Canad. Math. Bull. Vol. 18 (5), 1975

SOME PROPERTIES OF INDICATRICES IN A FINSLER SPACE[1]

BY
T. N. SRIVASTAVA AND SHÔJI WATANABE[2]

(Dedicated to Prof. J. Kanitani on the occasion of his 80th birthday)

1. **Introduction.** Let (M^n, F) be an n-dimensional Finsler space where M^n is the underlying n-dimensional manifold and $F=F(x^i, X^i)$[3] is the Finsler fundamental function. F being a differentiable function of the point $x=(x^i)\in M^n$ and element of support $X=(X^i)\in T_x(M^n)$ where $T_x(M^n)$ is the tangent space of M^n at x and is positively homogeneous of degree one with respect to X. Thus the fundamental function F determines at every point $x\in M^n$ an indicatrix in $T_x(M^n)$ defined by the equation $F(x^i, X^i)=1$ (X^i: variable).

From now and onwards we shall confine ourselves to a fixed point $x_0=(x_0^i)$ and to the corresponding tangent space $T_{x_0}(M^n)$ of M^n at the point x_0. Since $F(x_0, X)$ is positively homogeneous of degree one with respect to X hence $g_{ij}=\frac{1}{2}\partial^2F^2/\partial X^i\partial X^j$ are in general discontinuous at the origin $X=O=(0,0,\ldots,0)$ of $T_{x_0}(M^n)$. Clearly $T_{x_0}(M^n)-\{O\}$ is an n-dimensional Riemannian space with the Riemannian metric tensor $g_{ij}(X)$. We shall denote the space $T_{x_0}(M^n)-\{O\}$ by V^n.

Let M^{n-1} be a hypersurface in V^n represented by the equation $X^i=X^i(u^\alpha)=X^i(u)$.[3] If N^i denotes the unit normal vector to M^{n-1} and $B_\alpha^i=\partial X^i/\partial u^\alpha$ then with respect to the frame (B_α^i, N^i) the hypersurface M^{n-1} can be represented by the equation

$$X^i(u) = v^\beta(u)B_\beta^i+v(u)N^i. \tag{1.1}$$

It is easy to verify [3] that the vector field v^α and the scalar field v in (1.1) satisfy the equations

$$D_\alpha v_\beta = g_{\alpha\beta}+h_{\alpha\beta}v, \tag{1.2a}$$

$$D_\alpha v = -h_{\alpha\beta}v^\beta, \tag{1.2b}$$

where D_α is the covariant differentiation operator with respect to the Riemannian connection for the induced Riemannian metric tensor $g_{\alpha\beta}$, $v_\beta=g_{\alpha\beta}v^\alpha$, $g_{\alpha\beta}(u)=$

Received by the editors September 18, 1974 and, in revised form, January 29, 1975.

[1] This work was supported in part by the National Research Council of Canada under grant #A8788 and in part be the Science University of Tokyo.

[2] Permanent Address: Department of Mathematics, Science University of Tokyo, Tokyo, Japan.

[3] Latin indices run from 1 to n and Greek indices run from 1 to $n-1$.

$g_{ij}(x_0, X(u))B^i_\alpha B^j_\beta$ and $h_{\alpha\beta}$ is the second fundamental tensor of M^{n-1}.

Watanabe [3] proved the following result.

PROPOSITION 1.1. *Let M^{n-1} be a closed hypersurface in V^n, H_1 the first mean curvature of M^{n-1} and v the scalar function on M^{n-1} as defined by* (1.1). *If M^{n-1} satisfies the inequality $1+H_1v\geq 0$, then M^{n-1} is homothetic to the indicatrix $F(x_0, X)=1$ in $T_{x_0}(M^n)$.*

The inequality $1+H_1v\geq 0$ in the above proposition can also be replaced by $1+H_1v\leq 0$. (For more details see the proof of Theorem 2.1 in [3]).

The object of this paper is to discuss the properties of the indicatrix in $T_{x_0}(M^n)$ in some what more details.

2. **Integrability conditions.** In this section we will obtain the integrability conditions for the equations (1.2). Since $g_{ij}(X)$ is positively homogeneous of degree zero with respect to X in V^n hence C_{ijk} as defined by

(2.1) $$C_{ijk} = \tfrac{1}{2}\partial g_{ij}/\partial X^k = \tfrac{1}{4}\partial^3 F^2/\partial X^i \partial X^j\, \partial X^k$$

satisfies the conditions ([2], p. 15)

(2.2) $$C_{ijk}X^i = C_{ijk}X^j = C_{ijk}X^k = 0.$$

It can be shown [1] that in V^n the Riemannian connection $\{^{\,i}_{jk}\}$ and the Riemannian curvature tensor

$$R_{ijk}{}^l = \partial\begin{Bmatrix} l \\ jk \end{Bmatrix}\Big/\partial x^i - \partial\begin{Bmatrix} l \\ ik \end{Bmatrix}\Big/\partial x^j + \begin{Bmatrix} l \\ ir \end{Bmatrix}\begin{Bmatrix} r \\ jk \end{Bmatrix} - \begin{Bmatrix} l \\ jr \end{Bmatrix}\begin{Bmatrix} r \\ ik \end{Bmatrix}$$

based on $\{^{\,i}_{jk}\}$ have the following form

(2.3a) $$\begin{Bmatrix} i \\ jk \end{Bmatrix} = g^{ih}C_{hjk} \overset{def}{=} C^i_{jk},$$

(2.3b) $$R_{ijk}{}^l = C^l_{hj}C^h_{ki} - C^l_{hi}C^h_{kj}.$$

Accordingly, we can write

(2.4) $$R_{ijkl}X^i = R_{ijkl}X^j = R_{ijkl}X^k = R_{ijkl}X^l = 0,$$

where

(2.5) $$R_{ijkl} = g_{lh}R_{ijk}{}^h.$$

We have now the following theorem concerning the integrability conditions of (1.2).

THEOREM 2.1. *Let $M^{n-1}: X^i = X^i(u^\alpha)$ denote the hypersurface in V^n, then*

(2.6) $$[R_{\alpha\beta\gamma\delta} + h_{\alpha\gamma}h_{\beta\delta} - h_{\beta\gamma}h_{\alpha\delta}]v^\delta = [D_\beta h_{\alpha\gamma} - D_\alpha h_{\beta\gamma}]v,$$

(2.7) $$[D_\beta h_{\alpha\gamma} - D_\alpha h_{\beta\gamma}]v^\gamma = 0,$$

where $R_{\alpha\beta\gamma\delta}g^{\delta\varepsilon} = R_{\alpha\beta\gamma}{}^\varepsilon$ is the Riemannian curvature tensor of M^{n-1}.

https://doi.org/10.4153/CMB-1975-125-0 Published online by Cambridge University Press

Proof. On substituting (1.2) into the Ricci-identities for v and v_β respectively we obtain (2.6) and (2.7) after a little simplification.

Alternative Proof. This proof clarifies the relation between the equations of Gauss and Codazzi for M^{n-1} and the equation (2.6) and (2.7).

The equations of Gauss and Codazzi are given by

$$R_{\alpha\beta\gamma\delta} = R_{ijkl}B^i_\alpha B^j_\beta B^k_\gamma B^l_\delta + h_{\alpha\delta}h_{\beta\gamma} - h_{\beta\delta}h_{\alpha\gamma}, \tag{2.8}$$

$$R_{ijkl}B^i_\alpha B^j_\beta B^k_\gamma N^l = D_\alpha h_{\beta\gamma} - D_\beta h_{\alpha\gamma} \tag{2.9}$$

respectively. Now from (1.1) and (2.4), we have for M^{n-1},

$$\begin{aligned} 0 &= R_{ijkl}B^i_\alpha B^j_\beta B^k_\gamma X^l \\ &= R_{ijkl}B^i_\alpha B^j_\beta B^k_\gamma (B^l_\delta v^\delta + N^l v). \end{aligned} \tag{2.10}$$

On substituting (2.8) and (2.9) in (2.10) we obtain (2.6).

Also from (2.4) we can write

$$\begin{aligned} 0 &= R_{ijkl}B^i_\alpha B^j_\beta X^k N^l \\ &= R_{ijkl}B^i_\alpha B^j_\beta (B^k_\gamma v^\gamma + N^k v) N^l \\ &= R_{ijkl}B^i_\alpha B^j_\beta B^k_\gamma N^l v^\gamma. \end{aligned} \tag{2.11}$$

On substituting (2.9) in (2.11) we obtain (2.7).

3. **Hypersurfaces homothetic to the indicatrix in $T_{x_0}(M^n)$.** First of all we shall prove the converse of the proposition 1.1.

THEOREM 3.1. *Let $M^{n-1}: X^i = X^i(u^\alpha)$, be a hypersurface homothetic to the indicatrix in $T_{x_0}(M^n)$. Then M^{n-1} satisfies the condition* $1 + H_1 v = 0$.

Proof. Since M^{n-1} is homothetic to the indicatrix in $T_{x_0}(M^n)$ it is obviously closed and

$$g_{ij}(X(u))X^i(u)X^j(u) = c \qquad (= \text{constant} > 0). \tag{3.1}$$

On differentiating this equation with respect to u^α and using (2.2) we obtain $g_{ij}(X(u))B^i_\alpha X^j(u) = 0$ and hence from (1.1) we have $v_\alpha = 0$ and consequently $X^i(u) = v(u)N^i$. Now from (1.2) we can write $g_{\alpha\beta} + h_{\alpha\beta}v(u) = 0$.

The theorem now follows after contracting with $g_{\alpha\beta}$.

It should be noted that under the assumptions of the theorem 3.1, the hypersurface M^{n-1} is totally umbilical [2], $v = \text{constant} \neq 0$ and $H_1 = -1/v$.

THEOREM 3.2. *Let $M^{n-1}: X^i = X^i(u^\alpha)$ be a closed hypersurface in V^n. Then M^{n-1} is homothetic to the indicatrix in $T_{x_0}(M^n)$ if and only if $D_\alpha v_\beta = 0$ (or $v_\beta = 0$).*

Proof. If we assume that $D_\alpha v_\beta = 0$, then from (1.2a) we have $g_{\alpha\beta} + h_{\alpha\beta}v = 0$, and hence $1 + H_1 v = 0$. Now using proposition 1.1 we find that M^{n-1} is homothetic to the indicatrix in $T_{x_0}(M^n)$. The converse follows from the proof of theorem 3.1. The fact that the conditions $D_\alpha v_\beta = 0$ and $v_\beta = 0$ are equivalent is easy to verify.

THEOREM 3.3. *Let* $M^{n-1}:X^i=X^i(u^\alpha)$ *be a closed hypersurface in* V^n. *Then* M^{n-1} *is homothetic to the indicatrix in* $T_{x_0}(M^n)$ *if and only if*

$$1+2H_1v+(n-1)H_1^2v-(n-2)H_2v^2 = 0, \tag{3.2}$$

where H_2 *is the second mean curvature of* M^{n-1}.

Proof. From (1.2) we can write

$$\begin{aligned} g^{\alpha\gamma}g^{\beta\delta}(D_\gamma v_\delta)(D_\alpha v_\beta) &= (D^\alpha v^\beta)(D_\alpha v_\beta) \\ &= (g^{\alpha\beta}+h^{\alpha\beta}v)(g_{\alpha\beta}+h_{\alpha\beta}v) \\ &= 1+2H_1v+(n-1)H_1^2v-(n-2)H_2v^2. \end{aligned}$$

The result (3.2) now follows from theorem 3.2.

Next we shall consider totally umbilical hypersurfaces in V^n. In this case we have from (1.2),

$$(D_\alpha v_\beta+D_\beta v_\alpha) = 2(1+H_1v)g_{\alpha\beta}. \tag{3.3}$$

Now using theorem 3.1 and proposition 1.1 we can conclude that the vector v^α is a killing vector if and only if M^{n-1} is homothetic to the indicatrix in $T_{x_0}(M^n)$.

In the following paragraphs we shall confine ourselves to totally umbilical hypersurfaces with constant first mean curvature. Such a hypersurface may not necessarily be the hypersurface homothetic to the indicatrix in $T_{x_0}(M^n)$. For example, in the case where $g_{ij}=\delta_{ij}$ the hypersurfaces are hyperspheres. But the hyperspheres may not necessarily be homothetic to the indicatrix (the unit hypersphere with the origin as the centre) in $T_{x_0}(M^n)$. This is due to the fact that the centre of a hypersphere need not coincide with the origin of $T_{x_0}(M^n)$.

THEOREM 3.4. *Let* M^{n-1} *be a closed totally umbilical hypersurface with the constant first mean curvature* H_1 *in* V^n, *then* $H_1\neq 0$.

Proof. Let us assume that $H_1=0$. Then $h_{\alpha\beta}=0$ and from (1.2b) we have $D_\alpha v=0$. Hence $v=$constant and accordingly $1+H_1v=$constant. In this case, we have $1+{}_1Hv\geq 0$ or $1+H_1v\leq 0$. Now by proposition 1.1 this hypersurface M^{n-1} must be homothetic to the indicatrix in $T_{x_0}(M^n)$ and hence by theorem 3.1 we must have $1+H_1v=0$. But this is a contradiction. Hence, the result follows.

THEOREM 3.5. *Let* M^{n-1} *be a closed and totally umbilical hypersurface with the constant first mean curvature in* V^n. *If* M^{n-1} *satisfies the condition* $R_{\alpha\beta}v^\alpha v^\beta\leq 0$, *then* M^{n-1} *is homothetic to the indicatrix in* $T_{x_0}(M^n)$, *where* $n\geq 3$ *and* $R_{\alpha\beta}$ *is the Ricci tensor of* M^{n-1}.

Proof. Since M^{n-1} is totally umbilical, we have $h_{\alpha\beta}=H_1g_{\alpha\beta}$. Substituting this relation into (2.6) we obtain

$$[R_{\alpha\beta\gamma\delta}+H_1^2(g_{\alpha\gamma}g_{\beta\delta}-g_{\beta\gamma}g_{\alpha\delta})]v^\delta = 0. \tag{3.4}$$

On contracting (3.4) with $g^{\alpha\gamma}v^{\beta}$ we obtain

$$R_{\alpha\beta}v^{\alpha}v^{\beta} = (n-2)H_1^2 g_{\alpha\beta}v^{\alpha}v^{\beta} \geq 0. \tag{3.5}$$

Now according to the assumption $R_{\alpha\beta}v^{\alpha}v^{\beta}\leq 0$ and theorem 3.4 it is clear that $v^{\alpha}=0$. Thus by theorem 3.2 the hypersurface M^{n-1} is homothetic to the indicatrix in $T_{x_0}(M^n)$. (See also [4], Theorem 6.1, p. 46.)

THEOREM 3.6. *Let M^{n-1} be a closed totally umbilical hypersurface with the constant first mean curvature H_1 in V^n. Then M^{n-1} is homothetic to the indicatrix in $T_{x_0}(M^n)$ if and only if v=constant.*

Proof. Let M^{n-1} be homothetic to the indicatrix in $T_{x_0}(M^n)$. Then by theorem 3.1 we have $1+H_1v=0$ and hence v=constant. Conversely if v=constant then we should have $1+H_1v\geq 0$ or $1+H_1v\leq 0$. The hypersurface M^{n-1} is clearly homothetic to the indicatrix in $T_{x_0}(M^n)$ by proposition 1.1.

4. **Formulas and its applications.** We now mention an important result which will be used in the proceeding paragraph.

PROPOSITION 4.1. *Let M^{n-1} be a totally umbilical closed hypersurface with the constant first mean curvature in V^{n-1}. Then*

$$g^{\alpha\beta}D_{\alpha}D_{\beta}(1+H_1v)^m = mH_1^2(1+H_1v)^{m-2}[(m-1)H_1^2w-(n-1)(1+H_1v)^2], \tag{4.1}$$

$$g^{\alpha\beta}D_{\alpha}D_{\beta}w^m = 2mw^{m-1}[(2m+n-3)(1+H_1v)^2-H_1^2w], \tag{4.2}$$

where $m=1, 2, 3, \ldots$, and $w=g_{\alpha\beta}v^{\alpha}v^{\beta}$.

Proof. Using the conditions of the proposition and (1.2) the results (4.1) and (4.2) can be obtained after some calculations.

For the sake of brevity we will denote a totally umbilical closed hypersurface with the constant first mean curvature H_1 in V^n by $*M^{n-1}$.

THEOREM 4.1. *If $*M^{n-1}$ satisfies*

$$kH_1^2w \geq (1+H_1v)^2, \tag{4.3}$$

then $*M^{n-1}$ is homothetic to the indicatrix in $T_{x_0}(M^n)$, k being a positive number.

Proof. Let us choose an ever number m such that $m-1\geq(n-1)k$. Then from (4.1) we can write $g^{\alpha\beta}D_{\alpha}D_{\beta}(1+H_1v)^m\geq 0$. Thus by the Bochner's lemma [4, p. 39] we have $1+H_1v$=constant and by proposition 1.1 $*M^{n-1}$ is homothetic to the indicatrix in $T_{x_0}(M^n)$.

THEOREM 4.2. *For the hypersurface* M^{n-1} we have*

$$(1+H_1v)^2+H_1^2w = c, \tag{4.4}$$

https://doi.org/10.4153/CMB-1975-125-0 Published online by Cambridge University Press

where the constant c has the following properties;

(i) $c=0$ *implies that* $^*M^{n-1}$ *is homothetic to the indicatrix in* $T_{x_0}(M^n)$.

(ii) $v\neq 0$ *if and only if* $c<1$.

(iii) *There exists a point P on* $^*M^{n-1}$ *at which* $v=0$ *if and only if* $c>1$.

Proof. From (4.1) and (4.2) we can write

$$g^{\alpha\beta}D_\alpha D_\beta(1+H_1v)^2 = 2H_1^2[H_1^2w-(n-1)(1+H_1v)^2], \tag{4.5}$$

$$g^{\alpha\beta}D_\alpha D_\beta H_1^2 w = 2H_1^2[(n-1)(1+H_1v)^2-H_1^2w]. \tag{4.6}$$

On adding (4.5) and (4.6) we obtain

$$g^{\alpha\beta}D_\alpha D_\beta[(1+H_1v)^2+H_1^2w] \equiv 0.$$

Now using the Bochner's lemma the result (4.4) follows.

To prove (i) use proposition 1.1 in (4.4) and note $w\geq 0$.

To prove (ii) let $v\neq 0$. Since $^*M^{n-1}$ is a totally umbilical hypersurface with the constant first mean curvature H_1 we may use the relation $h_{\alpha\beta}=H_1g_{\alpha\beta}$ and write (1.2) in the form

$$D_\alpha v_\beta = g_{\alpha\beta}(1+H_1v), \qquad D_\alpha v = -H_1v_\alpha. \tag{4.7}$$

Now using (4.7) it is easy to verify that

$$D_\beta v^m = -mv^{m-1}H_1v^\beta$$

and

$$D_\alpha D_\beta v^m = m(m-1)v^{m-2}H_1^2v_\alpha v_\beta - mv^{m-1}H_1(1+H_1v)g_{\alpha\beta}. \tag{4.8}$$

On contracting (4.8) with $g^{\alpha\beta}$, using $w=g_{\alpha\beta}v^\alpha v^\beta=g^{\alpha\beta}v_\alpha v_\beta$, (4.4) and setting $m=-(n-2)$ we obtain after a little simplification

$$g^{\alpha\beta}D_\alpha D_\beta v^{-(n-2)} = -(n-2)v^{-n}(n-1)[1+H_1v-c]. \tag{4.9}$$

Also from (4.4) it is obvious that $(1+H_1v)^2\leq c$ or $-\sqrt{c}\leq 1+H_1v\leq\sqrt{c}$. If we assume $c\geq 1$ then $\sqrt{c}\leq c$ and we have $1+H_1v\leq c$. Now from (4.9) and the Bochner's lemma we conclude that $v=$constant. From theorem 3.6 it is clear that $^*M^{n-1}$ is homothetic to the indicatrix in $T_{x_0}(M^n)$. Now using theorems 3.1 and 3.2 in (4.4) we find $c=0$ which is a contradiction. Thus $c<1$. To prove the converse let $c<1$. If there exists a point P on $^*M^{n-1}$ at which $v=0$ then from (4.4) we must have $1+H_1^2w=c$ at P and consequently $c\geq 1$ which is a contradiction. Hence we conclude that there is no point P on $^*M^{n-1}$ at which $v=0$.

To prove (iii) observe that if there exists a point P on $^*M^{n-1}$ at which $v=0$ then from (4.4) we have $1+H_1^2w=c$ at P. But $H_1\neq 0$ by theorem 3.4 and also $w\neq 0$ (if $v=0$ and $w=0$ at the same time then from (1.1) $X^n(u)=0$ at P which is not possible). Hence $c>1$. To prove the converse observe that the second part of this problem suggests that if $c\geq 1$ then there exists at least one point P on $^*M^{n-1}$ at which $v=0$. We claim that if $c=1$ then there exists no point on the hypersurface with $v=0$. Because with $c=1$ and $v=0$ and the result (4.4) we have $H_1^2w=0$. But by theorem 3.4, $H_1\neq 0$ hence we conclude $w=0$. Since $v=0$ and $w=0$ implies

$X^n(u)=0$ which is absurd, hence we arrive at a contradiction. Thus if $c>1$ then there exists a point P on $^*M^{n-1}$ at which $v=0$.

THEOREM 4.3. *If* $^*M^{n-1}$ *satisfies*

$$k(1+H_1v)^2 \geq H_1^2 w, \tag{4.10}$$

then $^*M^{n-1}$ *is homothetic to the indicatrix in* $T_{x_0}(M^n)$, *k being a positive number.*

Proof. Let us choose a number m such that $(2m+n-3)\geq k$. Then from (4.2) we have $w=$constant. Thus from (4.4) we have $1+H_1v=$constant and hence $^*M^{n-1}$ is homothetic to the indicatrix in $T_{x_0}(M^n)$.

REFERENCES

1. S. Kikuchi, *Theory of Minkowski space and of non-linear connections in Finsler space*, Tensor N.S., **12** (1962), 47–60.
2. H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag (1959).
3. S. Watanabe, *On indicatrices of a Finsler space*, Tensor N.S., **27** (1973), 135–137.
4. K. Yano, *Integral formulas in Riemannian geometry*, Marcel Dekker (1970).

DEPARTMENT OF MATHEMATICS,
CONCORDIA UNIVERSITY (LOYOLA CAMPUS),
7141, SHERBROOKE STREET WEST,
MONTREAL, QUEBEC, CANADA

https://doi.org/10.4153/CMB-1975-125-0 Published online by Cambridge University Press