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Hardy–Poincaré inequalities with boundary singularities

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We are interested in variational problems involving weights that are singular at a point of the boundary of the domain. More precisely, we study a linear variational problem related to the Poincaré inequality and to the Hardy inequality for maps in $H^1_0(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with $0 \in \partial\Omega$. In particular, we give sufficient and necessary conditions so that the best constant is achieved.

1. Introduction

We are interested in linear variational problems involving weights that are singular at a point of the boundary of the domain. More precisely, let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geqslant 2$. We assume that $0 \in \partial\Omega$, and that $\partial\Omega$ is sufficiently smooth (hereafter, the assumptions that Ω is Lipschitz and $\partial\Omega$ is of class C^2 at the origin are sufficient for our purposes). We study the minimization problem

$$\mu_\lambda(\Omega) := \inf_{u \in H^1_0(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \lambda \int_\Omega |u|^2 \,\mathrm{d}x}{\int_\Omega |x|^{-2}|u|^2 \,\mathrm{d}x}, \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a varying parameter. For $\lambda = 0$ the *Ω-Hardy constant* $\mu_0(\Omega) \geqslant \frac{1}{4}(N-2)^2$ is the best constant in the Hardy inequality for maps supported by Ω. If $N = 2$, it has been proved [4, theorem 1.6] that $\mu_0(\Omega)$ is positive. Therefore, it always happens that $H^1_0(\Omega) \hookrightarrow L^2(\Omega; |x|^{-2}\,\mathrm{d}x)$ with a continuous embedding.

Problem (1.1) has some similarities with the questions studied by Brézis and Marcus [1], where the weight is the inverse square of the distance from the boundary of Ω. The work of Dávila and Dupaigne [5] is related to the minimization problem (1.1). Indeed, note that, for any fixed $\lambda \in \mathbb{R}$, any extremal for $\mu_\lambda(\Omega)$ is a weak solution to the linear Dirichlet problem

$$-\Delta u = \mu|x|^{-2}u + \lambda u \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $\mu = \mu_\lambda(\Omega)$. If $\mu_\lambda(\Omega)$ is achieved, then $\mu_\lambda(\Omega)$ is the first eigenvalue of the operator $-\Delta - \lambda$ on $H_0^1(\Omega) \hookrightarrow L^2(\Omega; |x|^{-2}\,dx)$. Starting from a different point of view, for $0 \in \Omega$, $N \geqslant 3$ and $\mu \leqslant \frac{1}{4}(N-2)^2$, Dávila and Dupaigne proved [5] the existence of the first eigenfunction φ_1 of the operator $-\Delta - \mu|x|^{-2}$ on a suitable functional space $H(\Omega) \hookrightarrow L^2(\Omega)$, such that $H(\Omega) \supseteq H_0^1(\Omega)$. Note that φ_1 solves (1.2), where the eigenvalue λ depends on the datum μ.

The problem of the existence of extremals for the Ω-Hardy constant $\mu_0(\Omega)$ was discussed in [4] for the case where $N = 2$ (with Ω possibly unbounded or having a conical singularity at $0 \in \partial\Omega$) and in [14], where Ω is a suitable compact perturbation of a cone in $\mathbb{R}^N$. Hardy–Sobolev inequalities with singularity at the boundary have been studied by several authors (see, for example, [3,6,7,10–12] and the references therein).

The minimization problem (1.1) is not compact, due to the group of dilations in $\mathbb{R}^N$. Actually, it may be that all minimizing sequences concentrate at 0. In this case $\mu_\lambda(\Omega)$ is not achieved and $\mu_\lambda(\Omega) = \mu^+$, where

$$\mu^+ = \tfrac{1}{4}N^2$$

is the best constant in the Hardy inequality for maps with support in a half-space. Indeed, in § 3 we show that

$$\sup_{\lambda \in \mathbb{R}} \mu_\lambda(\Omega) = \mu^+, \tag{1.3}$$

then we deduce that, provided $\mu_\lambda(\Omega) < \mu^+$, every minimizing sequence for $\mu_\lambda(\Omega)$ converges in $H_0^1(\Omega)$ to an extremal for $\mu_\lambda(\Omega)$.

We recall that Ω is said to be locally concave at $0 \in \partial\Omega$ if there exists $r > 0$ such that

$$\{x \in \mathbb{R}^N \mid x \cdot \nu > 0\} \cap B_r(0) \subset \Omega, \tag{1.4}$$

where ν is the interior normal of $\partial\Omega$ at 0. Note that if all the principal curvatures of $\partial\Omega$ at 0, with respect to ν, are strictly negative, then condition (1.4) is satisfied.

Our first main result is stated in the following theorem.

THEOREM 1.1. *Let $\Omega \in \mathbb{R}^N$ be a smooth bounded domain with $0 \in \partial\Omega$. Assume that Ω is locally concave at 0. Then $\mu_\lambda(\Omega)$ is attained if and only if $\mu_\lambda(\Omega) < \mu^+$.*

The 'only if' part, which is the most intriguing, is a consequence of corollary 4.2, where we provide local non-existence results for the problem

$$-\Delta u \geqslant \mu|x|^{-2}u + \lambda u \text{ on } \Omega, \quad u \geqslant 0 \text{ in } \Omega, \tag{1.5}$$

and also for negative values of the parameter λ.

At this point, several questions concerning the infimum $\mu_\lambda(\Omega)$ are still open. Set

$$\lambda^* := \inf\{\lambda \in \mathbb{R} \mid \mu_\lambda(\Omega) < \mu^+\}. \tag{1.6}$$

Since the map $\lambda \mapsto \mu_\lambda(\Omega)$ is non-increasing, $\mu_\lambda(\Omega)$ is achieved for any $\lambda > \lambda^*$ by the existence theorem 3.2. If $\lambda^* \in \mathbb{R}$, then, from (1.3), it follows that $\mu_\lambda(\Omega) = \mu^+$ for any $\lambda \leqslant \lambda^*$, and hence $\mu_\lambda(\Omega)$ is not achieved if $\lambda < \lambda^*$. We do not know whether there exist domains Ω for which $\lambda^* = -\infty$. On the other hand, we are able to prove the following facts (see § 6 for the precise statements).

(i) If Ω is locally convex at 0, that is, if there exists $r > 0$ such that $\Omega \cap B_r(0)$ is contained in a half-space, then $\lambda^* > -\infty$.

(ii) If Ω is contained in a half-space, then

$$\lambda^* \geqslant \frac{\lambda_1(\mathbb{D})}{|\mathrm{diam}(\Omega)|^2}, \tag{1.7}$$

where $\lambda_1(\mathbb{D})$ is the first Dirichlet eigenvalue of the unit ball $\mathbb{D}$ in $\mathbb{R}^2$ and $\mathrm{diam}(\Omega)$ is the diameter of Ω.

(iii) For any $\delta > 0$, there exists $\rho_\delta > 0$ such that if

$$\Omega \supseteq \{x \in \mathbb{R}^N \mid x \cdot \nu > -\delta|x|,\ \alpha < |x| < \beta\}$$

for some $\nu \in \mathbb{S}^{N-1}$, $\beta > \alpha > 0$ with $\beta/\alpha > \rho_\delta$, then $\lambda^* < 0$. In particular, the Hardy constant $\mu_0(\Omega)$ is achieved.

The relevance of the geometry of Ω at the origin is confirmed by theorem 1.1, by item (i) and by the existence theorems proved in [10–12] for a related superlinear problem. However, it should also be noted that the (conformal) 'size' of Ω (even far away from the origin) has some impact on the existence of compact minimizing sequences. Actually, no requirement on the curvature of Ω at 0 is needed in (iii). In particular, there exist smooth domains having strictly positive principal curvatures at 0, and such that the Hardy constant $\mu_0(\Omega)$ is achieved.

This paper is organized as follows. In § 2 we point out a few remarks on the Hardy inequality on dilation-invariant domains. In § 3 (see theorem 3.2 we give sufficient conditions for the existence of minimizers for (1.1). In § 4 we prove some non-existence theorems for solutions to (1.5) that might have an independent interest.

To prove inequality (1.7) for the case where Ω is contained in a half-space, in § 5 we provide computable remainder terms for the Hardy inequality on half-balls. We adopt here an argument by Brézis and Vázquez [2], where bounded domains $\Omega \subset \mathbb{R}^N$ with $N \geqslant 3$ and $0 \in \Omega$ are considered.

In § 6 we estimate λ^* from below and from above, under suitable assumptions on Ω.

Notation

- $\mathbb{R}^N_+$ and $\mathbb{S}^{N-1}_+$ denote any half-space and any hemisphere, respectively. More precisely,

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \cdot \nu > 0\}, \qquad \mathbb{S}^{N-1}_+ = \mathbb{S}^{N-1} \cap \mathbb{R}^N_+,$$

  where ν is any unit vector in $\mathbb{R}^N$.

- $B_R(x)$ is the open ball in $\mathbb{R}^N$ of radius r centred at x. If $x = 0$, we simply write B_R. If $N = 2$, we shall often write $\mathbb{D}_R$ and $\mathbb{D}$ instead of B_R and B_1, respectively.

- We denote by $H^1(\mathbb{S}^{N-1})$ the standard Sobolev space of maps on the unit sphere and we denote by ∇_σ and Δ_σ the gradient and the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$, respectively.

- Let Σ be a domain in $\mathbb{S}^{N-1}$. We denote by $H_0^1(\Sigma)$ the closure of $C_c^\infty(\Sigma)$ in the $H^1(\mathbb{S}^{N-1})$-space and by $\lambda_1(\Sigma)$ the first Dirichlet eigenvalue on Σ.

- A bounded domain $\Omega \subset \mathbb{R}^N$ with $0 \in \partial\Omega$ is said to be *smooth* if $\partial\Omega$ is of class C^2 in a neighbourhood of the origin.

  We denote by $L^2(\Omega; |x|^{-2}\,\mathrm{d}x)$ the space of measurable maps on Ω such that

$$\int_\Omega |x|^{-2}|u|^2\,\mathrm{d}x < \infty.$$

  We also set

$$\hat{H}^1(\Omega) := H^1(\Omega) \cap L^2(\Omega; |x|^{-2}\,\mathrm{d}x),$$

  where $H^1(\Omega)$ is the standard Sobolev space of maps on Ω.

2. Preliminaries

In this section we collect a few remarks on the Hardy inequality on dilation-invariant domains that are partially contained, for example, in [4] (in the case where $N = 2$) and in [14].

Via polar coordinates, to any domain Σ in $\mathbb{S}^{N-1}$ we associate a cone $\mathcal{C}_\Sigma \subset \mathbb{R}^{N-1}$ and a (half) cylinder $\mathcal{Z}_\Sigma \subset \mathbb{R}^{N+1}$ by setting

$$\mathcal{C}_\Sigma := \{t\sigma \mid t > 0,\ \sigma \in \Sigma\}, \qquad \mathcal{Z}_\Sigma := \mathbb{R}_+ \times \Sigma.$$

If Σ is a smooth domain in $\mathbb{S}^{N-1}$, then $\mathcal{C}_\Sigma$ is a Lipschitz dilation-invariant domain in $\mathbb{R}^{N-1}$. In particular, if Σ is a half-sphere, then $\mathcal{C}_\Sigma$ is a half-space. The map

$$\mathbb{R}^{N-1} \setminus \{0\} \to \mathbb{R}^{N+1}, \qquad x \mapsto \left(-\log|x|, \frac{x}{|x|}\right)$$

is a homeomorphism $\mathcal{C}_\Sigma \to \mathcal{Z}_\Sigma$. It induces the Emden–Fowler transform

$$T\colon C_c^\infty(\mathcal{C}_\Sigma) \to C_c^\infty(\mathcal{Z}_\Sigma), \qquad u(x) = |x|^{(2-N)/2}(Tu)\left(-\log|x|, \frac{x}{|x|}\right).$$

A direct computation based on the divergence theorem gives

$$\int_{\mathcal{C}_\Sigma} |\nabla u|^2\,\mathrm{d}x = \tfrac{1}{4}(N-2)^2 \int_0^\infty \int_\Sigma |Tu|^2\,\mathrm{d}s\,\mathrm{d}\sigma + \int_0^\infty \int_\Sigma |\nabla_{s,\sigma} Tu|^2\,\mathrm{d}s\,\mathrm{d}\sigma, \tag{2.1}$$

$$\int_{\mathcal{C}_\Sigma} |x|^{-2}|u|^2\,\mathrm{d}x = \int_0^\infty \int_\Sigma |Tu|^2\,\mathrm{d}s\,\mathrm{d}\sigma, \tag{2.2}$$

where $\nabla_{s,\sigma} = (\partial_s, \nabla_\sigma)$ denotes the gradient on $\mathbb{R}_+ \times \mathbb{S}^{N-1}$.

Now we introduce the Hardy constant on the cone $\mathcal{C}_\Sigma$:

$$\mu_0(\mathcal{C}_\Sigma) := \inf_{u \in C_c^\infty(\mathcal{C}_\Sigma),\, u \neq 0} \frac{\int_{\mathcal{C}_\Sigma} |\nabla u|^2\,\mathrm{d}x}{\int_{\mathcal{C}_\Sigma} |x|^{-2}|u|^2\,\mathrm{d}x}. \tag{2.3}$$

In the next proposition we note that the Hardy inequality on $\mathcal{C}_\Sigma$ is equivalent to the Poincaré inequality for maps supported by the cylinder $\mathcal{Z}_\Sigma$.

Proposition 2.1. *Let $\mathcal{C}_\Sigma$ be a cone. Then*

$$\mu_0(\mathcal{C}_\Sigma) = \tfrac{1}{4}(N-2)^2 + \lambda_1(\Sigma).$$

Proof. By (2.1) and (2.2), it turns out that

$$\mu_0(\mathcal{C}_\Sigma) - \tfrac{1}{4}(N-2)^2 = \inf_{v \in C_c^\infty(\mathcal{Z}_\Sigma),\, v \neq 0} \frac{\int_0^\infty \int_\Sigma |\nabla_{s,\sigma} v|^2 \,\mathrm{d}s\,\mathrm{d}\sigma}{\int_0^\infty \int_\Sigma |v|^2 \,\mathrm{d}s\,\mathrm{d}\sigma} =: \lambda_1(\mathcal{Z}_\Sigma).$$

The result follows by noting that $\lambda_1(\mathcal{Z}_\Sigma) = \lambda_1(\Sigma)$. □

The eigenvalue $\lambda_1(\Sigma)$ is explicitly known in few cases. For example, if $\Sigma = \mathbb{S}^{N-1}_+$ is a half-sphere, then $\lambda_1(\mathbb{S}^{N-1}_+) = N-1$. Thus, the Hardy constant of a half-space is given by

$$\mu_0(\mathbb{R}^N_+) = \mu^+ := \tfrac{1}{4}N^2. \tag{2.4}$$

If $N=2$ and if $\mathcal{C}_{\Sigma_\theta} \subset \mathbb{R}^2$ is a cone of amplitude $\theta \in (0, 2\pi]$ then $\lambda_1(\Sigma_\theta)$ coincide with the Dirichlet eigenvalue on the interval $(0,\theta)$. Hence, we obtain the conclusion, which was first pointed out in [4]:

$$\mu_0(\mathcal{C}_{\Sigma_\theta}) = \frac{\pi^2}{\theta^2} \geqslant \frac{1}{4}. \tag{2.5}$$

Let Σ be a domain in $\mathbb{S}^{N-1}$. If $N \geqslant 3$, the space $\mathcal{D}^{1,2}(\mathcal{C}_\Sigma)$ is defined in a standard way as a close subspace of $\mathcal{D}^{1,2}(\mathbb{R}^{N-1})$. Note that, in the case where $\Sigma = \mathbb{S}^{N-1}$, it turns out that

$$\mathcal{D}^{1,2}(\mathcal{C}_{\mathbb{S}^{N-1}}) = \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \{0\}) = \mathcal{D}^{1,2}(\mathbb{R}^N)$$

by a known density result.

If $N=2$ and if Σ is properly contained in $\mathbb{S}^1$, then $\mu_0(\mathcal{C}_\Sigma) > 0$ by (2.5). In this case we can introduce the space $\mathcal{D}^{1,2}(\mathcal{C}_\Sigma)$ by completing $C_c^\infty(\mathcal{C}_\Sigma)$ with respect to the Hilbertian norm $(\int_{\mathcal{C}_\Sigma} |\nabla u|^2 \,\mathrm{d}x)^{1/2}$.

The next result is an immediate consequence of the fact that the Dirichlet eigenvalue problem of $-\Delta$ in the strip $\mathcal{Z}_\Sigma$ is never achieved. The same conclusion was already noted in [4] in the case $N=2$ and in [14].

Proposition 2.2. *Let Σ be a domain in $\mathbb{S}^{N-1}$. Then $\mu_0(\mathcal{C}_\Sigma)$ is not achieved in $\mathcal{D}^{1,2}(\mathcal{C}_\Sigma)$.*

3. Existence

In this section we show that the condition $\mu_\lambda(\Omega) < \mu^+ = \frac{1}{4}N^2$ is sufficient to guarantee the existence of a minimizer for $\mu_\lambda(\Omega)$. We note here that, throughout this section, the regularity of Ω can be relaxed to Lipschitz domains that are of class C^2 at 0. We start with a preliminary result.

Lemma 3.1. *Let Ω be a smooth domain with $0 \in \partial\Omega$. Then*

$$\sup_{\lambda \in \mathbb{R}} \mu_\lambda(\Omega) = \mu^+.$$

Proof. The proof will be carried out in two steps.

(i) We claim that $\sup_{\lambda\in\mathbb{R}} \mu_\lambda(\Omega) \geqslant \mu^+$.

We denote by ν the interior normal of $\partial\Omega$ at 0. For $\delta > 0$, we consider the cone

$$\mathcal{C}^\delta_- := \{x \in \mathbb{R}^{N-1} \mid x\cdot\nu > -\delta|x|\}.$$

Now fix $\varepsilon > 0$. If δ is sufficiently small, then $\mu_0(\mathcal{C}^\delta_-) \geqslant \mu^+ - \varepsilon$. Since Ω is smooth at 0, there exists a small radius $r > 0$ (depending on δ) such that $\Omega \cap B_{r_\delta}(0) \subset \mathcal{C}^\delta_-$.

Next, let $\psi \in C^\infty(B_r(0))$ be a cut-off function, satisfying

$$0 \leqslant \psi \leqslant 1, \qquad \psi \equiv 0 \text{ in } \mathbb{R}^N \setminus B_{r/2}(0), \qquad \psi \equiv 1 \text{ in } B_{r/4}(0).$$

We write any $u \in H^1_0(\Omega)$ as $u = \psi u + (1-\psi)u$ to obtain

$$\int_\Omega |x|^{-2}|u|^2\,\mathrm{d}x \leqslant \int_\Omega |x|^{-2}|\psi u|^2\,\mathrm{d}x + c\int_\Omega |u|^2\,\mathrm{d}x, \tag{3.1}$$

where the constant c does not depend on u. Since $\psi u \in \mathcal{D}^{1,2}(\mathcal{C}^\delta_-)$, then

$$(\mu^+ - \varepsilon)\int_\Omega |x|^{-2}|\psi u|^2\,\mathrm{d}x \leqslant \mu_0(\mathcal{C}^\delta_-)\int_\Omega |x|^{-2}|\psi u|^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla(\psi u)|^2\,\mathrm{d}x \tag{3.2}$$

by our choice of the cone $\mathcal{C}^\delta_-$. In addition, we have

$$\int_\Omega |\nabla(\psi u)|^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \nabla(\psi^2)\cdot\nabla(u^2)\,\mathrm{d}x + c\int_\Omega |u|^2\,\mathrm{d}x.$$

Integrating by parts, we obtain

$$\int_\Omega |\nabla(\psi u)|^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega \Delta(\psi^2)|u|^2\,\mathrm{d}x + c\int_\Omega |u|^2\,\mathrm{d}x.$$

Comparing this with (3.1) and (3.2), we infer that there exists a positive constant c depending only on δ such that

$$(\mu^+ - \varepsilon)\int_\Omega |x|^{-2}|u|^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2\,\mathrm{d}x + c\int_\Omega |u|^2\,\mathrm{d}x \quad \forall u \in H^1_0(\Omega). \tag{3.3}$$

Hence, we obtain $(\mu^+ - \varepsilon) \leqslant \mu_{-c}(\Omega)$. Consequently, $(\mu^+ - \varepsilon) \leqslant \sup_\lambda \mu_\lambda(\Omega)$, and the conclusion follows by letting $\varepsilon \to 0$.

(ii) We claim that $\sup_\lambda \mu_\lambda(\Omega) \leqslant \mu^+$.

For $\delta > 0$ we consider the cone

$$\mathcal{C}^\delta_+ := \{x \in \mathbb{R}^{N-1} \mid x\cdot\nu > \delta|x|\}.$$

As in the first step, for any $\delta > 0$ there exists $r_\delta > 0$ such that $\mathcal{C}^\delta_+ \cap B_r(0) \subset \Omega$ for all $r \in (0, r_\delta)$. Clearly, by scale invariance,

$$\mu_0(\mathcal{C}^\delta_+ \cap B_r(0)) = \mu_0(\mathcal{C}^\delta_+).$$

For $\varepsilon > 0$, we let $\phi \in H^1_0(\mathcal{C}^\delta_+ \cap B_r(0))$ such that

$$\frac{\int_{\mathcal{C}^\delta_+\cap B_r(0)} |\nabla\phi|^2\,\mathrm{d}x}{\int_{\mathcal{C}^\delta_+\cap B_r(0)} |x|^{-2}|\phi|^2\,\mathrm{d}x} \leqslant \mu_0(\mathcal{C}^\delta_+) + \varepsilon.$$

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From this we deduce that

$$\mu_\lambda(\Omega) \leqslant \frac{\int_{\mathcal{C}_+^\delta \cap B_r(0)} |\nabla\phi|^2 \,\mathrm{d}x - \lambda \int_{\mathcal{C}_+^\delta \cap B_r(0)} |\phi|^2 \,\mathrm{d}x}{\int_{\mathcal{C}_+^\delta \cap B_{r_\delta}(0)} |x|^{-2}|\phi|^2 \,\mathrm{d}x}$$

$$\leqslant \mu_0(\mathcal{C}_+^\delta) + \varepsilon + |\lambda| \frac{\int_{\mathcal{C}_+^\delta \cap B_r(0)} |\phi|^2 \,\mathrm{d}x}{\int_{\mathcal{C}_+^\delta \cap B_r(0)} |x|^{-2}|\phi|^2 \,\mathrm{d}x}.$$

Since

$$\int_{\mathcal{C}_+^\delta \cap B_r(0)} |x|^{-2}|\phi|^2 \,\mathrm{d}x \geqslant r^{-2} \int_{\mathcal{C}_+^\delta \cap B_r(0)} |\phi|^2 \,\mathrm{d}x,$$

we obtain

$$\mu_\lambda(\Omega) \leqslant \mu_0(\mathcal{C}_+^\delta) + \varepsilon + r^2|\lambda|.$$

The conclusion follows immediately, since $\mu_0(\mathcal{C}_+^\delta) \to \mu^+$ when $\delta \to 0$. □

Note that if Ω is bounded, then by (3.3) and the Poincaré inequality,

$$\mu_0(\Omega) > 0. \tag{3.4}$$

This was shown in [4] for the case when $N = 2$ and for more general domains. We are now in a position to prove the main result of this section.

THEOREM 3.2. *Let $\lambda \in \mathbb{R}$ and let Ω be a smooth bounded domain of $\mathbb{R}^N$ with $0 \in \partial\Omega$. If $\mu_\lambda(\Omega) < \mu^+$, then $\mu_\lambda(\Omega)$ is attained.*

Proof. Let $u_n \in H_0^1(\Omega)$ be a minimizing sequence for $\mu_\lambda(\Omega)$. We can normalize it to obtain

$$\int_\Omega |\nabla u_n|^2 = 1, \tag{3.5}$$

$$1 - \lambda \int_\Omega |u_n|^2 = \mu_\lambda(\Omega) \int_\Omega |x|^{-2}|u_n|^2 + o(1). \tag{3.6}$$

We can assume that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $|x|^{-1}u_n \rightharpoonup |x|^{-1}u$ weakly in $L^2(\Omega)$, and $u_n \to u$ in $L^2(\Omega)$, by (3.4) and by the Rellich theorem. Setting $\theta_n := u_n - u$, from (3.5) and (3.6) we obtain

$$\left.\begin{aligned} \int_\Omega |\nabla\theta_n|^2 + \int_\Omega |\nabla u|^2 &= 1 + o(1), \\ 1 - \lambda \int_\Omega |u|^2 &= \mu_\lambda(\Omega)\biggl(\int_\Omega |x|^{-2}|\theta_n|^2 + \int_\Omega |x|^{-2}|u|^2\biggr) + o(1). \end{aligned}\right\} \tag{3.7}$$

By lemma 3.1, for any fixed positive $\delta < \mu^+ - \mu_\lambda(\Omega)$, there exists $\lambda_\delta \in \mathbb{R}$ such that $\mu_{\lambda_\delta}(\Omega) \geqslant \mu^+ - \delta$. Hence,

$$\int_\Omega |\nabla\theta_n|^2 + o(1) \geqslant (\mu^+ - \delta) \int_\Omega |x|^{-2}|\theta_n|^2,$$

as $\theta_n \to 0$ in $L^2(\Omega)$. Testing $\mu_\lambda(\Omega)$ with u, we obtain

$$\begin{aligned}\mu_\lambda(\Omega)\int_\Omega |x|^{-2}|u|^2 &\leqslant \int_\Omega |\nabla u|^2 - \lambda\int_\Omega |u|^2\\ &\leqslant 1 - \int_\Omega |\nabla\theta_n|^2 - \lambda\int_\Omega |u|^2 + o(1)\\ &\leqslant 1 - (\mu^+ - \delta)\int_\Omega |x|^{-2}|\theta_n|^2 - \lambda\int_\Omega |u|^2 + o(1)\\ &\leqslant (\mu_\lambda(\Omega) - \mu^+ + \delta)\int_\Omega |x|^{-2}|\theta_n|^2 + \mu_\lambda(\Omega)\int_\Omega |x|^{-2}|u|^2 + o(1)\end{aligned}$$

by (3.7). Therefore,

$$\int_\Omega |x|^{-2}|\theta_n|^2 \to 0,$$

since $\mu_\lambda(\Omega) - \mu^+ + \delta < 0$. In particular,

$$\mu_\lambda(\Omega)\int_\Omega |x|^{-2}|u|^2 = \int_\Omega |\nabla u|^2 - \lambda\int_\Omega |u|^2$$

and $u \neq 0$ by (3.7). Thus, u achieves $\mu_\lambda(\Omega)$. □

We conclude this section with a corollary of theorem 3.2.

Corollary 3.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ with $0 \in \partial\Omega$. Then*

$$\tfrac{1}{4}(N-2)^2 < \mu_0(\Omega) \leqslant \tfrac{1}{4}N^2.$$

Proof. It has already been proved in lemma 3.1 that $\mu_\lambda(\Omega) \leqslant \frac{1}{4}N^2$. If the strict inequality holds, then, by theorem 3.2, there exists $u \in H_0^1(\Omega)$ that achieves $\mu_0(\Omega)$. But then $\frac{1}{4}(N-2)^2 < \mu_0(\Omega)$, otherwise a null extension of u outside Ω would achieve the Hardy constant on $\mathbb{R}^N$. □

Remark 3.4. Following [4], for non-smooth domains Ω we can introduce the 'limiting' Hardy constant

$$\hat{\mu}_0(\Omega) = \sup_{r>0} \mu_0(\Omega \cap B_r).$$

Using similar arguments it can be proved that $\sup_\lambda \mu_\lambda(\Omega) = \hat{\mu}_0(\Omega)$, and that $\mu_\lambda(\Omega)$ is achieved provided $\mu_\lambda(\Omega) < \hat{\mu}_0(\Omega)$.

4. Non-existence

The main result in this section is stated in the following theorem.

Theorem 4.1. *Let Ω be a domain in $\mathbb{R}^N$, $N \geqslant 2$, and let $\lambda \in \mathbb{R}$. Assume that there exist $R > 0$ and a Lipschitz domain $\Sigma \subset \mathbb{S}^{N-1}$ such that $B_R \cap \mathcal{C}_\Sigma \subset \Omega$. If $u \in \hat{H}^1(\Omega)$ solves*

$$-\Delta u \geqslant (\tfrac{1}{4}(N-2)^2 + \lambda_1(\Sigma))|x|^{-2}u + \lambda u \quad \text{in } \mathcal{D}'(\Omega \setminus \{0\}), \quad u \geqslant 0, \tag{4.1}$$

then $u \equiv 0$ in Ω.

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Before proving theorem 4.1 we point out some of its consequences.

Corollary 4.2. *Let Ω be a smooth bounded domain containing a half-ball and such that $0 \in \partial\Omega$. If $\mu_\lambda(\Omega) = \mu^+$, then $\mu_\lambda(\Omega)$ is not achieved.*

Proof. Assume that u achieves $\mu_\lambda(\Omega) = \mu^+$. Then u is a weak solution to

$$-\Delta u = \mu^+|x|^{-2}u + \lambda u. \tag{4.2}$$

Test (4.2) with the negative and the positive part of u to conclude that u has constant sign. Now, by the maximum principle, $u > 0$ in Ω, contradicting theorem 4.1, since $\Omega \supset B_R \cap \mathcal{C}_{\mathbb{S}^{N-1}_+}$ and $\lambda_1(\mathbb{S}^{N-1}_+) = N-1$. □

We also point out the following consequence to theorem 4.1, which holds for smooth domains Ω with $0 \in \partial\Omega$.

Theorem 4.3. *Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geqslant 2$, with $0 \in \partial\Omega$ and let $\lambda \in \mathbb{R}$. If $u \in \hat{H}^1(\Omega)$ solves*

$$-\Delta u \geqslant \mu|x|^{-2}u + \lambda u \quad \text{in } \mathcal{D}'(\Omega), \quad u \geqslant 0,$$

for some $\mu > \mu^+$, then $u \equiv 0$ in Ω.

Proof. We start by noting that there exists a geodesic ball $\Sigma \subset \mathbb{S}^{N-1}$ contained in a hemisphere, and such that $\lambda_1(\Sigma) \leqslant N-1+\mu-\mu^+$. Since $0 \in \partial\Omega$ and since $\partial\Omega$ is smooth then, up to a rotation, we can find a small radius $r > 0$ such that $B_r \cap \mathcal{C}_\Sigma \subset \Omega$. The conclusion follows from theorem 4.1, as $\mu \geqslant \frac{1}{4}(N-2)^2 + \lambda_1(\Sigma)$. □

Remark 4.4. Theorem 4.1 also applies when the origin lies in the interior of the domain. More precisely, let Ω be any domain in $\mathbb{R}^N$, with $N \geqslant 2$ and $0 \in \Omega$. If $u \in \hat{H}^1_{\mathrm{loc}}(\Omega)$ is a non-negative solution to

$$-\Delta u \geqslant \tfrac{1}{4}(N-2)^2|x|^{-2}u + \lambda u \quad \text{in } \mathcal{D}'(\Omega \setminus \{0\})$$

for some $\lambda \in \mathbb{R}$, then $u \equiv 0$ in Ω.

In order to prove theorem 4.1 we need few preliminary results regarding maps of two variables. Recall that $\mathbb{D}_R \subset \mathbb{R}^2$ is the open disc of radius R centred at 0.

Lemma 4.5. *Let $\psi \in \hat{H}^1(\mathbb{D}_R)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{D}_R)$ for some $R > 0$. If ψ solves*

$$-\Delta\psi \geqslant f \quad \text{in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\}), \tag{4.3}$$

then $-\Delta\psi \geqslant f$ in $\mathcal{D}'(\mathbb{D}_R)$.

Proof. We start by noting that, from

$$\infty > \int_{\mathbb{D}_R} |z|^{-2}|\psi|^2 = \int_0^R \frac{1}{r}\left(r^{-1}\int_{\partial B_r}|\psi|^2\right),$$

it follows that there exists a sequence $r_h \to 0$, $r_h \in (0,R)$ such that

$$r_h^{-1}\int_{\partial B_{r_h}}|\psi|^2 \to 0, \qquad r_h^{-2}\int_{\partial B_{r_h^2}}|\psi|^2 \to 0 \tag{4.4}$$

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as $h \to \infty$. Next we introduce the following cut-off functions:

$$\eta_h(z) = \begin{cases} 0 & \text{if } |z| \leqslant r_h^2, \\ \dfrac{\log |z|/r_h^2}{|\log r_h|} & \text{if } r_h^2 < |z| < r_h, \\ 1 & \text{if } r_h \leqslant |z| \leqslant R. \end{cases}$$

Let $\varphi \in C_c^\infty(\mathbb{D}_R)$ be any non-negative function. We test (4.3) with $\eta_h\varphi$ to obtain

$$\int \nabla\psi \cdot \nabla(\eta_h\varphi) \geqslant \int f\eta_h\varphi.$$

Since $\psi \in H^1(\mathbb{D}_R)$ and since $\eta_h \rightharpoonup 1$ weakly* in L^∞, it is easy to check that

$$\int f \; \eta_h\varphi = \int f\varphi + o(1), \qquad \int \eta_h\nabla\psi \cdot \nabla\varphi = \int \nabla\psi \cdot \nabla\varphi + o(1)$$

as $h \to \infty$. Therefore,

$$\int \nabla\psi \cdot \nabla\varphi + \int \varphi\nabla\psi \cdot \nabla\eta_h \geqslant \int f\varphi + o(1). \tag{4.5}$$

To pass to the limit in the left-hand side, we note that $\nabla\eta_h$ vanishes outside the annulus $A_h := \{r_h^2 < |z| < r_h\}$, and that η_h is harmonic on A_h. Thus,

$$\begin{aligned} \int \varphi\nabla\psi \cdot \nabla\eta_h &= \int_{A_h} \nabla(\psi\varphi) \cdot \nabla\eta_h - \int_{A_h} \psi\nabla\varphi \cdot \nabla\psi \\ &= \mathcal{R}_h - \int_{A_h} \psi\nabla\varphi \cdot \nabla\eta_h, \end{aligned}$$

where

$$\mathcal{R}_h := -r_h^{-2} \int_{\partial B_{r_h^2}} (\nabla\eta_h \cdot z)\psi\varphi + r_h^{-1} \int_{\partial B_{r_h}} (\nabla\eta_h \cdot z)\psi\varphi.$$

Now

$$|\mathcal{R}_h| \leqslant c(r_h|\log r_h|)^{-1} \int_{\partial B_{r_h}} |\psi| + c\; (r_h^2|\log r_h|)^{-1} \int_{\partial B_{r_h^2}} |\psi|,$$

where $c > 0$ is a constant that does not depend on h, and

$$(r_h|\log r_h|)^{-1} \int_{\partial B_{r_h}} |\psi| \leqslant c|\log r_h|^{-1} \biggl(r_h^{-1} \int_{\partial B_{r_h}} |\psi|^2\biggr)^{1/2} = o(1)$$

by the Hölder inequality and by (4.4). In the same way, also

$$(r_h^2|\log r_h|)^{-1} \int_{\partial B_{r_h^2}} |\psi| \leqslant c|\log r_h|^{-1} \biggl(r_h^{-2} \int_{\partial B_{r_h^2}} |\psi|^2\biggr)^{1/2} = o(1),$$

and hence $\mathcal{R}_h = o(1)$. Moreover, from $\psi \in L^2(\mathbb{D}_R; |z|^{-2}\,\mathrm{d}z)$ it follows that

$$\biggl|\int_{A_h} \psi\nabla\varphi \cdot \nabla\eta_h\biggr| |\log r_h|^{-1} \int |z|^{-1}\psi|\nabla\varphi| = o(1).$$

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In conclusion, we have proved that

$$\int \varphi \nabla \psi \cdot \nabla \eta_h = o(1),$$

and therefore (4.5) gives

$$\int \nabla \psi \cdot \nabla \varphi \geqslant \int f \varphi.$$

Since φ was an arbitrary non-negative function in $C_c^\infty(\mathbb{D}_R)$, this proves that $-\Delta\psi \geqslant f$ in the distributional sense on $\mathbb{D}_R$, as desired. □

The same proof gives a similar result for subsolutions.

LEMMA 4.6. *Let $\varphi \in \hat{H}^1(\mathbb{D}_R)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{D}_R)$ for some $R > 0$. If φ solves*

$$\Delta\varphi \geqslant f \quad \text{in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\}),$$

then $\Delta\varphi \geqslant f$ in $\mathcal{D}'(\mathbb{D}_R)$.

The next result is crucial in our proof. We state it in a more general form than needed, as it could have an independent interest. Note that we do not need any *a priori* knowledge of the sign of ψ in the interior of its domain.

LEMMA 4.7. *For any $\lambda \in \mathbb{R}$ there exists $R_\lambda > 0$ such that for any $R \in (0, R_\lambda)$, $\varepsilon > 0$, problem*

$$\left.\begin{aligned} -\Delta\psi &\geqslant \lambda\psi \quad \text{in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\}), \\ \psi &\geqslant \varepsilon \qquad \text{on } \partial\mathbb{D}_R. \end{aligned}\right\} \tag{4.6}$$

has no solution $\psi \in \hat{H}^1(\mathbb{D}_R)$.

Proof. We fix sufficiently small $R_\lambda < \frac{1}{3}$ in such a way that

$$\lambda < \lambda_1(\mathbb{D}_{R_\lambda}) \qquad \text{if } \lambda \geqslant 0, \tag{4.7}$$

$$|\lambda||z|^2|\log|z||^2 \leqslant \tfrac{3}{4} \quad \text{for any } z \in \mathbb{D}_{R_\lambda} \quad \text{if } \lambda < 0. \tag{4.8}$$

We claim that the conclusion in lemma 4.7 holds with this choice of R_λ. We argue by contradiction. Let $R < R_\lambda$ and $\varepsilon > 0$, $\psi \in \hat{H}^1(\mathbb{D}_R)$ as in (4.6).

For any $\delta \in (\frac{1}{2}, 1)$ we introduce the following radially symmetric function on $\mathbb{D}_R$:

$$\varphi_\delta(z) = |\log|z||^{-\delta}.$$

By direct computation one can easily check that $\varphi_\delta \in \hat{H}^1(\mathbb{D}_R)$, and, in particular,

$$(2\delta - 1)\int_{\mathbb{D}_R} |z|^{-2}|\varphi_\delta|^2 = 2\pi + o(1) \quad \text{as } \delta \to \tfrac{1}{2}. \tag{4.9}$$

Since $\delta > \frac{1}{2}$, φ_δ is a smooth solution to

$$\Delta\varphi_\delta \geqslant \tfrac{3}{4}|z|^{-2}|\log|z||^{-2+\delta} = \tfrac{3}{4}|z|^{-2}|\log|z||^{-2}\varphi_\delta \tag{4.10}$$

in $\mathbb{D}_R \setminus \{0\}$. By lemma 4.6 we infer that φ_δ solves (4.10) in the dual of $\hat{H}^1(\mathbb{D}_R)$. Next we set

$$v := \varepsilon\varphi_\delta - \psi \in \hat{H}^1(\mathbb{D}_R),$$

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and we note that $v \leqslant 0$ on $\partial\mathbb{D}_R$, as $R < \frac{1}{3}$. Note also that

$$\begin{aligned}\Delta v &\geqslant \tfrac{3}{4}|z|^{-2}|\log|z||^{-2}(\varepsilon\varphi_\delta) + \lambda\psi \\ &= [\tfrac{3}{4}|z|^{-2}|\log|z||^{-2} + \lambda](\varepsilon\varphi_\delta) - \lambda v\end{aligned}$$

on the dual of $\hat{H}^1(\mathbb{D}_R)$, by (4.8). We use

$$v^+ := \max\{v, 0\} \in H^1_0(\mathbb{D}_R) \cap \hat{H}^1(\mathbb{D}_R)$$

as a test function to obtain

$$-\int_{\mathbb{D}_R} |\nabla v^+|^2 \geqslant \int_{\mathbb{D}_R} [\tfrac{3}{4}|z|^{-2}|\log|z||^{-2} + \lambda](\varepsilon\varphi_\delta)v^+ - \lambda\int_{\mathbb{D}_R} |v^+|^2.$$

If $\lambda \geqslant 0$, we infer that

$$\int_{\mathbb{D}_R} |\nabla v^+|^2 \leqslant \lambda\int_{\mathbb{D}_R} |v^+|^2$$

and hence $v^+ \equiv 0$ on $\mathbb{D}_R$ by (4.7). If $\lambda < 0$, we get

$$0 \geqslant -\int_{\mathbb{D}_R} |\nabla v^+|^2 \geqslant |\lambda|\int_{\mathbb{D}_R} |v^+|^2,$$

and hence again $v^+ = 0$ on $\mathbb{D}_R$, by (4.8). Thus $\psi \geqslant \varepsilon\varphi_\delta$ on $\mathbb{D}_R$, and therefore

$$\infty > \int_{\mathbb{D}_R} |z|^{-2}|\psi|^2 \geqslant \varepsilon\int_{\mathbb{D}_R} |z|^{-2}|\varphi_\delta|^2,$$

which contradicts (4.9). □

Proof of theorem 4.1. Without loss of generality, we may assume that $\lambda < 0$. Let $\Phi > 0$ be the first eigenfunction of $-\Delta_\sigma$ on Σ. Thus Φ solves

$$\left.\begin{aligned} -\Delta_\sigma\Phi = \lambda_1(\Sigma)\Phi \quad \text{in } \Sigma, \\ \Phi = 0, \quad \frac{\partial\Phi}{\partial\eta} \leqslant 0 \quad \text{on } \partial\Sigma,\end{aligned}\right\} \tag{4.11}$$

where $\eta \in T_\sigma(\mathbb{S}^{N-1})$ is the exterior normal to Σ at $\sigma \in \partial\Sigma$.

By density and the trace theorem, we can define the radially symmetric map ψ in $\mathbb{D}_R \setminus \{0\}$ as

$$\begin{aligned}\psi(z) &= |z|^{(N-2)/2}\int_\Sigma u(|z|\sigma)\Phi(\sigma)\,\mathrm{d}\sigma \\ &= |z|^{(N-2)/2}\int_{|z|\Sigma} u(\sigma')\Phi_{|z|}(\sigma')\,\mathrm{d}\sigma',\end{aligned} \tag{4.12}$$

where $\Phi_r(\sigma') = \Phi(\sigma'/r)$ for all $\sigma' \in r\Sigma$. Since, in polar coordinates $(r, \sigma) \in (0, \infty) \times \mathbb{S}^{N-1}$, it holds that

$$u_{rr} = -(N-1)r^{-1}u_r - r^{-2}\Delta_\sigma u;$$

direct computations based on (4.1) lead to

$$-\Delta\psi \geqslant \lambda\psi \quad \text{in } \mathcal{D}'(\mathbb{D}_R \setminus \{0\}).$$

We claim that $\psi \in \hat{H}^1(\mathbb{D}_R)$. Indeed, for $r = |z|$,

$$|\psi'| \leqslant cr^{(N-2)/2-1} \int_\Sigma |u(r\sigma)| + cr^{(N-2)/2} \int_\Sigma |\nabla u(r\sigma)|,$$

and, by the Hölder inequality,

$$\int_{\mathbb{D}_R} \left(r^{(N-2)/2-1} \int_\Sigma |u(r\sigma)| \right)^2 = c \int_0^R \int_\Sigma r^{N-3} u^2 \leqslant c \int_\Omega |x|^{-2} u^2 < \infty,$$

$$\int_{\mathbb{D}_R} \left(r^{(N-2)/2} \int_\Sigma |\nabla u(r\sigma)| \right)^2 \leqslant c \int_0^R r^{N-1} \int_\Sigma |\nabla u|^2 \leqslant c \int_\Omega |\nabla u|^2 < \infty.$$

Finally, $\psi \in L^2(R^2_R; |z|^{-2}\,\mathrm{d}z)$ as

$$\begin{aligned} \int_{\mathbb{D}_R} |z|^{-2} |\psi|^2 &= 2\pi \int_0^R r^{-1} |\psi|^2 \\ &\leqslant c \int_0^R r^{N-3} \int_\Sigma |u|^2 \\ &= c \int_\Omega |x|^{-2} |u|^2 < \infty. \end{aligned}$$

Thus, lemma 4.7 applies and since ψ is radially symmetric we obtain $\psi \equiv 0$ in a neighbourhood of 0. Hence, $u \equiv 0$ in $B_r \cap \mathcal{C}_\Sigma$ for sufficiently small $r > 0$. To conclude the proof for the case where Ω strictly contains $B_r \cap \mathcal{C}_\Sigma$, take any domain Ω' compactly contained in $\Omega \setminus \{0\}$ and such that Ω' intersects $B_r \cap \mathcal{C}_\Sigma$. Via a convolution procedure, approximate u in $H^1(\Omega')$ by a sequence of smooth maps u_ε that solve

$$-\Delta u_\varepsilon + |\lambda| u_\varepsilon \geqslant 0 \quad \text{in } \Omega'.$$

Since $u_\varepsilon \geqslant 0$ and $u_\varepsilon \equiv 0$ on $\Omega' \cap B_r \cap \mathcal{C}_\Sigma$, $u_\varepsilon \equiv 0$ on Ω' by the maximum principle. Thus also $u \equiv 0$ in Ω', and the conclusion follows. □

5. Remainder terms

We now prove some inequalities that will be used in the next section to estimate the infimum λ^* defined in (1.6).

Brézis and Vázquez proved [2] the following improved Hardy inequality:

$$\int_\Omega |\nabla u|^2 - \tfrac{1}{4}(N-2)^2 \int_\Omega |x|^{-2} |u|^2 \geqslant \omega_N \frac{\lambda_1(\mathbb{D})}{|\Omega|} \int_\Omega |u|^2, \tag{5.1}$$

which holds for any $u \in C^\infty_c(\Omega)$. Here $\Omega \subset \mathbb{R}^N$ is any bounded domain, $\lambda_1(\mathbb{D})$ is the first Dirichlet eigenvalue of the unit ball $\mathbb{D}$ in $\mathbb{R}^2$, and ω_N and $|\Omega|$ denote the measures of the unit ball in $\mathbb{R}^N$ and of Ω, respectively. If $0 \in \Omega$, then $\frac{1}{4}(N-2)^2$ is the Hardy constant $\mu_0(\Omega)$ relative to the domain Ω, by the invariance of the ratio

$$\frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega |x|^{-2}|u|^2\,\mathrm{d}x}$$

with respect to dilations in $\mathbb{R}^N$.

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We show that a Brézis–Vázquez-type inequality holds in cases where the singularity is placed at the boundary of the domain. We start with conic domains

$$\mathcal{C}_{R,\Sigma} = \{t\sigma \mid t \in (0,R),\ \sigma \in \Sigma\},$$

where $\Sigma \subset \mathbb{S}^{N-1}$ and $R > 0$.

PROPOSITION 5.1. *Let Σ be a domain in $\mathbb{S}^{N-1}$. Then*

$$\int_{\mathcal{C}_{R,\Sigma}} |\nabla u|^2 - \mu_0(\mathcal{C}_\Sigma) \int_{\mathcal{C}_{R,\Sigma}} |x|^{-2}|u|^2 \geqslant \frac{\lambda_1(\mathbb{D})}{R^2} \int_{\mathcal{C}_{R,\Sigma}} |u|^2 \quad \forall u \in C_c^\infty(\mathcal{C}_{1,\Sigma}). \tag{5.2}$$

Proof. By homogeneity, it suffices to prove the proposition for $R = 1$. Fix $u \in C_c^\infty(\mathcal{C}_{1,\Sigma})$ and compute in polar coordinates $t = |x|$, $\sigma = x/|x|$:

$$\int_{\mathcal{C}_{1,\Sigma}} |\nabla u|^2 = \int_0^1 \int_\Sigma \left|\frac{\partial u}{\partial t}\right|^2 t^{N-1}\,\mathrm{d}t\,\mathrm{d}\sigma + \int_0^1 \int_\Sigma |\nabla_\sigma u|^2 t^{N-3}\,\mathrm{d}t\,\mathrm{d}\sigma,$$

$$\int_{\mathcal{C}_{1,\Sigma}} |x|^{-2}|u|^2 = \int_0^1 \int_\Sigma |u|^2 t^{N-3}\,\mathrm{d}t\,\mathrm{d}\sigma.$$

Since, for every $t \in (0,1)$, it holds that

$$\int_\Sigma |\nabla_\sigma u|^2 t^{N-3}\,\mathrm{d}\sigma \geqslant \lambda_1(\Sigma) \int_\Sigma |u|^2 t^{N-3}\,\mathrm{d}\sigma,$$

by proposition 2.1, we only have to show that

$$\int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 t^{N-1}\,\mathrm{d}t - \tfrac{1}{4}(N-2)^2 \int_0^1 |u|^2 t^{N-3}\,\mathrm{d}t \geqslant \lambda_1(\mathbb{D}) \int_0^1 |u|^2 t^{N-1}\,\mathrm{d}t \tag{5.3}$$

for any fixed $\sigma \in \Sigma$. For that, we set $w(t) = t^{(N-2)/2}u(t\sigma)$, and we compute

$$\begin{aligned}
\int_0^1 \left|\frac{\partial u}{\partial t}\right|^2 t^{N-1}\,\mathrm{d}t - \mu_0(\mathbb{R}^N) \int_0^1 |u|^2 t^{N-3}\,\mathrm{d}t &= \int_0^1 \left|\frac{\partial w}{\partial t}\right|^2 t\,\mathrm{d}t + (2-N) \int_0^1 \frac{\partial w}{\partial t} w\,\mathrm{d}t \\
&= \int_0^1 \left|\frac{\partial w}{\partial t}\right|^2 t\,\mathrm{d}t + \tfrac{1}{2}(2-N) \int_0^1 \frac{\partial w^2}{\partial t}\,\mathrm{d}t \\
&= \int_0^1 \left|\frac{\partial w}{\partial t}\right|^2 t\,\mathrm{d}t \\
&\geqslant \lambda_1(\mathbb{D}) \int_0^1 w^2 t\,\mathrm{d}t \\
&= \lambda_1(\mathbb{D}) \int_0^1 |u|^2 t^{N-1}\,\mathrm{d}t.
\end{aligned}$$

This gives (5.3) and the proposition is proved. □

The main result of this section is contained in the next theorem.

THEOREM 5.2. *Let Ω be a bounded domain of $\mathbb{R}^N$ with $0 \in \partial\Omega$. If Ω is contained in a half-space, then*

$$\int_\Omega |\nabla u|^2 - \mu^+ \int_\Omega |x|^{-2}|u|^2 \geqslant \frac{\lambda_1(\mathbb{D})}{|\mathrm{diam}(\Omega)|^2} \int_\Omega |u|^2 \quad \forall u \in H_0^1(\Omega).$$

Proof. Let $R > 0$ be the diameter of Ω. Then $\Omega \subset B_R^+$, where B_R^+ is a half-ball of radius R centred at the origin. Take Σ to be a half-sphere in $\mathbb{S}^{N-1}$ in proposition 5.1 so that $\mathcal{C}_\Sigma$ is a half-space. Recalling (2.4), we conclude that

$$\int_{B_R^+} |\nabla u|^2 - \mu^+ \int_{B_R^+} |x|^{-2}|u|^2 \geqslant \frac{\lambda_1(\mathbb{D})}{R^2} \int_{B_R^+} |u|^2$$

for any $R > 0$, $u \in C_c^\infty(\Omega)$, and the theorem readily follows. □

Remark 5.3. Let Ω be a bounded domain of $\mathbb{R}^2$ with $0 \in \partial\Omega$ and assume that Ω does not intersect a half-line emanating from the origin. Then (2.5) and proposition 5.1 imply the following improved Hardy inequality:

$$\int_\Omega |\nabla u|^2 - \frac{1}{4}\int_\Omega |x|^{-2}|u|^2 \geqslant \frac{\lambda_1(\mathbb{D})}{|\mathrm{diam}(\Omega)|^2}\int_\Omega |u|^2 \quad \forall u \in H_0^1(\Omega).$$

Remark 5.4. As pointed out in [2, extension 4.3], the following Hardy–Sobolev inequality holds:

$$\int_{\mathcal{C}_{1,\Sigma}} |\nabla u|^2 - \mu_0(\mathcal{C}_{1,\Sigma}) \int_{\mathcal{C}_{1,\Sigma}} |x|^{-2}|u|^2 \geqslant c_p \biggl(\int_{\mathcal{C}_{1,\Sigma}} |u|^p\biggr)^{2/p} \quad \forall u \in C_c^\infty(\mathcal{C}_{1,\Sigma})$$

for all $p \in (2, 2N/(N-2))$, where c_p is a positive constant depending on p and N.

6. Estimates on λ^*

In this section we provide sufficient conditions to have $\lambda^* > -\infty$ or $\lambda^* < 0$.

6.1. Estimates from below

Let Ω be a smooth domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$. We say that Ω is *locally convex* at 0 if there exists a ball B centred at 0 such that $\Omega \cap B$ is contained in a half-space. In essence, for domains of class C^2, this means that all the principal curvatures of $\partial\Omega$ (with respect to the interior normal) at 0 are strictly positive.

In the case where Ω is locally convex at $0 \in \partial\Omega$, the supremum in lemma 3.1 is attained.

Proposition 6.1. *If Ω is locally convex at 0, then there exists $\lambda^*(\Omega) \in \mathbb{R}$ such that*

$$\mu_\lambda(\Omega) = \mu^+ \quad \forall \lambda \leqslant \lambda^*(\Omega),$$
$$\mu_\lambda(\Omega) < \mu^+ \quad \forall \lambda > \lambda^*(\Omega).$$

Proof. The local convexity assumption at 0 means that there exists $r > 0$ such that $B_r(0) \cap \Omega$ is contained in a half-space. We let $\psi \in C_c^\infty(\mathbb{R}^N)$ with $0 \leqslant \psi \leqslant 1$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_{r/2}(0)$ and $\psi \equiv 1$ in $B_{r/4}(0)$. Arguing in the same way as in the proof of lemma 3.1, for every $u \in H_0^1(\Omega)$ we obtain

$$\int_\Omega |x|^{-2}|u|^2 \,\mathrm{d}x \leqslant \int_\Omega |x|^{-2}|\psi u|^2 \,\mathrm{d}x + c\int_\Omega |u|^2 \,\mathrm{d}x \tag{6.1}$$

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for some constant $c = c(r) > 0$. Since $\psi u \in H_0^1(B_r(0) \cap \Omega)$, from the definition of μ^+ we infer

$$\mu^+ \int_\Omega |x|^{-2}|\psi u|^2 \, dx \leqslant \int_\Omega |\nabla(\psi u)|^2 \, dx.$$

As in lemma 3.1, we obtain

$$\int_\Omega |\nabla(\psi u)|^2 \, dx \leqslant \int_\Omega |\nabla u|^2 \, dx + c \int_\Omega |u|^2 \, dx.$$

Comparing this with (6.1), we infer that there exists a positive constant c such that

$$\mu^+ \int_\Omega |x|^{-2}|u|^2 \, dx \leqslant \int_\Omega |\nabla u|^2 \, dx + c \int_\Omega |u|^2 \, dx.$$

This proves that $\mu_{-c}(\Omega) \geqslant \mu^+$. Thus, $\mu_{-c}(\Omega) = \mu^+$ by lemma 3.1. Finally, noting that $\mu_\lambda(\Omega)$ is decreasing in λ, we can set

$$\lambda^*(\Omega) := \sup\{\lambda \in \mathbb{R} \colon \mu_\lambda(\Omega) = \mu^+\} \tag{6.2}$$

so that $\mu_\lambda(\Omega) < \mu^+$ for all $\lambda > \lambda^*(\Omega)$. □

Finally, we note that, by lemma 3.1, if Ω is contained in a half-space, then $\mu_0(\Omega) = \mu^+$, and therefore $\lambda^*(\Omega) \geqslant 0$. Thus, from theorem 5.2 we infer the following result.

Theorem 6.2. *Let Ω be a bounded smooth domain with $0 \in \partial\Omega$. If Ω is contained in a half-space, then*

$$\lambda^*(\Omega) \geqslant \frac{\lambda_1(\mathbb{D})}{|\mathrm{diam}(\Omega)|^2}.$$

It would be of interest to know whether it is possible to obtain lower bounds depending only on the measure of Ω, as in [2, 13].

6.2. Estimates from above

The local convexity assumption of Ω at 0 does not necessarily imply that $\lambda^*(\Omega) \geqslant 0$. Indeed, the following remark holds.

Proposition 6.3. *For any $\delta > 0$, there exists $\rho_\delta > 0$ such that if Ω is a smooth domain with $0 \in \partial\Omega$ and*

$$\Omega \supseteq \{x \in \mathbb{R}^N \mid x \cdot \nu > -\delta|x|, \ \alpha < |x| < \beta\}$$

for some $\nu \in \mathbb{S}^{N-1}$, $\beta > \alpha > 0$ with $\beta/\alpha > \rho_\delta$, then $\lambda^* < 0$. In particular, the Hardy constant $\mu_0(\Omega)$ is achieved.

Proof. Since the cone

$$\mathcal{C}_\delta = \{x \in \mathbb{R}^N \mid x \cdot \nu > -\delta|x|\}$$

contains a half-space, then its Hardy constant is smaller than μ^+. Thus, there exists $u \in C_c^\infty(\mathcal{C}_\delta)$ such that

$$\frac{\int_{\mathcal{C}_\delta} |\nabla u|^2 \, dx}{\int_{\mathcal{C}_\delta} |x|^{-2}|u|^2 \, dx} < \mu^+.$$

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Assume that the support of u is contained in an annulus of radii $b > a > 0$. Then the conclusion in proposition 6.3 holds, with $\rho := b/a$. □

Note that Ω can be locally strictly convex at 0.

REMARK 6.4. A similar remark holds for the following minimization problem, which is related to the Caffarelli–Kohn–Nirenberg inequalities:

$$\inf_{u\in H_0^1(\Omega),\, u\neq 0} \frac{\int_\Omega |\nabla u|^2 \, dx}{(\int_\Omega |x|^{-b}|u|^p \, dx)^{2/p}}, \tag{6.3}$$

where $2 < p < 2^*$, $b := N - p\frac{1}{2}(N-2)$. In the case where $0 \in \partial\Omega$, the minimization problem (6.3) was studied in [10–12].

REMARK 6.5. We do not know whether the strict local concavity of Ω at 0 implies that $\mu_0(\Omega) < \mu^+$ (see [10] for the minimization problem (6.3)).

Note added in proof

After this paper was submitted for publication, it was proved in [8] that $\lambda^*(\Omega) < \infty$ whenever Ω is a smooth bounded domain, and that the strict local concavity of Ω at 0 does not necessarily imply that $\mu_0(\Omega) < \mu^+$. We also cite [9] for some non-existence results related to theorem 4.1.

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