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A FLUID EOQ MODEL WITH A TWO-STATE RANDOM ENVIRONMENT

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We study a stochastic fluid EOQ-type model operating in a Markovian random environment of alternating good and bad periods determining the demand rate. We deal with the classical problem of "when to place an order" and "how big it should be," leading to the trade-off between the setup cost and the holding cost. The key functionals are the steady-state mean of the content level, the expected cycle length (which is the time between two large orders), and the expected number of orders in a cycle. These performance measures are derived in closed form by using the level crossing approach in an intricate way. We also present numerical examples and carry out a sensitivity analysis.

1. INTRODUCTION

The simple economic order quantity (EOQ) model is the most fundamental of all inventory models (see, e.g., [15]). It lucidly describes the trade-off between the

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constant setup cost and the variable holding cost. Its formula still serves as an effective approximation for much more complicated EOQ-type models. Although the simple EOE (SEOQ) is a deterministic model with constant demand rate, most of its ramifications include added factors of randomness. For these stochastic EOQ models, three types of approximation are generally used:

1. In *jump models* (see, e.g., [8,13,14]) discrete amounts of input and output enter and leave the buffer one by one or in small batches; the content level processes generated in this case are step functions.
2. In *heavy-traffic models* [3,4], both the gross input and the gross output are assumed to be very large (approaching infinity) over any time interval, with the ratio of input rate and output rate being close to unity and their difference being almost constant. Generally, the buffer content processes include Brownian components.
3. *Fluid models* [5,6] are characterized by large amounts of inflow and outflow over any time interval, with the ratio of input rate and output rate not equal to unity; the stochastic fluctuations of the input and the output streams are replaced by their local drifts, which might also depend on external factors changing randomly over time.

In this article we introduce a fluid EOQ (FEOQ) model with a two-state random environment that is modeled as a continuous-time Markov chain alternating between a *good* and a *bad* state. The net change rate of the system is assumed to be constantly equal to a in the good state and to b in the bad state. It can be assumed that $a > b$, which is intuitive since the sales during good periods are higher than those during bad ones, but this is not necessary for the analysis. Good and bad periods follow each other according to an alternating renewal process. In this article we restrict attention to the Markovian case of independent and exponentially distributed random variables with parameters λ and μ, respectively.

The two-state model reflects situations in which the demand rate for a certain commodity undergoes periodically recurring changes. For example, the demand for beef went down dramatically after the occurrence of BSE (mad cow disease) cases and then after some time of alertness, it swung back to its normal rate. Similar phenomena connected to diseases or health consciousness (e.g., popular diet plans) can be observed for poultry, pork, and other foodstuffs. The sales of furs are periodically influenced by campaigns of animal rights groups, and those of mineral water and canned food are influenced by terror alerts. Indeed, demand rates for many goods go up and down between different levels due to fashion or other recurring external effects. Models including a multistate Markovian environment can be suitable for such situations; the two-state case presented in this article could serve as a first approximation. We denote by $\mathbf{X} = \{X(t) : t \geq 0\}$ the content level process of the inventory operating under the alternating good and bad states. $\mathbf{X}$ decreases linearly at rates a or b depending on the state of the environment. Once the system becomes empty, a replenishment order, with a negligible lead time, of a certain controllable size is placed. Note that as demand is assumed to arrive in infinitesimal portions at rate a or b and the lead times are zero, there is no backlogging. Let $\mathbf{I} = \{I(t) : t \geq 0\}$ be the *status process* of the environment; that is, $I(t) = 1$ (0) if at time t the state of the environment is good (bad). The two-dimensional process $(\mathbf{X}, \mathbf{I})$ is Markovian and the content level process $\mathbf{X}$ alone is regenerative. There are several options for defining regeneration points for it. The cycle structure we will use is given by the periods between replenishments occurring in good periods. Let $(X(0), I(0)) = (q_g, 1)$ (i.e., $\mathbf{X}$ starts at level q_g in the good state). We define T to be the time of the first replenishment taking place in a good period; then $[0, T)$ is the first cycle. One expects $\mathbf{X}$ to proceed swiftly toward zero during good periods, whereas during bad periods, it will probably decrease more sluggishly. If this is the case, the marginal revenue is higher during good periods than during bad ones. Whenever the content level process hits zero at good (bad) periods, the controller places an order of size q_g (q_b), where q_g and q_b are controllable parameters. Note that the successive order sizes form a random sequence of values in $\{q_g, q_b\}$, determined in accordance with the random environment. However, in contrast to *random yield* models [10,11] in which the random order sizes are determined externally ("by nature"), in this model they are selected optimally.

In the monograph [17] a large variety of inventory models is presented in detail. The stochastic models are based on point processes for the demand arrivals in random environments (in the book called "world-driven"). The fluid systems in [17] are deterministic EOQ models with the classical extensions such as planned backorders, limited capacity, quantity discounts, and imperfect quality. In the deterministic setting, time-varying demands are considered also, however without multiple-order quantities. The stochastic FEOQ model with more than one order quantity expounded in this article seems to be new.

The controller's objective is to maximize the long-run average revenue by selecting q_g and q_b so as to properly balance between the setup cost K, due each time an order is placed, and the proportional holding cost h per unit time and per unit of stored items. Let p and c be the sale price and the purchase price of one unit, respectively. The long-run average holding cost is hEX, where X is a random variable having the steady-state distribution of $\mathbf{X}$. (Note that $X(t) \to X$ in distribution, by the limit theorem for regenerative processes [1, p. 170], and that $0 \leq X(t) \leq \max[q_g, q_b]$ so that also $EX(t) \to EX$.) Let N be the number of replenishment orders in one cycle (say, the first one) that are issued while the environment is in the bad state. Then the number of orders in this cycle is $N + 1$ and it is easily seen that N has a geometric distribution. The expected setup cost in a cycle is $KE(N+1)$; the mean cycle length is, of course, ET. The expected total sold output during a cycle is $q_g + q_b EN$. Combining all of these functionals, it follows from the renewal reward theorem [16] that the long-run average profit is given by

$$R(q_g, q_b) = (p - c)\,\frac{q_g + q_b EN}{ET} - hEX - \frac{KE(N+1)}{ET}. \tag{1.1}$$

Of course, the expected values EN, EX, and ET are all functions of q_g and q_b.

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Because the long-run average income generated by the sales depends only on the proportions of times in good and bad periods, it is independent of the order sizes, implying that the maximization of $R(q_g,q_b)$ is equivalent to the minimization of the cost functional

$$C(q_g,q_b) = hEX + \frac{KE(N+1)}{ET}. \tag{1.2}$$

It is easy to see that the long-run average profit and the long-run average cost are related by

$$R(q_g,q_b) = (p-c)\left[a\,\frac{\mu}{\lambda+\mu} + b\,\frac{\lambda}{\lambda+\mu}\right] - C(q_g,q_b). \tag{1.3}$$

An intuitive conjecture is that if $a > b$, the optimal (long-run average cost minimizing) replenishment levels q_g^* and q_b^* satisfy $q_g^* > q_b^*$. The numerical analysis in Section 4 confirms this expectation.

In order to maximize (1.1) (or minimize (1.2)) we first need to compute the functionals *EN*, *EX*, and *ET*. To this end, we use an extension of the so-called level crossing technique (see, e.g., [2,4,6]). In our context, this approach turns out to be more intricate than in most other applications because the exact calculation of the upcrossing rates requires some refined arguments.

The level crossing theory (LCT) was introduced in [7] for regenerative dam processes of the $GI/G/1$ type and generalized in [9] to stationary dam processes. The general approach makes it possible to construct Khintchine–Pollaczek formulas for dam processes by equating the long-run average number of upcrossings and downcrossings of an arbitrary level. It is based on the idea that the long-run average of upcrossings (or downcrossings) of level x gives the value of the steady-state density of the dam process at x multiplied by the local release rule of the dam. In [7,9] and the applications following these two pioneer studies, attention is restricted to the case that the release rule of the dam is a deterministic function. In the model considered here, the demand rate is not state dependent but stochastically changing according to the Markov environment, which makes a rigorous analysis more complicated.

The article is organized as follows. In Section 2 the dynamics of the FEOQ model are described in a formal manner. In Section 3 all relevant functionals are derived in closed form from a steady-state analysis of the model. Based on these explicit results, Section 4 provides numerical examples in which we consider, in particular, the sensitivity of the optimal ordering policies with respect to the model parameters.

2. MODEL DYNAMICS

Let $U_1, V_1, U_2, V_2, \ldots$ be the lengths of the alternating time periods in which the content decreases at rates a and b, respectively. For the analysis, we do not need the assumption $a > b$. We assume that $(U_1, U_2, \dots)$ and $(V_1, V_2, \dots)$ are two independent sequences of independent and identically distributed (i.i.d.) random variables, distributed according to $\exp(\lambda)$ and $\exp(\mu)$, respectively. The two threshold values q_g and q_b are positive numbers satisfying $q_g \geq q_b$. Let $Y(t)$ be the *accumulated output* at time $t \geq 0$ and set $\mathbf{Y} = \{Y(t) : t \geq 0\}$. Let $S_n = \sum_{i=1}^{n} U_i$ and $S'_n = \sum_{i=1}^{n} V_i$ for $n \geq 1$ and $S_0 = S'_0 = 0$. In terms of these sequences, $Y(t)$ is defined by

$$Y(t) = \begin{cases} aS_n + bS'_n + a(t - S_n - S'_n) & \text{if } S_n + S'_n \leq t < S_{n+1} + S'_n \text{ for some } n \geq 0 \\ aS_{n+1} + bS'_n + b(t - S_{n+1} - S'_n) & \text{if } S_{n+1} + S'_n \leq t < S_{n+1} + S'_{n+1} \text{ for some } n \geq 0. \end{cases} \tag{2.1}$$

The sample paths of $\mathbf{Y}$ increase strictly and continuously to infinity. The *status process* $\mathbf{I} = \{I(t) : t \geq 0\}$ is given by

$$I(t) = \begin{cases} 1 & \text{if } S_n + S'_n \leq t < S_{n+1} + S'_n \text{ for some } n \geq 0 \\ 0 & \text{if } S_{n+1} + S'_n \leq t < S_{n+1} + S'_{n+1} \text{ for some } n \geq 0. \end{cases}$$

The sample paths of $\mathbf{Y}$ increase strictly and continuously to infinity. Now we define recursively a sequence of level hitting times $\tau_1 < \tau_2 < \cdots \nearrow \infty$ and the *clearing process* $\mathbf{W} = \{W(t) : t \geq 0\}$. Let $\tau_1 = \inf\{t > 0 | Y(t) = q_g\}$ and $W(t) = Y(t)$ for $t \in [0, \tau_1)$. If τ_n and $W(t)$, $t \in [0, \tau_n)$, have already been defined, let

$$\tau_{n+1} = \begin{cases} \inf\{t > \tau_n | Y(t) - Y(\tau_n) = q_g\} & \text{if } I(\tau_n) = 1 \\ \inf\{t > \tau_n | Y(t) - Y(\tau_n) = q_b\} & \text{if } I(\tau_n) = 0 \end{cases}$$

and set

$$W(t) = (q_g - q_b) 1_{\{I(\tau_n) = 0\}} + Y(t) - Y(\tau_n), \qquad t \in [\tau_n, \tau_{n+1}).$$

By this construction, $W(t)$ is defined for all $t \geq 0$. The *content level process* $\mathbf{X} = \{X(t) : t \geq 0\}$ is given by

$$X(t) = q_g - W(t).$$

We will show that $\mathbf{W}$, and thus $\mathbf{X}$, has an absolutely continuous stationary distribution. The associated densities are denoted by f_X and f_W, respectively. Clearly,

$$f_W(x) = f_X(q_g - x)$$

and

$$EW = q_g - EX.$$

The processes $\mathbf{W}$ and $\mathbf{X}$ are regenerative. Typical realizations are depicted in Figures 1a and 1b. $\mathbf{W}$ starts to increase at time 0 from $W(0) = 0$ at slope a. Slopes alternate between a and b. Each time level q_g is reached by $\mathbf{W}$. The process has a negative jump leading to 0 if the current slope is a or to $q_g - q_b$ if it is b. Cycles are

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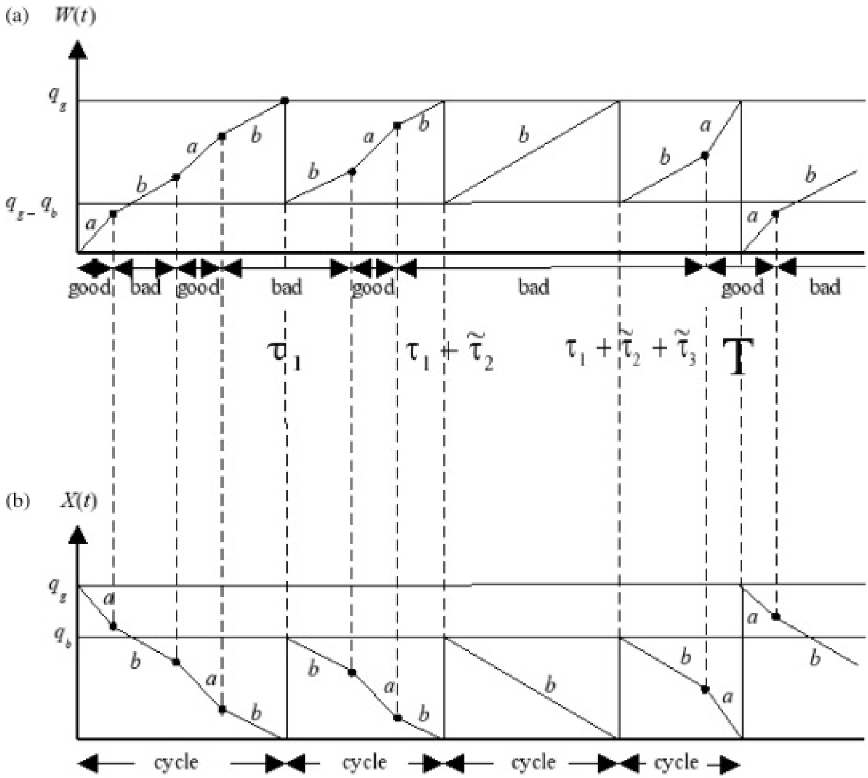

FIGURE 1. Typical realizations of W and X.

defined to be the intervals between jumps to zero. Let N be the number of jumps to $q_g - q_b$ before the first return to zero. Then $T = \tau_{N+1}$ is that value τ_n for which $I(\tau_n) = 1$ and $I(\tau_j) = 0$ for $j < n$. $[0, T)$ is the first cycle of **W**. The time intervals $[0, \tau_1), [\tau_1, \tau_2), \ldots, [\tau_n, \tau_{n+1})$ are called subcycles.

Let $\theta_1(x)$ be the probability that level x is upcrossed by **Y** while growing at rate a; similarly, let $\theta_0(x)$ be the probability that x is upcrossed by $\{Y(t - U_1) - aU_1 : t \geq U_1\}$ while the process grows at rate a. Note that $\{Y(t - U_1) - aU_1 : t \geq U_1\}$ is the accumulated output process starting in the bad state (i.e., growing at rate b). These probabilities will appear throughout our derivations.

The following facts are consequences of the strong Markov property of **W**. For any $n \geq 1$, the subcycle lengths $\tau_1, \tau_2 - \tau_1, \ldots, \tau_{n+1} - \tau_n$ are conditionally independent given that $N = n$. Moreover, $\tau_2 - \tau_1, \ldots, \tau_n - \tau_{n-1}$ are conditionally i.i.d. given that $N = n$, provided $n > 1$. The number $N + 1$ of subcycles has a modified geometric distribution with parameter $\theta_0(q_b)$: We have $P(N = 0) = \theta_1(q_g)$ and

$$P(N = n) = (1 - \theta_1(q_g))(1 - \theta_0(q_b))^{n-1}\theta_0(q_b), \qquad n = 1, 2, \ldots.$$

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3. STEADY-STATE ANALYSIS

In this section we derive ET and

$$EX = \int_0^{q_g} (q_g - w) f_W(w)\, dw.$$

For both, we need explicit formulas for $\theta_1(x)$ and $\theta_0(x)$.

LEMMA 1:

$$\theta_1(x) = \frac{\mu_b + \lambda_a e^{-(\lambda_a+\mu_b)x}}{\lambda_a + \mu_b} \tag{3.1}$$

and

$$\theta_0(x) = \frac{\mu_b}{\lambda_a + \mu_b}\left[1 - e^{-(\lambda_a+\mu_b)x}\right], \tag{3.2}$$

where $\lambda_a = \lambda/a$ *and* $\mu_b = \mu/b$.

PROOF. Consider the process $\mathbf{Z} = \{Z(t) : t \geq 0\}$ obtained from $\mathbf{Y}$ by replacing its linear pieces of slope b by independent $\exp(\mu_b)$-distributed upward jumps. Formally, let

$$Z(t) = at + bS'_{N(t)},$$

where $N(t) = \max\{n \geq 0 \mid S_n \leq t\}$. $\mathbf{Z}$ is the sum of a compound Poisson process and the linear function at and thus has the exponent

$$\varphi(\alpha) := \log Ee^{-\alpha Z(1)} = -\alpha a - \frac{\lambda\alpha}{\mu_b + \alpha}. \tag{3.3}$$

It is well known that the process $\mathbf{M} = \{M(t) : t \geq 0\}$ defined by

$$M(t) = \varphi(\alpha) \int_0^t e^{-\alpha Z(s)}\, ds + 1 - e^{-\alpha Z(t)} \tag{3.4}$$

is a martingale (see [12]). Fix $x > 0$ and define the stopping time

$$T_x = \inf\{t > 0 : Z(t) \geq x\}.$$

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Applying the martingale stopping theorem to $\mathbf{M}$ and T_x, we find that

$$\begin{aligned}\varphi(\alpha)E\left(\int_0^{T_x} e^{-\alpha Z(s)}\,ds\right) &= -1 + E(e^{-\alpha Z(T_x)}) \\ &= -1 + E(e^{-\alpha Z(T_x)}1_{\{Z(T_x)=x\}}) + E(e^{-\alpha Z(T_x)}1_{\{Z(T_x)>x\}}) \\ &= -1 + e^{-\alpha x}E(1_{\{Z(T_x)=x\}}) + e^{-\alpha x}\frac{\mu_b}{\mu_b+\alpha}E(1_{\{Z(T_x)>x\}}),\end{aligned} \tag{3.5}$$

where the third step of (3.5) follows from the lack-of-memory property of the jump size distribution: Given the event $\{Z(T_x) > x\}$, the random variable $Z(T_x) - x$ is $\exp(\mu_b)$ distributed for any x so that the conditional Laplace transform of $Z(T_x)$ is $e^{-\alpha x}\mu_b/(\mu_b + \alpha)$.

The right-hand side of (3.3) has the two roots 0 and

$$\alpha^* = -(\lambda_a + \mu_b). \tag{3.6}$$

If we can set $\alpha = \alpha^*$ in (3.5), we get

$$1 = e^{(\lambda_a+\mu_b)x}E(1_{\{Z(T_x)=x\}}) + e^{(\lambda_a+\mu_b)x}\frac{\mu_b}{\mu_b-(\lambda_a+\mu_b)}E(1_{\{Z(T_x)>x\}}). \tag{3.7}$$

However, the event $\{Z(T_x) = x\}$ is equivalent to the event that Y upcrosses level x during a good period. This means that

$$E(1_{\{Z(T_x)=x\}}) = \theta_1(x).$$

Solving for $\theta_1(x)$ in (3.7), we obtain (3.1).

There is a difficulty with inserting the root α^* in (3.5): $\alpha^* < -\mu_b$ is negative whereas the Laplace transforms of $Z(1)$ and of the $\exp(\mu_b)$-distribution are only defined for nonnegative arguments α and the defining integrals are not finite for $\alpha < -\mu_b$. This problem can be solved as follows. Being the logarithm of a Laplace transform, $\varphi(\alpha)$ is only defined for $\alpha \in \mathbb{R}_+$. However, $\varphi(\alpha)$ can be analytically extended to $\mathbb{C}\backslash\{-\mu_b\}$ by identity (3.3), taking the right-hand side of (3.3) as definition of φ. Next, the integral $\int_0^{T_x} e^{-\alpha Z(s)}\,ds$ is easily seen to be an analytic function of α for all $\alpha \in \mathbb{C}$; note that $0 \le Z(s) < x$ for $s \in [0, T_x)$. Now let

$$G(\alpha) = (\mu_b+\alpha)\left[-\alpha a - \frac{\lambda\alpha}{\mu_b+\alpha}\right]E\left(\int_0^{T_x} e^{-\alpha Z(s)}\,ds\right)$$

$$H(\alpha) = (\mu_b+\alpha)[e^{-\alpha x}E(1_{\{Z(T_x)=x\}}) - 1] + e^{-\alpha x}\mu_b E(1_{\{Z(T_x)>x\}}).$$

Then G is analytic on $\mathbb{C}\backslash\{-\mu_b\}$, H is analytic on $\mathbb{C}$, and, by (3.5), G and H coincide on $\mathbb{R}_+$. By the identity theorem for analytic functions, we obtain $G(\alpha) = H(\alpha)$ for all $\alpha \in \mathbb{C}\backslash\{-\mu_b\}$. Since $G(\alpha^*) = 0$, it follows that $H(\alpha^*) = 0$, i.e., (3.7).

To prove (3.2), let us make the dependence of $\theta_1(x)$ on λ_a and μ_b explicit by writing it as $\theta_1(x, \lambda_a, \mu_b)$. Then a little reflection shows that

$$\theta_0(x; \lambda_a, \mu_b) = 1 - \theta_1(x; \mu_b, \lambda_a)$$

and (3.2) follows from (3.1). ■

The following explicit formula for the steady-state density of W is the main analytical result of this article. It is derived by means of an intricate use of the LCT. The constant factor $f_W(0+)$ can be determined by the normalizing condition for the density; using another argument, an explicit formula for $f_W(0+)$ is given in Theorem 1.

THEOREM 1: *The stationary density f_W of* $\mathbf{W}$ *is given by*

$$f_W(x) = \Bigg[\left(\frac{\theta_1(x)}{a} + \frac{1-\theta_1(x)}{b}\right) + (1-\theta_1(q_g))\left(\frac{1-\theta_0(q_b)}{\theta_0(q_b)}\left(\frac{\gamma(x)}{a} + \frac{1-\gamma(x)}{b}\right) + \frac{\nu(x)}{a} + \frac{1-\nu(x)}{b}\right)1_{[q_g-q_b,q_g)}(x)\Bigg]af_W(0+),$$

$$x \in (0, q_b), \tag{3.8}$$

where $\theta_0(\cdot)$ and $\theta_1(\cdot)$ have been computed in Lemma 1,

$$\gamma(x) = \frac{\theta_0(x - q_g + q_b)(1 - \theta_1(q_g - x))}{(1 - \theta_0(q_b))}, \tag{3.9}$$

$$\nu(x) = \frac{\theta_0(x - q_g + q_b)\theta_1(q_g - x)}{\theta_0(q_b)}, \tag{3.10}$$

and $f_W(0+)$ can be determined from the normalizing condition $\int_0^{q_g} f_W(x)\,dx = 1$.

PROOF: By the limit theorem for regenerative processes [1, p. 170], the stationary distribution function F_W of $\mathbf{W}$ is given by

$$F_W(x) = \frac{E\left(\int_0^T 1_{\{W(t)\le x\}}\,dt\right)}{ET}, \tag{3.11}$$

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where 1_A denotes the indicator function of the set A. Fix $0 < x < q_g$. Then $0 < x < q_g - \varepsilon$ for sufficiently small $\varepsilon > 0$. We now use the basic arguments of the level crossing approach. Consider the integral

$$\int_0^T 1_{\{x \le W(t) < x+\varepsilon\}} \, dt,$$

which is equal to the amount of time that **W** spends in the interval $[x, x+\varepsilon)$ during the cycle $[0, T)$. Clearly,

$$\int_0^T 1_{\{x \le W(t) < x+\varepsilon\}} \, dt = \int_0^{\tau_1} 1_{\{x \le W(t) < x+\varepsilon\}} \, dt + \sum_{j=1}^{N} \int_{\tau_j}^{\tau_{j+1}} 1_{\{x \le W(t) < x+\varepsilon\}} \, dt \tag{3.12}$$

(where an empty sum is defined to be zero). For $x \in (0, q_g - q_b - \varepsilon)$, the sum on the right-hand side of (3.12) vanishes since all of the indicator variables in it are equal to zero. For $x \in [q_g - q_b, q_g - \varepsilon)$, each of them is equal to unity during an interval of length either ε/a or ε/b or between these numbers, because during each subcycle, the sample path of **W** is increasing and runs through $[x, x+\varepsilon)$ linearly at alternating slopes a and b. It follows that

$$\frac{1}{\varepsilon} \int_0^T 1_{\{x \le W(t) < x+\varepsilon\}} \, dt \le (N+1) \max[a, b] \tag{3.13}$$

for all $\varepsilon \in (0, q_g - x)$. As $W(t)$ is piecewise linear, the derivative

$$\frac{d}{dx}\left(\int_0^T 1_{\{W(t) < x\}} \, dt\right) \tag{3.14}$$

exists, and (3.13) and a similar estimate for negative ε together yield

$$\frac{d}{dx}\left(\int_0^T 1_{\{W(t) < x\}} \, dt\right) \le (N+1) \max[a, b]. \tag{3.15}$$

Since N has a modified geometric distribution (see Section 2), we have

$$EN = \frac{1 - \theta_1(q_g)}{\theta_0(q_b)} < \infty. \tag{3.16}$$

Thus, we can use dominated convergence to conclude that

$$E\left(\frac{d}{dx} \int_0^T 1_{\{W(t) < x\}} \, dt\right) = \frac{d}{dx} E\left(\int_0^T 1_{\{W(t) \le x\}} \, dt\right) \tag{3.17}$$

$$= \frac{d}{dx}\left(ET \, \frac{E\left(\int_0^T 1_{\{W(t) \le x\}} \, dt\right)}{ET}\right)$$

$$= ET \, \frac{d}{dx} F_W(x)$$

$$= ET f_W(x), \tag{3.18}$$

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where the third equality follows from (3.11). Hence, F_W is an absolutely continuous distribution whose density can be computed by taking the derivative of $x \mapsto E(\int_0^T 1_{\{W(t)\le x\}}\,dt)/ET$.

Let us compute $E(\int_0^T 1_{\{x\le W(t)<x+\varepsilon\}}\,dt)$. If $x \in (0, q_g - q_b)$, the interval $[x, x+\varepsilon)$ is crossed exactly once in the cycle $[0,T)$, and during this crossing, the sample path has slope a with probability $\theta_1(x) + O(\varepsilon)$ and slope b with probability $1 - \theta_1(x) + O(\varepsilon)$ as $\varepsilon \searrow 0$. (The terms $O(\varepsilon)$ are due to the possibility that $[x, x+\varepsilon)$ can also be crossed by a piece of sample path in which the slope changes.) Hence, the expected amount of time spent in $[x, x+\varepsilon)$ is

$$\frac{\varepsilon\theta_1(x)}{a} + \frac{\varepsilon(1-\theta_1(x))}{b} + O(\varepsilon^2) \quad \text{as } \varepsilon \searrow 0. \tag{3.19}$$

Relation (3.19) accounts for the term in square brackets in the corresponding assertion of (3.8) for $x \in (0, q_g - q_b)$.

Now let $x \in [q_g - q_b, q_g)$. In this case we can write

$$\begin{aligned}
&E\left(\int_0^T 1_{\{x\le W(t)<x+\varepsilon\}}\,dt\right)\\
&\quad = \varepsilon\left(\frac{\theta_1(x)}{a} + \frac{1-\theta_1(x)}{b} + O(\varepsilon)\right)\\
&\qquad + \sum_{n=0}^{\infty} P(N=n+1)\Bigg[\varepsilon E(Y|N=n+1)\\
&\qquad\qquad + \frac{\varepsilon\nu(x)}{a} + \frac{\varepsilon(1-\nu(x))}{b} + O(\varepsilon^2)\Bigg]\\
&\quad = \varepsilon\left(\frac{\theta_1(x)}{a} + \frac{1-\theta_1(x)}{b}\right) + \varepsilon\sum_{n=0}^{\infty}(1-\theta_1(q_g))(1-\theta_0(q_b))^n\theta_0(q_b)\\
&\qquad \times\left(\frac{n\gamma(x)+\nu(x)}{a} + \frac{n(1-\gamma(x))+1-\nu(x)}{b}\right) + O(\varepsilon^2),
\end{aligned} \tag{3.20}$$

where εY denotes the sojourn time of $\mathbf{W}$ in $[x, x+\varepsilon)$ during those subcycles of $[0,T)$ that start and end in the bad state. Indeed, the expected amount of time spent in $[x, x+\varepsilon)$ during the first subcycle is the same as in the case $x \in (0, q_g - q_b)$; that is, it is given by (3.19). The series in (3.20) gives the contribution of the other subcycles of $[0,T)$. With probability $1 - \theta_1(q_g)$, the first subcycle is followed by $n \ge 0$ consecutive subcycles starting and ending in the bad state and a final subcycle starting in the bad state and ending in the good state. The counting variable of the upcrossings of $[x, x+\varepsilon)$ at slope a (i.e., while being in the good state) during the n subcycles starting and ending in the bad state is a binomial random variable with success probability

$$\frac{\theta_0(x-q_g+q_b)\theta_1(q_g-x)}{\theta_0(q_b)}+O(\varepsilon)=\gamma(x)+O(\varepsilon)$$

and the probability that the upcrossing of $[x, x+\varepsilon)$ during the final subcycle (starting in the bad state and ending in the good state) occurs while growing at slope a is

$$\frac{\theta_0(x-q_g+q_b)\theta_1(q_g-x)}{\theta_0(q_b)}+O(\varepsilon)=\nu(x)+O(\varepsilon).$$

The corresponding values for slope b are $1-\gamma(x)+O(\varepsilon)$ and $1-\nu(x)+O(\varepsilon)$, respectively. The probability of having n consecutive subcycles leading from bad to bad followed by one leading from bad to good is $(1-\theta_0(q_b))^n\theta_0(q_b)$. These arguments explain (3.20). The series on the right-hand side of (3.20) can easily be calculated in closed form, leading to the term in square brackets in assertion (3.8) for $x \in [q_g-q_b, q_g)$.

To complete the analysis, it remains to show that

$$ET=\frac{1}{af_W(0+)}. \tag{3.21}$$

During a cycle level, x is upcrossed exactly once for every $x \in (0, q_g-q_b)$. Since the cycle $[0,T)$ starts in the good state, the initial slope is a, so that the slope while crossing $[x, x+\varepsilon)$ is a with probability $1-O(x)$ as $x \to 0$. Thus, by (3.18),

$$ETf_W(0+)=\lim_{x\searrow 0}\lim_{\varepsilon\searrow 0} E\left(\int_0^T 1_{\{x\le W(t)<x+\varepsilon\}}\,dt\right)=\frac{1}{a},$$

yielding (3.21).

Since f_W is a probability density on $(0, q_g)$, the value $f_W(0+)$ can be obtained by the normalizing condition $\int_0^{q_g} f_W(x)\,dx=1$. The proof is complete. ■

An explicit formula for ET and thus $f_W(0+)$ is given in the following theorem.

THEOREM 2:

$$ET=\left(a\frac{\mu}{\lambda+\mu}+b\frac{\lambda}{\lambda+\mu}\right)^{-1}\frac{q_b(1-\theta_1(q_g))+q_g\theta_0(q_b)}{\theta_0(q_b)}. \tag{3.22}$$

PROOF: From (1.1) and (1.3), we can conclude that

$$\frac{q_g+q_b EN}{ET}=a\frac{\mu}{\lambda+\mu}+b\frac{\lambda}{\lambda+\mu}; \tag{3.23}$$

(3.22) now follows from $EN=(1-\theta_1(q_g))/\theta_0(q_b)$. ■

4. NUMERICAL EXAMPLES AND SENSITIVITY ANALYSIS

We start with several asymptotic formulas. These results are quite intuitive and their proofs, while not difficult, are quite tedious in some cases and therefore omitted.

LEMMA 2:

(a)

$$\lim_{\lambda\to 0} EX = \lim_{\mu\to\infty} EX = \lim_{a\to\infty} EX = \frac{q_g}{2}$$

(b)

$$\lim_{\lambda\to\infty} EX = \lim_{\mu\to 0} EX = \frac{q_b}{2}$$

(c)

$$\lim_{\mu\to\infty} ET = \lim_{\lambda\to 0} ET = \frac{q_g}{a}$$

(d)

$$\lim_{\mu\to 0} ET = \lim_{\lambda\to\infty} ET = \infty$$

(e)

$$\lim_{a\to\infty} ET = 0$$

(f)

$$\lim_{\mu\to\infty} \frac{1+EN}{ET} = \lim_{\lambda\to 0} \frac{1+EN}{ET} = \frac{a}{q_g}$$

(g)

$$\lim_{\lambda\to\infty} \frac{1+EN}{ET} = \lim_{\mu\to 0} \frac{1+EN}{ET} = \frac{b}{q_b}$$

(h)

$$\lim_{a\to\infty} \frac{1+EN}{ET} = \infty.$$

Notice that parts (a), (c), and (g) provide the classical EOQ results for a system that is always in a good period. However, when $a \to \infty$, the expected cycle time is zero (part (e)), whereas when $\lambda \to 0$ or $\mu \to \infty$, it is positive and

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finite. When $a \to \infty$, an order of size $q_g/2$ is made an infinite number of times (part (h)). Parts (b) and (g) provide the classical EOQ result of expected inventory and expected number of orders for a system that is always in a bad period. However, when $\mu \to 0$ or $\lambda \to \infty$, the length of the cycle tends to infinity (part (d)).

Next we investigate the effect of parameter changes on the optimal solution. The following example is typical of many other examples studied. Consider a system with $\lambda = 0.4$/unit of time, $\mu = 0.6$/unit of time, $a = 10$/unit of time, and $b = 5$/unit of time. We let h vary in $\{0.5, 1, 1.5, 2, \ldots, 5\}$ and K vary in $\{5, 10, 15, 20, \ldots, 50\}$. The optimal values of q_g, q_b, EX, ET, EN, and $C(q_g, q_b)$ are given in Table 1, parts a–f, respectively. The following observations can be made from Table 1:

1. As expected, as K increases, the optimal values of q_g, q_b, EX, ET, and $C(q_g, q_b)$ increase and the optimal value of EN decreases.
2. As expected, as h increases, the optimal q_g, q_b, EX, ET, and $C(q_g, q_b)$ decrease and the optimal EN increases.
3. The expected number of orders $(1 + EN)/ET$ decreases when K increases, and it increases when h increases.
4. When K increases (h increases), both q_g and q_b increase (decrease), as noted in observations 1 and 2, but their difference $q_g - q_b$ increases (decreases). In other words, q_g is more sensitive to changes in K than q_b, and q_b is more sensitive to changes in h than q_g.

In Table 2, parts a–f, we set $h = \$1/(\text{unit} \times \text{unit of time})$ and $K = \$10$/order and let a vary in $\{12, 13, 14, \ldots, 22\}$ and b vary in $\{1, 2, 3, \ldots, 11\}$. The following observations can be made:

5. As a increases, q_g, EX, and $C(q_g, q_b)$ increase and ET decreases. Both q_b and EN decrease as a increases.
6. As b increases, q_b, EX, and $C(q_g, q_b)$ increase. Also, q_g, ET, and EN increase as b increases.
7. When a increases (b increases), the difference between q_g and q_b increases (decreases).
8. When either a or b increase, the expected number of orders increases.

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TABLE 1. Optimal Values as a Function of h and K

K/h	5	10	15	20	25	30	35	40	45	50
a. q_g as a function of h and K										
0.5	13.49607	18.87746	22.96775	26.39839	29.41122	32.12927	34.6251	36.94572	39.12354	41.18221
1	9.638926	13.49608	16.42454	18.87747	21.02898	22.96776	24.74624	26.39839	27.94777	29.41122
1.5	7.911063	11.08581	13.49608	15.51427	17.2837	18.87747	20.33882	21.69577	22.96776	24.16886
2	6.874215	9.638926	11.73831	13.49608	15.03693	16.42454	17.69657	18.87747	19.98418	21.02898
2.5	6.163358	8.646513	10.53258	12.11179	13.49608	14.74256	15.88509	16.94564	17.93942	18.87747
3	5.636793	7.911062	9.638926	11.08581	12.35408	13.49608	14.54276	15.51427	16.42454	17.2837
3.5	5.226407	7.337654	8.942069	10.28571	11.46354	12.52407	13.49608	14.39823	15.24347	16.04121
4	4.894849	6.874216	8.378775	9.638928	10.74361	11.73831	12.64996	13.49608	14.2888	15.03693
4.5	4.619685	6.489476	7.911063	9.101853	10.14579	11.08581	11.94735	12.74695	13.49608	14.20306
5	4.386521	6.16336	7.514561	8.646513	9.638928	10.53257	11.35163	12.1118	12.82398	13.49608
b. q_k as a function of h and K										
0.5	10.40142	15.03348	18.72394	21.91409	24.77331	27.38927	29.81528	32.08694	34.22987	36.2632
1	7.251524	10.40141	12.8887	15.03347	16.9579	18.72392	20.36808	21.91409	23.37846	24.77331
1.5	5.887175	8.416169	10.40141	12.10685	13.63332	15.03347	16.33638	17.56202	18.72393	19.83192
2	5.082171	7.251524	8.948041	10.40141	11.69997	12.88870	13.99415	15.03348	16.01853	16.9579
2.5	4.536155	6.464393	7.968459	9.254344	10.40141	11.45013	12.42442	13.33976	14.20687	15.03348
3	4.134758	5.887175	7.251524	8.416171	9.453768	10.40140	11.28103	12.10685	12.8887	13.63373
3.5	3.82373	5.440719	6.697821	7.769639	8.723562	9.594026	10.40141	11.15893	11.87575	12.55849
4	3.573596	5.082171	6.253657	7.25152	8.13889	8.948039	9.698079	10.40142	11.06663	11.69997
4.5	3.366787	4.786054	5.887175	6.824368	7.657211	8.416171	9.119308	9.778347	10.40141	10.99439
5	3.192095	4.536152	5.578125	6.464395	7.251526	7.968459	8.63235	9.254343	9.84216	10.40141
c. EX as a function of h and K										
0.5	6.254008	8.857438	10.86133	12.55431	14.04799	15.39964	16.64336	17.8014	18.88927	19.91832
1	4.418531	6.254007	7.665418	8.857441	9.909234	10.86134	11.73781	12.55431	13.32173	14.04799

(continued)

Table 1. *continued*

K/h	5	10	15	20	25	30	35	40	45	50
1.5	3.606624	5.103561	6.254007	7.225241	8.081997	8.857441	9.571257	10.23626	10.86134	11.45296
2	3.122937	4.418531	5.413909	6.254007	6.994926	7.665419	8.282553	8.857442	9.397798	9.909234
2.5	2.792969	3.951343	4.841108	5.591929	6.254007	6.853073	7.404407	7.917955	8.400625	8.857442
3	2.549453	3.606624	4.418531	5.103562	5.707554	6.254007	6.756873	7.225241	7.665418	8.081997
3.5	2.360223	3.338787	4.090241	4.724203	5.283116	5.788743	6.254007	6.687323	7.094535	7.479896
4	2.207706	3.122937	3.825697	4.418531	4.941148	5.413909	5.848904	6.254009	6.634687	6.994925
4.5	2.081386	2.944176	3.606624	4.165418	4.657999	5.103562	5.513512	5.895275	6.254007	6.593462
5	1.974531	2.79297	3.421331	3.951344	4.418532	4.841108	5.229892	5.591929	5.932113	6.254007
d. *ET* as a function of *h* and *K*										
0.5	2.159931	3.014618	3.67137	4.227349	4.719093	5.165055	5.576142	5.959454	6.31994	6.661237
1	1.550802	2.159931	2.623905	3.014619	3.359225	3.67137	3.959033	4.227349	4.47985	4.719093
1.5	1.27754	1.779259	2.159931	2.479435	2.760509	3.014619	3.248479	3.466407	3.67137	3.865508
2	1.113096	1.550802	1.882267	2.159931	2.403775	2.623905	2.826247	3.014619	3.191648	3.359225
2.5	1.000055	1.393943	1.691923	1.941237	2.159931	2.357147	2.538258	2.706728	2.864941	3.014619
3	0.916125	1.27754	1.550802	1.779259	1.979497	2.159931	2.325511	2.479435	2.623905	2.760509
3.5	0.850583	1.186654	1.44068	1.652947	1.838888	2.006344	2.159931	2.302632	2.436504	2.563034
4	0.797537	1.113096	1.351586	1.550802	1.725239	1.882267	2.026228	2.159931	2.285313	2.403775
4.5	0.753445	1.051949	1.27754	1.465938	1.630853	1.779259	1.915271	2.041549	2.159931	2.271745
5	0.716032	1.000055	1.214708	1.393944	1.550802	1.691923	1.821222	1.941237	2.053717	2.159931
e. *EN* as a function of *h* and *K*										
0.5	0.363736	0.348521	0.341979	0.338614	0.336714	0.335576	0.334863	0.334401	0.334094	0.333884
1	0.381643	0.363736	0.354318	0.348521	0.344667	0.34198	0.340043	0.338614	0.337537	0.336714
1.5	0.392253	0.374073	0.363736	0.356923	0.352095	0.348521	0.345794	0.343667	0.34198	0.340623
2	0.399544	0.381642	0.371011	0.363736	0.3584	0.354318	0.351104	0.348521	0.346411	0.344667
2.5	0.404986	0.387513	0.376836	0.36935	0.363736	0.359351	0.355829	0.352944	0.350543	0.348521
3	0.409264	0.392253	0.381642	0.374073	0.368307	0.363736	0.360014	0.356923	0.354318	0.352095

3.5	0.412752	0.396194	0.385703	0.378121	0.372276	0.367591	0.363736	0.360503	0.357751	0.355382
4	0.415672	0.399544	0.389199	0.381643	0.375764	0.371011	0.367069	0.363736	0.360879	0.3584
4.5	0.418166	0.402443	0.392253	0.384747	0.378863	0.374073	0.370074	0.366672	0.363736	0.361176
5	0.420331	0.404986	0.394954	0.387513	0.381642	0.376836	0.372801	0.36935	0.366359	0.363736
f. $C(q_g, q_k)$ as a function of h and K										
0.5	6.283901	8.901992	10.91355	12.61027	14.10541	15.4572	16.70027	17.85724	18.94381	19.97144
1	8.873138	12.5678	15.40761	17.80398	19.9165	21.8271	23.58451	25.22053	26.75726	28.21082
1.5	10.85889	15.37807	18.8517	21.78328	24.36798	26.70598	28.85684	30.85939	32.74065	34.52028
2	12.53259	17.74628	21.75357	25.13561	28.11763	30.81521	33.29706	35.60797	37.77906	39.83301
2.5	14.00697	19.83223	24.30932	28.08784	31.41951	34.43348	37.20652	39.78868	42.21471	44.50996
3	15.3398	21.71779	26.61941	30.75614	34.40366	37.70341	40.73945	43.56656	46.22282	48.73595
3.5	16.5654	23.45156	28.74344	33.20942	37.14723	40.7096	43.98731	47.0395	49.90729	52.62061
4	17.7061	25.06519	30.72021	35.49255	39.70044	43.50713	47.00965	50.27121	53.33575	56.23526
4.5	18.77745	26.58065	32.57668	37.63667	42.09814	46.13421	49.84778	53.30588	56.55511	59.62939
5	19.79072	28.01394	34.33245	39.66445	44.36569	48.61865	52.53176	56.17568	59.59952	62.83901

Table 2. Optimal Values as a Function of *a* and *b*

a/b	1	2	3	4	5	6	7	8	9	10	11
a. q_g as a function of a and b											
12	12.69754	13.36293	13.88956	14.2882	14.59134	14.8249	15.00718	15.15105	15.26565	15.35761	15.43181
13	13.16968	13.83332	14.36982	14.78196	15.09898	15.34567	15.54001	15.69479	15.8192	15.91996	16.00207
14	13.6246	14.28554	14.83037	15.25462	15.58438	15.8433	16.04895	16.21404	16.34779	16.45697	16.54667
15	14.06411	14.72165	15.27347	15.70862	16.05012	16.32044	16.53674	16.7116	16.85424	16.97148	17.06848
16	14.48972	15.14331	15.701	16.14599	16.49832	16.7793	17.00564	17.18978	17.3409	17.46587	17.56989
17	14.90271	15.55194	16.11452	16.56839	16.93075	17.22172	17.45756	17.65051	17.80973	17.94211	18.0529
18	15.30417	15.94873	16.51536	16.97727	17.34893	17.64928	17.8941	18.09544	18.26242	18.40192	18.51923
19	15.69502	16.3347	16.90463	17.37383	17.75413	18.06331	18.31664	18.52599	18.7004	18.84675	18.97035
20	16.07609	16.71074	17.28332	17.75914	18.14747	18.46496	18.72638	18.94337	19.12491	19.27785	19.40753
21	16.44809	17.07759	17.65227	18.13409	18.52992	18.85525	19.12435	19.34865	19.53703	19.69633	19.83188
22	16.81165	17.43595	18.01223	18.49949	18.90231	19.23505	19.51147	19.74277	19.93772	20.10315	20.24438
b. q_k as a function of a and b											
12	4.586706	6.646131	8.149368	9.372399	10.41985	11.34452	12.1774	12.93855	13.64175	14.29701	14.91181
13	4.524125	6.600389	8.119166	9.356486	10.41727	11.35448	12.19925	12.97169	13.68566	14.3512	14.97585
14	4.466962	6.555921	8.087599	9.337125	10.40948	11.35771	12.21299	12.99548	13.71908	14.39388	15.02742
15	4.414738	6.513091	8.055463	9.315432	10.39785	11.35577	12.22037	13.01183	13.74408	14.42721	15.06881
16	4.366962	6.472082	8.023304	9.292219	10.38341	11.34986	12.22274	13.02223	13.76226	14.45292	15.10181
17	4.323165	6.432961	7.991491	9.268078	10.36693	11.34092	12.22117	13.02785	13.77488	14.47236	15.12788
18	4.28292	6.395729	7.960274	9.243442	10.34899	11.32965	12.21648	13.02962	13.78297	14.48663	15.14819
19	4.245839	6.360338	7.929814	9.218632	10.33005	11.31661	12.20933	13.02828	13.78735	14.49662	15.16368
20	4.21158	6.326719	7.900212	9.193879	10.31044	11.30225	12.20023	13.02442	13.78867	14.50305	15.17513
21	4.179844	6.294786	7.871525	9.169353	10.29042	11.2869	12.18961	13.01852	13.78747	14.50651	15.18318
22	4.150366	6.264447	7.843781	9.145176	10.27021	11.27085	12.17779	13.01098	13.78421	14.50748	15.18836
c. EX as a function of a and b											
12	5.914148	6.065132	6.268809	6.473229	6.667173	6.849062	7.019677	7.180304	7.332212	7.476518	7.614171
13	6.140324	6.275563	6.470679	6.670427	6.861668	7.041911	7.211485	7.371429	7.52287	7.666842	7.804241
14	6.358994	6.479224	6.665879	6.8609	7.049359	7.227887	7.396363	7.555581	7.706527	7.850149	7.987289
15	6.570881	6.676798	6.855109	7.045374	7.230987	7.407735	7.575059	7.733509	7.883929	8.02718	8.16405

16	6.776595	6.868858	7.038955	7.224448	7.407163	7.582076	7.748197	7.905838	8.0557	8.198557	8.335142
17	6.976657	7.055885	7.217909	7.398626	7.578402	7.751431	7.916305	8.073098	8.222371	8.36481	8.501092
18	7.171516	7.238293	7.392388	7.568337	7.745142	7.916245	8.079832	8.235741	8.384395	8.526393	8.662354
19	7.361565	7.416437	7.562753	7.73395	7.907759	8.076902	8.239163	8.394156	8.542164	8.683696	8.819318
20	7.547148	7.590628	7.729317	7.895784	8.066579	8.233731	8.394636	8.548683	8.696018	8.837062	8.972326
21	7.728569	7.76114	7.892352	8.054118	8.221887	8.387022	8.546543	8.699616	8.846252	8.986789	9.121678
22	7.906101	7.928213	8.0521	8.209198	8.373933	8.537029	8.695141	8.847215	8.993131	9.13314	9.267637

d. *ET* as a function of *a* and *b*

12	1.706188	1.772774	1.836163	1.891063	1.938338	1.979496	2.015808	2.048247	2.077552	2.104283	2.128872
13	1.636075	1.696324	1.754664	1.805553	1.849498	1.887789	1.921568	1.951726	1.978944	2.003742	2.026525
14	1.573969	1.628847	1.682857	1.730307	1.771407	1.807258	1.838888	1.867115	1.89257	1.91574	1.937004
15	1.518453	1.568727	1.61898	1.663446	1.702084	1.73583	1.765612	1.792183	1.816131	1.83791	1.857879
16	1.468436	1.514725	1.561686	1.603536	1.640019	1.67193	1.700105	1.725239	1.747881	1.76846	1.787312
17	1.423066	1.465878	1.509927	1.549461	1.584042	1.614335	1.641098	1.664974	1.686475	1.706006	1.723885
18	1.381665	1.421418	1.462874	1.500342	1.533227	1.562084	1.587596	1.610358	1.630853	1.649461	1.666486
19	1.343684	1.380729	1.41986	1.455471	1.486835	1.514406	1.5388	1.560571	1.58017	1.59796	1.614227
20	1.308677	1.343311	1.380343	1.414275	1.444265	1.470676	1.494065	1.514946	1.533744	1.550802	1.566394
21	1.276275	1.308749	1.343879	1.376284	1.405025	1.430384	1.452864	1.472941	1.491016	1.507416	1.522401
22	1.246168	1.276699	1.310096	1.341105	1.368706	1.393105	1.414757	1.434104	1.451525	1.467329	1.481766

e. *EN* as a function of *a* and *b*

12	0.058755	0.123269	0.18826	0.251072	0.311076	0.368307	0.422988	0.475379	0.525732	0.574271	0.621192
13	0.054405	0.114398	0.17516	0.234088	0.290489	0.344338	0.395812	0.445139	0.492542	0.538226	0.582373
14	0.050667	0.106751	0.163838	0.219388	0.27266	0.323577	0.372276	0.418953	0.463811	0.507036	0.548795
15	0.047419	0.100086	0.153948	0.206529	0.257053	0.305398	0.351666	0.396027	0.438662	0.479743	0.519424
16	0.044571	0.094225	0.145227	0.195176	0.243263	0.289331	0.333449	0.375764	0.416439	0.45563	0.493483
17	0.042051	0.089028	0.137477	0.185071	0.23098	0.275013	0.317214	0.357706	0.396636	0.434149	0.470378
18	0.039805	0.084386	0.130539	0.176013	0.219962	0.262164	0.302641	0.341497	0.378863	0.414872	0.449648
19	0.037791	0.080214	0.124291	0.167844	0.210015	0.25056	0.289477	0.326854	0.362808	0.39746	0.430927
20	0.035974	0.076444	0.118631	0.160434	0.200985	0.240021	0.277518	0.31355	0.348221	0.381642	0.413922
21	0.034326	0.073019	0.113479	0.153679	0.192747	0.230401	0.2666	0.301402	0.334901	0.367199	0.398397
22	0.032825	0.069893	0.108769	0.147495	0.185198	0.22158	0.256585	0.290258	0.322681	0.353949	0.384155

(*continued*)

Table 2. *continued*

a/b	1	2	3	4	5	6	7	8	9	10	11
f. $C(q_g, q_b)$ as a function of a and b											
12	12.11953	12.40135	12.74024	13.08894	13.43109	13.76146	14.07882	14.38344	14.67611	14.95779	15.22944
13	12.58505	12.84505	13.16803	13.50538	13.83918	14.16314	14.47541	14.77584	15.06498	15.34361	15.61255
14	13.03427	13.27391	13.58173	13.90814	14.23382	14.55156	14.85889	15.15529	15.44104	15.71675	15.98312
15	13.46882	13.68941	13.98273	14.29856	14.61637	14.92804	15.23057	15.52304	15.80551	16.0784	16.34232
16	13.89009	14.09278	14.37223	14.67783	14.98795	15.29371	15.59153	15.88017	16.15945	16.42962	16.69117
17	14.29923	14.48507	14.75123	15.0469	15.34954	15.6495	15.94272	16.22762	16.50377	16.77128	17.03054
18	14.69726	14.8672	15.1206	15.40664	15.70196	15.99625	16.28495	16.56617	16.83925	17.10418	17.36119
19	15.08504	15.23994	15.48107	15.75777	16.04595	16.33466	16.61892	16.89652	17.1666	17.42897	17.68379
20	15.46334	15.60399	15.83332	16.10093	16.38212	16.66537	16.94526	17.21929	17.48641	17.74628	17.99894
21	15.83283	15.95996	16.17791	16.43669	16.71104	16.98891	17.2645	17.53502	17.79921	18.05661	18.30715
22	16.19411	16.30836	16.51536	16.76554	17.03319	17.30578	17.57713	17.84418	18.10549	18.36044	18.60889

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