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Sharp decay estimates in a bioconvection model with quadratic degradation in bounded domains

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The paper studies large time behaviour of solutions to the Keller–Segel system with quadratic degradation in a liquid environment, as given by

$$\left.\begin{aligned} &u_t + U\cdot\nabla u = \Delta u - \nabla\cdot(u\nabla v) - \mu u^2, && x\in\Omega,\ t>0,\\ &v_t + U\cdot\nabla v = \Delta v - v + u, && x\in\Omega,\ t>0,\end{aligned}\right\} \qquad (\star)$$

under Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^n$, where $n\geqslant 1$ is arbitrary. It is shown that whenever $U\colon \Omega\times(0,\infty)\to\mathbb{R}^n$ is a bounded and sufficiently regular solenoidal vector field any non-trivial global bounded solution of $(\star)$ approaches the trivial equilibrium at a rate that, with respect to the norm in either of the spaces $L^1(\Omega)$ and $L^\infty(\Omega)$, can be controlled from above and below by appropriate multiples of $1/(t+1)$. This underlines that, even up to this quantitative level of accuracy, the large time behaviour in $(\star)$ is essentially independent not only of the particular fluid flow, but also of any effect originating from chemotactic cross-diffusion. The latter is in contrast to the corresponding Cauchy problem, for which known results show that in the $n=2$ case the presence of chemotaxis can significantly enhance biomixing by reducing the respective spatial L^1 norms of solutions.

Keywords: chemotaxis; bioconvection; asymptotic behaviour; decay rate

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1. Introduction

We consider non-negative solutions of the boundary-value problem

$$\left.\begin{aligned} &u_t + U\cdot\nabla u = \Delta u - \chi\nabla\cdot(u\nabla v) - \mu u^2, && x\in\Omega, && t>0,\\ &v_t + U\cdot\nabla v = \Delta v - v + u, && x\in\Omega, && t>0,\\ &\nabla\cdot U\equiv 0, && x\in\Omega, && t>0,\\ &\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \quad u\equiv 0, && x\in\partial\Omega, && t>0,\end{aligned}\right\} \qquad (1.1)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $n \geqslant 1$, where $\chi > 0$ and μ are positive parameters and where $U\colon \Omega \times (0,\infty) \to \mathbb{R}^n$ is a prescribed solenoidal vector field. Systems of this type arise in the macroscopic modelling of chemotactic migration under the influence of a liquid environment by transport through a given fluid, and in the presence of quadratic degradation such as appears in logistic-type cell kinetics. Here we focus on situations in which cell proliferation in logistic models represented by linear production terms either can be neglected on the timescales considered, or is absent in principle. A prototypical example for the latter arises in the context of coral broadcast spawning processes [2, 6], during which eggs release a chemical signal, with concentration $v = v(x,t)$, which attracts sperms, where both eggs and sperms jointly constitute a population with density $u = u(x,t)$, and where the transporting incompressible ocean flow is represented through its velocity field $U = U(x,t)$.

In the fluid-free case when $U \equiv 0$, a variety of previous results indicate the cross-diffusive mechanism in (1.1) has a quite substantial effect that goes far beyond well-established knowledge on the ability of the classical Keller–Segel system obtained on letting $\mu = 0$, i.e. of

$$\left.\begin{aligned} u_t &= \Delta u - \chi\nabla\cdot(u\nabla v),\\ v_t &= \Delta v - v + u, \end{aligned}\right\} \tag{1.2}$$

to generate singularities in the sense of finite-time blow-up of some solutions in two- and higher-dimensional settings [5, 18]. Indeed, also in situations when $\mu > 0$ in

$$\left.\begin{aligned} u_t &= \Delta u - \chi\nabla\cdot(u\nabla v) - \mu u^2,\\ v_t &= \Delta v - v + u, \end{aligned}\right\} \tag{1.3}$$

and related systems, the destabilizing action of cross-diffusion may still enforce quite complex solution behaviour in comparison to the respective scalar absorptive parabolic equation, as indicated by numerical experiments [11] and rigorously confirmed by results on the spontaneous emergence of large population densities at intermediate timescales (see [20]; see also [8, 19] for similar findings on associated parabolic–elliptic simplifications). In fact, even the drastic phenomenon of finite-time blow-up has been shown to be suppressed by the presence of quadratic degradation only when either $n \leqslant 2$ [9, 10] or $n \geqslant 3$ and μ is suitably large (see [16]; see also [14] for a precedent). The question of how far such systems are globally solvable when $n \geqslant 3$ and $\mu > 0$ is small has so far only been partially been answered by a statement on the global existence of weak solutions, possibly unbounded but, at least in the $n = 3$ case, eventually bounded and smooth and asymptotically decaying in both components [8]. Strong cross-diffusive effects also manifest in blow-up examples despite certain subquadratic but superlinear degradation terms in some appropriately high-dimensional chemotaxis systems [17].

In the light of these premises, for the investigation of common large-scale qualitative features of solutions to (1.1) in general n-dimensional frameworks, it seems adequate to resort explicitly to situations when solutions are globally regular. With a time shift if necessary, this will in fact cover widely arbitrary solutions to (1.3) in all physically relevant $n \leqslant 3$ cases, but will also capture more complex frameworks in which the fluid evolution itself is unknown, affected, for example by the cell population, and governed by appropriate equations from fluid mechanics (see [1] for corresponding modelling aspects), at least in situations when the chemotaxis–fluid system obtained is globally solvable by suitably regular functions [12, 13]. Accordingly, the aim of this work is to describe the large time behaviour of arbitrary global bounded solutions to (1.1) in bounded domains for any $n \geqslant 1$, ignoring the question of whether such solutions exist under particular assumptions on supposedly prescribed initial data $(u_0, v_0) \equiv (u(\cdot,0), v(\cdot,0))$. Hence, assuming a sufficiently smooth vector field U and a non-trivial global bounded classical solution (u, v) of (1.1) as given, we shall focus on deriving optimal estimates for the decay rate of $u(\cdot,t)$ with respect to the norms both in $L^\infty(\Omega)$ and in $L^1(\Omega)$, bearing in mind the particular biological relevance of the latter as representing the total mass of the population considered.

Previous work in this direction addresses the Cauchy problem in $\Omega = \mathbb{R}^2$ for a simplified parabolic–elliptic variant of (1.1) that can be rewritten in the form of a scalar non-local parabolic equation:

$$u_t + U \cdot \nabla u = \Delta u + \chi \nabla \cdot (u \nabla (\Delta)^{-1} u) - \mu u^q, \tag{1.4}$$

with an additional parameter $q \geqslant 2$. For this problem with initial condition $u(\cdot,0) = u_0 \in L^1(\mathbb{R}^2)$, in the $q > 2$ case any sufficiently regular non-negative global solution u is known to satisfy

$$\int_{\mathbb{R}^2} u(\cdot,t) \to m_\infty(\chi, u_0, U) \quad \text{as } t \to \infty$$

with some $m_\infty(\chi, u_0, U) > 0$ satisfying $m_\infty(\chi, u_0, U) \to 0$ as $\chi \to \infty$ [6]. In the critical case $q = 2$, the influence of chemotaxis on the evolution of the total mass functional, which then decays to zero when $\chi > 0$ and $\chi = 0$, has been shown to exist but to be more subtle in character, and mainly relevant on finite time intervals [7].

1.1. Main results

We shall show that in our considered framework of bounded domains, unlike in the latter Cauchy problem, the solution behaviour in (1.1) is essentially unaffected by chemotaxis, at least on large timescales. Indeed, throughout the paper, assuming for simplicity that

$$\begin{aligned} &U \in C^{1,0}(\bar{\Omega} \times [0,\infty); \mathbb{R}^n) \cap L^\infty(\Omega \times (0,\infty); \mathbb{R}^n) \\ &\qquad \text{is such that } \nabla \cdot U \equiv 0 \text{ in } \Omega \times (0,\infty) \text{ and } U \equiv 0 \text{ on } \delta\Omega \times (0,\infty), \end{aligned} \tag{1.5}$$

we shall see that, for any given non-trivial and sufficiently regular bounded solution of (1.1), with respect to the norms in either $X := L^1(\Omega)$ or $L^\infty(\Omega)$, the quantity $\|u(\cdot,t)\|_X$ can be estimated from above and below by positive multiples, possibly depending on the solution, e.g. through its norm in $L^\infty(\Omega \times (0,\infty))$, of $1/(t+1)$. More precisely, our main results read as follows.

THEOREM 1.1. *Let $n \geqslant 1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume that $\mu > 0$ and that U satisfies (1.5), and suppose that $(u, v) \in (C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)))^2$ is a classical solution of (1.1) for which both u and v are non-negative, and which is bounded in the sense that u belongs to $L^\infty(\Omega \times (0,\infty))$.*

(i) *There exists $C_1 > 0$ with the property that*

$$\frac{1}{|\Omega|}\|u(\cdot,t)\|_{L^1(\Omega)} \leqslant \|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \frac{C_1}{t+1} \quad \textit{for all } t > 0. \tag{1.6}$$

(ii) *If, furthermore, $u \not\equiv 0$, then one can find $C_2 > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \geqslant \frac{1}{|\Omega|}\|u(\cdot,t)\|_{L^1(\Omega)} \geqslant \frac{C_2}{t+1} \quad \textit{for all } t > 0. \tag{1.7}$$

Note that here we do not explore how the constants in the above statements depend on χ and μ, but not on the function U, thus leaving open the question of whether chemotactic cross-diffusion may influence a fine structure in the large time asymptotics of solutions.

In corresponding chemotaxis–fluid systems in which the fluid evolution itself is affected by the presence of the other quantities, e.g. through buoyant forces, the above results can be directly applied to solutions known *a priori* to enjoy the above regularity and boundedness properties; for two- and three-dimensional examples of situations when the latter is in fact guaranteed for all reasonably regular initial data, we refer the reader to [12,13]. However, theorem 1.1 is actually more general, as it considers widely arbitrary fluid fields that do not necessarily receive any feedback from the taxis components.

1.2. Plan of the paper

The main idea underlying our approach is directly motivated by the ultimate result: our analysis aims to show appropriate negligibility of the cross-diffusive action in (1.1) in comparison to the further mechanisms therein. After establishing preliminary but fundamental decay information on solutions in $L^1(\Omega) \times L^1(\Omega)$ in § 2, we show appropriate negligibility in § 3 on the basis of the fundamental decay information using a series of arguments that rely on the smoothing action of the heat semigroup in the second equation in (1.1). In § 4 a primary application of the outcome in will yield the estimate from theorem 1.1(i). A second application in § 5 will show that in the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \ln u \geqslant -\frac{\chi^2}{4}\int_\Omega |\nabla v|^2 - \mu\int_\Omega u, \quad t > 0,$$

which constitutes the key step in our proof of theorem 1.1(ii), the summand originating from the taxis term in (1.1) decays quickly enough that it is asymptotically irrelevant.

2. Upper decay estimates for u and v in $L^1(\Omega)$

The following basic one-sided decay estimates for the spatial L^1 norms of both solution components can be gained in a quite simple way, and similar observations have been made in [12, lemma 5.1]. Since these estimates will be fundamental to our analysis, and since, in particular, they quantitatively underline the difference between the bounded Ω and $\Omega = \mathbb{R}^n$ cases, we include a short proof here.

Lemma 2.1. *Let (u,v) be a non-negative global classical solution of (1.1). Then*

$$\int_\Omega u(\cdot,t) \leqslant \frac{|\Omega|}{\mu}\frac{1}{t+\gamma} \quad \textit{for all } t>0 \tag{2.1}$$

and

$$\int_\Omega v(\cdot,t) \leqslant \frac{K}{t+2} \quad \textit{for all } t>0, \tag{2.2}$$

where

$$\gamma := \frac{|\Omega|}{\mu\int_\Omega u(\cdot,0)} \tag{2.3}$$

and

$$K := \max\left\{2\int_\Omega v(\cdot,0), 4\int_\Omega u(\cdot,0), \frac{2|\Omega|}{\mu}\right\}. \tag{2.4}$$

Proof. We only need to consider the case when $u(\cdot,0)\not\equiv 0$, in which, according to (1.1) and the Cauchy–Schwarz inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = -\mu\int_\Omega u^2 \leqslant -\frac{\mu}{|\Omega|}\left\{\int_\Omega u\right\}^2 \quad \text{for all } t>0,$$

which on integration readily implies (2.1) with γ as in (2.3).

Since from (1.1) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v = -\int_\Omega v + \int_\Omega u \quad \text{for all } t>0,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v \leqslant -\int_\Omega v + \frac{|\Omega|}{\mu(t+\gamma)} \quad \text{for all } t>0. \tag{2.5}$$

Now, with K as given by (2.4), $\bar{y}(t) := K/(t+2)$, $t\geqslant 0$, satisfies

$$\bar{y}(0) = \frac{K}{2} \geqslant \int_\Omega v(\cdot,0)$$

by (2.4), and therefore

$$\begin{aligned}
\bar{y}'(t) + \bar{y}(t) - \frac{|\Omega|}{\mu(t+\gamma)} &= -\frac{K}{(t+2)^2} + \frac{K}{t+2} - \frac{|\Omega|}{\mu(t+\gamma)} \\
&= \frac{K}{t+2}\left\{1 - \frac{1}{t+2} - \frac{|\Omega|}{K\mu}\frac{t+2}{t+\gamma}\right\} \\
&\geqslant \frac{K}{t+2}\left\{1 - \frac{1}{2} - \frac{|\Omega|}{K\mu}\max\left\{\frac{2}{\gamma},1\right\}\right\} \\
&= \frac{K}{2(t+2)}\left\{1 - \frac{1}{K}\max\left\{4\int_\Omega u(\cdot,0), \frac{2|\Omega|}{\mu}\right\}\right\} \\
&\geqslant 0 \quad \text{for all } t>0,
\end{aligned}$$

due to (2.3) and the second and third restrictions in (2.4). By an ordinary differential equation comparison, we thus conclude from (2.5) that $\int_\Omega v(\cdot,t) \leqslant \bar{y}(t)$ for all $t>0$, and that (2.2) indeed holds. □

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3. Boundedness and decay properties of ∇v

A crucial step towards both parts of theorem 1.1 is to identify the cross-diffusive term in (1.1) as being asymptotically negligible relative to the diffusive action in (1.1), by deriving appropriate quantitative bounds for the chemotactic gradient ∇v. This will be done in this section by making use of the L^1 decay property of u from lemma 2.1 to obtain the decay of ∇v at an apparently optimal rate in an unfavourable topology, and then establishing an assumption on the boundedness of u to establish boundedness of v in certain higher norms without any decay information. Interpolating these two extremal results will yield a decay result for ∇v in arbitrary L^p spaces at a rate that is probably far from optimal but is sufficient for our purposes.

For what follows, recall that for $p \in (1,\infty)$ the realization $A = A_p$ of $-\Delta + 1$ under homogeneous Neumann boundary conditions, i.e. the operator defined by letting $A_p\varphi := -\Delta\varphi + \varphi$ for $\varphi \in D(A_p) := \{\varphi \in W^{2,p}(\Omega) \mid \partial\varphi/\partial\nu = 0 \text{ on } \partial\Omega\}$, is sectorial in the space $L^p(\Omega)$, with its spectrum contained in the half-line $[1,\infty)$. Accordingly, A possesses closed and densely defined fractional powers A^β for all $\beta \in \mathbb{R}$, and A^β is bounded whenever $\beta < 0$ [4, theorem 1.4.2].

Now, the space $L^1(\Omega)$ is continuously embedded into a suitable space $D(A^{-\beta})$ obtained analogously, an explicit definition of which is unnecessary and thus omitted here. We focus rather on an associated embedding inequality.

Lemma 3.1. *Let $p > 1$ and $\beta > n(p-1)/2p$. Then there exists $C > 0$ such that*

$$\|A^{-\beta}\varphi\|_{L^p(\Omega)} \leqslant C\|\varphi\|_{L^1(\Omega)} \quad \textit{for all } \varphi \in L^1(\Omega). \tag{3.1}$$

Proof. Since $\beta > n(p-1)/2p$ implies that $p' := p/(p-1)$ satisfies $2\beta - n/p' > 0$, it follows from known embedding results [4, theorem 1.6.1] that $D(A_{p'}^{\beta}) \hookrightarrow L^\infty(\Omega)$, whence there exists $c_1 > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)} \leqslant c_1\|A^\beta\phi\|_{L^{p'}(\Omega)} \quad \text{for all } \phi \in D(A_{p'}^{\beta}). \tag{3.2}$$

Thus, given any $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(\Omega)$, using the self-adjointness of $A^{-\beta}$ in $L^2(\Omega)$ we can estimate

$$\left|\int_\Omega A^{-\beta}\varphi\cdot\psi\right| = \left|\int_\Omega \varphi\cdot A^{-\beta}\psi\right| \leqslant \|\varphi\|_{L^1(\Omega)}\|A^{-\beta}\psi\|_{L^\infty(\Omega)} \leqslant c_1\|\varphi\|_{L^1(\Omega)}\|\psi\|_{L^{p'}(\Omega)}.$$

Therefore,

$$\|A^{-\beta}\varphi\|_{L^p(\Omega)} = \sup_{\varphi\in C_0^\infty(\Omega),\|\psi\|_{L^{p'}(\Omega)}\leqslant 1}\left|\int_\Omega A^{-\beta}\varphi\psi\right| \leqslant c_1\|\varphi\|_{L^1(\Omega)},$$

as claimed. □

By appropriately making use of lemma 3.1 in the course of an argument based on a variation-of-constants representation of v, we see that, with respect to the norm in $L^p(\Omega)$ for suitably small $p > 1$, ∇v inherits the decay rate of the mass functional $\int_\Omega u$ from lemma 2.1.

Lemma 3.2. *Let (u, v) be a non-negative global classical solution of (1.1). Then, for all $p \in (1, n/(n-1))$, one can find $C(p) > 0$ such that*

$$\|\nabla v(\cdot, t)\|_{L^p(\Omega)} \leqslant \frac{C(p)}{t} \quad \textit{for all } t \geqslant 2. \tag{3.3}$$

Proof. Since

$$\frac{n}{n - 2(1-\alpha)} \to \frac{n}{n-1} > p \quad \text{as } \alpha \searrow \frac{1}{2},$$

it is possible to fix $\alpha \in (\frac{1}{2}, 1)$ such that $p < n/(n - 2(1-\alpha))$, which means that

$$\alpha + \frac{n}{2}\left(1 - \frac{1}{p}\right) < 1. \tag{3.4}$$

We choose an arbitrary $\varepsilon \in (0, \alpha - \frac{1}{2})$ and pick $\beta > n(p-1)/2p$, so that, since $D(A_p^{1/2+\varepsilon}) \hookrightarrow W^{1,p}(\Omega)$ [4, theorem 1.6.1], by employing a well-known interpolation argument [3, theorem 14.1] we can find $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \|\nabla v(\cdot, t)\|_{L^p(\Omega)} &\leqslant c_1 \|A^{1/2+\varepsilon} v(\cdot, t)\|_{L^p(\Omega)} \\ &\leqslant c_2 \|A^{\alpha} v(\cdot, t)\|_{L^p(\Omega)}^{a} \|A^{-\beta} v(\cdot, t)\|_{L^p(\Omega)}^{1-a} \quad \text{for all } t > 0, \end{aligned} \tag{3.5}$$

where

$$a := \frac{\frac{1}{2} + \varepsilon + \beta}{\alpha + \beta} \in (0, 1).$$

Here the fact that $\beta > n(p-1)/2p$ enables us to invoke lemma 3.1 and then apply lemma 2.1 to find $c_3 > 0$ and $c_4 > 0$ such that

$$\|A^{-\beta} v(\cdot, t)\|_{L^p(\Omega)} \leqslant c_3 \|v(\cdot, t)\|_{L^1(\Omega)} \leqslant \frac{c_3 c_4}{t} \quad \text{for all } t > 0. \tag{3.6}$$

Now, in order to derive (3.3), by means of a variation-of-constants representation of v we write

$$\begin{aligned} v(\cdot, t) = \mathrm{e}^{-A} v(\cdot, t-1) &+ \int_{t-1}^{t} \mathrm{e}^{-(t-s)A} u(\cdot, s)\, \mathrm{d}s \\ &+ \int_{t-1}^{t} \mathrm{e}^{-(t-s)A} U(\cdot, s) \cdot \nabla v(\cdot, s)\, \mathrm{d}s, \quad t \geqslant 1, \end{aligned}$$

and apply A^α to both sides to see that

$$\begin{aligned} \|A^\alpha v(\cdot, t)\|_{L^p(\Omega)} &\leqslant \|A^\alpha \mathrm{e}^{-A} v(\cdot, t-1)\|_{L^p(\Omega)} \\ &+ \int_{t-1}^{t} \|A^\alpha \mathrm{e}^{-(t-s)A} u(\cdot, s)\|_{L^p(\Omega)}\, \mathrm{d}s \\ &+ \int_{t-1}^{t} \|A^\alpha \mathrm{e}^{-(t-s)A} U(\cdot, s) \cdot \nabla v(\cdot, s)\|_{L^p(\Omega)}\, \mathrm{d}s \quad \text{for all } t \geqslant 1. \end{aligned} \tag{3.7}$$

Here, according to the known smoothing properties of $(\mathrm{e}^{-\tau A})_{\tau\geqslant 0}$ and by lemma 2.1, there exist $c_5>0$ and $c_6>0$ satisfying

$$\|A^\alpha \mathrm{e}^{-A}v(\cdot,t-1)\|_{L^p(\Omega)} \leqslant c_5\|v(\cdot,t-1)\|_{L^1(\Omega)} \leqslant \frac{c_6}{t-1} \quad \text{for all } t\geqslant 2. \tag{3.8}$$

Making use of lemma 2.1 and (3.4), we can find $c_7>0$ and $c_8>0$, again using a standard semigroup estimate such that

$$\begin{aligned}\int_{t-1}^t \|A^\alpha \mathrm{e}^{-(t-s)A}u(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s &\leqslant c_7\int_{t-1}^t (t-s)^{-\alpha-n(1-1/p)/2}\|u(\cdot,s)\|_{L^1(\Omega)}\,\mathrm{d}s\\ &\leqslant c_8\int_{t-1}^t (t-s)^{-\alpha-n(1-1/p)/2}\frac{1}{s}\,\mathrm{d}s\\ &\leqslant c_8\frac{1}{t-1}\int_{t-1}^t (t-s)^{-\alpha-n(1-1/p)/2}\,\mathrm{d}s\\ &= \frac{c_8}{1-\alpha-n(1-1/p)/2}\frac{1}{t-1} \quad \text{for all } t\geqslant 2.\end{aligned} \tag{3.9}$$

Finally, to treat the last summand in (3.7) appropriately, we introduce the numbers

$$M(T) := \sup_{t\in(1,T)}\{t\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}\}, \quad T>2,$$

which are all finite due to our overall assumption that $v\in C^{2,1}(\bar\Omega\times(0,\infty))$. In terms of $M(T)$, by the boundedness of U, (3.5) and (3.6), with some $c_9>0$ and $c_{10}>0$, the integral in question can be estimated according to

$$\begin{aligned}&\int_{t-1}^t \|A^\alpha \mathrm{e}^{-(t-s)A}U(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s\\ &\quad\leqslant c_9\int_{t-1}^t (t-s)^{-\alpha}\|U(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s\\ &\quad\leqslant c_{10}\int_{t-1}^t (t-s)^{-\alpha}\|\nabla v(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s\\ &\quad\leqslant c_2c_{10}\int_{t-1}^t (t-s)^{-\alpha}\left\{\frac{M(T)}{s}\right\}^a\left\{\frac{c_3c_4}{s}\right\}^{1-a}\,\mathrm{d}s\\ &\quad= c_2c_3c_4c_{10}M^a(T)\int_{t-1}^t (t-s)^{-\alpha}\frac{1}{s}\,\mathrm{d}s\\ &\quad\leqslant c_2(c_3c_4)^{1-a}c_{10}M^a(T)\frac{1}{t-1}\int_{t-1}^t (t-s)^{-\alpha}\,\mathrm{d}s\\ &\quad= \frac{c_2(c_3c_4)^{1-a}c_{10}}{1-\alpha}M^a(T)\frac{1}{t-1} \quad \text{for all } t\in[2,T].\end{aligned}$$

Combined with (3.7)–(3.9), in view of the fact that $1/(t-1)\leqslant 2/t$ for all $t\geqslant 2$, this shows that there exists $c_{11}>0$ such that, for each $T>2$,

$$t\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \leqslant c_{11}+c_{11}M^a(T) \quad \text{for all } t\in[2,T].$$

Hence, with the number

$$c_{12} := \max\Big\{c_{11}, \sup_{t\in(1,2)} \{t\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}\}\Big\}$$

finite, again by the inclusion $v \in C^{2,1}(\bar{\Omega}\times(0,\infty))$ and the fact that $\alpha < 1$, we also have

$$M(T) \leqslant c_{12} + c_{12}M^a(T) \quad \text{for all } T > 2.$$

As $a < 1$, by an elementary argument this implies that

$$M(T) \leqslant c_{13} := \max\{1, (2c_{12})^{1/(1-a)}\} \quad \text{for all } T > 2$$

and thus proves (3.3), because, for example, by (3.5) and (3.6) it yields the inequality

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \leqslant c_2 \Big\{\frac{c_{13}}{t}\Big\}^a \Big\{\frac{c_3c_4}{t}\Big\}^{1-a}$$

for arbitrary $t \geqslant 1$. □

We next modify the above argument but make use of different ingredients (in particular, the boundedness of u) to derive the following higher-order boundedness property of v.

LEMMA 3.3. *Let (u,v) be a non-negative global classical solution of (1.1) with the property that u is bounded in $\Omega\times(0,\infty)$. Then, for all $p > 1$ and each $\alpha \in (\frac{1}{2},1)$, there exists $C(p,\alpha) > 0$ such that*

$$\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \leqslant C(\alpha,p) \quad \textit{for all } t \geqslant 1. \tag{3.10}$$

Proof. Following a variant of the strategy pursued in lemma 3.2, we let

$$M(T) := \sup_{t\in(1,T)} \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}, \quad T > 2,$$

and note that, since $\alpha < 1$, the inclusion $v \in C^{2,1}(\bar{\Omega}\times(0,\infty))$ again warrants that $M(T) < \infty$ for all $T > 2$.

To prepare an adequate estimation of $M(T)$ on the basis of a Duhamel formula associated with the second equation in (1.1), we once more invoke standard smoothing estimates for $(\mathrm{e}^{-\tau A})_{\tau\geqslant 0}$ to find $c_1 > 0$ and $c_2 > 0$ such that

$$\|A^\alpha \mathrm{e}^{-A} v(\cdot,t-1)\|_{L^p(\Omega)} \leqslant c_1\|v(\cdot,t-1)\|_{L^1(\Omega)} \leqslant c_2 \quad \text{for all } t \geqslant 1, \tag{3.11}$$

as lemma 2.1 in particular warrants that $(v(\cdot,t))_{t\geqslant 0}$ is bounded in $L^1(\Omega)$. Next, as u is assumed to be bounded in $\Omega\times(0,\infty)$, there exist $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned}\int_{t-1}^t \|A^\alpha \mathrm{e}^{-(t-s)A} u(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s &\leqslant c_3\int_{t-1}^t (t-s)^{-\alpha}\|u(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s \\ &\leqslant c_3c_4\int_{t-1}^t (t-s)^{-\alpha}\,\mathrm{d}s \\ &= \frac{c_3c_4}{1-\alpha} \quad \text{for all } t \geqslant 1, \end{aligned} \tag{3.12}$$

because $\alpha < 1$. Moreover, once more fixing any $\varepsilon \in (0, \alpha - \frac{1}{2})$ and $\beta > n(p-1)/2p$, we may apply known embedding and interpolation estimates along with lemma 3.1 to gain positive constants c_5, c_6, c_7, c_8, c_9 and c_{10} such that with $a := (\frac{1}{2} + \varepsilon + \beta)/(\alpha + \beta) \in (0, 1)$ we have

$$\begin{aligned}
\int_{t-1}^{t} \|A^{\alpha} \mathrm{e}^{-(t-s)A} U \cdot \nabla v(\cdot, s)\|_{L^p(\Omega)} \, \mathrm{d}s
&\leqslant c_5 \int_{t-1}^{t} (t-s)^{-\alpha} \|U(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla v(\cdot, s)\|_{L^p(\Omega)} \, \mathrm{d}s \\
&\leqslant c_6 \int_{t-1}^{t} (t-s)^{-\alpha} \|\nabla v(\cdot, s)\|_{L^p(\Omega)} \, \mathrm{d}s \\
&\leqslant c_7 \int_{t-1}^{t} (t-s)^{-\alpha} \|A^{1/2+\varepsilon} v(\cdot, s)\|_{L^p(\Omega)} \, \mathrm{d}s \\
&\leqslant c_8 \int_{t-1}^{t} (t-s)^{-\alpha} \|A^{\alpha} v(\cdot, s)\|_{L^p(\Omega)}^{a} \|A^{-\beta} v(\cdot, s)\|_{L^p(\Omega)}^{1-a} \, \mathrm{d}s \\
&\leqslant c_9 \int_{t-1}^{t} (t-s)^{-\alpha} M^a(T) \|v(\cdot, s)\|_{L^1(\Omega)}^{1-a} \, \mathrm{d}s \\
&\leqslant c_{10} M^a(T) \int_{t-1}^{t} (t-s)^{-\alpha} \, \mathrm{d}s \\
&\leqslant \frac{c_{10}}{1-a} M^a(T) \quad \text{for all } t \in [1, T],
\end{aligned} \tag{3.13}$$

again due to the fact that v belongs to $L^\infty((0, \infty); L^1(\Omega))$ by lemma 2.1.

Now, using (3.11)–(3.13), we can estimate

$$\begin{aligned}
\|A^{\alpha} v(\cdot, t)\|_{L^p(\Omega)} &\leqslant \|A^{\alpha} \mathrm{e}^{-A} v(\cdot, t-1)\|_{L^p(\Omega)} \\
&\quad + \int_{t-1}^{t} \|A^{\alpha} \mathrm{e}^{-(t-s)A} u(\cdot, s)\|_{L^p(\Omega)} \, \mathrm{d}s \\
&\quad + \int_{t-1}^{t} \|A^{\alpha} \mathrm{e}^{-(t-s)A} U(\cdot, s) \cdot \nabla v(\cdot, s)\|_{L^p(\Omega)} \, \mathrm{d}s \\
&\leqslant c_2 + \frac{c_3 c_4}{1-\alpha} + c_8 M^a(T) \quad \text{for all } t \in [2, T],
\end{aligned}$$

so that with the evidently finite constant

$$c_{11} := \max\left\{c_2 + \frac{c_3 c_4}{1-\alpha}, \frac{c_{10}}{1-\alpha}, \sup_{t \in (1,2)} \|A^{\alpha} v(\cdot, t)\|_{L^p(\Omega)}\right\}$$

we have

$$M(T) \leqslant c_{11} + c_{11} M^a(T) \quad \text{for all } T > 1,$$

and therefore

$$M(T) \leqslant \max\{1, (2c_{11})^{1/(1-a)}\} \quad \text{for all } T > 1,$$

which proves the lemma. □

A straightforward interpolation shows that the two lemmata above imply decay of ∇v in Lebesgue spaces with high summability powers, but at rates slower than that in lemma 3.2. The following statement on this will be applied to some large value of p and $\kappa := 0$ in proving the upper estimate claimed in theorem 1.1(i), and to $p := 2$ with some $\kappa > \frac{1}{2}$ in corollary 5.1, preparing the proof of the lower bound for $\int_\Omega u$ in theorem 1.1(ii).

LEMMA 3.4. *Let (u, v) be a non-negative global classical solution of (1.1) such that u is bounded, and let $p > 1$. Then for all $\kappa < \min\{1, n/(n-1)p\}$ there exists $C(p, \kappa) > 0$ such that*

$$\|\nabla v(\cdot, t)\|_{L^p(\Omega)} \leqslant \frac{C(p,\kappa)}{t^\kappa} \quad \textit{for all } t \geqslant 2. \tag{3.14}$$

Proof. If $p < n/(n-1)$, the claim immediately results from lemma 3.2. In the case $p \geqslant n/(n-1)$, our assumption ensures that $\kappa < n/(n-1)p$, so that we can fix $r \in [1, n/(n-1))$ such that κ is still less than r/p. Thus, writing

$$q := \frac{(1-\kappa)pr}{r - p\kappa},$$

we can easily verify that $q > p > r$, and that

$$\left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} - \frac{1}{q}\right)^{-1} = 1 - \kappa.$$

Therefore, the Hölder inequality says that

$$\|\nabla v(\cdot, t)\|_{L^p(\Omega)} \leqslant \|\nabla v(\cdot, t)\|_{L^q(\Omega)}^{1-\kappa} \|\nabla v(\cdot, t)\|_{L^r(\Omega)}^{\kappa} \quad \text{for all } t > 0, \tag{3.15}$$

where, picking any $\alpha \in (\frac{1}{2}, 1)$, we infer from the continuity of the embedding $D(A_q^\alpha) \hookrightarrow W^{1,q}(\Omega)$ [4] and from lemma 3.3 that

$$\|\nabla v(\cdot, t)\|_{L^q(\Omega)} \leqslant c_1 \|A^\alpha v(\cdot, t)\|_{L^q(\Omega)} \leqslant c_2 \quad \text{for all } t \geqslant 2$$

with some $c_1 > 0$ and $c_2 > 0$. Moreover, as the inequality $r < n/(n-1)$ along with lemma 3.2 yields $c_3 > 0$ satisfying

$$\|\nabla v(\cdot, t)\|_{L^r(\Omega)} \leqslant \frac{c_3}{t} \quad \text{for all } t \geqslant 2,$$

we readily derive (3.14) from (3.15). □

4. An upper bound for u in $L^\infty(\Omega)$: the proof of theorem 1.1(i)

On the basis of the Duhamel formula now associated with the first equation in (1.1), and knowing that the cross-diffusive gradient ∇v is bounded in $L^\infty((0,\infty); L^p(\Omega))$ for any finite $p > 1$, we can now turn the L^1 decay information on u from lemma 2.1 into a corresponding estimate in $L^\infty(\Omega)$.

Proof of theorem 1.1(i). We fix an arbitrary $p > n$ and recall that, by standard regularization properties of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau \geqslant 0}$ on Ω [15], one can pick $c_1 > 0$ and $c_2 > 0$ such that for all $\tau \in (0, 1)$ we have

$$\|e^{\tau\Delta}\varphi\|_{L^\infty(\Omega)} \leqslant c_1 \tau^{-n/2} \|\varphi\|_{L^1(\Omega)} \quad \text{for all } \varphi \in L^1(\Omega) \tag{4.1}$$

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and

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \leqslant c_2\tau^{-1/2-n/2p}\|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi\in C^1(\bar{\Omega};\mathbb{R}^n) \text{ such that } \varphi\nu=0 \quad \text{on } \partial\Omega. \tag{4.2}$$

Now, in order to estimate the numbers

$$M(T) := \sup_{t\in(0,T)}\{(t+1)\|u(\cdot,t)\|_{L^\infty(\Omega)}\}, \quad T>2,$$

we use that $\nabla\cdot U\equiv 0$ in representing $u(\cdot,t)$ according to

$$u(\cdot,t) = e^{\Delta}u(\cdot,t-1) - \chi\int_{t-1}^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\,ds - \mu\int_{t-1}^t e^{(t-s)\Delta}u^2(\cdot,s)\,ds - \int_{t-1}^t e^{(t-s)\Delta}\nabla(U(\cdot,s)u(\cdot,s))\,ds \quad \text{for all } t\geqslant 1.$$

Since $e^{(t-s)\Delta}u^2(\cdot,s)$ is non-negative in Ω for all $t>0$ and $s\in(0,t)$ due to the maximum principle, by the non-negativity of u we therefore see that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \|e^{\Delta}u(\cdot,t-1)\|_{L^\infty(\Omega)} + \chi\int_{t-1}^t\|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\|_{L^\infty(\Omega)}\,ds + \int_{t-1}^t\|e^{(t-s)\Delta}\nabla\cdot(U(\cdot,s)u(\cdot,s))\|_{L^\infty(\Omega)}\,ds \quad \text{for all } t\geqslant 1, \tag{4.3}$$

where, combining (4.1) with lemma 2.1, we can find $c_3>0$ such that

$$\|e^{\Delta}u(\cdot,t-1)\|_{L^\infty(\Omega)} \leqslant c_1\|u(\cdot,t)\|_{L^1(\Omega)} \leqslant \frac{c_3}{t-1} \leqslant \frac{2c_3}{t} \quad \text{for all } t\geqslant 2. \tag{4.4}$$

To relate the two integrals in (4.3) to $M(T)$, we first invoke (4.2) to obtain

$$\chi\int_{t-1}^t\|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\|_{L^\infty(\Omega)}\,ds \leqslant c_2\chi\int_{t-1}^t(t-s)^{-1/2-n/2p}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)}\,ds \quad \text{for all } t\geqslant 1,$$

and then use the Hölder inequality twice to infer that, again due to lemma 2.1, and as a consequence of the boundedness of ∇v in $\Omega\times(1,2)$ and lemma 3.4 when applied to $\kappa:=0$, with some $c_4>0$ and $c_5>0$ and $a:=1-1/2p$, we have

$$\begin{aligned}
&\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)}\\
&\quad\leqslant \|u(\cdot,s)\|_{L^{2p}(\Omega)}\|\nabla v(\cdot,s)\|_{L^{2p}(\Omega)} \leqslant \|u(\cdot,s)\|^a_{L^\infty(\Omega)}\|u(\cdot,s)\|^{1-a}_{L^1(\Omega)}\|\nabla v(\cdot,s)\|_{L^{2p}(\Omega)}\\
&\quad\leqslant \left\{\frac{M(T)}{s+1}\right\}^a\left\{\frac{c_4}{s+1}\right\}^{1-a}c_5 = c_4^{1-a}c_5M^a(T)\frac{1}{s+1} \quad \text{for all } s\in(1,T)
\end{aligned}$$

and hence

$$\begin{aligned}\chi\int_{t-1}^{t}\|\mathrm{e}^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\leqslant c_2c_4^{1-a}c_5\chi M^a(T)\int_{t-1}^{t}(t-s)^{-1/2-n/2p}\frac{1}{s+1}\,\mathrm{d}s\\ &\leqslant c_2c_4^{1-a}c_5\chi M^a(T)\frac{1}{t}\int_{t-1}^{t}(t-s)^{-1/2-n/2p}\,\mathrm{d}s\\ &= (c_2c_4^{1-a}c_5\chi)\left(\frac{1}{2}-\frac{n}{2p}\right)^{-1}M^a(T)\frac{1}{t}\quad\text{for all } t\in[2,T], \qquad (4.5)\end{aligned}$$

because $p > n$.

Likewise, combining (4.2) with the boundedness of U we obtain $c_6 > 0$ such that

$$\begin{aligned}\int_{t-1}^{t}\|\mathrm{e}^{(t-s)\Delta}\nabla\cdot(U(\cdot,s)u(\cdot,s))\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\leqslant c_2\int_{t-1}^{t}(t-s)^{-1/2-n/2p}\|U(\cdot,s)u(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s\\ &\leqslant c_6\int_{t-1}^{t}(t-s)^{-1/2-n/2p}\|u(\cdot,s)\|_{L^p(\Omega)}\,\mathrm{d}s\quad\text{for all } t\geqslant 1, \qquad (4.6)\end{aligned}$$

where, again by the Hölder inequality and lemma 2.1, there exists $c_7 > 0$ such that

$$\begin{aligned}\|u(\cdot,s)\|_{L^p(\Omega)} &\leqslant \|u(\cdot,s)\|_{L^\infty(\Omega)}^{b}\|u(\cdot,s)\|_{L^1(\Omega)}^{1-b}\\ &\leqslant \left\{\frac{M(T)}{s+1}\right\}^b\left\{\frac{c_7}{s+1}\right\}^{1-b}\\ &= \frac{c_7^{1-b}M^b(T)}{s+1}\quad\text{for all } s\in(1,T)\end{aligned}$$

with $b := 1 - 1/p$. Therefore, (4.6) implies that

$$\begin{aligned}\int_{t-1}^{t}\|\mathrm{e}^{(t-s)\Delta}\nabla\cdot(U(\cdot,s)u(\cdot,s))\|_{L^\infty(\Omega)}\,\mathrm{d}s \\ &\leqslant c_6c_7^{1-b}M^b(T)\int_{t-1}^{t}(t-s)^{-1/2-n/2p}\frac{1}{s+1}\,\mathrm{d}s\\ &\leqslant c_6c_7^{1-b}M^b(T)\frac{1}{t}\int_{t-1}^{t}(t-s)^{-1/2-n/2p}\,\mathrm{d}s\\ &= (c_6c_7^{1-b})\left(\frac{1}{2}-\frac{n}{2p}\right)^{-1}M^b(T)\frac{1}{t}\quad\text{for all } t\in[2,T],\end{aligned}$$

so that summarizing (4.3), (4.4) and (4.5) and using Young's inequality yields $c_8 > 0$ and $c_9 > 0$ such that

$$\begin{aligned}t\|u(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant c_8 + c_8M^a(T) + c_8M^b(T)\\ &\leqslant c_9 + c_9M^a(T)\quad\text{for all } t\in[2,T],\end{aligned}$$

because $b < a$. Since u is bounded in $\Omega \times (0,2)$, this entails that for some $c_{10} > 0$ we have

$$M(T) \leqslant c_{10} + c_{10} M^a(T) \quad \text{for all } T > 2$$

and thus

$$M(T) \leqslant \max\{1, (2c_{10})^{1/(1-a)}\} \quad \text{for all } T > 2,$$

which readily yields (1.6), as $T > 2$ was arbitrary. □

5. A lower bound for u in $L^1(\Omega)$: proof of theorem 1.1(ii)

In deriving the lower bound for $\int_\Omega u$ claimed in theorem 1.1(ii), we shall make use of the following consequence of lemma 3.4, which relies strongly on the fact that the decay exponent κ appearing therein can be chosen to be favourably large, at least in the $p = 2$ case.

Corollary 5.1. *There exist $\lambda > 1$ and $C > 0$ such that*

$$\int_\Omega |\nabla v(\cdot,t)|^2 \leqslant \frac{C}{t^\lambda} \quad \textit{for all } t \geqslant 2. \tag{5.1}$$

Proof. This immediately results from an application of lemma 3.4 to any $\kappa > \frac{1}{2}$ satisfying $\kappa < \min\{1, n/2(n-1)\}$. □

Now the fact that the function on the right of (5.1) is integrable over $t \in (2,\infty)$ enables us to make sure that the taxis term in (1.1) becomes asymptotically negligible in the framework of the following testing procedure.

Lemma 5.2. *There exists $C > 0$ such that*

$$\int_\Omega \ln u(\cdot,t) \geqslant -|\Omega| \ln(t+\gamma) - C \quad \textit{for all } t \geqslant 2, \tag{5.2}$$

where $\gamma > 0$ is the constant defined in (2.3).

Proof. As u is positive in $\bar{\Omega} \times (0,\infty)$ according to the strong maximum principle, we may test the first equation in (1.1) against $1/u$ in order to see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \ln u &= \int_\Omega \frac{1}{u} u_t \\ &= \int_\Omega \frac{1}{u} \Delta u - \chi \int_\Omega \frac{1}{u} \nabla \cdot (u \nabla v) - \mu \int_\Omega u \\ &= \int_\Omega \frac{|\nabla u|^2}{u^2} - \chi \int_\Omega \frac{\nabla u}{u} \cdot \nabla v - \mu \int_\Omega u \quad \text{for all } t > 0, \end{aligned} \tag{5.3}$$

where, by Young's inequality,

$$-\chi \int_\Omega \frac{\nabla u}{u} \cdot \nabla v \geqslant - \int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{\chi^2}{4} \int_\Omega |\nabla v|^2 \quad \text{for all } t > 0. \tag{5.4}$$

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Now, from lemma 2.1 we know that

$$\mu \int_\Omega u \leqslant \frac{|\Omega|}{t+\gamma} \quad \text{for all } t>0,$$

whereas corollary 5.1 provides $\lambda > 1$ and $c_1 > 0$ satisfying

$$\frac{\chi^2}{4} \int_\Omega |\nabla v|^2 \leqslant \frac{c_1}{t^\lambda} \quad \text{for all } t \geqslant 2.$$

From (5.3) and (5.4) we therefore obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \ln u \geqslant -\frac{|\Omega|}{t+\gamma} - \frac{c_1}{t^\lambda} \quad \text{for all } t \geqslant 2,$$

which on direct integration shows that

$$\begin{aligned} \int_\Omega \ln u(\cdot,t) - \int_\Omega \ln u(\cdot,2) & \\ &\geqslant -|\Omega| \int_2^t \frac{\mathrm{d}s}{s+\gamma} - c_1 \int_2^t \frac{\mathrm{d}s}{s^\lambda} \\ &= -|\Omega|\ln(t+\gamma) + |\Omega|\ln(2+\gamma) - \frac{c_1}{2^{\lambda-1}(\lambda-1)} + \frac{c_1}{(\lambda-1)t^{\lambda-1}} \\ &\geqslant -|\Omega|\ln(t+\gamma) - \frac{c_1}{2^{\lambda-1}(\lambda-1)} \quad \text{for all } t \geqslant 2. \end{aligned}$$

As $\int_\Omega \ln u(\cdot,2)$ is finite by the strict positivity of $u(\cdot,2)$ throughout $\bar{\Omega}$, this establishes (5.2). □

Thanks to the precise information on the multiple of $\ln(t+\gamma)$ appearing in (5.2), upon a simple application of Jensen's inequality we can turn this into a lower estimate for $\int_\Omega u$ that involves exactly the desired decay rate.

Lemma 5.3. *There exists $C > 0$ such that*

$$\int_\Omega u(\cdot,t) \geqslant \frac{C}{t+1} \quad \textit{for all } t>0. \tag{5.5}$$

Proof. From lemma 5.2 we know that, with $\gamma > 0$ from (2.3), for some $c_1 > 0$ we have

$$\int_\Omega \ln u \geqslant -|\Omega|\ln(t+\gamma) - c_1 \quad \text{for all } t \geqslant 2.$$

Since by Jensen's inequality we can estimate

$$\begin{aligned} \int_\Omega \ln u = |\Omega| \left\{ \frac{1}{|\Omega|} \int_\Omega \ln u \right\} &\leqslant |\Omega| \ln \left\{ \frac{1}{|\Omega|} \int_\Omega u \right\} \\ &= |\Omega| \ln \left\{ \int_\Omega u \right\} - |\Omega| \ln |\Omega| \quad \text{for all } t>0, \end{aligned}$$

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this implies that

$$\begin{aligned}\int_\Omega u &\geqslant |\Omega| \exp\left(\frac{1}{|\Omega|}\int_\Omega \ln u\right)\\ &\geqslant |\Omega| \exp\left(\frac{1}{|\Omega|}\{-|\Omega|\ln(t+\gamma)-c_1\}\right)\\ &= |\Omega| \mathrm{e}^{-c_1/|\Omega|}\frac{1}{t+\gamma}\\ &\geqslant |\Omega| \mathrm{e}^{-c_1/|\Omega|}\min\left\{\frac{1}{\gamma},1\right\}\frac{1}{t+1} \quad \text{for all } t \geqslant 2.\end{aligned}$$

Therefore, the proof is completed upon the observation that

$$\min_{t\in[0,2]}\left\{(t+1)\int_\Omega u(\cdot,t)\right\}$$

must be positive by continuity of u and the fact that $u \not\equiv 0$. □

We can now complete the proof of our main results.

Proof of theorem 1.1(ii). For appropriately large $C > 0$, the second inequality in (1.7) is precisely asserted by lemma 5.3, whereas the first is obvious. □

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References

1 N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler. Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. *Math. Models Meth. Appl. Sci.* **25** (2015), 1663–1763.

2 J. P. Crimaldi, J. R. Hartford and J. B. Weiss. Reaction enhancement of point sources due to vortex stirring. *Phys. Rev.* E **74** (2006), 016307.

3 A. Friedman. *Partial differential equations* (New York: Holt, Rinehart and Winston, 1969).

4 D. Henry. *Geometric theory of semilinear parabolic equations* (Berlin: Springer, 1981).

5 M. A. Herrero and J. J. L. Velázquez. A blow-up mechanism for a chemotaxis model. *Annali Scuola Norm. Sup. Pisa IV* **24** (1997), 663–683.

6 A. Kiselev and L. Ryzhik. Biomixing by chemotaxis and enhancement of biological reactions. *Commun. PDEs* **37** (2012), 298–318.

7 A. Kiselev and L. Ryzhik. Biomixing by chemotaxis and efficiency of biological reactions: the critical reaction case. *J. Math. Phys.* **53** (2012), 115609.

8 J. Lankeit. Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source. *J. Diff. Eqns* **258** (2015), 1158–1191.

9 K. Osaki and A. Yagi. Finite dimensional attractor for one-dimensional Keller–Segel equations. *Funkcial. Ekvac.* **44** (2001), 441–469.

10 K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura. Exponential attractor for a chemotaxis-growth system of equations. *Nonlin. Analysis* **51** (2002), 119–144.

11 K. J. Painter and T. Hillen. Spatio-temporal chaos in a chemotaxis model. *Physica* D **240** (2011), 363–375.

12 Y. Tao and M. Winkler. Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis–fluid system. *Z. Angew. Math. Phys.* **66** (2015), 2555–2573.

13 Y. Tao and M. Winkler. Blow-up prevention by quadratic degradation in a two-dimensional Keller–Segel–Navier–Stokes system. *Z. Angew. Math. Phys.* **67** (2016), 138. (Available online at http://doi.org/f9fcsp.)

14 J. I. Tello and M. Winkler. A chemotaxis system with logistic source. *Commun. PDEs* **32** (2007), 849–877.

15 M. Winkler. Aggregation vs global diffusive behavior in the higher-dimensional Keller–Segel model. *J. Diff. Eqns* **248** (2010), 2889–2905.

16 M. Winkler. Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source. *Commun. PDEs* **35** (2010), 1516–1537.

17 M. Winkler. Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. *J. Math. Analysis Applic.* **384** (2011), 261–272.

18 M. Winkler. Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system. *J. Math. Pures Appl.* **100** (2013), 748–767.

19 M. Winkler. How far can chemotactic cross-diffusion enforce exceeding carry capacities? *J. Nonlin. Sci.* **24** (2014), 809–855.

20 M. Winkler. Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems. *Discrete Contin. Dynam. Syst.* B **22** (2017), 2777–2993.