize, amongst free Banach lattices, those with a countable number of generators. Before doing so, though, we note that once there are infinitely many generators then there is an immediate connection between the number of generators and the cardinality of dense subsets. Perhaps not entirely unexpectedly, given corollary 6.6, the same result holds for order intervals. Recall that the *density character* of a topological space is the least cardinal of a dense subset.

Theorem 8.3. *If $\mathfrak{a}$ is an infinite cardinal, then the following conditions on a set A are equivalent.*

- $\mathrm{card}(A) = \mathfrak{a}$.
- $\mathrm{FBL}(A)$ *has density character $\mathfrak{a}$.*
- *The smallest cardinal $\mathfrak{b}$ such that every order interval in* $\mathrm{FBL}(A)$ *has density character at most $\mathfrak{b}$ is $\mathfrak{a}$.*

Proof. Let $\mathfrak{a} = \mathrm{card}(A)$, let $\mathfrak{b}$ be the density character of $\mathrm{FBL}(A)$ and let $\mathfrak{c}$ be the smallest cardinal that is at least as large as the density character of every order interval in $\mathrm{FBL}(A)$. We need to show that $\mathfrak{a} = \mathfrak{b} = \mathfrak{c}$.

The free vector lattice over $\mathbb{Q}$ with $\mathfrak{a}$ many generators has cardinality precisely $\mathfrak{a}$, given that $\mathfrak{a}$ is infinite. It is dense in $\mathrm{FVL}(A)$, and hence in $\mathrm{FBL}(A)$, for the free norm, so $\mathfrak{b} \leqslant \mathfrak{a}$. Clearly, $\mathfrak{c} \leqslant \mathfrak{b}$. Let K be a compact Hausdorff space such that the smallest cardinality of a dense subset of $C(K)$, and hence of the unit ball in $C(K)$, is $\mathfrak{a}$. For example, we could take $K = [0,1]^{\mathfrak{a}}$. There exists a bounded lattice homomorphism $T\colon \mathrm{FBL}(A) \to C(K)$ that maps the generators of A onto a dense subset of the unit ball of $C(K)$. The proof of proposition 6.8 shows that T is onto. Let $\mathbf{1}_K$ denote the constantly-one function on K. The order interval

$[-T^{-1}\mathbf{1}_K, T^{-1}\mathbf{1}_K]$ has a dense subset of cardinality at most $\mathfrak{c}$. As T is a surjective lattice homomorphism, $T([-T^{-1}\mathbf{1}_K, T^{-1}\mathbf{1}_K]) = [-\mathbf{1}_K, \mathbf{1}_K]$, and this will have a dense subset of cardinality at most $\mathfrak{c}$. Hence, $\mathfrak{a} \leqslant \mathfrak{c}$. This establishes that $\mathfrak{a} = \mathfrak{b} = \mathfrak{c}$. □

For the statement of the next result, which characterizes a free Banach lattice having countably many generators, we need to recall some definitions. A *topological order unit* e of a Banach lattice E is an element of the positive cone such that the closed order ideal generated by e is the whole of E. These are also referred to as *quasi-interior points*. Separable Banach lattices always possess topological order units. The *centre* of E, $Z(E)$, is the space of all linear operators on E lying between two real multiples of the identity. The centre is termed *topologically full* if, whenever $x, y \in E$ with $0 \leqslant x \leqslant y$, there exists a sequence (T_n) in $Z(E)$ with $T_n y \to x$ in norm. If E has a topological order unit, then its centre is topologically full. At the other extreme there are AM-spaces in which the centre is *trivial*, i.e. it consists only of multiples of the identity.

Theorem 8.4. *If A is a non-empty set, then the following are equivalent.*

(1) *A is finite or countably infinite.*

(2) FBL(A) *is separable.*

(3) *Every order interval in* FBL(A) *is separable.*

(4) FBL(A) *has a topological order unit.*

(5) $Z(\mathrm{FBL}(A))$ *is topologically full.*

(6) $Z(\mathrm{FBL}(A))$ *is non-trivial.*

Proof. If A is finite, then it follows from the isomorphism seen in theorem 8.2 that FBL(A), and hence its order intervals, is separable. Combining this observation with the preceding theorem shows that (1), (2) and (3) are equivalent.

We noted earlier that separable Banach lattices always have a topological order unit. The fact that Banach lattices with a topological order unit have a topologically full centre is also widely known, but finding a complete proof in the literature is not easy. The earliest is in [15, example 1][2], but that proof is more complicated than it need be. A simpler version is in [27, proposition 1.1]; see also [16, lemma 1].

Even if $\mathfrak{a} = 1$, FBL$(\mathfrak{a})$ is not one dimensional, so if the centre is topologically full, it cannot be trivial.

We know from proposition 5.7 that we may identify FBL(A) with a sublattice of $H(\Delta_A)$. It is clear, as it contains the coordinate projections, that it separates points of Δ_A. If $|A|$ is uncountable, then $\{0\}$ is not a G_δ subset of Δ_A. It follows from [25, theorem 3.1] that the centre of this sublattice, and therefore of FBL$(\mathfrak{a})$, is then trivial. □

Corollary 8.5. *If A is an uncountable set, then* FBL(A) *has trivial centre.*

[2]This preprint should not be confused with a paper with the same title by the same author in *Positivity*.

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Note that this would seem to be the first 'natural' example of a Banach lattice with a trivial centre. If $\mathfrak{a} > 1$, then $\mathrm{FVL}(A)$ always has trivial centre. The details are left to the interested reader.

9. Lifting disjoint families in quotient Banach lattices

In [23], Weinberg asked what are the projective objects in the category of abelian ℓ-groups, pointing out, for example, that a summand of a free ℓ-group is projective. Topping studied projective vector lattices in [22], but the reader should be warned that theorem 8, claiming that countable positive disjoint families in quotients L/J of vector lattices L lift to positive disjoint families in L, is false. In fact, that is only possible for an Archimedean Riesz space if the space is a direct sum of copies of the reals (see [5] and [14]).

Later in this paper we study projective Banach lattices, which are intimately connected with quotient spaces. We need to know when disjoint families in a quotient Banach lattice X/J can be lifted to disjoint families in X. As this is a question of considerable interest in its own right, and also because the results that we need do not seem to be in the literature already, we present them in a separate section here.

It is well known, although we know of no explicit reference, that any finite disjoint family $(y_k)_{k=1}^n$ in a quotient Riesz space X/J can be lifted to a disjoint family $(x_k)_{k=1}^n$ in X with $Qx_k = y_k$, where $Q\colon X \to X/J$ is the quotient map.

Proposition 9.1. *If X is a vector lattice, J is a vector lattice ideal in X, $Q\colon X \to X/J$ is the quotient map and $(y_k)_{k=1}^n$ is a disjoint family in X/J, then there exists a disjoint family $(x_k)_{k=1}^n$ in X with $Qx_k = y_k$ for $1 \leqslant k \leqslant n$.*

Proof. It suffices to consider the case when each $y_k \geqslant 0$. The proof is by induction, the case $n = 1$ being trivial. Assume that the result is true for $n = m$ and we verify it for $n = m + 1$. If $(y_k)_{k=1}^{m+1}$ is a disjoint non-negative family in X/J, we may find $(\tilde{x}_k)_{k=1}^{m+1}$ in X with $Q\tilde{x}_k = y_k$ $(1 \leqslant k \leqslant m+1)$ and $\tilde{x}_j \perp \tilde{x}_k$ for $j \neq k$ and $1 \leqslant j,\, k \leqslant m$. Let $x_k = \tilde{x_k} = \tilde{x}_k \wedge \tilde{x}_{m+1}$, for $1 \leqslant k \leqslant m$, and let $x_{m+1} = \tilde{x}_{m+1}$. For $i \leqslant k \leqslant m$ we then have

$$Qx_k = Q\tilde{x}_k - Q\tilde{x}_k \wedge Q(\tilde{x}_{m+1}) = y_k - y_k \wedge y_{m+1} = y_k.$$

Clearly, if $1 \leqslant k \leqslant m$, then $x_k \perp x_{m+1}$, while if $j \neq k$ and $1 \leqslant j,\, k \leqslant m$, then $0 \leqslant x_j \wedge x_k \leqslant \tilde{x}_j \tilde{x}_k = 0$. This establishes the result for $n = m + 1$. □

If we restrict our attention to norm closed ideals in Banach lattices, then, unlike the vector lattice case, we can handle countably infinite disjoint liftings, but not larger ones. This does not contradict the vector lattice result cited above, as there are many non-closed ideals in a Banach lattice.

Theorem 9.2. *If X is a Banach lattice, J is a closed ideal in X, $Q\colon X \to X/J$ is the quotient map and $(y_k)_{k=1}^\infty$ is a disjoint sequence in X/J, then there exists a disjoint sequence (x_k) in X with $Qx_k = y_k$ for all $k \in \mathbb{N}$.*

Proof. It suffices to consider the case when each $y_n \geqslant 0$ and $\| \sum_{k=1}^\infty \|y_k\| < \infty$. Define $z_n = \sum_{k=n+1}^\infty y_k \in X/J$ and note that z_n is disjoint from $y_1, \dots, y_n$. The sequence (x_n) will be constructed inductively.

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For $n=1$ we start by choosing $x_1, u_1 \in X$ with $x_1 \perp u_1$, $Qx_1 = y_1$ and $Qu_1 = z_1$ using proposition 9.1.

Now, suppose that we have constructed a disjoint system $\{x_1, \dots, x_n, u_n\}$ with $Qx_j = y_j$ $(1 \leqslant j \leqslant n)$ and $Qu_n = z_n$. As $0 \leqslant y_{n+1} \perp z_{n+1}$, there exist disjoint $\tilde{x}_{n+1}, \tilde{u}_{n+1} \in X_+$ with $Q\tilde{x}_{n+1} = y_{n+1}$ and $q\tilde{u}_{n+1} = z_{n+1}$. Let $x_{n+1} = \tilde{x}_{n+1} \wedge u_n$ and $u_{n+1} = \tilde{u}_{n+1} \wedge u_n$, so that, for example,

$$Qx_{n+1} = Q\tilde{x}_{n+1} \wedge Qu_n = y_{n+1} \wedge z_n = y_{n+1}.$$

Obviously, $x_{n+1} \perp u_{n+1}$, while if $1 \leqslant k \leqslant n$, we have, for example, that

$$0 \leqslant x_k \wedge u_{n+1} \leqslant x_k \wedge u_n = 0.$$

□

Even in Banach lattices, theorem 9.2 is as far as we can go.

Example 9.3. Given any uncountable disjoint family in a Banach lattice X, we know from proposition 6.8 that there exist a free Banach lattice $\mathrm{FBL}(\mathfrak{a})$ and a closed ideal J in $\mathrm{FBL}(\mathfrak{a})$ such that X is isometrically order isomorphic to $\mathrm{FBL}(\mathfrak{a})/J$. As a disjoint family in a free Banach lattice has to be countable (see corollary 6.6), the disjoint family cannot possibly be lifted to $\mathrm{FBL}(\mathfrak{a})$.

A slightly more concrete example may be found using [9, problem 6S] where it is shown that $\beta\mathbb{N} \setminus \mathbb{N}$ contains continuum many disjoint non-empty open and closed subsets. I.e. ℓ_∞/c_0 contains continuum many non-zero disjoint positive elements. As ℓ_∞ contains only countably many disjoint elements, we cannot possibly lift each of this continuum of disjoint elements in ℓ_∞/c_0 to disjoint elements in ℓ_∞. The same is true of any uncountable subset of these disjoint positive elements of ℓ_∞/c_0, so this shows that lifting of disjoint positive families of cardinality $\aleph_1$ is not possible.

An apparently simpler problem is to start with two subsets A and B in X/J with $A \perp B$ and seek subsets A', B' of X with $A' \perp B'$, $Q(A') = A$ and $Q(B') = B$. Again, countability is vital to the success of this attempt; in fact it allows us to do much more.

Proposition 9.4. *If X is a Banach lattice, J is a closed ideal in X, $Q\colon X \to X/J$ is the quotient map and $(A_n)_{n=1}^\infty$ is a sequence of countable subsets of X/J with $A_m \perp A_n$ if $m \neq n$, then there exist subsets (B_n) of X, with $B_m \perp B_n$ if $m \neq n$ and $Q(B_n) = A_n$ for each $n \in \mathbb{N}$.*

Proof. As above, there is no loss of generality in assuming that each $A_n \subset (X/J)_+$. Enumerate each set as $A_n = \{a_k^n \colon k \in \mathbb{N}\}$ (there is no difference, apart from notation, if one or both of the sets are finite). Let $v_n = \sum_{k=1}^\infty a_k^n/(2^k\|a_k\|)$, so $v_m \perp v_n$ if $m \neq n$ and $0 \leqslant a_k^n \leqslant 2^k\|a_k\|v_n$ for $k, n \in \mathbb{N}$. We know from theorem 9.2 that there exists a disjoint sequence (u_n) in X_+ with $Q(u_n) = v_n$. For any $a_k^n \in A_n$ we can find $c_k^n \in X_+$ with $Q(c_k^n) = a_k^n$. Now set $b_k^n = c_k^n \wedge (2^k\|a_k\|u_n)$ so that we still have that

$$Q(b_k^n) = Q(c_k^n) \wedge (2^k\|a_k\|Q(u_n)) = a_k^n \wedge (2^k\|a_k\|v_n) = a_k^n.$$

Also, each $b_k^n \in u_n^{\perp\perp}$, so if $m \neq n$, then for any choice of j and k we see that $b_j^m \perp b_k^n$ as $u_m \perp u_n$. Now, defining $B_n = \{b_k^n \colon k \in \mathbb{N}\}$ gives the required sets. □

Considering the case of singleton sets, the example above shows that we cannot allow an uncountable number of disjoint families. Nor can we allow even one of the families to be uncountable.

In the case when $X = C(K)$, for K a compact Hausdorff space, a closed ideal J is of the form $F = \{f \in C(K)\colon f|_A \equiv 0\}$ for some closed subset $A \subset K$, and the quotient X/J may be identified with $C(A)$ in the obvious manner. For two elements $f, g \in C(K)$, $f \perp g$ if and only if the two sets $f^{-1}(\mathbb{R} \setminus \{0\})$ and $g^{-1}(\mathbb{R} \setminus \{0\})$ are disjoint.

Example 9.5. The *Tychonoff plank* K is the topological space $[0, \omega] \times [0, \omega_1] \setminus \{(\omega, \omega_1)\}$, where ω is the first infinite ordinal and ω_1 the first uncountable ordinal. This is renowned as an example of a non-normal Hausdorff space. The sets $U = [0, \omega) \times \{\omega_1\}$ and $V = \{\omega\} \times [0, \omega_1)$ are disjoint closed subsets that cannot be separated by disjoint open sets. See, for example, [9, § 8.20]. If we add back in the removed corner point, and define $A = U \cup V \cup \{(\omega, \omega_1)\}$, then U and V become open subsets of A.

Each point of U is isolated, so their characteristic functions lie in $C(A)$ giving a (countable) family F with $U = \bigcup\{f^{-1}(\mathbb{R} \setminus \{0\})\colon f \in F\}$. Let G be a family of functions in $C(A)$ such that $V = \bigcup\{g^{-1}(\mathbb{R} \setminus \{0\})\colon g \in G\}$, which is certainly possible using Urysohn's lemma. If these could be lifted to disjoint families L and M in $C(K)$, then $\bigcup\{f^{-1}(\mathbb{R} \setminus \{0\})\colon f \in L\}$ and $\bigcup\{f^{-1}(\mathbb{R} \setminus \{0\})\colon f \in M\}$ would be disjoint open subsets of K that intersected A in the disjoint open sets U and V.

But, any disjoint open subsets of the whole product space that intersected A in U and V, respectively, would, with the corner point removed if necessary, separate the closed sets U and V in the plank. This contradiction shows that the lifting is not possible.

10. Projective Banach lattices

Definition 10.1. A Banach lattice P is *projective* if, whenever X is a Banach lattice, J is a closed ideal in X and $Q\colon X \to X/J$ is the quotient map, for every linear lattice homomorphism $T\colon P \to X/J$ and $\varepsilon > 0$ there exists a linear lattice homomorphism $\hat{T}\colon P \to X$ such that

(1) $T = Q \circ \hat{T}$,

(2) $\|\hat{T}\| \leqslant (1 + \varepsilon)\|T\|$.

Even if we take $P = \mathbb{R}$, which is easily seen to be projective given this definition, it is clear that we cannot replace $1 + \varepsilon$ by 1, as the quotient norm is an infimum that need not be attained. There are projective Banach lattices, as shown in the following.

Proposition 10.2. *A free Banach lattice is projective.*

Proof. Let $(\delta_a)_{a \in \mathfrak{a}}$ be the generators of the free Banach lattice F. Suppose that X is a Banach lattice, J is a closed ideal in X, $Q\colon X \to X/J$ is the quotient map, $T\colon F \to X/J$ is a lattice homomorphism and $\varepsilon > 0$. For each $\alpha \in \mathfrak{a}$, there exists $x_a \in X$ with $Qx_a = T\delta_a$ and $\|x_a\| \leqslant (1+\varepsilon)\|T\delta_a\| \leqslant (1+\varepsilon)\|T\|$, using the definition

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of the quotient norm. As F is free, there exists a linear lattice homomorphism $\hat{T}\colon F \to X$ with $\hat{T}\delta_a = x_a$ for all $a \in \mathfrak{a}$ and

$$\|\hat{T}\| \leqslant \sup\{\|x_a\| \colon a \in \mathfrak{a}\} \leqslant (1+\varepsilon)\|T\|.$$

As $(Q \circ \hat{T})\delta_a = T\delta_a$ for all $a \in \mathfrak{a}$ and both $Q \circ \hat{T}$ and T are linear lattice homomorphisms, they must coincide on the vector lattice generated by the δ_a and, by continuity, on F. □

We can characterize projective Banach lattices in a reasonably familiar manner.

Theorem 10.3. *The following conditions on a Banach lattice P are equivalent.*

(1) *P is projective.*

(2) *For all $\varepsilon > 0$ there exist*

  (a) *a free Banach lattice F,*

  (b) *a closed sublattice H of F and a lattice isomorphism $\mathcal{I}\colon H \to P$ with $\|\mathcal{I}\|, \|\mathcal{I}^{-1}\| \leqslant 1+\varepsilon$,*

  (c) *a lattice homomorphism projection $R\colon F \to H$ with $\|R\| \leqslant 1+\varepsilon$.*

(3) *For all $\varepsilon > 0$ there exist*

  (a) *a projective Banach lattice F,*

  (b) *a closed sublattice H of F and a lattice isomorphism $\mathcal{I}\colon H \to P$ with $\|\mathcal{I}\|, \|\mathcal{I}^{-1}\| \leqslant 1+\varepsilon$,*

  (c) *a lattice homomorphism projection $R\colon F \to H$ with $\|R\| \leqslant 1+\varepsilon$.*

Proof. To see that (1) ⇒ (2), suppose that P is projective, let F be a free Banach lattice and let J be a closed ideal in F such that P is isometrically order isomorphic to the quotient F/J *via* the linear lattice isomorphism $\mathcal{I}\colon P \to F/J$, which is always possible using proposition 6.8. Let $Q\colon F \to F/J$ be the quotient map. As P is projective, for any $\varepsilon > 0$ there exists a linear lattice homomorphism $\hat{\mathcal{I}}\colon P \to F$ with $Q \circ \hat{\mathcal{I}} = \mathcal{I}$ and $\|\hat{\mathcal{I}}\| \leqslant (1+\varepsilon)\|\mathcal{I}\| = 1+\varepsilon$. As $Q \circ \hat{\mathcal{I}}$ is injective, $\hat{\mathcal{I}}$ is also injective and $\hat{\mathcal{I}}P$ is a closed sublattice of F as $\|\hat{\mathcal{I}}p\| \geqslant \|Q(\hat{\mathcal{I}}p)\| = \|\mathcal{I}p\| = \|p\|$. The map $\hat{\mathcal{I}} \circ \mathcal{I}^{-1} \circ Q$ is a lattice homomorphism that projects F onto $\hat{\mathcal{I}}(P)$ and $\|\hat{\mathcal{I}} \circ \mathcal{I}^{-1} \circ Q\| \leqslant \|\hat{\mathcal{I}}\| \leqslant 1+\varepsilon$, so (2)(b) holds. We know that $\|\hat{\mathcal{I}}\| \leqslant 1+\varepsilon$ and $\hat{\mathcal{I}}^{-1} = \mathcal{I}^{-1} \circ Q$, so $\|\hat{\mathcal{I}}^{-1}\| = 1$, and (2)(c) holds.

In view of proposition 10.2, clearly (2) ⇒ (3).

Suppose that (3) holds, and, in particular, that (a), (b) and (c) hold for the real number η. Suppose that X is any Banach lattice, J is a closed ideal in X, $Q\colon X \to X/J$ is the quotient map, $\eta > 0$ and that $T\colon P \to X/J$ is a linear lattice homomorphism. The map $T \circ \mathcal{I} \circ R\colon F \to X/J$ is also a linear lattice homomorphism with $\|T \circ \mathcal{I} \circ R\| \leqslant \|T\|\|\mathcal{I}\|\|R\| \leqslant (1+\eta)^2\|T\|$. As F is projective, there exists a linear lattice homomorphism $S\colon F \to X$ with $Q \circ S = T \circ \mathcal{I} \circ R$ and $\|S\| \leqslant (1+\eta)\|T \circ \mathcal{I} \circ R\| \leqslant (1+\eta)^3\|T\|$. Now let $\hat{T} = S \circ \mathcal{I}^{-1}\colon P \to X$, which is also a linear lattice homomorphism, so $\|\hat{T}\| \leqslant \|S\|\|\mathcal{I}^{-1}\| \leqslant (1+\eta)^4\|T\|$ and

$$Q \circ \hat{T} = Q \circ (S \circ \mathcal{I}^{-1}) = (Q \circ S) \circ \mathcal{I}^{-1} = (T \circ \mathcal{I} \circ R) \circ \mathcal{I}^{-1} = T.$$

By choosing η small enough we can ensure that $(1+\eta)^4 \leqslant 1+\varepsilon$, and we have shown that P is projective. □

In particular, in light of corollary 6.11, all the separable projective Banach lattices that we produce later will (almost) embed in $\mathrm{FBL}(\aleph_0)$ reinforcing the richness of its structure.

Combining theorem 10.3 with corollary 5.6 we have the following.

Corollary 10.4. *The real-valued lattice homomorphisms on a projective Banach lattice separate points.*

In particular, this tells us that, for finite p, the Banach lattice $L_p([0,1])$ is *not* projective.

Similarly, from corollary 6.6 and theorem 10.3, using the lattice homomorphism projection from a free Banach lattice onto a projective, we see the following.

Corollary 10.5. *Every disjoint system in a projective Banach lattice is at most countable.*

Although, in a sense, theorem 10.3 gives a complete description of projective Banach lattices, given that we know little about free Banach lattices it actually tells us very little. One immediate consequence, given that $\mathrm{FBL}(1)$ may be identified with $\ell_\infty(2)$, is the following.

Corollary 10.6. *The one-dimensional Banach lattice $\mathbb{R}$ is projective.*

Of course, this is easy to verify directly, but it does show that there are projective Banach lattices that are not free.

We also note one rather simple consequence of the characterization of projectives in theorem 10.3.

Corollary 10.7. *If X is a projective Banach lattice, H is a closed sublattice of X for which there exists a contractive lattice homomorphism projecting X onto H, then H is a projective Banach lattice.*

11. Which Banach lattices are projective?

We now approach matters from the other end. We try to find out as much as we can about projective Banach lattices and deduce information about the structure of free Banach lattices. We start by identifying some 'small' Banach lattices, apart from free ones, that are projective. We then show that certain AL-sums of projectives are again projective.

Our first positive result may be slightly surprising, given that when dealing with Banach spaces the free and projective objects are precisely the spaces $\ell_1(I)$ (see [19, theorem 27.4.2]).

Theorem 11.1. *Every finite-dimensional Banach lattice is projective.*

Proof. Let P be a finite-dimensional Banach lattice, let X be an arbitrary Banach lattice, let J be a closed ideal in X, let $Q\colon X \to X/J$ be the quotient map, let $T\colon P \to X/J$ be a lattice homomorphism and let $1 \geqslant \varepsilon > 0$. We identify

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P with $\mathbb{R}^n$ with the pointwise order and normed by some lattice norm $\|\cdot\|_P$. Without loss of generality we may assume that the standard basic vectors in $\mathbb{R}^n$, e_k, all have $\|e_k\|_P = 1$. Let $\{p_k \colon 1 \leqslant k \leqslant m\}$ be an ε-net for the compact set $\{p \in \mathbb{R}^n_+ \colon \|p\|_P = 1\}$. We write $p_k = (p_k^1, p_k^2, \dots, p_k^n)$.

As T is a lattice homomorphism, the family $(Te_k)_{k=1}^n$ is a disjoint family in $(X/J)_+$, so by proposition 9.1 there exists a disjoint family $(s_k)_{k=1}^n$ in X_+ with $Qs_k = Te_k$ for $1 \leqslant k \leqslant n$. By the definition of the quotient norm, for each k there exists $t_k \in X$ with $Qt_k = Te_k$ and $\|t_k\| \leqslant \|Te_k\| + \varepsilon \leqslant \|T\| + \varepsilon$. Now, let $x_k = s_k \wedge t_k^+$, so the family (x_k) remains disjoint. As Q is a lattice homomorphism,

$$Qx_k = Qs_k \wedge Qt_k^+ = (Te_k) \wedge (Te_k)^+ = Te_k.$$

Also, we now have $\|x_k\| \leqslant \|t_k^+\| \leqslant \|t_k\| \leqslant \|T\| + \varepsilon$.

Also, for each $i \in \{1, 2, \dots, m\}$ there exists $q_i \in X_+$ with $Qq_i = Tp_i$ and $\|q_i\| \leqslant \|Tp_i\| + \varepsilon \leqslant \|T\| + \varepsilon$.

Define $z_k = x_k \wedge \bigwedge{}'^{m}_{i=1} (p_i^k)^{-1} q_i$ where the prime indicates that terms where $p_i^k = 0$ are omitted. As the family (x_k) is disjoint, the same is true for the family (z_k). If $p_i^k > 0$, then $(p_i^k)^{-1} p_i \geqslant e_k$, so $(p_i^k)^{-1} Qq_i = (p_i^k)^{-1} Tp_i \geqslant Te_k$, and thus $Qz_k = Qx_k = Te_k$.

Define $Se_k = z_k$ and extend S linearly to a lattice homomorphism (because (z_k) is a disjoint sequence) of $\mathbb{R}^n \to X$. Clearly, $Q \circ S_k = T$. As $\mathbb{R}^n$ is finite dimensional, there exists a constant $K \in \mathbb{R}_+$ such that $\|x\|_1 \leqslant K\|x\|_P$ for all $x \in \mathbb{R}^n$. It follows that

$$\begin{aligned}
\biggl\| S\biggl(\sum_{k=1}^n \lambda_k e_k\biggr)\biggr\| &\leqslant \sum_{k=1}^n |\lambda_k| \|Se_k\| \\
&= \sum_{k=1}^n |\lambda_k| \|z_k\| \\
&\leqslant \sum_{k=1}^n |\lambda_k| \|x_k\| \\
&\leqslant \biggl\| \sum_{k=1}^n \lambda_k e_k \biggr\| (\|T\| + 1) \\
&\leqslant K(\|T\| + 1) \biggl\| \sum_{k=1}^n \lambda_k e_k \biggr\|_P,
\end{aligned}$$

so $\|S\| \leqslant K(\|T\| + 1)$. Note that this estimate is independent of the choice of ε.

In order to better estimate the norm of S, we write $p_i = \sum_{k=1}^n p_i^k e_k$ and see that

$$\begin{aligned}
Sp_i &= \sum_{k=1}^n S(p_i^k e_k) \\
&= \sum_{k=1}^n p_i^k Se_k \\
&= \sum_{k=1}^n p_i^k z_k.
\end{aligned}$$

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Also, if $p_i^k = 0$, then certainly $p_i^k z_k \leqslant q_i$, while if $p_i^k > 0$, then

$$p_i^k z_k \leqslant p_i^k (p_i^k)^{-1} q_i = q_i.$$

As $p_i^j z_j \perp p_i^k z_k$ if $j \neq k$, we see that $\sum_{k=1}^n p_i^k z_k \leqslant q_i$, so

$$Sp_i \leqslant q_i \quad \text{and} \quad \|Sp_i\| \leqslant \|q_i\| \leqslant \|T\| + \varepsilon.$$

If we now take an arbitrary $p \in \{P_+ \colon \|p\| = 1\}$, then we can choose i with $\|p - p_i\|_P < \varepsilon$, so

$$\begin{aligned} \|Sp\| &\leqslant \|Sp_i\| + \|S\| \|p - p_i\|_P \\ &\leqslant \|T\| + \varepsilon + K(\|T\| + 1)\varepsilon, \end{aligned}$$

which can be made as close to $\|T\|$ as we desire. □

The spaces $C(K)$, for K a compact Hausdorff space, play a distinguished role in the general theory of Banach lattices, so it is worth knowing which $C(K)$ spaces are projective. We give a partial answer here, which is already of substantial interest. We refer the reader to [4] for basic concepts about retracts, but include the basic definitions here for the convenience of the reader.

Definition 11.2. If X is a topological space and K a subset of X, then the following hold.

(1) K is a *retract* of X if there exists a continuous function $\pi \colon X \to K$ with $\pi(k) = k$ for all $k \in K$.

(2) K is a *neighbourhood retract* of X if there exists a neighbourhood U of K in X and a continuous function $\sigma \colon U \to K$ with $\sigma(k) = k$ for all $k \in K$.

Definition 11.3. In a category $\mathcal{C}$ of topological spaces, the following hold.

(1) A space K is an *absolute retract* if K is a retract of X whenever $K \subseteq X \in \mathcal{C}$.

(2) A space K is an *absolute neighbourhood retract* if K is a neighbourhood retract of X whenever $K \subseteq X \in \mathcal{C}$.

Theorem 11.4. *If K is a compact subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then the following are equivalent.*

(1) *$C(K)$ is a projective Banach lattice under some norm.*

(2) *$C(K)$ is projective under the supremum norm.*

(3) *K is a neighbourhood retract of $\mathbb{R}^n$.*

Proof. Without loss of generality we may suppose that K is a subset of the unit ball in $\mathbb{R}^n$ for the supremum norm. We write p_k for the restriction to K of the kth coordinate projection in $\mathbb{R}^n$ and p_0 for the constantly-one function on K. The vector sublattice generated by the $\{p_k \colon 0 \leqslant k \leqslant n\}$ is certainly dense in $C(K)$ by the Stone–Weierstrass theorem. As $\mathrm{FBL}(n+1)$ is free, there exists a bounded vector lattice homomorphism $T \colon \mathrm{FBL}(n+1) \to C(K)$ with $T(\delta_k) = p_{k-1}$. We know that, algebraically, we may identify $\mathrm{FBL}(n+1)$ with $C(F_{n+1})$, and that the constantly-one function on F_{n+1} is precisely $\bigvee_{k=1}^{n+1}|\delta_k|$. As $\bigvee_{k=0}^{n}|p_k| = p_0$, because K is a subset of the supremum norm unit ball, we may regard T as a unital lattice homomorphism from $C(F_{n+1})$ to $C(K)$. Such maps are of the form $f \mapsto f \circ \phi$ where $\phi\colon K \to F_{n+1}$ is continuous. The image of $C(F_{n+1})$ is dense in $C(K)$ and it is well known that the image of such composition maps is closed so that T is onto. This is equivalent to ϕ being injective. I.e. we have a topological embedding of K into F_{n+1}, and we may regard T as simply being the restriction map from $C(F_{n+1})$ to $C(K)$. So far we have not used the assumption that $C(K)$ is projective.

If J is the kernel of T, then $C(F_{n+1})/J$ is isomorphic to $C(K)$. If $C(K)$ is projective (even in a purely algebraic sense), then there exists a vector lattice homomorphism $U\colon C(K) \to C(F_{n+1})$ with $Uf|_K = f$ for all $f \in C(K)$. But, U is of the form

$$Uf(p) = \begin{cases} w(p)f(\pi p) & (p \in U), \\ 0 & (p \notin U), \end{cases}$$

where w is a non-negative continuous real-valued function on F_{n+1} and $\pi\colon F_{n+1} \setminus w^{-1}(0) \to K$, so we must have $w(p) = 1$ and $\pi p = p$ for $p \in K$. Thus, $F_{n+1} \setminus w^{-1}(0)$ is open and contains K, so π is a neighbourhood retract of F_{n+1} onto K. If we remove any single point from F_{n+1} that is not in K, then what remains is homeomorphic to $\mathbb{R}^n$, so we have a neighbourhood retraction from $\mathbb{R}^n$ onto K. This only fails to be possible if $K = F_{n+1}$, and that is not homeomorphic to a subset of $\mathbb{R}^n$ by the Borsuk–Ulam theorem; see, for example, [20, theorem 5.8.9]. Thus, (1) implies (3).

Clearly, (2) implies (1), so we need only prove that (3) implies (2). The blanket assumption on K tells us that it is homeomorphic to a subset of one face G of F_{n+1}. By scaling it if necessary, we may assume that it is a neighbourhood retract of G, and therefore of the whole of F_{n+1}. That allows us to construct a continuous $w\colon F_{n+1} \to [0,1]$ with $U = \{p \in F_{n+1}\colon w(p) > 0\} \subset G$, $K \subset w^{-1}(1)$ and a continuous retract $\pi\colon U \to K$. The vector lattice homomorphism $U\colon C(F_{n+1}) \to C(F_{n+1})$ defined by

$$Uf(p) = \begin{cases} w(p)f(\pi p) & (p \in U), \\ 0 & (p \notin U) \end{cases}$$

is certainly a projection. For any $p \in F_{n+1}$ we have, writing

$$J^K = \{f \in C(F_{n+1})\colon f|_K \equiv 0\},$$

that

$$\begin{aligned} \|Uf\|_F &= \|Uf\|_\infty && \text{(corollary 7.2)} \\ &= \sup\{|w(p)f(\pi p)|\colon p \in U\} \\ &\leqslant \sup\{|f(\pi p)|\colon p \in U\} \\ &\leqslant \sup\{|f(k)|\colon k \in K\} = \|f|_K\|_\infty \\ &= \|f + J^K\| && \text{(corollary 7.4)} \\ &\leqslant \|f\|_F, \end{aligned}$$

so U is a contraction.

We also claim that the image $UC(F_{n+1})$ is isometrically order isomorphic to $C(K)$ under its supremum norm. To prove this, it suffices to prove that $Uf \mapsto F|_K$ is an isometry for the free norm on Uf, which is equal to its supremum norm, and the supremum norm on $f|_K$. The calculation above shows that $\|Uf\|_\infty \leqslant \|f|_K\|_\infty$. We also have, for $p \in U$, that $|Uf(p)| = |w(p)||f(\pi p)| \leqslant \|f|_K\|_\infty$ as $|w(p)| \leqslant 1$ and $\pi p \in K$. Thus, $\|Uf\|_\infty \leqslant \|f|_K\|_\infty$ and we have our desired isometry.

In view of theorem 10.3, this shows that $C(K)$ is projective. □

The reader will note that the first implication would actually work for an isomorphic version of projectivity. We allude to this further in § 12.

Corollary 11.5. *Under the usual supremum norm, $C([0,1])$ is a projective Banach lattice.*

Note that some $C(K)$-spaces can be projective for different (necessarily equivalent) Banach lattice norms. For example, $C(F_n)$ will be projective under both the free and supremum norms.

Recall that, as closed bounded convex subsets of $\mathbb{R}^n$ are absolute retracts in the category of compact Hausdorff spaces, any compact neighbourhood retract of $\mathbb{R}^n$ will necessarily be an absolute neighbourhood retract in the category of compact Hausdorff spaces, and therefore certainly in the category of compact metric spaces.

Descriptions of absolute neighbourhood retracts in the category of compact metric spaces may be found in [4, ch. V]. We note two particular properties that they have. Firstly, absolute neighbourhood retracts have only finitely many components (see [4, V.2.7]), and if K is an absolute neighbourhood retract subset of $\mathbb{R}^n$, then $\mathbb{R}^n \setminus K$ has only finitely many components (see [4, V.2.20]).

In particular, we have the following.

Corollary 11.6. *The sequence space c is not projective.*

Proof. We can identify c with $C(K_0)$, where $K_0 = \{1/n \colon n \in \mathbb{N}\} \cup \{0\}$. As $K_0 \subset \mathbb{R}$ and K_0 has infinitely many components, it is not an absolute neighbourhood retract. □

There seems little hope of removing the assumption of finite dimensionality from K in theorem 11.4. We can rescue one implication.

Proposition 11.7. *If $C(K)$ is a projective Banach lattice under the supremum (or an equivalent) norm, then K is an absolute neighbourhood retract in the category of compact Hausdorff spaces.*

Proof. Suppose that K is a closed subset of a compact Hausdorff space X. We need to show that there exists a continuous retraction π of U onto K, where U is an open subset of X with $K \subset U$.

The restriction map $R \colon C(X) \to C(K)$ may be identified with the canonical quotient map of $C(X)$ onto $C(X)/J$, where J is the closed ideal $\{f \in X(K) \colon f|_K \equiv 0\}$. If $C(K)$ is projective, then the identity on $C(K)$ lifts to a lattice homomorphism $T \colon C(K) \to C(X)$ with $R \circ T - I_{C(K)}$. There exist a continuous function w from

X into $\mathbb{R}_+$ and a continuous map $\pi\colon U = \{x \in X : w(x) > 0\} \to K$ such that

$$Tf(x) = \begin{cases} w(x)f(\pi x) & [w(x) > 0], \\ 0 & [w(x) = 0]. \end{cases}$$

If $k \in K$, then $Tf(k) = f(k)$, so $\pi k = k$ and $w(k) = 1$ showing that $K \subset U$ and that π is a retraction of the open set U onto K. □

Without knowledge of the properties of absolute neighbourhood retracts in the category of compact Hausdorff spaces, this does not tell us a lot. There seems to be very little material in the literature on absolute neighbourhood retracts in this setting, so we make our own modest contribution here.

Lemma 11.8. *If C is a compact convex subset of a locally convex space, K is a closed subset of C and U is an open subset of C with $K \subseteq U \subset C$, then there exists an open set V with $K \subseteq V \subseteq U \subseteq C$ such that V has finitely many components.*

Proof. As U is open, if $k \in K$, there is a convex (and therefore connected) open set W_k with $k \in W_k \subseteq U$, using local convexity. The open sets W_k, for $k \in K$, cover the compact set K, so there exists a finite subcover $W_1, W_2, \dots, W_n$. Take $V = \bigcup_{k=1}^{n} W_k$. □

Proposition 11.9. *If K is an absolute neighbourhood retract in the category of compact Hausdorff spaces, then K has only finitely many components.*

Proof. Let $C = P(K)$, the space of probability measures on K, with the weak* topology induced by $C(K)$, which is a locally convex topology under which C is compact as well as certainly being convex. The mapping that takes k to the point mass at k is a homeomorphism of K onto the set of extreme points of C. If K is an absolute neighbourhood retract, then there exists a retraction $\pi\colon U \to K$, where U is an open subset of C with $K \subseteq U$. By the preceding lemma, there exists an open set V, with finitely many components, such that $K \subseteq V \subseteq U$. The image of each component of V under π is connected and their union is K, so K has only finitely many components. □

Thus, if $C(K)$ is a projective Banach lattice under any norm, then K has only finitely many components. In particular, we have the following.

Corollary 11.10. *The sequence space ℓ_∞ is not a projective Banach lattice.*

In [1] Baker characterized projective vector lattices with n generators as being quotients of FVL(n) by a principal ideal. If we embed K_0 into one of the faces of F_2, then we know that c is isometrically order isomorphic to FBL$(2)/J^{K_0}$. It is clear that J^{K_0} is a principal closed ideal of FBL(2) and that c has two generators as a Banach lattice, so the natural analogue of Baker's result fails in the Banach lattice setting.

The obvious candidate for a projective Banach lattice, as in the Banach space case, is $\ell_1(I)$ for an arbitrary index set I; however, corollary 10.5 tells us that if I is an uncountable index set, then $\ell_1(I)$ is definitely *not* a projective Banach lattice. Similarly $\ell_p(I)$ $(1 \leqslant p < \infty)$ and $c_0(I)$ are not projective if I is uncountable.

Given that we can lift disjoint sequences, it is not difficult to show that ℓ_1 is projective. In fact we can show much more.

THEOREM 11.11. *If, for each $n \in \mathbb{N}$, P_n is a projective Banach lattice with a topological order unit, then the countable sum $\ell_1(P_n)$, under the coordinate-wise order and normed by $\|(p_n)\|_1 = \sum_{n=1}^{\infty} \|p_n\|$, is a projective Banach lattice.*

Proof. Let e_n be a topological order unit for P_n. We identify P_n with the subspace of $\ell_1(P_n)$ in which all entries apart from the nth are 0, and e_n with the corresponding member of that subspace so that the e_n are all disjoint. If X is a Banach lattice, J is a closed ideal in X, $Q\colon X \to X/J$ is the quotient map, $T\colon \oplus_1 (P_n) \to X/J$ is a lattice homomorphism and $\varepsilon > 0$, then we start by noting that the Te_n are disjoint, so by theorem 9.2 we can find disjoint u_n in X_+ with $Qu_n = Te_n$. If we write X_n for the closed ideal in X generated by u_n, then the family (X_n) is disjoint in X.

Note that the natural embedding of $X_n/(J \cap X_n)$ into X/J is an isometry onto an ideal, and that $T(P_n) \subset X_n/(J \cap X_n)$ as e_n is a topological order unit for P_n and T is a lattice homomorphism. The projectivity of P_n allows us to lift T_n to a lattice homomorphism $\hat{T}_n\colon P_n \to X_n$ with $\|\hat{T}_n\| \leqslant \|T_n\| + \varepsilon \leqslant \|T\| + \varepsilon$ with $Q \circ \hat{T}_n = T_n$. Piecing together this sequence of operators in the obvious way gives us the desired lifting of T. □

Recall that if $\mathfrak{a}$ is finite or countably infinite, then FBL($\mathfrak{a}$) has a topological order unit, as do finite-dimensional Banach lattices and $C(K)$-spaces. This gives us a source of building blocks to create other projectives.

We already have some examples of Banach lattices that are not projective. It is interesting to note that the free Banach lattices on uncountably many generators seem to be, in some sense at least, maximal projectives.

EXAMPLE 11.12. If $\mathfrak{a}$ is uncountable, then there is no non-zero Banach lattice X for which $X \oplus \mathrm{FBL}(\mathfrak{a})$ is projective under any norm.

Proof. Suppose that, under some norm, $\mathrm{FBL}(\mathfrak{a}) \oplus X$ is projective, where X is a Banach lattice and $\mathfrak{a}$ is uncountable.

Consider $C(K)$, where $K = [0, \omega] \times [0, \omega_1]$, and (with the notation of example 9.5) $J = \{f \in C(K)\colon f|_A \equiv 0\}$ so that $C(K)/J$ is isometrically order isomorphic to $C(A)$.

For each $v \in V$ there exists $f_v \in C(A)$ with $0 \leqslant f_v(a) \leqslant 1$ for all $a \in V$, $f_v(v) = 1$ and f_v identically 0 on $A \setminus V$. As V has cardinality $\aleph_1$, there exists a map of the set of generators $\{\delta_a\colon a \in \mathfrak{a}\}$ of FBL($\mathfrak{a}$) onto $\{f_v\colon v \in V\}$, which extends to a lattice homomorphism of FBL($\mathfrak{a}$) into $C(A)$. The image of every generator vanishes on U, hence the same is true for elements of $T(\mathrm{FVL}(\mathfrak{a}))$ and, by continuity, for elements of $T(\mathrm{FBL}(\mathfrak{a}))$. Note that $\bigcup_{f \in \mathrm{FBL}(\mathfrak{a})} \{a \in A\colon f(a) \neq 0\} = V$.

As U is an F_σ, there exists $g \in C(A)$ with $g(u) > 0$ for all $u \in U$ and with g identically 0 on $A \setminus U$. If $X \oplus \mathrm{FBL}(\mathfrak{a})$ were projective and $x_0 \in X_+ \setminus \{0\}$, there would exist a real-valued lattice homomorphism on $X \oplus \mathrm{FBL}(\mathfrak{a})$ with $\phi(x_0) > 0$ (and necessarily $\phi|_{\mathrm{FBL}(\mathfrak{a})} \equiv 0$). Define $Sx = \phi(x)g$ for $x \in X$ so that S is a lattice homomorphism of X into $C(A)$. The disjointness of the images of $S(X)$ and

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$T(\mathrm{FBL}(\mathfrak{a}))$ shows that the direct sum operator

$$S \oplus T \colon X \oplus \mathrm{FBL}(\mathfrak{a}) \to C(A) = C(K)/J$$

is also a lattice homomorphism. If $X \oplus \mathrm{FBL}(\mathfrak{a})$ were projective, we could find a lattice homomorphism $\hat{S} \oplus \hat{T} \colon X \oplus \mathrm{FBL}(\mathfrak{a}) \to C(K)$ with $Q \circ (\hat{S} \oplus \hat{T}) = S \oplus T$. The images of $X \oplus \{0\}$ and $\{0\} \oplus \mathrm{FBL}(\mathfrak{a})$ will be disjoint in $C(K)$, and their open supports will give disjoint open sets with traces on A equal to U and V, respectively, which we know is impossible. □

The family of projective Banach lattices seems to possess very few stability properties beyond those that we have already noted. In particular, closed sublattices of projectives need not be projective, as the non-projective c may be isometrically embedded as a closed sublattice of the projective Banach lattice $C([0,1])$, by mapping the sequence (a_n) to the function that is linear on each interval $[1/(n+1), 1/n]$ and takes the value a_n at $1/n$. Similarly, we may realize c as the quotient of $C([0,1])$ by the closed ideal $\{f \in C([0,1]) \colon f(1/n) = 0 \ \forall n \in \mathbb{N}\}$, showing that the class of projective Banach lattices is not closed under quotients.

12. Some open problems

We start with a few questions on free Banach lattices.

Question 12.1. Must the norm on a free Banach lattice be Fatou, or even Nakano? See [26] for the definition of a Nakano norm. We are not sure of the answer even when there are only finitely many generators.

The following question is rather a long shot as we have very little evidence for it beyond the case of a finite number of generators (see below).

Question 12.2. If the free Banach lattice $\mathrm{FBL}(\mathfrak{a})$ is embedded as a closed ideal in a Banach lattice, must it be a projection band?

The reason that this holds in the case of a finite number of generators is because this (isomorphic) property of Banach lattices is possessed by Banach lattices with a strong order unit. The following is undoubtedly well known, but we know of no convenient reference for it.

Proposition 12.3. *Let Y be a Banach lattice with the property that every upward directed norm bounded subset of Y_+ is bounded above. If Y is embedded as a closed ideal in a Banach lattice X, then it must be a projection band.*

Proof. It suffices to prove that if $x \in X_+$, then the set $B = \{y \in Y \colon 0 \leqslant y \leqslant x\}$ has a supremum in Y. As B is upward directed and norm bounded, it has an upper bound $u \in Y_+$. As $u \wedge x \in Y_+$, since Y is an ideal, $u \wedge x$ is an upper bound for B in Y. As we also have $0 \leqslant u \wedge x \leqslant x$, $u \wedge x \in B$, so it is actually the maximum element of B. □

We have seen that, unless $|A| = 1$, $\mathrm{FBL}(A)^*$ is not an injective Banach lattice. However, in the case of *finite* A, $\mathrm{FBL}(A)^*$ is isomorphic to an AL-space, and therefore to an injective Banach lattice. We suspect that the following question might lead to another characterization of finitely generated free Banach lattices.

Question 12.4. When is $\mathrm{FBL}(\mathfrak{a})^*$ isomorphic to an injective Banach lattice?

In theorem 8.3 we showed that the density character of $\mathrm{FBL}(A)$ was equal to the cardinality of A and related this to the density character of order intervals in $\mathrm{FBL}(A)$. This is of importance in the study of regular operators between Banach lattices, so the answer to the following question has implications in that field.

Question 12.5. Does every order interval in $\mathrm{FBL}(A)$ have the same density character?

In the light of theorem 8.3, that density character would have to be the cardinality of A.

Question 12.6. What is the structure of the symmetric free norm on $\mathrm{FBL}(n)$?

Question 12.7. Can the construction of a free Banach lattice be generalized to give a free Banach lattice over a metric space? Here, a metric space S embeds in a 'free' Banach lattice in some sense, and any isometry of the generators into a Banach lattice extends to a lattice homomorphism with some restriction on the norm. See [17] for the Banach space case.

We have seen in corollary 6.10 that $\mathrm{FBL}(\mathfrak{a})^*$ contains a disjoint family of cardinality $\mathfrak{a}$, which contrasts strongly with the fact that disjoint families in $\mathrm{FBL}(A)$ itself can only be at most countably infinite.

Question 12.8. How large can disjoint families of non-zero elements in $\mathrm{FBL}(\mathfrak{a})^*$ be?

At present we have no feel at all for what kinds of Banach lattice are likely to be projective. Clearly, there are a lot of 'small' ones, where small means either separable or having a topological order unit. A major and obvious question to pose is the following.

Question 12.9. What is the structure of the class of projective Banach lattices?

In particular, we ask the following three questions.

Question 12.10. Are separable atomic Banach lattices with an order continuous norm projective?

Question 12.11. Is c_0, under the supremum norm, a projective Banach lattice?

Question 12.12. For what compact Hausdorff spaces K is $C(K)$ projective under the supremum norm?

We know the answer to the preceding question for compact subsets of $\mathbb{R}^n$ by theorem 11.4.

The following two questions were posed by Buskes. An apparently simple question to answer is the following.

Question 12.13. If P_k $(1 \leqslant k \leqslant n)$ are projective Banach lattices with topological order units, then is their ℓ_∞ sum also projective?

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It is not difficult to lift a lattice homomorphism $T\colon \bigoplus_{k=1} nP_k \to Y/J$ to a lattice homomorphism $\hat{T}\colon \bigoplus_{k=1}^n P_k \to Y$ by lifting the images of the topological order units first. The problem seems to be the norm condition on $\hat{T}$.

It is clear that the Fremlin tensor product (see [8]) of two projective Banach lattices need not be projective in general. Example 11.12 shows that this cannot be true for the product of ℓ_1 and $\mathrm{FBL}(\mathfrak{a})$ when $\mathfrak{a}$ is uncountable. There seems to be no good structural reason to expect a positive result to the next question, but a counter-example has eluded us so far.

Question 12.14. If X and Y are projective Banach lattices with topological order units, is their Fremlin tensor product projective?

The building blocks that we can use in theorem 11.11 to build new projectives include finite-dimensional spaces, $\mathrm{FBL}(\mathfrak{a})$ for $\mathfrak{a}$ either finite or countably infinite and certain $C(K)$-spaces. Any of these, and the space that is produced by that theorem, will be separable, and hence will have a topological order unit. The following are some (possibly rather rash) conjectures that we might make.

Conjecture 12.15. *If a projective Banach lattice has a topological order unit, then it is separable.*

Conjecture 12.16. *A projective Banach lattice that does not have a topological order unit must be free.*

Even if this conjecture were to fail, we can look for an improvement of example 11.12 by asking the following question.

Conjecture 12.17. *If $\mathfrak{a}$ is uncountable and a projective Banach lattice X contains a closed ideal isomorphic to $\mathrm{FBL}(\mathfrak{a})$, do we actually have that $X = \mathrm{FBL}(\mathfrak{a})$?*

Question 12.18. The ℓ_1 sum of a sequence of finite-dimensional Banach lattices is a Dedekind complete projective. Are these the only Dedekind (σ-)complete projectives?

Conjecture 12.19. *All order continuous functionals on a projective Banach lattice are determined by its atoms.*

Question 12.20. Assuming a positive answer to question 12.10, is there a result similar to theorem 11.11 for ℓ_p sums ($1 < p < \infty$) or for c_0 sums?

The whole of this paper has been written in an isometric setting. All of our results may be re-proved in an isomorphic setting, where we replace an (almost) isometric condition on operators with mere norm boundedness. It is not difficult to see that there will automatically be uniform bounds to the norms of operators, and that isometrically free (respectively, projective) Banach lattices will be isomorphically free (respectively, projective). Isomorphically free Banach lattices will certainly be isomorphic to isometrically free Banach lattices. At present it does not seem worth recording such a theory, unless there is a negative answer to the following question.

Question 12.21. Is every isomorphically projective Banach lattice isomorphic to an isometrically projective Banach lattice?

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