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AN ANALYTIC PROOF OF A THEOREM OF FELIX KLEIN

BY
GARETH J. GRIFFITH

In 1876, Klein published the following result: "If a crunode of a real, irreducible, plane, algebraic curve changes into an acnode via the intermediary stage of a real cusp, two real inflexions are introduced in a neighborhood of the double points" [2].

Klein's approach to the proof of his result is a synthetic one and no other, non-synthetic proof seems to have been published. It shall be the purpose of this paper to provide such a proof.

Proof of the theorem. We note that the rational cubic curve:

$$ax^3+bx^2y+cxy^2+dy^3+\eta x^2+y^2 = 0 \tag{1}$$

where a, b, c, d, η are real and $a\neq 0$, is crunodal, acnodal or cuspidal according as η is $<$, $>$, or $=0$. It has

$$\eta^2cx^3+(3\eta^2d-2\eta b)x^2y+(3a-2\eta c)xy^2+by^3+\eta^2x^2+\eta y^2 = 0 \tag{2}$$

as its associated Hessian curve, and, upon eliminating ηx^2+y^2 from these equations, we obtain:

$$(\eta c-a)x(\eta x^2-3y^2)+(\eta d-b)y(3\eta x^2-y^2) = 0 \tag{3}$$

which represents the three lines from the origin to the three inflexion points ($\eta\neq 0$).

If $\eta=0$, the origin is a cusp of the curve, and (3) reduces to

$$3axy^2+by^3 = y^2(3ax+by) = 0.$$

Now, $y^2=0$ represents the cuspidal tangent, and therefore

$$3ax+by = 0$$

is the equation of the line from the origin to the inflexion point, I, say. If we write $x=-by/3a$ ($a\neq 0$) in (1), we obtain:

$$x = -b\mu \quad \text{and} \quad y = 3a\mu,$$

where $\mu=9a/\{9abc-27a^2d-2b^3\}$. Hence the coordinates of I are $(-b\mu, 3a\mu)$. (We assume, without loss of generality, that $9abc-27a^2d-2b^3\neq 0$ so that I is a finite point.)

Let η increase from zero to a value η_0 which is considered to be small. Then the inflexion, I_0, will have coordinates $(-b\mu+\eta_1, 3a\mu+\eta_2)$ where η_1 and η_2 are also small, due to the continuity of the second derivative. We consider only those values of η_0 such that $\eta_i \cdot \eta_j$ $(i, j=0, 1, 2)$ are so small that they may be ignored. Equation (3) then becomes

$$(4) \qquad -a\eta_0 x^3 - 3\eta_0 cxy^2 + 3axy^2 - \eta_0\, dy^3 - 3\eta_0 bx^2y + by^3 = 0.$$

One of these lines passes through I_0. The quotient obtained as a result of dividing (4) by $(-b\mu+\eta_1)y-(3a\mu+\eta_2)x$ may be written as the product of the quadratic form:

$$(5) \qquad \begin{aligned} &b^2(b\mu-2\eta_1-d\mu\eta_0)y^2 \\ &\quad +3(3a^2\eta_1+ab\eta_2+3abc\mu\eta_0-3a^2d\mu\eta_0-b^3\mu\eta_0)x^2 \\ &\quad -b(3a\eta_1+b\eta_2+3bc\mu\eta_0-3ad\mu\eta_0)xy \end{aligned}$$

and $\mu/(-b\mu+\eta_1)^3$.

Therefore, the form (5) equated to zero, must represent the two lines from the origin to the two inflexions associated with the double point.

If we now equate the form (5) to zero, we obtain:

$$y = x\{\lambda \pm 2\sqrt{3b\mu\nu}\}/\{2b(b\mu-2\eta_1-d\mu\eta_0)\},$$

where $$\lambda = 3a\eta_1+b\eta_2+3bc\mu\eta_0-3ad\mu\eta_0$$

and $$\nu = b^3\mu\eta_0 - a\lambda$$

so that λ and ν are of the same order of smallness as η_i, $i=0, 1, 2$.

The points of intersection of these lines with the given curve (1), other than the origin are:

$$x = \frac{-2(a\lambda+4\nu)}{2ab^2(b\mu-6\eta_1-3d\mu\eta_0)+b^2(\lambda\pm 2\sqrt{3b\mu\nu})+6c\nu}$$

and

$$y = \frac{x(\lambda\pm 2\sqrt{3b\mu\nu})}{2b(b\mu-2\eta_1-d\mu\eta_0)}.$$

The fact that the acnodal cubic has three real inflexions is well known and may be easily demonstrated, either by means of Klein's equation for real, plane, algebraic curves [1], or by considering the discriminant of (3). Hence, the two inflexions in question are real when $\eta_0 > 0$. Therefore when η_0 is small, these lie in a neighborhood of the origin, and this neighborhood may be as small as we please.

Therefore, since all curves whose equations are of the form

$$\varphi(x, y)+\eta x^2+y^2 = 0,$$

where φ is a given real polynomial containing terms of degree three and higher only, does not have ηx^2+y^2 as a factor and such that the coefficient of x^3 is non-zero, behave like a cubic curve with a double point in a sufficiently small neighborhood of the origin, due to the fact that those terms involving x and y which have weight greater than three have such little effect that they may be ignored, it follows that, corresponding to a small change of η from zero to a positive quantity, two real inflexions are introduced for all such curves. This completes the proof.

If, for some value of the parameter, η, a real inflexion fails to remain real, and becomes imaginary, then another inflexion must do so simultaneously, since the number of imaginary inflexions of all real, plane, algebraic curves is even.

Suppose that for some value of η, say η_r, the inflexions in question are real, while for some (necessarily larger) value, say η_i, the inflexions are imaginary. Then there exists some η_c, such that $\eta_r<\eta_c<\eta_i$, for which the inflexional tangents coincide.

References

1. J. L. Coolidge, *A treatise on algebraic, plane curves*, Dover, New York (1959), 113–114.

2. F. Klein, *Eine neue Relation zwischen den Singulariten einer algebraischen Curve*, Math. Ann. Vol. X, 1876.

3. J. G. Semple and L. Roth, *Introduction to algebraic geometry*, Oxford Univ. Press, London (1959), 103–104.

University of Saskatchewan,
Saskatoon, Saskatchewan