


The Daugavet and Delta-constants of points in Banach spaces

Geunsu Choi 

Department of Mathematics Education, Sunchon National University,
57922 Jeollanam-do, Republic of Korea (gschoi@scnu.ac.kr)

Mingu Jung

June E Huh Center for Mathematical Challenges, Korea Institute for
Advanced Study, 02455 Seoul, Republic of Korea (jmingoo@kias.re.kr)
(corresponding author)

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We introduce two new notions called the *Daugavet constant* and Δ -*constant* of a point, which measure quantitatively how far the point is from being Daugavet point and Δ -point and allow us to study Daugavet and Δ -points in Banach spaces from a quantitative viewpoint. We show that these notions can be viewed as a localized version of certain global estimations of Daugavet and diameter two properties such as Daugavet indices of thickness. As an intriguing example, we present the existence of a Banach space X in which all points on the unit sphere have positive Daugavet constants despite the Daugavet indices of thickness of X being zero. Moreover, using the Daugavet and Δ -constants of points in the unit sphere, we describe the existence of almost Daugavet and Δ -points, as well as the set of denting points of the unit ball. We also present exact values of the Daugavet and Δ -constant on several classical Banach spaces, as well as Lipschitz-free spaces. In particular, it is shown that there is a Lipschitz-free space with a Δ -point, which is the furthest away from being a Daugavet point. Finally, we provide some related stability results concerning the Daugavet and Δ -constant.

Keywords: Banach space; Daugavet point; Daugavet property; Delta-point; diameter two property

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1. Introduction

Let X be a Banach space with closed unit ball B_X , unit sphere S_X , and topological dual space X^* . We say that X has the *Daugavet property* if the equation

$$\|\text{Id} + T\| = 1 + \|T\| \tag{DE}$$

holds for every rank-one bounded linear operator from X to itself, where Id denotes the identity operator on X . Given $x \in S_X$ and $\varepsilon > 0$, let us denote by $\Delta_\varepsilon(x) = \{y \in$

$B_X : \|y-x\| \geq 2-\varepsilon\}$. By the geometric characterization in [19], X has the Daugavet property if and only if, for every $x \in S_X$ and $\varepsilon > 0$, we have $B_X \subseteq \overline{\text{co}} \Delta_\varepsilon(x)$.

The following related (weaker) property is introduced in [14]: a Banach space X is *with bad projections* if, for every rank-one projection P , we have

$$\|\text{Id} - P\| \geq 2. \tag{\Delta E}$$

Let us mention that this property was studied in [2, 6] under the names of *local diameter two property +* and *diametral local diameter two property*. There is a similar characterization for this property [14]: a Banach space X is with bad projections if and only if for every $x \in S_X$ and $\varepsilon > 0$, we have $x \in \overline{\text{co}} \Delta_\varepsilon(x)$.

There have been numerous efforts in the study of the Daugavet property and its related properties as it turned out that they could significantly affect the isomorphic structure of a Banach space. For instance, a Banach space with the Daugavet property contains an isomorphic copy of ℓ_1 [19]; it does not have an unconditional basis [18]. Moreover, if a Banach space is with bad projections and has an unconditional basis, then the unconditional suppression basis constant must be at least 2. Classical examples of Banach spaces having the Daugavet property include $C(K)$, $L_1(\mu)$, and $L_\infty(\mu)$, provided that K is perfect and μ is non-atomic [9, 22, 29], the space of Lipschitz functions $\text{Lip}_0(M)$ over a metrically convex space M [15], and the Banach algebra of bounded holomorphic functions H^∞ [16, 30, 32]. For background on the Daugavet property and its related properties including being with bad projections, we refer to [20, 24, 31] and references therein.

The pointwise version of the Daugavet property and the diametral local diameter two property were first introduced in [3] and studied in [5, 8, 17, 26, 27]: given a Banach space X and a point $x \in S_X$, x is said to be

- (i) a *Daugavet point* if, for every $\varepsilon > 0$, $\delta > 0$, and $x^* \in S_{X^*}$, there exists $y \in S(x^*, \delta)$ such that $\|y - x\| > 2 - \varepsilon$,
- (ii) a Δ -point if, for every $\varepsilon > 0$, $\delta > 0$ and $x^* \in S_{X^*}$ with $x \in S(x^*, \delta)$, there exists $y \in S(x^*, \delta)$ such that $\|y - x\| > 2 - \varepsilon$,

where $S(x^*, \delta)$ is a *slice* of B_X , i.e., $S(x^*, \delta) = \{y \in B_X : \text{Re } x^*(y) > 1 - \delta\}$. For the spaces $L_1(\mu)$ and L_1 -predual spaces, the notions of Daugavet point and Δ -point coincide [3, 24]. However, in general, there exists a Δ -point, which is not a Daugavet point (see, for instance, [3, example 4.7] and [17, example 4.4]).

In this article, we introduce the concepts of Daugavet constant and Δ -constant (see definition 2.1 for definitions) of points in order to analyse how close or how far the points in Banach spaces are from becoming Daugavet points or Δ -points. The aim of this work is to initiate a systematic investigation of Daugavet and Δ -constants, as well as to extend several known properties and results on Daugavet and Δ -points from a quantitative point of view.

The Daugavet index of thickness, which quantifies how far a Banach space is from having the Daugavet property, was first introduced in [25], and its variations are deeply studied in [10]. Motivated by the question of whether or not a super-reflexive space admits Δ -points, the authors of [4] investigated a Banach space that admits *almost Daugavet (or, Δ -) points*. These are weaker than assuming the existence

of Daugavet and Δ -points in Banach spaces. For instance, c_0 does not have any Δ -points, while it admits almost Δ -points. It turns out that Daugavet indices of thickness and the existence of almost Daugavet or Δ -points are closely related to the quantitative approach of Daugavet and Δ -points, so these will be one of our main aim of the study.

Throughout the article, $\text{ext}(B_X)$ and $\text{dent}(B_X)$ denote the set of all extreme points and all denting points, respectively, of B_X , where a point $x \in S_X$ is said to be a *denting point* if there is a slice of B_X containing x with an arbitrary small diameter. Recall that a Banach space X has the *Radon-Nikodým property* if every non-empty bounded subset $B \subseteq X$ is *dentable*, that is, B contains a slice with an arbitrary small diameter. It is immediate that a Banach space with Daugavet property (or, diametral local diameter two property) fails the Radon-Nikodým property as it has no denting points. In contrast to this, a surprising example of a Banach space with the Radon-Nikodým property and a Daugavet point was constructed in [26]. Recently, it turned out that every infinite-dimensional Banach space can be renormed with a Δ -point [1], and a Daugavet point provided that it has an unconditional weakly null Schauder basis [12] (hence, ℓ_2 can be renormed with a Daugavet point). Let us also mention that a Banach space with a sub-symmetric basis cannot have Δ -points [5].

The outline of the article is as follows. In §2, we investigate the general properties of Daugavet and Δ -constants. We give the definitions of Daugavet and Δ -constant and obtain some general observations on them in §2.1. Moreover, we study their relation with some operator-norm inequalities, which extends the well-known relation [3] between Daugavet points (resp. Δ -points) and the equality (DE) (resp. (Δ E)). §2.2 is devoted to study the relation between our new notions and extremal points of the unit ball. Namely, we show that the set of points with vanishing Δ -constant coincides with the set of denting points of B_X . Furthermore, we provide precise values of the Daugavet constant of a locally uniformly rotund point of the unit ball and Δ -constant of a point lying in the unit ball of a Hilbert space. In §2.3, we link our two new notions to global estimations of Daugavet and diametral local diameter two properties introduced in [10, 25]. More precisely, we first observe that the characterization for X to admit almost Daugavet points can be done using the supremum of Daugavet constants of x over all points in S_X . Also, the infimum of Daugavet constants of x over all points in S_X is shown to be, by its nature, the same as the Daugavet index of thickness of X . Finally, we construct a Banach space whose infimum of Δ -constants is zero, while there are no points with vanishing Daugavet constants.

In §3, we present a number of estimations on Daugavet and Δ -constants of a point in certain classical Banach spaces, including c_0 , L_1 -spaces, uniform algebras, and Lipschitz-free spaces. We present some optimal calculations of those values, while for others, we discuss possible bounds on the values that the Daugavet and Δ -constant can have. It is observed as a consequence that behaviours of Daugavet constant and Δ -constant are totally different in the case of c_0 spaces, while those two constants coincide in certain L_1 -spaces. Finally, through the ideas used to study Daugavet and Δ -points in Lipschitz-free spaces, we are able to construct a Banach space with a Δ -point whose Daugavet constant is zero.

In §4, we explore some related stability results. These results from a quantitative standpoint generalize several stability results that have already been discussed in many literatures. As an application, we estimate lower bounds for Daugavet and Δ -constants of points in spaces of absolute sums such as ℓ_1 - and ℓ_∞ -sums.

2. General results on Daugavet and Δ -constants

2.1. The Daugavet and Δ -constant

In the following, all Banach spaces are assumed to be over the **real** field:

DEFINITION 2.1. *Let X be a Banach space and $x \in B_X$.*

(i) *The Daugavet constant of the point x is*

$$dc(x) := \sup \left\{ \alpha \geq 0 \mid \begin{array}{l} \forall \varepsilon, \delta > 0, y^* \in S_{X^*}, \exists y \in S(y^*, \delta) \\ \text{such that } \|y - x\| > \alpha - \varepsilon \end{array} \right\}.$$

(ii) *The Δ -constant of the point x is*

$$\delta c(x) := \sup \left\{ \alpha \geq 0 \mid \begin{array}{l} \forall \varepsilon, \delta > 0, y^* \in S_{X^*} \text{ with } x \in S(y^*, \delta), \\ \exists y \in S(y^*, \delta) \text{ so that } \|y - x\| > \alpha - \varepsilon \end{array} \right\}.$$

Notice that the definition remains equivalent without the use of ε . However, we include it in the definition so that the supremums are indeed maximums. Also, it is clear by definition that $x \in S_X$ is a Daugavet point (resp. Δ -point) if and only if $dc(x) = 2$ (resp. $\delta c(x) = 2$).

Observe from the definition 2.1 of Daugavet and Δ -constant that

$$1 - \|x\| \leq dc(x) \leq \delta c(x) \leq 1 + \|x\|$$

and $dc(x) = dc(-x)$ and $\delta c(x) = \delta c(-x)$ for every $x \in B_X$. By definition, we have that

$$\sup_{y \in S} \|x - y\| \geq dc(x) \text{ for any slice } S \text{ of } B_X. \tag{2.1}$$

On the other hand, for any slice S_0 of B_X and $\varepsilon > 0$, there exists $y_0 \in S_0$ so that $\|x - y_0\| \geq \sup_{y \in S_0} \|x - y\| - \varepsilon$. This, combined with (2.1), shows that

$$dc(x) = \inf_{S \subseteq B_X \text{ slice}} \sup_{y \in S} \|x - y\|.$$

Similarly, one can show that

$$\delta c(x) = \inf_{\substack{S \subseteq B_X \text{ slice} \\ x \in S}} \sup_{y \in S} \|x - y\|.$$

In this regard, following the terminology in [23, §5], we can say a point $x \in B_X$ is a Daugavet point (resp. Δ -point) if and only if $dc(x) = 1 + \|x\|$ (resp. $\delta c(x) = 1 + \|x\|$).

REMARK 2.2. Note that $x \in B_X \mapsto \text{dc}(x)$ is a 1-Lipschitz map. In fact, let $x, y \in B_X$ be given. Given a slice S and ε , choose $u \in S$ so that $\|x - u\| > \text{dc}(x) - \varepsilon$. Then,

$$\|y - u\| \geq \|x - u\| - \|x - y\| > \text{dc}(x) - \|x - y\| - \varepsilon.$$

This implies that $\text{dc}(y) \geq \text{dc}(x) - \|x - y\|$. By changing the role of x and y , we obtain $\text{dc}(x) \geq \text{dc}(y) - \|x - y\|$. It follows that $|\text{dc}(x) - \text{dc}(y)| \leq \|x - y\|$.

Let us observe the following as a direct application of the Hahn–Banach theorem, which may be considered as a quantitative counterpart of the geometric characterization of being a Daugavet and a Δ -point.

PROPOSITION 2.3. *Let X be a Banach space and $0 < \alpha \leq 2$, $x \in B_X$ be given.*

- (i) $\text{dc}(x) \geq \alpha$ if and only if $B_X \subseteq \overline{\text{co}} \Delta_{2-\alpha+\varepsilon}(x)$ for every $\varepsilon > 0$.
- (ii) $\delta c(x) \geq \alpha$ if and only if $x \in \overline{\text{co}} \Delta_{2-\alpha+\varepsilon}(x)$ for every $\varepsilon > 0$.

Recall from [3] that:

- (i) $x \in S_X$ is a Daugavet point if and only if, for every non-zero $x^* \in X^*$, the rank-one operator $T = x^* \otimes x$ satisfies $\|\text{Id} + T\| = 1 + \|T\|$.
- (ii) $x \in S_X$ is a Δ -point if and only if, for every $x^* \in X^*$ with $x^*(x) = 1$, the projection $P = x^* \otimes x$ satisfies $\|\text{Id} - P\| \geq 2$.

In the following theorem, we present generalizations of these results in terms of the Daugavet constant and the Δ -constant.

PROPOSITION 2.4. *Let X be a Banach space with $\dim(X) \geq 2$ and $x \in B_X$.*

- (i) For every rank-one operator $T = x^* \otimes x$ with $\|x^*\| \geq 1$, we have

$$\|\text{Id} + T\| \geq (\text{dc}(x) - 1)\|x^*\| + 1.$$

- (ii) For every rank-one projection $P = x^* \otimes x$ with $x^* \in X^*$ satisfying $x^*(x) = 1$, we have

$$\|\text{Id} - P\| \geq \delta c(x).$$

Proof. (i): If $\text{dc}(x) \leq 1$, then the inequality is trivial since

$$\|\text{Id} + T\| \geq 1 \geq (\text{dc}(x) - 1)\|x^*\| + 1,$$

so we may assume that $\text{dc}(x) > 1$. Let $x^* \in X^*$ with $\|x^*\| \geq 1$, $\delta > 0$ and $0 < \varepsilon < \text{dc}(x) - 1$ be given. Suppose first that $\|x^*\| = 1$. Since $\text{dc}(-x) = \text{dc}(x)$, we can find $y \in S(x^*, \delta)$ so that $\|y + x\| > \text{dc}(x) - \varepsilon > 1$. This implies that

$$\|\text{Id} + x^* \otimes x\| \geq \|y + x^*(y)x\| \geq \|y + x\| - \delta > \text{dc}(x) - \varepsilon - \delta.$$

Since $\delta > 0$ and $\varepsilon > 0$ were chosen arbitrarily, we conclude that $\|\text{Id} + x^* \otimes x\| \geq \text{dc}(x)$.

Next, suppose that $\|x^*\| > 1$. Fix $y \in S(\frac{x^*}{\|x^*\|}, \delta)$ so that $\|x + y\| > dc(x) - \varepsilon > 1$. For a small enough $\delta > 0$ such that $x^*(y) > 1$, we have

$$\begin{aligned} \|\text{Id} + x^* \otimes x\| &\geq \|y + x^*(y)x\| \geq x^*(y)\|y + x\| - x^*(y) + 1 \\ &> x^*(y)(dc(x) - \varepsilon - 1) + 1 > \|x^*\|(1 - \delta)(dc(x) - \varepsilon - 1) + 1. \end{aligned}$$

Again from the choice of $\delta > 0$ and $\varepsilon > 0$, it follows that $\|\text{Id} + x^* \otimes x\| \geq (dc(x) - 1)\|x^*\| + 1$.

(ii): First note that in at least 2-dimensional Banach space X , the inequality is trivial if $\delta c(x) \leq 1$. For $\delta c(x) > 1$, fix $\varepsilon, \delta > 0$ small and find $y \in B_X$ such that $x^*(y) > 1 - \delta$ and $\|x - y\| > \delta c(x) - \varepsilon > 1$. If $x^*(y) \leq 1$, then

$$\|\text{Id} - P\| \geq \|y - x^*(y)x\| \geq \|y - x\| - \delta > \delta c(x) - \varepsilon - \delta.$$

If $x^*(y) > 1$, then

$$\begin{aligned} \|\text{Id} - P\| &\geq \|y - x^*(y)x\| \\ &\geq x^*(y)\|x - y\| - x^*(y) + 1 \\ &> x^*(y)(\delta c(x) - \varepsilon - 1) + 1 > (1 - \delta)(\delta c(x) - \varepsilon - 1) + 1. \end{aligned}$$

As ε and δ were arbitrary, this concludes the proof. □

In order to show that the notion of Daugavet and Δ -points coincide for an L_1 -predual space, the equality $S_X \cap (\cap_{\varepsilon>0} \overline{\text{co}} \Delta_\varepsilon^X(x)) = S_X \cap (\cap_{\varepsilon>0} \overline{\text{co}} \Delta_\varepsilon^{X^{**}}(x))$ for any $x \in S_X$ is proved in [3, lemma 3.5]. Here, $\Delta_\varepsilon^X(x)$ is the set $\Delta_\varepsilon(x)$ and $\Delta_\varepsilon^{X^{**}}(x) = \{z \in B_{X^{**}} : \|z - x\| \geq 2 - \varepsilon\}$. We would like to extend this result by replacing the constant 2 in the definition of Δ_ε with $\alpha \in (0, 2]$.

Given $x \in B_X$, the notation $dc_{X^{**}}(x)$ and $\delta c_{X^{**}}(x)$ denotes the Daugavet constant and Δ -constant, respectively, of the point x , where x is considered as an element of X^{**} . Just to emphasize, let us denote by $dc_X(x) = dc(x)$ and $\delta c_X(x) = \delta c(x)$ in the proposition below.

PROPOSITION 2.5. *Let X be a Banach space, $x \in B_X$, and $0 < \alpha \leq 2$. Then,*

$$B_X \cap \overline{\text{co}} \Delta_{2-\alpha}^{X^{**}}(x) \subseteq \cap_{\varepsilon>0} \overline{\text{co}} \Delta_{2-\alpha+\varepsilon}^X(x). \tag{2.2}$$

In particular, we have the following:

- (i) $dc(x) \geq dc_{X^{**}}(x)$.
- (ii) $\delta c(x) = \delta c_{X^{**}}(x)$.

Proof. The inclusion (2.2) can be proved similarly as in [3, lemma 3.5]. For the sake of completeness, we present the proof. Let $\varepsilon > 0$ be given. Let $y \in B_X$, $y_1^{**}, \dots, y_m^{**} \in \Delta_{2-\alpha}^{X^{**}}(x)$, and $\lambda_1, \dots, \lambda_m > 0$ be such that $\sum_{i=1}^m \lambda_i = 1$ and $\|y - \sum_{i=1}^m \lambda_i y_i^{**}\| < \frac{\varepsilon}{2}$. Put $E = \text{span}\{x, y, y_1^{**}, \dots, y_m^{**}\} \subseteq X^{**}$ and let $0 < \eta < 1$ be

such that $(1 - \eta)\alpha > \alpha - \varepsilon$. By Principle of Local Reflexivity, there exists a linear operator $T : E \rightarrow X$ such that

- (i) $Te = e$ for every $e \in E \cap X$;
- (ii) $(1 - \eta)\|e\| \leq \|Te\| \leq (1 + \eta)\|e\|$ for every $e \in E$.

It follows that $\|x - Ty_i^{**}\| = \|Tx - Ty_i^{**}\| \geq (1 - \eta)\alpha > \alpha - \varepsilon$. That is, $Ty_i^{**} \in \Delta_{2-\alpha+\varepsilon}^X(x)$ for $i = 1, \dots, m$. Moreover,

$$\left\| y - \sum_{i=1}^m \lambda_i Ty_i^{**} \right\| = \left\| Ty - \sum_{i=1}^m \lambda_i Ty_i^{**} \right\| \leq (1 + \eta) \frac{\varepsilon}{2} < \varepsilon.$$

It follows that $y \in \overline{\text{co}} \Delta_{2-\alpha+\varepsilon}^X(x)$.

- (i) By [proposition 2.3](#), if $\text{dc}_{X^{**}}(x) \geq \alpha$, then $B_{X^{**}} \subseteq \overline{\text{co}} \Delta_{2-\alpha+\varepsilon}^{X^{**}}(x)$ for every $\varepsilon > 0$. Let us fix $\varepsilon > 0$. Note that if $y \in B_X$, then $y \in \overline{\text{co}} \Delta_{2-\alpha+\varepsilon}^{X^{**}}(x)$. By [\(2.2\)](#), we have that $y \in \overline{\text{co}} \Delta_{2-\alpha+\varepsilon+\eta}^X(x)$ for every $\eta > 0$. In particular, $y \in \overline{\text{co}} \Delta_{2-\alpha+2\varepsilon}^X(x)$, which shows that $B_X \subseteq \Delta_{2-\alpha+2\varepsilon}^X(x)$. As this holds for every $\varepsilon > 0$, by [proposition 2.3](#), we conclude that $\text{dc}(x) \geq \alpha$.
- (ii) Note from [proposition 2.3](#) that $x \in \overline{\text{co}} \Delta_{2-\delta c(x)+\varepsilon}^X(x)$ for every $\varepsilon > 0$, which is, by [\(2.2\)](#), equivalent to that $x \in \overline{\text{co}} \Delta_{2-\delta c(x)+\varepsilon}^{X^{**}}(x)$ for every $\varepsilon > 0$. It follows that $\delta c(x) \geq \alpha$ if and only if $\delta c_{X^{**}}(x) \geq \alpha$.

□

REMARK 2.6. Let Y be a subspace of X and let $x \in B_Y$. Let us denote by $\delta c_Y(x)$ the Δ -constant of x when the point x is considered as a point of Y . Given $x \in B_Y$, $\varepsilon > 0$ and a slice $S(x^*, \delta)$ of B_X containing x , consider $x^*|_Y \in Y^*$, which satisfies clearly that $x^*|_Y(x) > 1 - \delta$. Then, we may find $u \in B_Y \cap S(x^*, \delta)$ such that $\|u - x\| > \delta c_Y(x) - \varepsilon$. This implies that $\delta c_X(x) \geq \delta c_Y(x)$, which, in particular, shows that $\delta c_{X^{**}}(x) \geq \delta c_X(x)$ and that Δ -points pass to superspaces.

2.2. Relationship to extremal points

By nature, the concept of a Δ -point can be understood as being diametrically opposed to that of a denting point. The following confirms the intuition by showing that the set of denting points is precisely the set of points with vanishing Δ -constant.

PROPOSITION 2.7. *Let X be a Banach space. Then, we have*

$$\text{dent}(B_X) = \{x \in S_X : \delta c(x) = 0\}.$$

Proof. Suppose first that $x \in S_X$ satisfies $\delta c(x) = 0$. Let $\eta > 0$ be given. Then, there exist $\varepsilon > 0$, $\delta > 0$, and $y^* \in S_{X^*}$ such that $x \in S(y^*, \delta)$ and $\|y - x\| \leq \eta - \varepsilon$ for every $y \in S(y^*, \delta)$. Thus, $S(y^*, \delta)$ is a slice of B_X containing x such that $\text{diam} S(y^*, \delta) \leq 2(\eta - \varepsilon)$. Since $\eta > 0$ was arbitrary, it follows that x is a denting point of B_X .

On the other hand, let $x \in \text{dent}(B_X)$ and $\eta > 0$ be given. Fix $\delta > 0$ and $y^* \in S_{X^*}$ such that $x \in S(y^*, \delta)$ and $\text{diam}S(y^*, \delta) < \eta/2$. Then $\|y - x\| \leq \eta/2$ for every $y \in S(y^*, \delta)$, so $\delta c(x) \leq \eta$. This finishes the proof. \square

We refer to [theorem 3.14](#) for the remark that the same kind of result cannot be guaranteed in the case of Daugavet constants. On the other hand, it is observed in [[17](#), proposition 3.1] that Daugavet points have to be far from the set of denting points. In general, this result can be interpreted as follows by using the Daugavet constant, which will be used in [§3.4](#).

LEMMA 2.8. *Let X be a Banach space and $x \in B_X$. If $y \in \text{dent}(B_X)$, then $\|x - y\| \geq dc(x)$.*

Proof. Suppose that $y \in \text{dent}(B_X)$ and $\|x - y\| < dc(x)$. Take a slice $S = S(x^*, \delta)$ containing y such that $\text{diam}(S) < \eta$, where $0 < 2\eta < dc(x) - \|x - y\|$. Find $u \in S$ so that $\|x - u\| > dc(x) - \eta$. Then,

$$\|x - y\| \geq \|x - u\| - \|u - y\| > dc(x) - \eta - \eta > \|x - y\|,$$

which is a contradiction. \square

Recall that a Banach space X is said to have the *Kadec property* if the norm topology and the weak topology coincide on S_X . It is known that X is strictly convex with the Kadec property if and only if every point in S_X is a denting point of B_X [[21](#)]. It is well known that X has the Radon–Nikodým property if and only if every bounded closed convex subset of X is the closed convex hull of its denting points. As a direct consequence of [proposition 2.7](#), we obtain the following:

COROLLARY 2.9. *Let X be a Banach space.*

- (i) *If X is strictly convex with the Kadec property (in particular, X is locally uniformly rotund), then $\delta c(x) = 0$ for every $x \in S_X$.*
- (ii) *If B_X is dentable, then we have $\inf_{x \in S_X} \delta c(x) = 0$. Moreover, if X has the Radon–Nikodým property, then $B_X = \overline{\text{co}}\{x \in S_X : \delta c(x) = 0\}$.*

In the case of locally uniformly rotund (for short, LUR) points $x \in B_X$, we can compute the exact value of $dc(x)$. Recall that $x \in B_X$ is a *LUR point* if $\|x_n - x\| \rightarrow 0$ whenever a sequence $(x_n) \subseteq X$ with $\|x_n\| \rightarrow \|x\|$ satisfies that $\|x_n + x\| \rightarrow 2\|x\|$. Observe that $x \in X \setminus \{0\}$ is a LUR point if and only if $\|x_n - x/\|x\|\| \rightarrow 0$ whenever a sequence $(x_n) \subseteq S_X$ satisfies that $\|x_n + x\| \rightarrow 1 + \|x\|$.

PROPOSITION 2.10. *Let X be a Banach space, and let $x \in B_X$ be a LUR point. Then, we have $dc(x) = 1 - \|x\|$.*

Proof. It is clear that $dc(x) \geq 1 - \|x\|$ from the definition. In order to show that $dc(x) \leq 1 - \|x\|$, let $\varepsilon > 0$ be given and fix $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$. We claim that there exists $\delta > 0$ such that if $x^*(y) > 1 - \delta$ for some $y \in B_X$, then

$\|y - x/\|x\|\| < \varepsilon$. Otherwise, we may choose $y_n \in S(x^*, 1/n)$ for each $n \in \mathbb{N}$ such that $\|y_n - x/\|x\|\| \geq \varepsilon$ for all $n \in \mathbb{N}$. But notice that

$$\|y_n + x\| \geq x^*(y_n) + x^*(x) > 1 + \|x\| - \frac{1}{n};$$

thus, since x is a LUR point, we can deduce that $\|y_n - x/\|x\|\| \rightarrow 0$. This contradicts the choice of (y_n) . Now, observe that for every $y \in S(x^*, \delta)$,

$$\|y - x\| \leq \left\| y - \frac{x}{\|x\|} \right\| + 1 - \|x\| < 1 - \|x\| + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that $dc(x) \leq 1 - \|x\|$. □

Dealing with Hilbert spaces, we can compute the Δ -constant of points in the unit ball. Note that for the 1-dimensional Hilbert space H , it is obvious that $\delta c(x) = 1 - \|x\|$ for $x \in B_H$.

PROPOSITION 2.11. *Let H be a Hilbert space with $\dim(H) \geq 2$, and $x \in B_H$. Then, we have $\delta c(x) = \sqrt{1 - \|x\|^2}$.*

Proof. Consider the slice $S = \{u \in B_H : \langle u, \frac{x}{\|x\|} \rangle > \|x\| - \delta\}$ with $\delta > 0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on H . It is clear that $x \in S$. For $y \in S$, observe that

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle \leq \|x\|^2 + 1 - 2\|x\|(\|x\| - \delta). \tag{2.3}$$

By letting $\delta \rightarrow 0$, the right-hand side of (2.3) tends to $1 - \|x\|^2$. This implies that $\delta c(x) \leq \sqrt{1 - \|x\|^2}$. For the reverse inequality, let $S(y, \alpha)$ be a slice containing the point x . Since $\dim(H) \geq 2$, we can find $z \in S_H$, which is orthogonal to x , i.e., $\langle z, x \rangle = 0$. By changing the sign if necessary, we may assume that $\langle z, y \rangle \geq 0$. Note that $x + \sqrt{1 - \|x\|^2}z \in S_H$ and $x + \sqrt{1 - \|x\|^2}z \in S(y, \alpha)$ since

$$\langle y, x + \sqrt{1 - \|x\|^2}z \rangle \geq \langle y, x \rangle > 1 - \alpha.$$

Note from $\|x - (x + \sqrt{1 - \|x\|^2}z)\| = \sqrt{1 - \|x\|^2}$ that $\delta c(x) \geq \sqrt{1 - \|x\|^2}$. □

2.3. On global estimations of Daugavet and Δ -constants

Recall from [4] that X admits almost Daugavet points (resp. almost Δ -points) if, for every $\varepsilon > 0$, there exists $x \in B_X$ such that $B_X \subseteq \overline{co} \Delta_\varepsilon(x)$ (resp. $x \in \overline{co} \Delta_\varepsilon(x)$). From proposition 2.3, we have the following characterization, which shows that the behaviour of the Daugavet constant of points in the unit sphere S_X completely determines whether X admits almost Daugavet points or not:

PROPOSITION 2.12. *Let X be a Banach space. Then, X admits almost Daugavet points if and only if $\sup_{x \in S_X} dc(x) = 2$.*

Proof. The proof of the ‘if’ part is clear. Assume that X admits almost Daugavet points. Let $\varepsilon > 0$ and $x_0 \in B_X$ be so that $B_X \subseteq \overline{co} \Delta_{\varepsilon/2}(x_0)$. By proposition 2.3, it is clear that $dc(x_0) \geq 2 - \frac{\varepsilon}{2}$. Note that $\|x_0\| \geq 1 - \frac{\varepsilon}{2}$. Thus, putting $z := \frac{x_0}{\|x_0\|} \in S_X$,

we have that $\|z - x_0\| < \frac{\varepsilon}{2}$. Since the mapping $x \mapsto dc(x)$ is 1-Lipschitz (see [remark 2.2](#)), we have

$$dc(z) \geq dc(x_0) - |dc(z) - dc(x_0)| \geq dc(x_0) - \|z - x_0\| > \left(2 - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} = 2 - \varepsilon,$$

which implies $\sup_{x \in S_X} dc(x) = 2$. □

Let us point out that it is unclear whether or not an analog result for almost Δ -points holds in general. In fact, given a point $x \in B_X$ satisfying that $x \in \overline{co} \Delta_\varepsilon(x)$, we cannot expect the normalization $z = \|x\|^{-1}x$ of the point x to satisfy $z \in \overline{co} \Delta_\varepsilon(z)$. Moreover, the mapping $x \in B_X \mapsto \delta c(x)$ is not a Lipschitz map in general (see [remark 3.4](#) for instance). Nevertheless, we note for certain Banach spaces that a result similar to [proposition 2.12](#) holds.

PROPOSITION 2.13. *Let X be a Banach space such that $X \stackrel{1}{=} X \oplus_\infty \mathbb{R}$. Then, X admits almost Δ -points if and only if $\sup_{x \in S_X} \delta c(x) = 2$.*

Proof. It suffices to show that for $x \in B_X$, there exists $z \in S_X$ such that $\delta c(z) \geq \delta c(x)$. Let $x \in B_X \setminus \{0\}$ be given. We claim that $\delta c((x, 1)) \geq \delta c(x)$. Let $\varepsilon > 0$, $\delta > 0$ and $(x^*, \beta) \in S_{X^* \oplus_1 \mathbb{R}}$ with $(x, 1) \in S := S((x^*, \beta), \delta)$ be given. Put $z := (-\|x\|^{-1}x, 1)$ and observe that $z \in S$. Also, $\|z - (x, 1)\| = 1 + \|x\| \geq \delta c(x) > \delta c(x) - \varepsilon$. Next, assume that $x^* \neq 0$. Note that we have

$$\frac{x^*}{\|x^*\|}(x) > \frac{1}{\|x^*\|}(1 - \delta - \beta).$$

Thus, there exists $y \in B_X$ such that $\|x^*\|^{-1}x^*(y) > \|x^*\|^{-1}(1 - \delta - \beta)$ and $\|y - x\| > \delta c(x) - \varepsilon$. Set $z := (y, 1)$ and observe that $z \in S$ since $(x^*, \beta)(y, 1) > 1 - \delta - \beta + \beta = 1 - \delta$. It is clear that

$$\|z - (x, 1)\| = \|y - x\| > \delta c(x) - \varepsilon.$$

This proves that $\delta c((x, 1)) \geq \delta c(x)$. □

Recall from [\[23\]](#) that a point $x \in B_X$ is called a *super Daugavet point* if $\sup_{y \in V} \|x - y\| = 1 + \|x\|$ for every non-empty relatively weakly open subset V of B_X . Thus, it is also natural to consider the *super Daugavet constant* of a point x in B_X , denoted by $sdc(x)$. That is, $sdc(x)$ is the constant given by

$$sdc(x) := \sup \left\{ \alpha \geq 0 \mid \begin{array}{l} \forall \varepsilon > 0 \text{ and a relatively weakly open } V \subseteq B_X, \\ \exists y \in V \text{ such that } \|y - x\| > \alpha - \varepsilon \end{array} \right\}.$$

By definition, it is clear that $dc(x) \geq sdc(x)$ for every $x \in B_X$. Although we introduced the notion of super Daugavet constant, we will focus on Daugavet and Δ -constants in the following sections.

In [25], the Daugavet index of thickness of a Banach space X , denoted by $\mathcal{T}(X)$, is introduced. Namely,

$$\mathcal{T}(X) = \inf \left\{ r > 0 \mid \begin{array}{l} \exists x \in S_X \text{ and a relatively weakly open } W \text{ of } B_X \\ \text{such that } \emptyset \neq W \subseteq B(x, r) := \{u \in B_X : \|u - x\| \leq r\} \end{array} \right\}.$$

Afterwards, a related Daugavet index $\mathcal{T}^s(X)$ is introduced in [10] as follows:

$$\mathcal{T}^s(X) = \inf \left\{ r > 0 \mid \begin{array}{l} \exists x \in S_X \text{ and a slice } S \text{ of } B_X \\ \text{such that } S \subseteq B(x, r) \end{array} \right\}.$$

The following shows that those indexes are indeed the infimum of (super) Daugavet constant of points over the unit sphere:

PROPOSITION 2.14. *Let X be a Banach space. Then,*

- (i) $\inf_{x \in S_X} dc(x) = \mathcal{T}^s(X)$.
- (ii) $\inf_{x \in S_X} sdc(x) = \mathcal{T}(X)$.

Proof. As the statements (i) and (ii) can be proved in a very similar way, we only prove (i). Suppose that $dc(x) < r$ for some $x \in S_X$. Then, there exist $\varepsilon > 0$ and $S = S(x^*, \delta)$ such that $\|x - y\| \leq r - \varepsilon$ for every $y \in S$. This implies that $\mathcal{T}^s(X) \leq r$. Conversely, suppose that $\mathcal{T}^s(X) < r$. Let $\varepsilon > 0$ be such that $\mathcal{T}^s(X) < r - \varepsilon < r$, and find $\mathcal{T}^s(X) < t < r - \varepsilon$ such that there exist $x \in S_X$ and $S = S(x^*, \delta)$ satisfying $S \subseteq B(x, t)$. That is, $\|x - y\| \leq t < r - \varepsilon$ for every $y \in S$. This implies that $dc(x) < r$. □

REMARK 2.15. It is also natural to think of the Δ -version of the Daugavet index of thickness, which was once considered in [10]. Given a Banach space X , let us define

$$\mathcal{T}^\delta(X) := \inf \left\{ r > 0 \mid \begin{array}{l} \exists \text{ a slice } S \text{ of } B_X \text{ and } x \in S \cap S_X \\ \text{such that } S \subseteq B(x, r) \end{array} \right\}.$$

Notice that $\mathcal{T}^s(X) \leq \mathcal{T}^\delta(X)$ and, arguing as in proposition 2.14, observe the following:

$$\inf_{x \in S_X} \delta c(x) = \mathcal{T}^\delta(X).$$

One may ask whether it can happen that $\inf_{x \in S_X} dc(x) = 0$ while $dc(x) > 0$ for every $x \in S_X$, and a similar question for the Δ -constant. The following shows that there is indeed such a Banach space X , and its construction relies on the fact [10, theorem 2.7].

THEOREM 2.16 *There exists a Banach space X such that $\inf_{x \in S_X} \delta c(x) = 0$ while $dc(x) > 0$ for every $x \in S_X$.*

Proof. The result [10, proposition 2.8] guarantees that for any $n \in \mathbb{N}$, there exists a Banach space X_n satisfying that $\mathcal{T}^s(X_n) = 1/2^n$. As a matter of fact, a careful

investigation of the proof shows that $\mathcal{T}^s(X_n)$ in fact coincides with $\mathcal{T}^\delta(X_n)$. Observe also that $\mathcal{T}^\delta(E_1 \oplus_1 E_2) \leq \mathcal{T}^\delta(E_1)$ for any Banach spaces E_1 and E_2 , which is a consequence of [proposition 4.11](#) in §4 and [remark 2.15](#).

Now, set $X = (\oplus_n X_n)_{\ell_1}$. Then, we have that

$$0 \leq \mathcal{T}^s(X) \leq \mathcal{T}^\delta(X) \leq \min\{\mathcal{T}^\delta(X_n), \mathcal{T}^\delta(Y_n)\} \leq \frac{1}{2^n},$$

where $Y_n = (\oplus_{k \neq n} Y_k)_{\ell_1}$ for each $n \in \mathbb{N}$. This shows that $\mathcal{T}^\delta(X) = 0$.

Next, we claim that $\text{dc}(x) > 0$ for every $x \in S_X$. Let $x = (x_n) \in S_X$ be given. If there exists $m \in \mathbb{N}$ such that $\|x_m\| = 1$, then we have

$$x = (0, \dots, 0, x_m, 0, \dots) =: (x_m, 0) \in X_m \oplus_1 Y_m.$$

Note that $\text{dc}(x) = \text{dc}(x_m, 0) \geq \text{dc}(x_m) \geq 2^{-m}$ (see [proposition 4.2](#) for its justification). So, suppose that $\|x_n\| < 1$ for every $n \in \mathbb{N}$. Let $y^* = (y_n^*) \in S_{X^*}$, $\varepsilon > 0$, $\delta > 0$ be given. Find $k \in \mathbb{N}$ so that $\|y_k^*\| > 1 - \delta$; so there is $y_k \in S_{X_k}$ such that $y_k^*(y_k) > 1 - \delta$. Letting ι_k be the embedding from X_k into X , observe that $\iota_k(y_k) \in S(y^*, \delta)$ and

$$\|\iota_k(y_k) - x\| = \|y_k - x_k\| + \sum_{n \neq k} \|x_n\| \geq 1 - \|x_k\| + (1 - \|x_k\|) \geq 2 - 2 \sup_{n \in \mathbb{N}} \|x_n\|.$$

This implies that $\text{dc}(x) \geq 2 - 2 \sup_{n \in \mathbb{N}} \|x_n\| > 0$. □

3. Daugavet and Δ -constants of points in some classical spaces

3.1. On the Banach space c_0

The following example shows that even a Banach space with the Radon–Nikodým property (indeed, a finite dimensional Banach space) may have only finitely many elements whose Daugavet constant is zero.

PROPOSITION 3.1. *Let $N \in \mathbb{N}$. For every $x = (x_n)_{n=1}^N \in B_{\ell_\infty^N}$, we have*

$$\text{dc}(x) = \max\{1 - |x_n| : n = 1, \dots, N\}.$$

In particular, when $N = 2$, we have

$$\delta c(x) = \text{dc}(x) = \max\{1 - |x_1|, 1 - |x_2|\}.$$

Proof. Let $x = (x_n)_{n=1}^N \in B_{\ell_\infty^N}$ be given. Let $\varepsilon > 0$, $\delta > 0$, and $y^* = (y_n^*)_{n=1}^N \in S_{\ell_1^N}$ be given. If we let $y := (\text{sgn}(y_n^*))_{n=1}^N \in S_{\ell_\infty^N}$, then $y^*(y) = 1$; hence $y \in S(y^*, \delta)$. Notice that

$$|x_n - \text{sgn}(y_n^*)| \geq 1 - |x_n|.$$

Thus, $\|x - y\| \geq \max\{1 - |x_n| : n = 1, \dots, N\}$. This argument shows that $\text{dc}(x) \geq \max\{1 - |x_n| : n = 1, \dots, N\}$.

Next, assume to the contrary that $dc(x) > \max\{1 - |x_n| : n = 1, \dots, N\}$. Without loss of generality, we may assume that $x_i \geq 0$ for every $i = 1, \dots, N$. Then, we can find $\eta > 0$ such that $x_i > 1 - \eta > 1 - dc(x)$ for every $i = 1, \dots, N$. Let $\delta > 0$ be sufficiently small so that $\delta < N^{-1}\eta$, and $y^* = N^{-1}(1, \dots, 1) \in S_{\ell_1^N}$. If $y = (y_1, \dots, y_N) \in S(y^*, \delta)$, then by a standard argument, we have

$$y_i > 1 - N\delta > 1 - \eta \quad \text{for } i = 1, \dots, N.$$

It follows that $|x_i - y_i| < \eta$ for every $i = 1, \dots, N$, that is, $\|x - y\| < \eta$. This implies that $dc(x) \leq \eta < dc(x)$, which is a contradiction.

Finally, when $N = 2$, it suffices to show that $\delta c(x) \leq \max\{1 - |x_1|, 1 - |x_2|\}$. Without loss of generality, we may assume that $x_1 \geq x_2 \geq 0$. Thus, $\max\{1 - |x_1|, 1 - |x_2|\} = 1 - x_2$. Again, assume to the contrary that $\delta c(x) > 1 - x_2$. Find $\eta > 0$ so that $x_2 > 1 - \eta > 1 - \delta c(x)$. If $y^* = (1/2, 1/2) \in S_{\ell_2^2}$, then

$$y^*(x) = \frac{x_1 + x_2}{2} > \frac{x_1}{2} + \frac{1 - \eta}{2} =: 1 - \delta.$$

Observe that if $y = (y_1, y_2) \in S(y^*, \delta)$, then

$$y_1 + y_2 > x_1 + (1 - \eta).$$

This implies that

$$x_1 - y_1 < \eta \quad \text{and} \quad x_2 - y_2 \leq x_1 - y_2 < \eta.$$

If $y_1 > x_1$, then $|x_1 - y_1| = y_1 - x_1 \leq 1 - x_1 < \eta$. Also, if $y_1 \leq x_1$, then $|x_1 - y_1| = x_1 - y_1 < \eta$; hence $|x_1 - y_1| < \eta$ in any case. Similarly, $|x_2 - y_2| < \eta$, therefore $\|x - y\| < \eta$. Since $y \in S(y^*, \delta)$ is chosen arbitrarily, we obtain that $\|x - y\| < \eta$, which contradicts that $\delta c(x) > \eta$. □

It should be noted that computing the precise value of the Δ -constant of points in ℓ_∞^N is significantly more difficult, even for $N = 3$. In particular, it is not true in general for $N \geq 3$ that $\delta c(x) = \max\{1 - |x_n| : n = 1, \dots, N\}$ when $x \in B_{\ell_\infty^N}$ (see example 3.4 and figure 1).

As an infinite dimensional version of proposition 3.1, we shall see that the Daugavet constant of a point x in the space c_0 is precisely computed as follows, which can be considered as an improvement of [4, lemma 6.5].

PROPOSITION 3.2. *For every $x \in B_{c_0}$, we have $dc(x) = 1$.*

Proof. We first show that $dc(x) \geq 1$ for every $x \in B_{c_0}$. Observe that it suffices to prove that $dc(x) \geq 1$ for every $x \in B_{c_{00}}$. Fix $x \in B_{c_{00}}$ and we may write $x = (x_1, \dots, x_m, 0, \dots)$ for some $m \in \mathbb{N}$. Let $\varepsilon > 0$, $\delta > 0$ and $y^* = (y_n^*) \in S_{\ell_1}$ be

given. Pick any $y = (y_n) \in S(y^*, \delta/3) \cap S_{c_0}$ and choose $k > m$ such that $|y_k^*| < \delta/3$. If we define $z = (z_n) \in S_{c_0}$ by $z_n = y_n$ for each $n \in \mathbb{N} \setminus \{k\}$ and $z_k = 1$, then

$$y^*(z) \geq \sum_{i \neq k} y_i^* y_i - \frac{\delta}{3} \geq y^*(y) - \frac{2\delta}{3} > 1 - \delta,$$

which yields $z \in S(y^*, \delta)$. The conclusion follows from $\|z - x\| \geq |z_k - x_k| = 1$.

Now, suppose that $\text{dc}(x_0) > 1$ for some $x_0 = (x_n) \in B_{c_0}$. Let $A \subseteq \mathbb{N}$ be the finite subset of all indices such that $|x_i| \geq (\text{dc}(x_0) - 1)/4$ for each $i \in A$, where A is non-empty since $\|x_0\| \geq \text{dc}(x_0) - 1$. For

$$\varepsilon_0 = \frac{\text{dc}(x_0) - 1}{2}, \delta_0 = \frac{\text{dc}(x_0) + 1}{2|A|} \text{ and } y_0^* = \frac{1}{|A|} \sum_{i \in A} \text{sgn}(x_i) e_i^* \in S_{\ell_1},$$

there must exist $y_0 = (y_n) \in S(y_0^*, \delta_0)$ such that

$$\|y_0 - x_0\| > \text{dc}(x_0) - \varepsilon_0 = \frac{1}{2} (\text{dc}(x_0) + 1).$$

However, observe that, for each $i \notin A$, we have

$$|y_i - x_i| \leq \frac{1}{4} (\text{dc}(x_0) - 1) + 1 < \frac{1}{2} (\text{dc}(x_0) + 1).$$

It follows that there must exist $k \in A$ such that

$$|y_k - x_k| = \|y_0 - x_0\| > \frac{1}{2} (\text{dc}(x_0) + 1) > 1.$$

This, in particular, implies that $\text{sgn}(y_k) \neq \text{sgn}(x_k)$. Moreover,

$$|y_k| > \frac{1}{2} (\text{dc}(x_0) - 1).$$

Thus, we have

$$\begin{aligned} y_0^*(y_0) &= \frac{1}{|A|} \sum_{i \in A} \text{sgn}(x_i) y_i = \frac{1}{|A|} \sum_{i \in A \setminus \{k\}} \text{sgn}(x_i) y_i - \frac{1}{|A|} |y_k| \\ &\leq \frac{|A| - 1}{|A|} - \left(\frac{\text{dc}(x_0) - 1}{2|A|} \right) = 1 - \left(\frac{\text{dc}(x_0) + 1}{2|A|} \right). \end{aligned}$$

This contradicts that $y_0 \in S(y_0^*, \delta_0)$. □

Recall from [proposition 2.13](#) and [4, lemma 6.4] that $\sup_{x \in S_{c_0}} \text{dc}(x) = 2$. This, compared with [proposition 3.2](#), shows that the Daugavet constant and the Δ -constant of a point behave totally differently. We present an estimation of the Δ -constant for points in c_0 .

THEOREM 3.3 Let $x = (x_n) \in B_{c_0}$. Define $f_n : [0, 1] \rightarrow \mathbb{R}$ for each $n \geq 3$ by

$$f_n(t) = \begin{cases} 1 + |t| & \text{if } 0 \leq |t| \leq 1 - \frac{2}{n} \\ (n - 1)(1 - |t|) & \text{if } 1 - \frac{2}{n} \leq |t| \leq 1. \end{cases}$$

Then, we have

$$\delta c(x) \geq \min\{f_n(x_{i_1}), \dots, f_n(x_{i_n})\}$$

for every $n \geq 3$ and for every set of indices $\{i_1, \dots, i_n\} \subseteq \mathbb{N}$.

Proof. We may assume $x_n \geq 0$ for each $n \in \mathbb{N}$ for convenience. Let $n_0 \geq 3$ and $\{i_1, \dots, i_{n_0}\} \subseteq \mathbb{N}$ be given. For simplicity, let us say for some $n_1 \leq n_0$ that

- (i) $i_n = n$ for $n = 1, \dots, n_0$;
- (ii) $0 < x_n \leq 1 - \frac{2}{n_0}$ for $n = 1, \dots, n_1$;
- (iii) $1 - \frac{2}{n_0} < x_n \leq 1$ for $n = n_1 + 1, \dots, n_0$.

Define $w_1, \dots, w_{n_0} \in c_0$ by

$$\begin{aligned} w_1 &= \left(-1, \frac{n_0 x_2 + 1}{n_0 - 1}, \dots, \frac{n_0 x_{n_1} + 1}{n_0 - 1}, 1, 1, \dots, 1, x_{n_0+1}, x_{n_0+2}, \dots \right), \\ w_2 &= \left(\frac{n_0 x_1 + 1}{n_0 - 1}, -1, \dots, \frac{n_0 x_{n_1} + 1}{n_0 - 1}, 1, 1, \dots, 1, x_{n_0+1}, x_{n_0+2}, \dots \right), \\ &\quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ w_{n_1} &= \left(\frac{n_0 x_1 + 1}{n_0 - 1}, \frac{n_0 x_2 + 1}{n_0 - 1}, \dots, -1, 1, 1, \dots, 1, x_{n_0+1}, x_{n_0+2}, \dots \right), \\ w_{n_1+1} &= \left(\frac{n_0 x_1 + 1}{n_0 - 1}, \frac{n_0 x_2 + 1}{n_0 - 1}, \dots, \frac{n_0 x_{n_1} + 1}{n_0 - 1}, n_0 x_{n_1+1} - (n_0 - 1), 1, \dots, 1, x_{n_0+1}, x_{n_0+2}, \dots \right), \\ &\quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \\ w_{n_0} &= \left(\frac{n_0 x_1 + 1}{n_0 - 1}, \frac{n_0 x_2 + 1}{n_0 - 1}, \dots, \frac{n_0 x_{n_1} + 1}{n_0 - 1}, 1, 1, \dots, n_0 x_{n_0} - (n_0 - 1), x_{n_0+1}, x_{n_0+2}, \dots \right). \end{aligned}$$

Then, we have $\frac{1}{n_0} \sum_{n=1}^{n_0} w_n = x$ and $\|w_n - x\| \geq f_{n_0}(x_n)$ for $n = 1, \dots, n_0$. Indeed, one can see that $\|w_n - x\| \geq 1 + x_n$ for $n = 1, \dots, n_1$ and $\|w_n - x\| \geq (n_0 - 1)(1 - x_n)$ for $n = n_1 + 1, \dots, n_0$. This shows that $x \in \text{co} \Delta_{2-\min\{f_{n_0}(x_1), \dots, f_{n_0}(x_{n_0})\}}(x)$, and thus $\delta c(x) \geq \min\{f_{n_0}(x_1), \dots, f_{n_0}(x_{n_0})\}$. \square

As a consequence of [theorem 3.3](#) (or a similar technique), we obtain the following:

REMARK 3.4.

- (i) Let $x = (x_1, x_2, x_3) \in B_{\ell_\infty^3}$ be given. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} 1 + |t| & \text{if } 0 \leq |t| \leq \frac{1}{3} \\ 2(1 - |t|) & \text{if } \frac{1}{3} \leq |t| \leq 1. \end{cases}$$

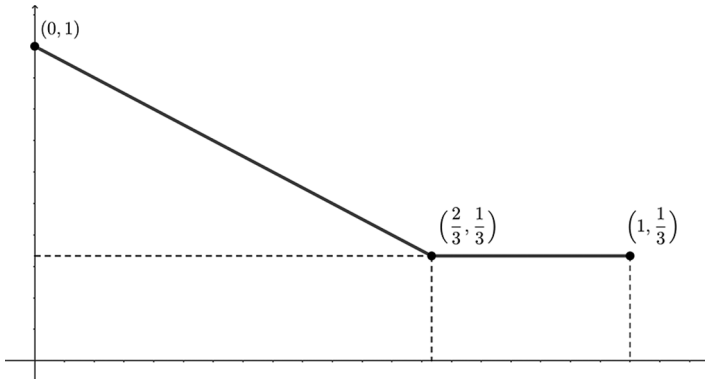


Figure 1. The value of $dc((\frac{2}{3}, \frac{2}{3}, t))$, $0 \leq t \leq 1$.

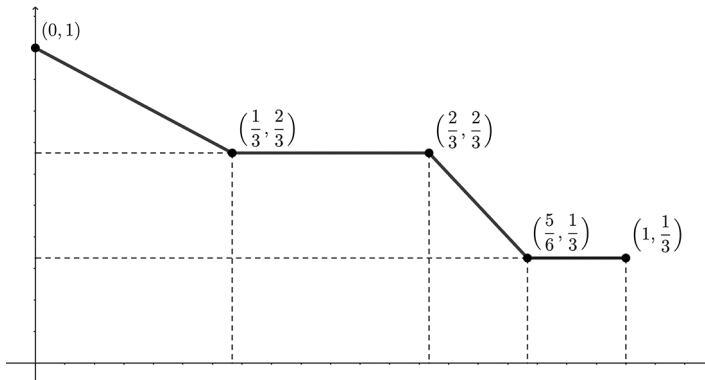


Figure 2. The lower bound for $\delta c((\frac{2}{3}, \frac{2}{3}, t))$, $0 \leq t \leq 1$.

Then, we have

$$\delta c(x) \geq \max \left\{ \min\{f(x_1), f(x_2), f(x_3)\}, 1 - |x_1|, 1 - |x_2|, 1 - |x_3| \right\}.$$

Note that this, in particular, shows that a version of [proposition 3.1](#) for the Δ -constant does not hold (see [figure 2](#)).

- (ii) Recall again from [proposition 2.13](#) that $\sup_{x \in S_{c_0}} \delta c(x) = 2$. This can be shown explicitly by [theorem 3.3](#): let $n \geq 3$ be given and consider

$$u = e_1 + \sum_{k=2}^{n+1} \left(1 - \frac{2}{n}\right) e_k \in S_{c_0}.$$

A direct computation shows that the corresponding vector z in [theorem 3.3](#) has norm $2 - \frac{2}{n}$; thus, $\delta c(u) \geq 2 - \frac{2}{n}$.

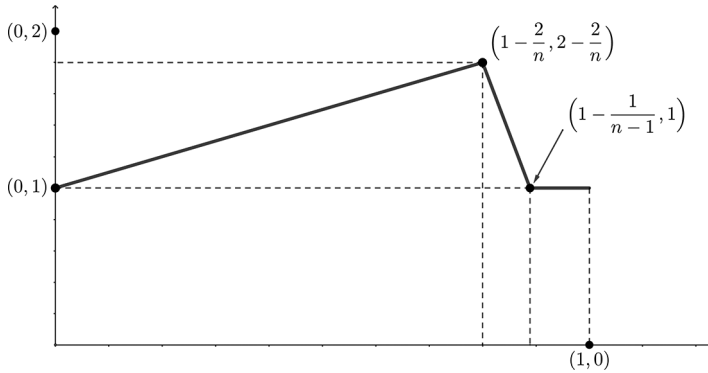


Figure 3. The graph of $f(t) = \delta c(\sum_{k=1}^n te_k)$, $0 \leq t \leq 1$.

(iii) Let $n \in \mathbb{N}$ and $t \in [-1, 1]$ be given. Considering the element $\sum_{k=1}^n te_k \in B_{c_0}$, we have

$$\delta c \left(\sum_{k=1}^n te_k \right) = \min \left\{ 1 + |t|, \max \{ 1, (1 - |t|)(n - 1) \} \right\}. \quad (\diamond)$$

Let us mention that this example shows that the mapping $x \in B_X \mapsto \delta c(x)$ does not need to be Lipschitz (see figure 3). In order to obtain the equality in (\diamond) , we distinguish three cases. Without loss of generality, assume that $t \geq 0$. First, if $0 \leq t \leq 1 - \frac{2}{n}$, it is clear that $\delta c(\sum_{k=1}^n te_k) \leq 1 + t$. Second, suppose that $1 - \frac{2}{n} \leq t \leq 1$. Consider

$$y^* = \left(\frac{1}{n}, \dots, \frac{1}{n}, 0, \dots \right) \in S_{\ell_1}.$$

Then, $y^*(\sum_{k=1}^n te_k) = t > t - \delta$ for any $\delta > 0$. By a standard convex combination argument, if $y = (y_n) \in S(y^*, 1 - t + \delta)$, then $y_i \geq 1 - (1 - t + \delta)n$ for every $i = 1, \dots, n$. It follows that

$$\left\| y - \left(\sum_{k=1}^n te_k \right) \right\| \leq \max \{ 1, (1 - t)(n - 1) + n\delta \}. \quad (\dagger)$$

Letting $\delta \rightarrow 0$, the right-hand side of (\dagger) tends to $\max \{ 1, (1 - t)(n - 1) \}$. This implies that $\delta c(\sum_{k=1}^n te_k) \leq \max \{ 1, (1 - t)(n - 1) \}$.

We finish this section considering the space $\ell_\infty = \ell_1^*$. For a dual Banach space X^* , we can consider natural weak* versions of the Daugavet and Δ -constant of a point by considering weak* slices instead of slices. For a point $x^* \in S_{X^*}$, the weak* Daugavet constant of x is defined as follows:

$$w^* dc(x^*) := \sup \{ \alpha \geq 0 \mid \forall \varepsilon, \delta > 0, y \in S_X, \exists y^* \in S(y, \delta) \text{ so that } \|y^* - x^*\| > \alpha - \varepsilon \},$$

where $S(y, \delta) = \{z^* \in B_{Y^*} : z^*(y) > 1 - \delta\}$. The weak* Δ -constant of x , denoted by $w^*dc(x^*)$, can be defined in a similar way. It is clear that for a point $x^* \in B_{X^*}$,

$$dc(x^*) \leq w^*dc(x^*) \leq w^*\delta c(x^*) \text{ and } dc(x^*) \leq \delta c(x^*) \leq w^*\delta c(x^*). \tag{3.1}$$

Moreover, if we write J , the canonical isometric embedding from X into X^{**} , then

$$dc(x) = w^*dc(J(x)). \tag{3.2}$$

As a matter of fact, it is clear by definition that $dc(x) \leq w^*dc(J(x))$. For the reversed inequality, let $\varepsilon > 0$ and S be a slice of B_X . Considering S be a weak* slice of $B_{X^{**}}$, we find $z \in S_{X^{**}} \cap S$ such that $\|z - J_X(x)\| > w^*dc(J_X(x)) - \varepsilon$. By Goldstine’s theorem and using the lower weak* semicontinuity of the norm in dual spaces, we can find $u \in B_X \cap S$ such that $\|u - x\| > w^*dc(J_X(x)) - \varepsilon$; hence $dc(x) \geq w^*dc(J_X(x))$.

Combining the observations (3.1) and (3.2), we reprove the item (i) of proposition 2.5 that $dc(x) \geq dc(J(x))$. We now present the following result concerning the weak* Daugavet constant of points in $\ell_\infty = \ell_1^*$.

PROPOSITION 3.5. *For every $x \in B_{\ell_\infty}$, we have $w^*dc(x) = 1 + \limsup_n |x_n|$.*

Proof. Let $x \in B_{\ell_\infty}$ be given. First, let $S = S(y, \delta)$ be a weak*-slice of B_{ℓ_∞} with $y = (y_n) \in S_{\ell_1}$. Find $z = (z_n) \in S_{c_0}$ so that $z \in S(y, \delta/3)$. Choose $m \in \mathbb{N}$ such that $\sum_{n=m+1}^\infty |y_n| < \delta/3$. Consider $w \in B_{\ell_\infty}$ given by

$$w = (z_1, \dots, z_m, -\text{sgn}(x_{m+1}), \text{sgn}(x_{m+2}), \dots).$$

Then, $\langle y, w \rangle \geq \sum_{n=1}^m y_n z_n - \sum_{n=m+1}^\infty |y_n| \geq \langle y, z \rangle - 2 \sum_{n=m+1}^\infty |y_n| > 1 - \delta/3 - 2\delta/3$; thus $w \in S(y, \delta)$. Moreover,

$$\|x - w\| \geq \sup_{n \geq m+1} |x_n + \text{sgn}(x_n)| = 1 + \sup_{n \geq m+1} |x_n| \geq 1 + \limsup_n |x_n|.$$

This proves that $w^*dc(x) \geq 1 + \limsup_n |x_n|$.

Next, assume that $w^*dc(x) > 1 + \limsup_n |x_n|$. Find $\delta > 0$ sufficiently small so that

$$w^*dc(x) - \delta > 1 + \limsup_n |x_n|.$$

Let $N_1 \in \mathbb{N}$ and consider the vector

$$y^{(1)} := \left(\frac{\text{sgn}(x_1)}{N_1}, \dots, \frac{\text{sgn}(x_{N_1})}{N_1}, 0, 0, \dots \right) \in S_{\ell_1}.$$

Then, there exists $z \in S(y^{(1)}, 1/N_1)$ such that $\|x - z\| > w^*dc(x) - \delta > 1 + \limsup_n |x_n|$. Note that if there exists $i \in \{1, \dots, N_1\}$ so that $\text{sgn}(x_i) \neq \text{sgn}(z_i)$, then

$$\frac{N_1 - 1}{N_1} \geq \frac{1}{N_1} \sum_{n \in \{1, \dots, N_1\} \setminus \{i\}} \text{sgn}(x_n)z_n > \frac{1}{N_1} \sum_{n=1}^{N_1} \text{sgn}(x_n)z_n > 1 - \frac{1}{N_1},$$

which is a contradiction. Thus, $\text{sgn}(x_i) = \text{sgn}(z_i)$ for every $i = 1, \dots, N_1$. In particular, this implies that there exists $j_1 > N_1$ such that $|x_{j_1} - z_{j_1}| > w^*dc(x) - \delta$. Thus, $|x_{j_1}| > w^*dc(x) - \delta - 1$. Now, take $N_2 > j_1 > N_1$ and consider the element

$$y^{(2)} := \left(\frac{\text{sgn}(x_1)}{N_2}, \dots, \frac{\text{sgn}(x_{N_2})}{N_2}, 0, 0, \dots \right) \in S_{\ell_1}.$$

Arguing in the same way, we end up with finding $j_2 > N_2$ such that $|x_{j_2}| > w^*dc(x) - \delta - 1$. In this way, we find a sequence of natural numbers $\{N_1 < j_1 < N_2 < j_2 < \dots\}$. Passing to a subsequence, we may assume that $\lim_n |x_{j_n}|$ exists, so

$$1 + \lim_n |x_{j_n}| \geq w^*dc(x) - \delta > 1 + \limsup_n |x_n|,$$

which is a contradiction. This shows that $w^*dc(x) \leq 1 + \limsup_n |x_n|$ and completes the proof. □

3.2. On a Banach space L_1

Let μ be a measure on some σ -algebra Σ on a set Ω . Recall that a set $A \in \Sigma$ is called an *atom* if $0 < \mu(A) < \infty$ and whenever $B \in \Sigma$ with $B \subseteq A$ satisfies $\mu(B) < \mu(A)$, then $\mu(B) = 0$. A σ -finite measure on (Ω, Σ) is called *atomic* if every measurable set of positive measure contains an atom. In this case, there exists a countable partition of Ω formed by atoms up to a null set. In this subsection, we shall assume that all measures are σ -finite.

For simplicity, we denote by $L_1(\mu)$ the space $L_1(\Omega, \Sigma, \mu)$. Notice that a measurable function is almost everywhere constant on an atom. We start by giving the exact value of the Daugavet constant of an element in $L_1(\mu)$ provided that μ is atomic.

PROPOSITION 3.6. *Given a σ -finite atomic measure μ on (Ω, Σ) , let (A_n) be a countable partition of Ω formed by atoms up to a null set. For every $f \in L_1(\mu)$ with $\|f\| \leq 1$, we have*

$$dc(f) = 1 + \|f\| - 2 \sup \{ |f|_{A_n} | \mu(A_n) : n \in \mathbb{N} \}.$$

Proof. Let $f \in L_1(\mu)$ with $\|f\| \leq 1$ and $\delta > 0$. Without loss of generality, let each A_n be an atom for μ . Note first that $|f|_{A_n} | \mu(A_n) | \leq \|f\|$ for every $n \in \mathbb{N}$. Since μ is σ -finite, $L_1(\mu)^* = L_\infty(\mu)$. Pick $g \in L_\infty(\mu)$ with $\|g\| = 1$. For each $n \in \mathbb{N}$, let $I_n := \{x \in A_n : |g(x)| > 1 - \delta/2\}$. Then, there exists $m \in \mathbb{N}$ such that I_m has a positive measure, which implies that $\chi_{I_m} = \chi_{A_m}$ almost everywhere.

Note that $h := \frac{\text{sgn}(g|_{A_m})}{\mu(A_m)} \chi_{A_m}$ satisfies that

$$g(h) = \int_{A_m} \frac{\text{sgn}(g|_{A_m})}{\mu(A_m)} g \, d\mu = \int_{I_m} \frac{|g|_{A_m}}{\mu(I_m)} \, d\mu \geq 1 - \frac{\delta}{2} > 1 - \delta.$$

Observe that

$$\begin{aligned} \|f - h\| &= \int_{A_m} \left| f - \frac{\text{sgn}(g|_{A_m})}{\mu(A_m)} \right| \, d\mu + \int_{\cup_{n \neq m} A_n} |f| \, d\mu \\ &= |\text{sgn}(g|_{A_m}) - f|_{A_m} \mu(A_m)| + \|f\| - |f|_{A_m} \mu(A_m) \\ &\geq 1 - |f|_{A_m} \mu(A_m) + \|f\| - |f|_{A_m} \mu(A_m) = 1 + \|f\| - 2|f|_{A_m} \mu(A_m). \end{aligned}$$

This shows that $\text{dc}(f) \geq 1 + \|f\| - 2|f|_{A_m} \mu(A_m)$.

Next, assume that

$$\text{dc}(f) > 1 + \|f\| - 2 \sup \{ |f|_{A_n} \mu(A_n) : n \in \mathbb{N} \}. \tag{3.3}$$

For simplicity, we may assume that $|f|_{A_1} \mu(A_1) = \sup \{ |f|_{A_n} \mu(A_n) : n \in \mathbb{N} \}$. If $|f|_{A_1} \mu(A_1) = \|f\|$, then this would imply that

$$f = |f|_{A_1} \chi_{A_1}.$$

It is not difficult to check that $\text{dc}(f) = 1 - \|f\|$ in such a case, which contradicts to (3.3).

Thus, we assume that $|f|_{A_1} \mu(A_1) < \|f\|$. Take $S = S(\text{sgn}(f|_{A_1}) \chi_{A_1}, \delta)$ with $\delta > 0$ sufficiently small so that

$$\delta < \text{dc}(f) - (1 + \|f\| - 2|f|_{A_1} \mu(A_1)) \text{ and } \delta < \|f\| - |f|_{A_1} \mu(A_1).$$

Let us pick $h \in S$ with $\|h\| = 1$ such that $\|f - h\| > \text{dc}(f) - \delta$. That is,

$$\text{sgn}(f|_{A_1}) h|_{A_1} \mu(A_1) > 1 - \delta \geq \|f\| - \delta,$$

and

$$\|f - h\| > \text{dc}(f) - \delta > 1 + \|f\| - 2|f|_{A_1} \mu(A_1).$$

Note, in particular, that $\text{sgn}(f|_{A_1}) h|_{A_1} \mu(A_1) > |f|_{A_1} \mu(A_1)$. Observe that

$$\begin{aligned} \|f - h\| &= \int_{A_1} |f - h| \, d\mu + \int_{\cup_{n \geq 2} A_n} |f - h| \, d\mu \\ &= |\text{sgn}(f|_{A_1}) h|_{A_1} \mu(A_1) - |f|_{A_1} \mu(A_1)| + \int_{\cup_{n \geq 2} A_n} |f - h| \, d\mu \\ &\leq \text{sgn}(f|_{A_1}) h|_{A_1} \mu(A_1) - |f|_{A_1} \mu(A_1) + \|f\| - |f|_{A_1} \mu(A_1) + 1 - |h_{A_1}| \mu(A_1) \\ &\leq 1 + \|f\| - 2|f|_{A_1} \mu(A_1), \end{aligned}$$

which is a contradiction. □

Next, we obtain an upper bound for the Δ -constant of an element $f \in L_1(\mu)$ in terms of an atom A contained in $\text{supp}(f)$, and the idea of the proof comes from

[3, theorem 3.1], where it is proved that $\text{supp}(f)$ does not contain any atom for μ if and only if $\text{dc}(f)$ (or, $\delta c(f)$) = 2.

PROPOSITION 3.7. *Let $f \in L_1(\mu)$ with $\|f\| \leq 1$. If A is an atom in $\text{supp}(f)$, then*

$$\delta c(f) \leq 1 + \|f\| - 2|f|_A \mu(A).$$

Proof. By considering $-f$ instead of f when $c := f|_A$ less than or equal to 0, we may assume that $c \geq 0$. Assume to the contrary that $\delta c(f) > 1 + \|f\| - 2c\mu(A)$. Let $\varepsilon > 0$ be such that $\varepsilon < \delta c(f) - (1 + \|f\| - 2c\mu(A))$. Suppose that $g \in L_1(\mu)$ satisfies that $\|g\| \leq 1$ and $\|f - g\| \geq \delta c(f) - \varepsilon$. Put $g|_A =: d$ and observe that

$$\begin{aligned} \delta c(f) - \varepsilon &\leq \int_{\Omega} |f - g| d\mu = \int_A |f - g| d\mu + \int_{\Omega \setminus A} |f - g| d\mu \\ &\leq \int_A |f - g| d\mu + \left(\|f\| - \int_A |f| d\mu \right) + \left(1 - \int_A |g| d\mu \right) \\ &\leq |c - d| \mu(A) + \|f\| + 1 - c\mu(A) - |d| \mu(A). \end{aligned}$$

This implies that

$$c\mu(A) + d\mu(A) \leq 1 + \|f\| - \delta c(f) + \varepsilon + |c - d| \mu(A). \tag{3.4}$$

If $c \leq d$, then (3.4) implies that

$$2c\mu(A) \leq 1 + \|f\| - \delta c(f) + \varepsilon,$$

which contradicts the choice of ε . Thus, $c \geq d$, and we have from (3.4) that

$$c\mu(A) + d\mu(A) \leq 1 + \|f\| - \delta c(f) + \varepsilon + (c - d)\mu(A).$$

It follows that

$$g|_A = d \leq \frac{1}{2\mu(A)} (1 + \|f\| - \delta c(f) + \varepsilon). \tag{3.5}$$

Now, by (iii) of proposition 2.3, we obtain that $f \in \overline{\text{co}} \Delta_{2-\delta c(f)+\eta}(f)$ for every $\eta > 0$. In particular, $f \in \overline{\text{co}} \Delta_{2-\delta c(f)+\varepsilon}(f) = \overline{\text{co}} \{g \in B_{L_1(\mu)} : \|f - g\| \geq \delta c(f) - \varepsilon\}$. However, if $g_1, \dots, g_m \in \Delta_{2-\delta c(f)+\varepsilon}(f)$ and $\lambda_1, \dots, \lambda_m$ are positive scalars so that $\sum_{i=1}^m \lambda_i = 1$, then

$$\begin{aligned} \left\| f - \sum_{i=1}^m \lambda_i g_i \right\| &\geq \int_A \left| \sum_{i=1}^m \lambda_i (c - d_i) \right| d\mu \quad (d_i := g_i|_A, i = 1, \dots, m) \\ &= \int_A \sum_{i=1}^m \lambda_i (c - d_i) d\mu \quad (\text{since } c \geq d_i, i = 1, \dots, m) \\ &\geq \int_A c - \frac{1}{2\mu(A)} (1 + \|f\| - \delta c(f) + \varepsilon) d\mu \quad (\text{by (3.5)}) \\ &= \frac{1}{2} [2c\mu(A) - (1 + \|f\| - \delta c(f) + \varepsilon)] > 0. \end{aligned}$$

This contradicts $f \in \overline{\text{co}} \Delta_{2-\delta c(f)+\varepsilon}(f)$. □

REMARK 3.8. Observe that there is another way to complete the proof of [proposition 3.7](#). For instance, note that the assumption on $\varepsilon > 0$ is equivalent to say

$$\int_A f \, d\mu > \frac{1}{2}(1 + \|f\| - \delta c(f) + \varepsilon).$$

Thus, by definition, there exists $g \in S_{L_1(\mu)}$ such that $\|f - g\| > \delta c(f) - \varepsilon$ and

$$d\mu(A) = \int_A g \, d\mu > \frac{1}{2}(1 + \|f\| - \delta c(f) + \varepsilon),$$

where $d = g|_A$, which contradicts [\(3.5\)](#).

Combining [propositions 3.6](#) and [3.7](#), we obtain the following.

COROLLARY 3.9. *Let $I = \mathbb{N}$ or $\{1, \dots, N\}$ with $N \in \mathbb{N}$. For every $x = (x_n) \in B_{\ell_1(I)}$, we have*

$$dc(x) = \delta c(x) = 1 + \|x\| - 2 \sup\{|x_n| : n \in I\}.$$

Moreover,

$$w^*dc(x) = w^*\delta c(x) = 1 + \|x\| - 2 \sup\{|x_n| : n \in I\}.$$

Proof. The equalities for $dc(x)$ and $\delta c(x)$ are immediate from [propositions 3.6](#) and [3.7](#). For their weak* constants, it suffices to observe that $w^*\delta c(x) \leq 1 + \|x\| - 2 \sup\{|x_n| : n \in I\}$. Indeed, note that in the case of ℓ_1 , the slice used in [remark 3.8](#) is nothing but a weak* slice determined by $e_k \in c_0$ for some $k \in \mathbb{N}$. □

3.3. On a uniform algebra

Given an infinite compact Hausdorff space K and $f \in S_{C(K)}$, it is known that f is a Δ -point (or, Daugavet point) if and only if $|f(t)| = 1$ for some limit point $t \in K$ [[3](#), theorem 3.4]. Afterwards, it is generalized in [[24](#), theorem 3.2] to L_1 -predual cases.

Under certain assumptions about extreme points of B_{X^*} , the following result provides an upper bound for the Δ -constant of a point x in a general Banach space X , allowing us to estimate an upper bound for the Δ -constant of a certain element of uniform algebras.

PROPOSITION 3.10. *Let X be a Banach space and $x \in S_X$. Suppose that*

$$\exists \varepsilon > 0 \text{ such that } \{e^* \in \text{ext}(B_{X^*}) : |e^*(x)| > 1 - \varepsilon\} \text{ is finite.} \tag{3.6}$$

Then,

$$\delta c(x) \leq 1 + \sup\{|e^*(x)| : e^* \in \text{ext}(B_{X^*}), |e^*(x)| < 1\} (< 2). \tag{3.7}$$

Proof. By the assumption that the set in (3.6) is finite, and by the Krein–Milman theorem, observe that $\{e^* \in \text{ext}(B_{X^*}) : e^*(x) = 1\}$ is non-empty and finite. Let us say

$$\{e^* \in \text{ext}(B_{X^*}) : e^*(x) = 1\} = \{e_1^*, \dots, e_n^*\}.$$

It follows that

$$\alpha := \sup\{|e^*(x)| : e^* \in \text{ext}(B_{X^*}) \setminus \{\pm e_i^* : i = 1, \dots, n\}\} < 1.$$

Consider the slice $S = S(g, \frac{1-\alpha}{2n})$, where $g = \frac{1}{n} \sum_{i=1}^n e_i^*$. If $y \in S$, then it is not difficult to check that $y \in S(e_i^*, \frac{1-\alpha}{2})$ for every $i = 1, \dots, n$. In particular, $|e_i^*(x - y)| < \frac{1+\alpha}{2}$ for every $i = 1, \dots, n$. Thus, we observe that

$$\begin{aligned} \|y - x\| &= \sup\{|e^*(y - x)| : e^* \in \text{ext}(B_{X^*})\} \\ &\leq \max\{1 + \alpha, \max\{|e_i^*(y - x)| : i = 1, \dots, n\}\} = 1 + \alpha < 2. \end{aligned}$$

Since this holds for every $y \in S$, we conclude (3.7). □

Let K be an infinite compact Hausdorff space. We say $A \subseteq C(K)$ is a uniform algebra if it is a closed subalgebra of A , which separates points of K and contains the constant functions. Recall that the *Choquet boundary* for A is defined by

$$\partial A = \{t \in A : \delta_t|_A \text{ is an extreme point of } B_{A^*}\}. \tag{◆}$$

Given $f \in S_A$, notice from (◆) that the assumption (3.6) in proposition 3.10 is equivalent to say that

$$\exists \varepsilon > 0 \text{ such that } \{t \in \partial A : |f(t)| > 1 - \varepsilon\} \text{ is finite,}$$

which is, in turn, equivalent to that there exists no limit point $t \in \partial A$ so that $|f(t)| = 1$. In this case, $H := \{t \in \partial A : |f(t)| = 1\}$ is a set of isolated points and

$$\sup\{|f(t)| : t \in \partial A \setminus H\} = \max\{|f(t)| : t \in \partial A \setminus H\} < 1.$$

Having this in mind, we obtain the following as a consequence of proposition 3.10:

COROLLARY 3.11. *Let $A \subseteq C(K)$ be a uniform algebra and $f \in S_A$. Suppose that there is no limit point $t \in \partial A$ such that $|f(t)| = 1$. If $H = \{t \in K : |f(t)| = 1\}$, then*

$$\delta_c(f) \leq 1 + \max\{|f(t)| : t \in \partial A \setminus H\} < 2. \tag{3.11}$$

3.4. On a Lipschitz-free space $\mathcal{F}(M)$

Let us start by recalling some basic definitions about Lipschitz-free spaces. Given a pointed metric space M with a distinguished point 0 , we denote by $\text{Lip}_0(M)$ the space of all real-valued Lipschitz functions f on M , which vanish at 0 equipped with the norm

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in M, x \neq y \right\}.$$

For each point $x \in M$, let δ_x be the evaluation functional on $\text{Lip}_0(M)$, that is, $\delta_x(f) = f(x)$ for every $f \in \text{Lip}_0(M)$. The norm closed linear span of $\{\delta_x : x \in M\}$

in $\text{Lip}_0(M)^*$ is called the *Lipschitz-free space over M* and denoted by $\mathcal{F}(M)$. We refer the reader to [28] for background on Lipschitz-free spaces.

In the following, we investigate a quantitative version of the result [26, theorem 2.1] by following the original idea and finally obtain the converse of lemma 2.8 is valid in the case of Lipschitz-free spaces. Recall that $[u, v]_\delta := \{p \in M : d(u, p) + d(v, p) < d(u, v) + \delta\}$.

THEOREM 3.12 *Let M be a metric space, $\mu \in S_{\mathcal{F}(M)}$, and $0 < \alpha \leq 2$ be given. Then, the following are equivalent:*

- (i) $dc(\mu) \geq \alpha$,
- (ii) for every $\nu \in \text{dent}(B_{\mathcal{F}(M)})$, we have $\|\mu - \nu\| \geq \alpha$,
- (iii) if for $u \neq v \in M$ and $r, s > 0$, there exists $\delta > 0$ such that

$$[u, v]_\delta \subseteq B(u, rd(u, v)) \cup B(v, sd(u, v)),$$

$$\text{then } \|\mu - m_{u,v}\| \geq \alpha - 2(r + s).$$

Before we present the proof, we recall some necessary lemmas.

LEMMA 3.13. *Let M be a metric space.*

- (i) [17, theorem 2.6] *Let $\mu \in S_{\mathcal{F}(M)}$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $u \neq v \in M$ satisfy $d(u, v) < \delta$, then $\|\mu - m_{u,v}\| \geq 2 - \varepsilon$.*
- (ii) [26, lemma 1.2] *Given $x, y, u, v \in M$ with $x \neq y$, $u \neq v$, and $\varepsilon > 0$, the following are equivalent:*
 - (a) $\|m_{x,y} + m_{u,v}\| \geq 2 - \varepsilon$,
 - (b) $d(x, v) + d(u, y) \geq d(x, y) + d(u, v) - \varepsilon \max\{d(x, y), d(u, v)\}$.

Furthermore, the equalities in (i) and (ii) hold simultaneously and in that case

$$\|m_{x,y} + m_{u,v}\| = \frac{d(x, v) + d(u, y) + |d(x, y) - d(u, v)|}{\max\{d(x, y), d(u, v)\}}.$$

- (iii) [26, lemma 2.4] *If M is complete, $u, v \in M$ and $r, s, \delta > 0$ with $r + s < d(u, v)$ are such that*

$$[u, v]_\delta \subseteq B(u, r) \cup B(v, s),$$

then there exists $x \in B(u, r)$ and $y \in B(v, s)$ such that $m_{x,y}$ is a denting point.

Proof. *Proof of theorem 3.12.* (i) \Rightarrow (ii): It is a direct consequence of lemma 2.8.

(ii) \Rightarrow (iii): Arguing as in the proof of (ii) \Rightarrow (iii) in [26, theorem 2.1], it is enough to prove the case when M is complete. Let $u \neq v \in M$, $r, s > 0$ such that there exists $\delta > 0$ satisfying

$$[u, v]_\delta \subseteq B(u, rd(u, v)) \cup B(v, sd(u, v)).$$

If $r + s \geq \alpha/2$, then there is nothing to prove; so we assume that $r + s < \alpha/2$. By lemma 3.13(iii), there exist $x \in B(u, rd(u, v))$ and $y \in B(v, sd(u, v))$ such that

$m_{x,y}$ is a denting point. By the assumption, we then have that $\|\mu - m_{x,y}\| \geq \alpha$. Note that

$$d(u, x) + d(v, y) \leq rd(u, v) + sd(u, v) < \frac{\alpha}{2}d(u, v) \leq d(u, v).$$

Thus, we may find $\eta > 0$ such that

$$d(x, u) + d(v, y) = d(u, v) + d(x, y) - \eta \max\{d(x, y), d(u, v)\}.$$

By applying lemma 3.13(ii), we obtain that

$$\begin{aligned} \|m_{x,y} - m_{u,v}\| &= \|m_{x,y} + m_{v,u}\| \\ &= \frac{d(u, x) + d(v, y) + |d(u, v) - d(x, y)|}{\max\{d(u, v), d(x, y)\}} \\ &\leq \frac{2d(u, x) + 2d(v, y)}{\max\{d(u, v), d(x, y)\}} \leq 2 \frac{rd(u, v) + sd(u, v)}{\max\{d(u, v), d(x, y)\}} \leq 2(r + s). \end{aligned}$$

Consequently,

$$\|\mu - m_{u,v}\| \geq \|\mu - m_{x,y}\| - \|m_{x,y} - m_{u,v}\| \geq \alpha - 2(r + s).$$

(iii) \Rightarrow (i): Fix $\varepsilon > 0$ and a slice $S = S(f, \xi)$ of $B_{\mathcal{F}(M)}$. Let $u_0 \neq v_0 \in M$ be such that $f(u_0) - f(v_0) > (1 - \xi)d(u_0, v_0)$. By lemma 3.13(i), there exists $\gamma > 0$ such that if $x \neq y \in M$ satisfy $d(x, y) < \gamma$, then $\|\mu - m_{x,y}\| \geq 2 - \varepsilon$. Let $n \in \mathbb{N}$ and $\delta > 0$ be such that

- (i) $f(u_0) - f(v_0) > (1 - \xi)(1 + \delta)^n d(u_0, v_0)$;
- (ii) $\left(1 - \frac{\varepsilon}{4} + \delta\right)^n d(u_0, v_0) < \gamma$.

If $\|\mu - m_{u_0,v_0}\| \geq \alpha - \varepsilon$, then there is nothing to prove. Suppose that $\|\mu - m_{u_0,v_0}\| < \alpha - \varepsilon$. Then, by the assumption, there exists

$$p \in [u_0, v_0]_{\delta d(u_0,v_0)} \setminus \left(B\left(u_0, \frac{\varepsilon}{4}d(u_0, v_0)\right) \cup B\left(v_0, \frac{\varepsilon}{4}d(u_0, v_0)\right) \right).$$

This allows us to choose $u_1, v_1 \in M$ such that

- (i) $f(u_1) - f(v_1) > (1 - \xi)(1 + \delta)^{n-1}d(u_1, v_1)$;
- (ii) $d(u_1, v_1) < \left(1 - \frac{\varepsilon}{4} + \delta\right)d(u_0, v_0)$

(for its detail, see the proof of (iii) \Rightarrow (i) in [26, theorem 2.1]). Assume for $k \in \{1, \dots, n - 1\}$ that

- (i) $f(u_k) - f(v_k) > (1 - \xi)(1 + \delta)^{n-k}d(u_k, v_k)$;
- (ii) $d(u_k, v_k) < \left(1 - \frac{\varepsilon}{4} + \delta\right)^k d(u_0, v_0)$.

If $\|\mu - m_{u_k, v_k}\| \geq \alpha - \varepsilon$ for some k , then we finish the proof. If not, we end up with the case:

- (i) $f(u_n) - f(v_n) > (1 - \xi)d(u_n, v_n)$;
- (ii) $d(u_n, v_n) < \left(1 - \frac{\varepsilon}{4} + \delta\right)^n d(u_0, v_0) < \gamma$; implying that $\|\mu - m_{u_n, v_n}\| \geq 2 - \varepsilon$.

Therefore, in any case, we find $u \neq v \in M$ so that $m_{u, v} \in S$ and $\|\mu - m_{u, v}\| \geq \alpha - \varepsilon$. \square

With the help of the structure of Lipschitz-free space, we can construct a Banach space in which a particular point exhibits diametrically opposed behaviours concerning its Daugavet constant and Δ -constant. It should be noted that a similar discussion has already taken place in [1, §3], where the authors constructed a Banach space X with a Δ -point, which is a limit of strongly exposed points of B_X (thus, answering [23, question 7.13] in the positive).

THEOREM 3.14 *There exists a Banach space X with a point $x \in S_X$ such that $dc(x) = 2$ and $dc(x) = 0$.*

Proof. Let us consider

$$M := ([0, 1] \times \{0\}) \cup \left\{ \left(0, \frac{1}{n}\right), \left(1, \frac{1}{n}\right) : n \in \mathbb{N} \right\} \subseteq (\mathbb{R}^2, \|\cdot\|_2).$$

Put $x := (1, 0)$, $y := (0, 0)$, $u_n = (1, \frac{1}{n})$, and $v_n = (0, \frac{1}{n})$ for each $n \in \mathbb{N}$. As there is a (1-Lipschitz) path connecting x and y , by [17, proposition 4.2], the molecule $m_{x, y}$ is a Δ -point, i.e., $\delta c(m_{x, y}) = 2$.

Let us fix $n \in \mathbb{N}$, and note that $[u_n, v_n] = \{u_n, v_n\}$. Thus, it is clear to see that there exist sufficiently small $r, s, \delta > 0$ with $r + s < d(u_n, v_n) = 1$ such that

$$[u_n, v_n]_\delta := \{p \in M : d(u_n, p) + d(v_n, p) < d(u_n, v_n) + \delta\} = \{u_n, v_n\},$$

$B(u_n, r) = \{u_n\}$, and $B(v_n, s) = \{v_n\}$. Applying (3) of lemma 3.13, we obtain that m_{u_n, v_n} is a denting point.

Suppose that $dc(m_{x, y}) > 0$. Take $\alpha > 0$ so that $dc(m_{x, y}) \geq \alpha > 0$. By theorem 3.12, it follows that we have $\|m_{u_n, v_n} - m_{x, y}\| \geq \alpha$ for every $n \in \mathbb{N}$. However, the distance between molecules can be estimated as follows (see [7, lemma 1.3]):

$$\|m_{u_n, v_n} - m_{x, y}\| \leq 2 \frac{d(x, u_n) + d(y, v_n)}{\max\{d(x, y), d(u_n, v_n)\}} = \frac{4}{n} \rightarrow 0.$$

This contradicts $dc(m_{x, y}) \geq \alpha > 0$; hence we conclude that $dc(m_{x, y}) = 0$. \square

4. Stability results

A norm N on \mathbb{R}^2 is called *absolute* if

$$N(a, b) = N(|a|, |b|) \quad \text{for all } (a, b) \in \mathbb{R}^2,$$

and *normalized* if $N(1, 0) = N(0, 1) = 1$. Let X and Y be Banach spaces and N an absolute normalized norm on \mathbb{R}^2 . We denote by $X \oplus_N Y$ the product space $X \times Y$ endowed with the norm

$$\|(x, y)\|_N = N(\|x\|, \|y\|) \quad \text{for all } x \in X \text{ and } y \in Y.$$

The behaviour of Daugavet and Δ -points while taking direct sums with absolute normalized norm N is first analysed in [3]. Afterwards, the study of the stability for Daugavet and Δ -points is continued in [13]. In this section, we study how Daugavet and Δ -constants behave under taking absolute sums. As the Daugavet and Δ -constants have been shown to be closely related to the notions of certain Daugavet indexes of thickness (see proposition 2.14), we find several results in [10] concerning the stability of Daugavet indexes of thickness useful in our scenario. For simplicity, we shall consider only points in the sphere of $X \oplus_N Y$.

4.1. Stability for the Daugavet constant

We start with the following result, which is motivated by [10, proposition 2.2]. The proof is similar, but we present the proof for the sake of completeness.

PROPOSITION 4.1. *Let X and Y be Banach spaces, N be an absolute normalized norm on \mathbb{R}^2 , and $\gamma > 0$ is such that $N(\cdot) \geq \gamma \|\cdot\|_1$. Then, we have the following: For $(x, y) \in S_{X \oplus_N Y}$,*

$$dc((x, y)) \geq \begin{cases} 2\gamma \left(\min \left\{ dc \left(\frac{x}{\|x\|} \right), dc \left(\frac{y}{\|y\|} \right) \right\} - 1 \right) & \text{when } x \neq 0, y \neq 0, \\ 2\gamma(dc(x) - 1) & \text{when } y = 0, \\ 2\gamma(dc(y) - 1) & \text{when } x = 0. \end{cases}$$

Proof. Assume first that $x \neq 0$ and $y \neq 0$. Assume also $\min \left\{ dc \left(\frac{x}{\|x\|} \right), dc \left(\frac{y}{\|y\|} \right) \right\} > 1 + \varepsilon > 1$ for some $\varepsilon > 0$ (otherwise, there is nothing to prove). Let $S = S((x^*, y^*), \delta)$ be a slice of $B_{X \oplus_N Y}$ with $\|(x^*, y^*)\| = 1$. Take $(u, v) \in S_{X \oplus_N Y}$ so that $x^*(u) + y^*(v) > 1 - \delta' > 1 - \delta$ for some $0 < \delta' < \delta$. Fix $\eta > 0$ so that $2\eta < \delta - \delta'$, and consider

$$S_1 = \begin{cases} \{z \in B_X : x^*(z) > x^*\left(\frac{u}{\|u\|}\right) - \eta\} & \text{when } u \neq 0 \\ B_X & \text{when } u = 0 \end{cases}$$

and

$$S_2 = \begin{cases} \{w \in B_Y : y^*(w) > y^*\left(\frac{v}{\|v\|}\right) - \eta\} & \text{when } v \neq 0 \\ B_Y & \text{when } v = 0. \end{cases}$$

Observe that there exist $z \in S_1$ and $w \in S_2$ such that $\|z - \frac{x}{\|x\|}\| > dc\left(\frac{x}{\|x\|}\right) - \frac{\varepsilon}{2}$ and $\|w - \frac{y}{\|y\|}\| > dc\left(\frac{y}{\|y\|}\right) - \frac{\varepsilon}{2}$. Note that

$$\left\| z - \frac{x}{\|x\|} \right\| > dc\left(\frac{x}{\|x\|}\right) - \frac{\varepsilon}{2} > (1 + \varepsilon) - \frac{\varepsilon}{2} = 1 + \frac{\varepsilon}{2}.$$

Then,

$$\|x - \|u\|z\| = \left\| \|x\| \frac{x}{\|x\|} + \|u\|(-z) \right\| \geq (\|x\| + \|u\|) \left(\text{dc} \left(\frac{x}{\|x\|} \right) - \frac{\varepsilon}{2} - 1 \right),$$

where we used the fact: if $e_1, e_2 \in B_X$ satisfy $\|e_1 + e_2\| \geq 1 + \alpha$ for some $\alpha \in [0, 1]$, then $\|\lambda e_1 + \mu e_2\| \geq (\lambda + \mu)\alpha$ for all $\lambda, \mu \geq 0$. Similarly, we obtain

$$\|y - \|v\|w\| \geq (\|y\| + \|v\|) \left(\text{dc} \left(\frac{y}{\|y\|} \right) - \frac{\varepsilon}{2} - 1 \right).$$

Note that $(\|u\|z, \|v\|w) \in B_{X \oplus_N Y}$ and if $u \neq 0, v \neq 0$, then

$$\begin{aligned} (x^*, y^*)(\|u\|z, \|v\|w) &= \|u\|x^*(z) + \|v\|y^*(w) \\ &> \|u\| \left(x^* \left(\frac{u}{\|u\|} \right) - \eta \right) + \|v\| \left(y^* \left(\frac{v}{\|v\|} \right) - \eta \right) \\ &> 1 - \delta' - 2\eta > 1 - \delta. \end{aligned}$$

If one of u and v is 0, say $v = 0$, then $\|u\| = 1$ and

$$(x^*, y^*)(\|u\|z, \|v\|w) = x^*(z) > x^*(u) - \eta > 1 - \delta' - \eta > 1 - \delta.$$

Thus, in any case, $(\|u\|z, \|v\|w) \in S$. Now, observe that

$$\begin{aligned} &\|(x, y) - (\|u\|z, \|v\|w)\|_N \\ &= N(\|x - \|u\|z\|, \|y - \|v\|w\|) \\ &\geq N \left((\|x\| + \|u\|) \left(\text{dc} \left(\frac{x}{\|x\|} \right) - \frac{\varepsilon}{2} - 1 \right), (\|y\| + \|v\|) \left(\text{dc} \left(\frac{y}{\|y\|} \right) - \frac{\varepsilon}{2} - 1 \right) \right) \\ &\geq \min \left\{ \text{dc} \left(\frac{x}{\|x\|} \right) - \frac{\varepsilon}{2} - 1, \text{dc} \left(\frac{y}{\|y\|} \right) - \frac{\varepsilon}{2} - 1 \right\} N(\|x\| + \|u\|, \|y\| + \|v\|) \\ &\geq 2\gamma \min \left\{ \text{dc} \left(\frac{x}{\|x\|} \right) - \frac{\varepsilon}{2} - 1, \text{dc} \left(\frac{y}{\|y\|} \right) - \frac{\varepsilon}{2} - 1 \right\}. \end{aligned}$$

Next, assume that $x = 0$ or $y = 0$. We only prove the case when $y = 0$. Assume that $\text{dc}(x) > 1 + \varepsilon$ for some $\varepsilon > 0$, otherwise there is nothing to prove. Let $S = S((x^*, y^*), \delta)$ be a slice of $B_{X \oplus_N Y}$ with $\|(x^*, y^*)\| = 1$. Take $(u, v) \in S_{X \oplus_N Y}$ as above with some $0 < \delta' < \delta$. Fix $\eta > 0$ and consider S_1, S_2 as above too. Find $z \in S_1$ such that $\|z - x\| > \text{dc}(x) - \frac{\varepsilon}{2}$. Let $w \in S_2 \cap S_Y$ be arbitrarily given. Arguing as above, $(\|u\|z, \|v\|w) \in S$. Moreover,

$$\begin{aligned} \|(x, 0) - (\|u\|z, \|v\|w)\|_N &= N(\|x - \|u\|z\|, \|v\|) \\ &\geq N \left((1 + \|u\|) \left(\text{dc}(x) - \frac{\varepsilon}{2} - 1 \right), \|v\| \right) \\ &\geq \left(\text{dc}(x) - \frac{\varepsilon}{2} - 1 \right) N(1 + \|u\|, \|v\|) \\ &\geq 2\gamma \left(\text{dc}(x) - \frac{\varepsilon}{2} - 1 \right). \end{aligned}$$

This finishes the proof. □

When the absolute normalized norm N on \mathbb{R}^2 in [proposition 4.1](#) is actually ℓ_1 -norm, then for $x \neq 0, y \neq 0$, we have

$$dc((x, y)) \geq 2 \left(\min \left\{ dc \left(\frac{x}{\|x\|} \right), dc \left(\frac{y}{\|y\|} \right) \right\} - 1 \right).$$

However, in the case of ℓ_1 -norm, the following improved lower bound for $dc(x, y)$ can be obtained.

PROPOSITION 4.2. *Let X and Y be Banach spaces, and $(x, y) \in S_{X \oplus_1 Y}$. Then, we have the following:*

- (i) $dc((x, y)) \geq \min \left\{ dc \left(\frac{x}{\|x\|} \right), dc \left(\frac{y}{\|y\|} \right) \right\}$ when $x \neq 0$ and $y \neq 0$.
- (ii) $dc((x, y)) \geq dc(y)$ when $x = 0$.
- (iii) $dc((x, y)) \geq dc(x)$ when $y = 0$.

Proof. (i): Let $S = S((x^*, y^*), \delta)$ be a slice of $B_{X \oplus_1 Y}$ with $\|(x^*, y^*)\| = 1$ and $\varepsilon > 0$. Without loss of generality, we may assume that $\|x^*\| = 1$. Find $u \in S(x^*, \delta)$ so that $\|u - \frac{x}{\|x\|}\| \geq dc(\frac{x}{\|x\|}) - \varepsilon$. Note that $(u, 0) \in S$ and

$$\begin{aligned} \|(u, 0) - (x, y)\| &= \|u - x\| + \|y\| \geq \left\| u - \frac{x}{\|x\|} \right\| - \left\| \frac{x}{\|x\|} - x \right\| + \|y\| \\ &\geq dc \left(\frac{x}{\|x\|} \right) - \varepsilon - (1 - \|x\|) + \|y\| \\ &= dc \left(\frac{x}{\|x\|} \right) - \varepsilon. \end{aligned}$$

This implies that $dc((x, y)) \geq dc(\frac{x}{\|x\|})$. Notice that if $\|y^*\| = 1$, then, by changing the role of x and y , we observe $dc((x, y)) \geq dc(\frac{y}{\|y\|})$.

As the proofs of (ii) and (iii) are very similar, we only prove (ii): Let $S = S((x^*, y^*), \delta)$ be a slice of $B_{X \oplus_1 Y}$ and $\varepsilon > 0$. Suppose that $\|x^*\| = 1$. Then, there exists $u \in S_X$ so that $x^*(u) > 1 - \delta$. In other words, $(u, 0) \in S$. In this case, $\|(0, y) - (u, 0)\| = \|u\| + \|y\| = 2$. If $\|y^*\| = 1$, then we can pick $v \in S(y^*, \delta)$ so that $\|v - y\| > dc(y) - \varepsilon$. This implies that $(0, v) \in S$ and $\|(0, y) - (0, v)\| = \|v - y\| > dc(y) - \varepsilon$. In any case, we obtain a point in S , which is a distance of $dc(y) - \varepsilon$ from $(0, y)$. This proves that $dc((0, y)) \geq dc(y)$. □

If the norm N on \mathbb{R}^2 in [proposition 4.1](#) is ℓ_∞ -norm, then we may obtain the result on $dc((x, y))$ for $(x, y) \in S_{X \oplus_\infty Y}$ by taking $\gamma = \frac{1}{2}$. However, the following direct computation produces a better lower bound.

PROPOSITION 4.3. *Let X and Y be Banach spaces, and $(x, y) \in S_{X \oplus_\infty Y}$. Then, we have the following:*

$$dc((x, y)) \geq \begin{cases} dc(x) & \text{when } \|x\| = 1, \\ dc(y) & \text{when } \|y\| = 1. \end{cases}$$

Proof. We only prove the case when $\|x\| = 1$: Let $S = S((x^*, y^*), \delta)$ be a slice of $B_{X \oplus_\infty Y}$ with $\|(x^*, y^*)\| = 1$ and $\varepsilon > 0$. Note that $\|x^*\| + \|y^*\| = 1$. If $x^* \neq 0$, then find $u \in S(\frac{x^*}{\|x^*\|}, \delta)$ such that $\|u - x\| > dc(x) - \varepsilon$. Pick $v \in S(\frac{y^*}{\|y^*\|}, \delta)$ if $y^* \neq 0$ and $v \in B_Y$ arbitrarily if $y^* = 0$. In any case, $(x^*, y^*)(u, v) = x^*(u) + y^*(v) > 1 - \delta$, that is, $(u, v) \in S$. Note that

$$\|(x, y) - (u, v)\| = \max\{\|x - u\|, \|y - v\|\} > dc(x) - \varepsilon.$$

It remains to consider the case when $x^* = 0$. In this case, $\|y^*\| = 1$. Then $(u, v) \in S$ for any $u \in B_X$ and $v \in S(y^*, \delta)$. Note that

$$\|(x, y) - (-x, v)\| = \max\{\|2x\|, \|y - v\|\} = 2.$$

Consequently, whether x^* is 0 or not, we can always find a point in S , which is $(dc(x) - \varepsilon)$ -away from (x, y) . So, $dc((x, y)) \geq dc(x)$. □

In [13], the authors generalized the notion of *positively octahedrality* [11] of an absolute norm on X by introducing the so-called *A-octahedrality* for a subset A of S_X . With this notion, they were able to complete describing the class of absolute norms that admit a Daugavet point. Following their ideas, we will provide a lower bound for the Daugavet constant of a point lying in an absolute sum $X \oplus_N Y$ given that the absolute norm N is *A-OH*.

DEFINITION 4.4. *Let X be a Banach space and $A \subseteq S_X$. We say the norm on X is *A-octahedral* (for short, *A-OH*) if, for every $x_1, \dots, x_n \in A$ and $\varepsilon > 0$, there exists $y \in S_X$ such that $\|x_i + y\| \geq 2 - \varepsilon$ for every $i = 1, \dots, n$.*

Let N be an absolute normalized norm on \mathbb{R}^2 . Put $c = \max\{s : N(s, 1) = 1\}$ and $d = \max\{t : N(1, t) = 1\}$, and define $A = \{(c, 1), (1, d)\}$. We will consider from now on this specific set A . If an absolute norm N is *A-OH*, then, by definition, we can find $(a, b) \in \mathbb{R}^2$ with $N(a, b) = 1$, $a, b \geq 0$ such that

$$N((c, 1) + (a, b)) = 2 \quad \text{and} \quad N((1, d) + (a, b)) = 2. \tag{*}$$

PROPOSITION 4.5. *Let X and Y be Banach spaces, $x \in S_X$, $y \in S_Y$, and let N be an *A-OH* norm with (a, b) as in (*). Then, $(ax, by) \in S_{X \oplus_N Y}$ satisfies that*

$$dc((ax, by)) \geq 2(\min\{dc(x), dc(y)\} - 1).$$

Proof. Let $(x^*, y^*) \in S_{(X \oplus_N Y)^*}$, $\alpha > 0$ and $\varepsilon > 0$. Choose $\delta > 0$ so that $\delta N(1, 1) < \varepsilon$. Find $u \in B_X$ and $v \in B_Y$ such that

$$x^*(u) \geq \left(1 - \frac{\alpha}{2}\right) \|x^*\| \quad \text{and} \quad y^*(v) \geq \left(1 - \frac{\alpha}{2}\right) \|y^*\|,$$

$\|x - u\| > dc(x) - \delta$ and $\|y - v\| > dc(y) - \delta$. Find $k \geq 0, \ell \geq 0$ such that $N(k, \ell) = 1$, $N((a, b) + (k, \ell)) = 2$, and $k\|x^*\| + \ell\|y^*\| = 1$. Note that

$$(x^*, y^*)(ku, \ell v) = kx^*(u) + \ell y^*(v) \geq \left(1 - \frac{\alpha}{2}\right) (k\|x^*\| + \ell\|y^*\|) = 1 - \frac{\alpha}{2},$$

which implies that $(ku, \ell v) \in S((x^*, y^*), \alpha)$. By triangle inequality, we obtain $\|ax - ku\| \geq (a + k)(dc(x) - \delta - 1)$ and $\|by - \ell v\| \geq (b + \ell)(dc(y) - \delta - 1)$. Thus,

$$\begin{aligned} \|(ax, by) - (ku, \ell v)\|_N &= N(\|ax - ku\|, \|by - \ell v\|) \\ &\geq N((a + k)(dc(x) - \delta - 1), (b + \ell)(dc(y) - \delta - 1)) \\ &\geq N((a + k)(dc(x) - 1), (b + \ell)(dc(y) - 1)) - 2\varepsilon \\ &\geq \min\{dc(x) - 1, dc(y) - 1\}N(a + k, b + \ell) - 2\varepsilon \\ &= 2 \min\{dc(x) - 1, dc(y) - 1\} - 2\varepsilon, \end{aligned}$$

as desired. □

Arguing very similarly, we can obtain the following result in the special case when $a = 0$ or $b = 0$.

PROPOSITION 4.6. *Let X and Y be Banach spaces, $x \in S_X$, $y \in S_Y$, and let N be an A -OH norm with (a, b) as in (\star) .*

- (i) *If $a = 0$, then $dc((0, y)) \geq 2(dc(y) - 1)$.*
- (ii) *If $b = 0$, then $dc((x, 0)) \geq 2(dc(x) - 1)$.*

REMARK 4.7. Note that ℓ_1 - and ℓ_∞ -norm are A -OH. For instance, if we consider ℓ_1 -norm on \mathbb{R}^2 , then the corresponding set A turns out to be $\{(0, 1), (1, 0)\}$. In this case, any point $(a, b) \in \mathbb{R}^2$ with $a + b = 1, a, b \geq 0$ satisfies (\star) . Given $(x, y) \in S_{X \oplus_1 Y}$, writing $(x, y) = (\|x\| \frac{x}{\|x\|}, \|y\| \frac{y}{\|y\|})$ if $x \neq 0, y \neq 0$, [propositions 4.5](#) and [4.6](#) show that

$$dc((x, y)) \geq \begin{cases} 2 \left(\min \left\{ dc \left(\frac{x}{\|x\|} \right), dc \left(\frac{y}{\|y\|} \right) \right\} - 1 \right) & \text{when } x \neq 0, y \neq 0. \\ 2(dc(x) - 1) & \text{when } y = 0. \\ 2(dc(y) - 1) & \text{when } x = 0. \end{cases}$$

In the case of ℓ_∞ -norm on \mathbb{R}^2 , given $(x, y) \in S_{X \oplus_\infty Y}$, one can obtain a similar lower bound for $dc((x, y))$.

We finish this subsection by presenting the following result, which gives an upper bound for $dc(x, y)$. It can be obtained by simply arguing as in [[10](#), proposition 2.4], so we omit the proof.

PROPOSITION 4.8. *Let X and Y be Banach spaces, N be an absolute normalized norm on \mathbb{R}^2 , and $\Gamma > 0$ is such that $N(\cdot) \leq \Gamma \|\cdot\|_\infty$. Then, we have the following:*

- (i) *If $(0, 1)$ is an extreme point of $B_{(\mathbb{R}^2, N)}$, then $dc((x, 0)) \leq \Gamma$ for every $x \in S_X$.*
- (ii) *If $(1, 0)$ is an extreme point of $B_{(\mathbb{R}^2, N)}$, then $dc((0, y)) \leq \Gamma$ for every $y \in S_Y$.*

In particular, for $1 < p < \infty$, $dc((x, 0)) \leq 2^{1/p}$ and $dc((0, y)) \leq 2^{1/p}$ whenever $(x, 0) \in S_{X \oplus_p Y}$ and $(0, y) \in S_{X \oplus_p Y}$.

As a consequence of [proposition 4.8](#) combined with [proposition 4.1](#), we have that for $1 < p < \infty$, $x \in S_X$ and $y \in S_Y$,

$$2^{1/p}(\text{dc}(x) - 1) \leq \text{dc}((x, 0)) \leq 2^{1/p} \text{ and } 2^{1/p}(\text{dc}(y) - 1) \leq \text{dc}((0, y)) \leq 2^{1/p}.$$

4.2. Stability for the Δ -constant

In the proof of [proposition 4.1](#), if we start with a slice S containing the given point (x, y) , then we may take $(u, v) = (x, y)$ and proceed as before. Thus, a similar result with respect to Δ -constants can be obtained. However, we will see that an even stronger claim holds in this case.

Recall that the diametral local diameter two property (for short, DLD2P) behaves well under absolute sums, that is, for any absolute normalized norm N on \mathbb{R}^2 , both X and Y have the DLD2P if and only if $X \oplus_N Y$ has the DLD2P [[14](#), theorem 3.2]. In this regard, it is natural to expect some stability result of the Δ -constant.

PROPOSITION 4.9. *Let X and Y be Banach spaces, N an absolute normalized norm on \mathbb{R}^2 , and $a, b \geq 0$ with $N(a, b) = 1$. Given $x \in B_X$ and $y \in B_Y$, we have*

$$\delta c((ax, by)) \geq \min\{\delta c(x), \delta c(y)\}.$$

Moreover, we have the following:

- (i) If $b = 0$, then $\delta c((x, 0)) \geq \delta c(x)$.
- (ii) If $a = 0$, then $\delta c((0, y)) \geq \delta c(y)$.

Proof. Let $\varepsilon > 0$ and $0 < \eta < \varepsilon$. By [proposition 2.3](#), we can find $x_1, \dots, x_m \in \Delta_{2-\delta c(x)+\varepsilon}(x)$ and $y_1, \dots, y_m \in \Delta_{2-\delta c(y)+\varepsilon}(y)$ such that

$$\left\| x - \frac{1}{m} \sum_{i=1}^m x_i \right\| < \eta \text{ and } \left\| y - \frac{1}{m} \sum_{i=1}^m y_i \right\| < \eta,$$

where we applied [[3](#), lemma 4.1] to choose the same number of vectors in X and Y . Note that

$$\left\| (ax, by) - \frac{1}{m} \sum_{i=1}^m (ax_i, by_i) \right\|_N = N \left(a \left\| x - \frac{1}{m} \sum_{i=1}^m x_i \right\|, b \left\| y - \frac{1}{m} \sum_{i=1}^m y_i \right\| \right) \leq \eta.$$

Moreover, for each $i = 1, \dots, m$,

$$\|(ax, by) - (ax_i, by_i)\|_N = N(a\|x - x_i\|, b\|y - y_i\|) \geq \min\{\delta c(x) - \varepsilon, \delta c(y) - \varepsilon\}.$$

Thus, by again [proposition 2.3](#), we conclude that $\delta c((ax, by)) \geq \min\{\delta c(x), \delta c(y)\}$.

When $a = 0$ or $b = 0$, the above argument proves the estimates (i) and (ii). \square

In case of ℓ_1 -sum or ℓ_∞ -sum, we have the following direct consequences:

COROLLARY 4.10. *Let X and Y be Banach spaces. Then, we have the following:*

(i) For $(x, y) \in S_{X \oplus_1 Y}$,

$$\delta c((x, y)) \geq \begin{cases} \min \left\{ \delta c \left(\frac{x}{\|x\|} \right), \delta c \left(\frac{y}{\|y\|} \right) \right\} & \text{when } x \neq 0, y \neq 0, \\ \delta c(x) & \text{when } y = 0, \\ \delta c(y) & \text{when } x = 0. \end{cases}$$

(ii) For $(x, y) \in S_{X \oplus_\infty Y}$,

$$\delta c((x, y)) \geq \min\{\delta c(x), \delta c(y)\}.$$

Proof. For the assertion (i), we apply proposition 4.9 with $a = \|x\|$ and $b = \|y\|$. For (ii), use again proposition 4.9 with $a = 1$ and $b = 1$. □

If one of the components is zero in case of ℓ_1 -sum, then the above inequality is indeed an equality.

PROPOSITION 4.11. *Let X and Y be Banach spaces. Then, $\delta c((x, 0)) = \delta c(x)$ for every $(x, 0) \in S_{X \oplus_1 Y}$.*

Proof. Thanks to corollary 4.10, it suffices to show for every $(x, 0) \in S_{X \oplus_1 Y}$ that $\delta c((x, 0)) \leq \delta c(x)$. Assume that $\delta c(x) \leq \alpha$, then there exists a slice $S(x^*, \delta)$ with $x \in S(x^*, \delta)$ such that $\|y - x\| < \alpha$ for every $y \in S(x^*, \delta)$. Given $0 < \eta < \delta$, by [14, lemma 2.1], we can find $y_1^* \in S_{X^*}$ satisfying that

$$x \in S(y_1^*, \eta) \subseteq S(x^*, \delta).$$

Note that for $(u, v) \in S((y_1^*, 0), \eta)$, we have $y_1^*(u) > 1 - \eta$ and $\|v\| < \eta$. This implies that

$$\|(u, v) - (x, 0)\| = \|u - x\| + \|v\| < \|u - x\| + \eta,$$

where $u, x \in S(x^*, \delta)$. As it is clear that $(x, 0) \in S((y_1^*, 0), \eta)$ and η can be chosen arbitrarily small, we conclude that $\delta c((x, 0)) \leq \alpha$. □

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Conflicts of interest

There is no conflict of interest.

Data availability

No data were used for the research described in the article.

References

- [1] T. A. Abrahamsen, R. Aliaga, V. Lima, A. Martiny, Y. Perreau, A. Prochazka and T. Veeorg. Delta-points and their implications for the geometry of Banach spaces. *J. London Math. Soc.* **109** e12913.
- [2] T. A. Abrahamsen, P. Hájek, O. Nygaard, J. Talponen and S. Troyanski. Diameter 2 properties and convexity. *Studia Math.* **232** (2016), 227–242.
- [3] T. A. Abrahamsen, R. Haller, V. Lima and K. Pirk. Delta- and Daugavet points in Banach spaces. *Proc. Edinb. Math. Soc.* **63** (2020), 475–496.
- [4] T. A. Abrahamsen, V. Lima, A. Martiny and Y. Perreau. Asymptotic geometry and Delta-points. *Banach J. Math. Anal.* **16** (2022), 57.
- [5] T. A. Abrahamsen, V. Lima, A. Martiny and S. Troyanski. Daugavet- and Delta-points in Banach spaces with unconditional bases. *Trans. Amer. Math. Soc. Ser. B.* **8** (2021), 379–398.
- [6] J. Becerra Guerrero, G. López-Pérez and A. Rueda Zoca. Diametral diameter two properties in Banach spaces. *J. Convex Anal.* **25** (2018), 817–840.
- [7] B. Cascales, R. Chiclana, L. García-Lirola, M. Martín and A. Rueda Zoca. On strongly norm attaining Lipschitz maps. *J. Funct. Anal.* **277** (2019), 1677–1717.
- [8] S. Dantas, M. Jung and A. Rueda Zoca. Daugavet points in projective tensor products. *Q. J. Math.* **73** (2022), 443–459.
- [9] I. K. Daugavet. On a property of completely continuous operators in the space. *C. Uspekhi Mat. Nauk.* **18.5** (1963), 157–158(Russian).
- [10] R. Haller, J. Langemets, V. Lima, R. Nadel and A. Rueda Zoca. On Daugavet indices of thickness. *J. Funct. Anal.* **280** (2021), 108846.
- [11] R. Haller, J. Langemets and R. Nadel. Stability of average roughness, octahedrality, and strong diameter 2 properties of Banach spaces with respect to absolute sums. *Banach J. Math. Anal.* **12** (2018), 222–239.
- [12] R. Haller, J. Langemets, Y. Perreau and T. Veeorg. Unconditional bases and Daugavet renormings. *J. Funct. Anal.* **286** (2024), 110421.
- [13] R. Haller, K. Pirk and T. Veeorg. Daugavet- and Delta-points in absolute sums of Banach spaces. *J. Convex Anal.* **28** (2021), 41–54.
- [14] Y. Ivakhno and V. Kadets. Unconditional sums of spaces with bad projections. *Visn. Khark. Univ. Ser. Mat. Prykl. Mat. Mekh.* **645** (2004), 30–35.
- [15] Y. Ivakhno, V. Kadets and D. Werner. The Daugavet property for spaces of Lipschitz functions. *Math. Scand.* **101** (2007), 261–279.
- [16] M. Jung. Daugavet property of Banach algebras of holomorphic functions and norm-attaining holomorphic functions. *Adv. Math.* **421** (2023), 109005.
- [17] M. Jung and A. Rueda Zoca. Daugavet points and Δ -points in Lipschitz-free spaces. *Stud. Math.* **265** (2022), 37–55.
- [18] V. M. Kadets. Some remarks concerning the Daugavet equation. *Quaestiones Math.* **19** (1996), 225–235.
- [19] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin and D. Werner. Banach spaces with the Daugavet property. *Trans. Amer. Math. Soc.* **352** (2000), 855–873.
- [20] V. Kadets and D. Werner. A Banach space with the Schur and the Daugavet property. *Proc. Amer. Math. Soc.* **132** (2004), 1765–1773.
- [21] B. L. Lin, P. K. Lin and S. L. Troyanski. Some geometric and topological properties of the unit sphere in a Banach space. *Math. Ann.* **274** (1986), 613–616.
- [22] G. Y. Lozanovskii. On almost integral operators in KB-spaces. *Vestnik Leningrad Univ. Mat. Mekh. Astr.* **21.7** (1966), 35–44(Russian).
- [23] M. Martín, Y. Perreau and A. Rueda Zoca. Diametral notions for elements of the unit ball of a Banach space. *Dissertationes Math.* **594** (2024), 1–61.
- [24] M. Martín and A. Rueda Zoca. Daugavet property in projective symmetric tensor products of Banach spaces. *Banach J. Math. Anal.* **16** (2022), 35.
- [25] A. Rueda Zoca. Daugavet property and separability in Banach spaces. *Banach J. Math. Anal.* **12** (2018), 68–84.

- [26] T. Veeorg. Characterizations of Daugavet points and Delta-points in Lipschitz-free spaces. *Studia Math.* **268** (2023), 213–233.
- [27] T. Veeorg. Daugavet- and Delta-points in spaces of Lipschitz functions. arXiv:2206.03475.
- [28] N. Weaver. *Lipschitz algebras*. Second edition, (World Scientific Publishing Co. Inc., River Edge, NJ, 2018).
- [29] D. Werner. An elementary approach to the Daugavet equation, *Interaction between functional analysis, harmonic analysis, and probability (Columbia, MO, 1994)* Lecture Notes in Pure and Appl. Math., (Dekker, New York, 1996), pp. 449–454Vol175.
- [30] D. Werner. The Daugavet equation for operators on function spaces. *J. Funct. Anal.* **143** (1997), 117–128.
- [31] D. Werner. Recent progress on the Daugavet property. *Irish Math. Soc. Bull.* **46** (2001), 77–97.
- [32] P. Wojtaszczyk. Some remarks on the Daugavet equation. *Proc. Amer. Math. Soc.* **115** (1992), 1047–1052.