



Weak random periodic solutions of random dynamical systems

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Abstract. We first introduce the concept of weak random periodic solutions of random dynamical systems. Then, we discuss the existence of such periodic solutions. Further, we introduce the definition of weak random periodic measures and study their relationship with weak random periodic solutions. In particular, we establish the existence of invariant measures of random dynamical systems by virtue of their weak random periodic solutions. We use concrete examples to illustrate the weak random periodic phenomena of dynamical systems induced by random and stochastic differential equations.

1 Introduction

Periodic solutions are a very active research topic of the qualitative theory of ordinary differential equations. Given a dynamical system, it is important to investigate the existence, number, and positions of periodic solutions as well as the behavior of their nearby trajectories. For example, for polynomial vector fields in the plane, it is essential to derive an upper bound of the number of limit cycles and discuss their relative positions. This is the second part of Hilbert's 16th problem. Another example is the famous Poincaré–Bendixson theorem. It plays a fundamental role in the qualitative theory of differential equations in the plane because it provides a useful method to check the existence of periodic solutions and to find their positions (cf. [3, 10, 18, 19]). In the past 50 years, a lot of progress has been made for periodic solutions and the global structure of dynamical systems (cf. [9, 11, 13, 15, 21–23, 26]).

When we study random dynamical systems, it is natural to consider the counterparts of fixed points and periodic solutions. In the literature, the counterparts are called stationary solutions and random periodic solutions, respectively. Stationary solutions have attracted lots of attention, and a series of results have been obtained (cf. [2, 14, 17, 24]). In [25], Zhao and Zheng introduced for the first time the concept of random periodic solutions and gave a sufficient condition for their existence.

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In [7], Feng and Zhao introduced random periodic measures and discussed the close relationship between random periodic solutions and random periodic measures. We call the reader’s attention to [5, 6, 8] for some other recent works on random periodic solutions.

Note that the period T of the random periodic solutions introduced in [25] is deterministic and uniform for all random paths ω . However, for many random dynamical systems induced by random or stochastic differential equations, the solutions exhibit some periodic behaviors, while the periods depend on ω . To deal with these phenomena, we introduce in this paper the novel concept of weak random periodic solutions. It is easy to see that any random periodic solution is a weak random periodic solution. But, in general, a weak random periodic solution might not be a random periodic solution.

The remainder of this paper is organized as follows. In Section 2, we introduce the concept of weak random periodic solutions and present a useful criterion for their existence. In Section 3, we give the definition of weak random periodic measures and show that the existence of weak random periodic solutions implies the existence of weak random periodic measures. Further, we establish the existence of invariant measures for random dynamical systems by virtue of their weak random periodic solutions. In Section 4, we use concrete examples to illustrate the weak random periodic phenomena of dynamical systems induced by random and stochastic differential equations.

2 Definition and existence of weak random periodic solutions

First, let us recall the concepts of fixed point and periodic solution. Let E be a Polish space with Borel σ -algebra $\mathcal{B}(E)$. For a deterministic dynamical system $\Psi : \mathbb{R} \times E \rightarrow E$, a fixed point is a point $x \in E$ such that

$$\Psi(t)x = x, \quad \forall t \in \mathbb{R}.$$

A periodic solution with period $T > 0$ is a $\mathcal{B}(\mathbb{R})$ -measurable function $Y : \mathbb{R} \rightarrow E$ such that

$$\Psi(t)Y(s) = Y(t + s), \quad Y(t + T) = Y(t), \quad \forall s, t \in \mathbb{R}.$$

Suppose that $\Psi : \mathbb{R} \times \Omega \times E \rightarrow E$ is a measurable random dynamical system on $(E, \mathcal{B}(E))$ over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. Then, for $\omega \in \Omega$,

$$(2.1) \quad \Psi(0, \omega) = \text{id}_E, \quad \Psi(t + s, \omega) = \Psi(t, \theta_s \omega) \Psi(s, \omega), \quad \forall s, t \in \mathbb{R}.$$

A stationary solution (cf. [2]) of Ψ is a random variable $Y : \Omega \rightarrow E$ such that for almost all $\omega \in \Omega$,

$$\Psi(t, \omega)Y(\omega) = Y(\theta_t \omega), \quad \forall t \in \mathbb{R}.$$

A random periodic solution with period $T > 0$ (see [7, 25]) is a $\mathcal{B}(\mathbb{R}) \times \mathcal{F}$ -measurable function $Y : \mathbb{R} \times \Omega \rightarrow E$ such that for almost all $\omega \in \Omega$,

$$\Psi(t, \theta_s \omega)Y(s, \omega) = Y(t + s, \omega), \quad Y(s + T, \omega) = Y(s, \theta_T \omega), \quad \forall s, t \in \mathbb{R}.$$

The period T in the above definition is nonrandom. For many applications, this is not satisfactory. Here is a simple example. Suppose that $X(\bar{\omega})$ is a positive random variable. Consider the following random differential equation (RDE):

$$(2.2) \quad \frac{d^2x(t)}{dt^2} = \sin(X(\bar{\omega})(t+s)),$$

where $s \in \mathbb{R}$. The periodic solution of (2.2) is given by

$$x(t) = \frac{-\sin(X(\bar{\omega})(t+s))}{X(\bar{\omega})^2} + r, \quad r \in \mathbb{R},$$

whose period $T\bar{\omega} = \frac{2\pi}{X(\bar{\omega})}$ is random.

To deal with the phenomenon of random periods, we now introduce the concept of weak random periodic solution of a random dynamical system.

Definition 2.1 A weak random periodic solution of Ψ is a pair of measurable maps $Y : \mathbb{R} \times \Omega \rightarrow E$ and $T : \Omega \rightarrow (0, \infty)$ such that for almost all $\omega \in \Omega$,

$$(2.3) \quad \Psi(t, \theta_s \omega) Y(s, \omega) = Y(t+s, \omega), \quad Y(s+T\omega, \theta_{-T\omega} \omega) = Y(s, \omega), \quad \forall s, t \in \mathbb{R}.$$

Obviously, if T is a constant map, then the weak random periodic solution is reduced to the random periodic solution. For the existence of weak random periodic solutions, we have the following useful criterion.

Proposition 2.2 If there exist measurable maps $Y_0 : \Omega \rightarrow E$ and $T : \Omega \rightarrow (0, \infty)$ such that for almost all $\omega \in \Omega$,

$$(2.4) \quad Y_0(\omega) = \Psi(T\omega, \theta_{-T\omega} \omega) Y_0(\theta_{-T\omega} \omega),$$

then the random dynamical system Ψ has a weak random periodic solution.

Proof For $\omega \in \Omega$, define $Y(0, \omega) = Y_0(\omega)$ and

$$(2.5) \quad Y(t, \omega) := \Psi(t, \omega) Y(0, \omega), \quad t \in \mathbb{R}.$$

Then, by (2.1), (2.4), and (2.5), we obtain that for almost all $\omega \in \Omega$,

$$\begin{aligned} \Psi(t, \theta_s \omega) Y(s, \omega) &= \Psi(t, \theta_s \omega) \Psi(s, \omega) Y(0, \omega) \\ &= \Psi(t+s, \omega) Y(0, \omega) \\ &= Y(t+s, \omega), \quad \forall s, t \in \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} Y(s, \omega) &= \Psi(s, \omega) Y(0, \omega) \\ &= \Psi(s, \omega) \Psi(T\omega, \theta_{-T\omega} \omega) Y(0, \theta_{-T\omega} \omega) \\ &= \Psi(s+T\omega, \theta_{-T\omega} \omega) Y(0, \theta_{-T\omega} \omega) \\ &= Y(s+T\omega, \theta_{-T\omega} \omega), \quad \forall s \in \mathbb{R}. \end{aligned}$$

Therefore, (Y, T) is a weak random periodic solution of Ψ . ■

We next consider the weak random periodic solution of a stochastic semiflow. Denote

$$\Delta = \{(t, s) \in \mathbb{R}^2 : s \leq t\}.$$

Let $\varphi : \Delta \times \Omega \times E \rightarrow E$ be a stochastic semiflow. Then, for $\omega \in \Omega$,

$$(2.6) \quad \varphi(t, s, \omega) = \varphi(t, u, \omega) \circ \varphi(u, s, \omega), \quad \forall s \leq u \leq t,$$

and

$$\varphi(s, s, \omega) = \text{id}_E, \quad \forall s \in \mathbb{R}.$$

Definition 2.3 A weak random periodic solution of φ is a pair of measurable maps $Y : \mathbb{R} \times \Omega \rightarrow E$ and $T : \Omega \rightarrow (0, \infty)$ such that for almost all $\omega \in \Omega$,

$$(2.7) \quad \varphi(t, s, \omega)Y(s, \omega) = Y(t, \omega), \quad Y(s + T\omega, \theta_{-T\omega}\omega) = Y(s, \omega), \quad \forall s \leq t.$$

3 Weak random periodic measures and invariant measures

Let Ψ be a measurable random dynamical system. Define

$$Y_t(\omega, x) = (\theta_t\omega, \Psi(t, \omega)x), \quad \omega \in \Omega, x \in E, t \in \mathbb{R}.$$

Denote by $\mathcal{P}(\Omega \times E)$ the set of all probability measures on $(\Omega \times E, \mathcal{F} \otimes \mathcal{B}(E))$.

Definition 3.1 A weak random periodic probability measure of Ψ is a pair of measurable maps $\mu : \mathbb{R} \times \Omega \rightarrow \mathcal{P}(\Omega \times E)$ and $T : \Omega \rightarrow (0, \infty)$ such that for almost all $\omega \in \Omega$,

$$Y_t\mu(s, \omega) = \mu(t + s, \omega), \quad \mu(s + T\omega, \theta_{-T\omega}\omega) = \mu(s, \omega), \quad \forall s, t \in \mathbb{R}.$$

Theorem 3.2 If a random dynamical system $\Psi : \mathbb{R} \times \Omega \times E \rightarrow E$ has a weak random periodic solution $Y : \mathbb{R} \times \Omega \rightarrow E$ and $T : \Omega \rightarrow (0, \infty)$, then it has a weak random periodic probability measure. Additionally, if for almost all $\omega \in \Omega$,

$$(3.1) \quad T\omega = T(\theta_s\omega), \quad \forall s \in \mathbb{R},$$

then Ψ has an invariant probability measure whose random factorization is supported by

$$L^\omega := \{Y(s, \theta_{-s}\omega) : s \in [0, T\omega]\}.$$

Proof For $s \in \mathbb{R}$ and $\omega \in \Omega$, define

$$\mu(s, \omega)(A) = \delta_{Y(s, \omega)}(A_{\theta_s\omega}), \quad A \in \mathcal{F} \otimes \mathcal{B}(E),$$

where A_ω is the ω -section of A . Then, $\mu(s, \omega) \in \mathcal{P}(\Omega \times E)$.

We have

$$(Y_t^{-1}(A))_\omega = \{x : (\theta_t\omega, \Psi(t, \omega)x) \in A\} = \{x : \Psi(t, \omega)x \in A_{\theta_t\omega}\} = \Psi^{-1}(t, \omega)A_{\theta_t\omega}.$$

Then, by (2.3), we obtain that for almost all $\omega \in \Omega$,

$$\begin{aligned} \Upsilon_t \mu(s, \omega)(A) &= \mu(s, \omega)(\Upsilon_t^{-1}(A)) \\ &= \delta_{Y(s, \omega)}((\Upsilon_t^{-1}(A))_{\theta_s \omega}) \\ &= \delta_{\Psi(t, \theta_s \omega)Y(s, \omega)}(A_{\theta_{t+s} \omega}) \\ &= \delta_{Y(t+s, \omega)}(A_{\theta_{t+s} \omega}) \\ &= \mu(t+s, \omega)(A) \end{aligned}$$

and

$$\begin{aligned} \mu(s+T\omega, \theta_{-T\omega} \omega)(A) &= \delta_{Y(s+T\omega, \theta_{-T\omega} \omega)}(A_{\theta_s \omega}) \\ &= \delta_{Y(s, \omega)}(A_{\theta_s \omega}) \\ &= \mu(s, \omega)(A). \end{aligned}$$

Thus, μ is a weak random periodic probability measure of Ψ .

For $A \in \mathcal{F} \otimes \mathcal{B}(E)$, define

$$\tilde{\mu}(A) := \int_{\Omega} \frac{1}{T\omega} \int_0^{T\omega} \mu(s, \omega)(A) ds P(d\omega).$$

By (2.3), (3.1), and the measure preserving property of $\{\theta_t\}$, we get

$$\begin{aligned} \tilde{\mu}(A) &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([0, T\omega]) \cdot \delta_{Y(s, \omega)}(A_{\theta_s \omega})}{T\omega} P(d\omega) ds \\ &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([0, T\omega]) \cdot \delta_{Y(s+T\omega, \theta_{-T\omega} \omega)}(A_{\theta_s \omega})}{T\omega} P(d\omega) ds \\ &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([0, T\omega]) \cdot \delta_{Y(s+T\omega, \theta_{-s-T\omega} \omega)}(A_{\omega})}{T\omega} P(d\omega) ds \\ &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([0, T\omega]) \cdot \delta_{Y(s+T\omega, \theta_{-s-T\omega} \omega)}(A_{\omega})}{T\omega} ds P(d\omega) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta_s([T\omega, 2T\omega]) \cdot \delta_{Y(s, \theta_{-s} \omega)}(A_{\omega})}{T\omega} ds P(d\omega) \\ &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([T\omega, 2T\omega]) \cdot \delta_{Y(s, \theta_{-s} \omega)}(A_{\omega})}{T\omega} P(d\omega) ds \\ &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([T\omega, 2T\omega]) \cdot \delta_{Y(s, \omega)}(A_{\theta_s \omega})}{T\omega} P(d\omega) ds \\ &= \int_{\Omega} \frac{1}{T\omega} \int_{T\omega}^{2T\omega} \mu(s, \omega)(A) ds P(d\omega). \end{aligned}$$

Repeating this argument, we can show that

$$\tilde{\mu}(A) = \int_{\Omega} \frac{1}{T\omega} \int_{k(T\omega)}^{(k+1)(T\omega)} \mu(s, \omega)(A) ds P(d\omega), \quad \forall k \in \mathbb{N},$$

which implies that

$$\begin{aligned} \tilde{\mu}(A) &= \int_{\Omega} \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \delta_{Y(s, \omega)}(A_{\theta_s \omega}) ds P(d\omega) \\ (3.2) \quad &= \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\Omega} \int_0^N \delta_{Y(s, \omega)}(A_{\theta_s \omega}) P(d\omega) ds. \end{aligned}$$

By (2.3), (3.2), and the measure preserving property of $\{\theta_t\}$, we obtain that

$$\begin{aligned} Y_t \tilde{\mu}(A) &= \tilde{\mu}(Y_t^{-1}(A)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \int_{\Omega} \delta_{Y(s,\omega)}((Y_t^{-1}(A))_{\theta_s\omega}) P(d\omega) ds \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \int_{\Omega} \delta_{\Psi(t,\theta_s\omega)Y(s,\omega)}(A_{\theta_{t+s}\omega}) P(d\omega) ds \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \int_{\Omega} \delta_{Y(t+s,\omega)}(A_{\theta_{t+s}\omega}) P(d\omega) ds \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \int_{\Omega} \delta_{Y(s,\omega)}(A_{\theta_s\omega}) P(d\omega) ds \\ &= \tilde{\mu}(A). \end{aligned}$$

Let $\pi_{\Omega} : \Omega \times E \rightarrow \Omega$, $\pi_{\Omega}(\omega, x) = \omega$, be the projection onto Ω . By (3.2) and the measure preserving property of $\{\theta_t\}$, we get $\tilde{\mu} \circ \pi_{\Omega}^{-1} = P$. Hence, $\tilde{\mu}$ is an invariant probability measure of Ψ (cf. [2, Definition 1.4.1]).

By (3.1) and the measure preserving property of $\{\theta_t\}$, we get

$$\begin{aligned} \tilde{\mu}(A) &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([0, T\omega]) \cdot \delta_{Y(s,\omega)}(A_{\theta_s\omega})}{T\omega} P(d\omega) ds \\ &= \int_{\mathbb{R}} \int_{\Omega} \frac{\delta_s([0, T\omega]) \cdot \delta_{Y(s,\theta_{-s}\omega)}(A_{\omega})}{T\omega} P(d\omega) ds \\ &= \int_{\Omega} \frac{1}{T\omega} \int_0^{T\omega} \delta_{Y(s,\theta_{-s}\omega)}(A_{\omega}) ds P(d\omega). \end{aligned}$$

Then, the random factorization of $\tilde{\mu}$ is given by

$$(\tilde{\mu})_{\omega} = \frac{1}{T\omega} \int_0^{T\omega} \delta_{Y(s,\theta_{-s}\omega)} ds,$$

which is supported by L^{ω} . Therefore, the proof is complete. ■

We now consider weak random periodic measures and invariant measures of a semiflow φ . Define $\bar{E} := \mathbb{R} \times E$ and

$$\bar{\Psi}(t, \omega)(s, x) = (t + s, \varphi(t + s, s, \theta_{-s}\omega)x), \quad \omega \in \Omega, s \in \mathbb{R}, x \in E, t \geq 0.$$

Then, $\bar{\Psi} : [0, \infty) \times \Omega \times \bar{E} \rightarrow \bar{E}$ is a measurable random dynamical system on $(\bar{E}, \mathcal{E}(\bar{E}))$ over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$. Assume that φ has a weak random periodic solution (Y, T) . Define

$$\bar{Y}(s, \omega) = (s, Y(s, \omega)), \quad \omega \in \Omega, s \in \mathbb{R},$$

and

$$\eta_t(s, x) = (t + s, x), \quad \omega \in \Omega, s, t \in \mathbb{R}, x \in E.$$

Then, by (2.7), we obtain that for almost all $\omega \in \Omega$,

$$(3.3) \quad \bar{\Psi}(t, \theta_s\omega) \bar{Y}(s, \omega) = \bar{Y}(t + s, \omega), \quad \bar{Y}(s + T\omega, \theta_{-T\omega}\omega) = \eta_{T\omega} \circ \bar{Y}(s, \omega), \quad s \in \mathbb{R}, t \geq 0.$$

Define

$$\bar{Y}_t(\omega, s, x) = (\theta_t \omega, \bar{\Psi}(t, \omega)(s, x)), \quad \omega \in \Omega, s \in \mathbb{R}, x \in E, t \geq 0,$$

and

$$\tilde{\eta}_t(\omega, s, x) = (\omega, t + s, x), \quad \omega \in \Omega, s, t \in \mathbb{R}, x \in E.$$

Denote by $\mathcal{P}(\Omega \times \bar{E})$ the set of all probability measures on $(\Omega \times \bar{E}, \mathcal{F} \otimes \mathcal{B}(\bar{E}))$. Let $\pi_\Omega : \Omega \times \bar{E} \rightarrow \Omega, \pi_\Omega(\omega, (s, x)) = \omega$, be the projection onto Ω .

Definition 3.3 A weak random periodic probability measure of φ is a pair of measurable maps $\mu : \mathbb{R} \times \Omega \rightarrow \mathcal{P}(\Omega \times \bar{E})$ and $T : \Omega \rightarrow (0, \infty)$ such that for almost all $\omega \in \Omega$,

$$\bar{Y}_t \mu(s, \omega) = \mu(t + s, \omega), \quad \mu(s + T\omega, \theta_{-T}\omega) \circ \tilde{\eta}_{T\omega} = \mu(s, \omega), \quad \forall s \in \mathbb{R}, t \geq 0.$$

Theorem 3.4 If a stochastic semiflow $\varphi : \Delta \times \Omega \times E \rightarrow E$ has a weak random periodic solution $Y : \mathbb{R} \times \Omega \rightarrow E$ and $T : \Omega \rightarrow (0, \infty)$, then it has a weak random periodic probability measure. If in addition (3.1) holds for almost all $\omega \in \Omega$, then there exists a weak-invariant probability measure $\tilde{\mu}$ on $\mathcal{F} \otimes \mathcal{B}(\bar{E})$ satisfying $\tilde{\mu} \circ \pi_\Omega^{-1} = P$,

$$\bar{Y}_t \tilde{\mu}(A) = \tilde{\mu}(A), \quad \forall A \in \mathcal{F} \otimes \{\emptyset, \mathbb{R}\} \times \mathcal{B}(E), t \geq 0,$$

and its random factorization is supported by

$$L^\omega := \{\bar{Y}(s, \theta_{-s}\omega) : s \in [0, T\omega]\}.$$

Proof For $s \in \mathbb{R}$ and $\omega \in \Omega$, define

$$\mu(s, \omega)(A) = \delta_{\bar{Y}(s, \omega)}(A_{\theta_s \omega}), \quad A \in \mathcal{F} \otimes \mathcal{B}(\bar{E}),$$

where A_ω is the ω -section of A . Then, $\mu(s, \omega) \in \mathcal{P}(\Omega \times \bar{E})$.

We have

$$\begin{aligned} (\bar{Y}_t^{-1}(A))_\omega &= \{(s, x) : (\theta_t \omega, \bar{\Psi}(t, \omega)(s, x)) \in A\} \\ &= \{(s, x) : \bar{\Psi}(t, \omega)(s, x) \in A_{\theta_t \omega}\} \\ &= \bar{\Psi}^{-1}(t, \omega)A_{\theta_t \omega}. \end{aligned}$$

Then, by (3.3), we obtain that for almost all $\omega \in \Omega$,

$$\begin{aligned} \bar{Y}_t \mu(s, \omega)(A) &= \mu(s, \omega)(\bar{Y}_t^{-1}(A)) \\ &= \delta_{\bar{Y}(s, \omega)}((\bar{Y}_t^{-1}(A))_{\theta_s \omega}) \\ &= \delta_{\bar{\Psi}(t, \theta_s \omega) \bar{Y}(s, \omega)}(A_{\theta_{t+s} \omega}) \\ &= \delta_{\bar{Y}(t+s, \omega)}(A_{\theta_{t+s} \omega}) \\ &= \mu(t + s, \omega)(A) \end{aligned}$$

and

$$\begin{aligned} \mu(s + T\omega, \theta_{-T}\omega)(\tilde{\eta}_{T\omega}A) &= \delta_{\bar{Y}(s+T\omega, \theta_{-T}\omega)}((\tilde{\eta}_{T\omega}A)_{\theta_{s+\omega}}) \\ &= \delta_{\tilde{\eta}_{T\omega} \circ \bar{Y}(s, \omega)}(\eta_{T\omega}A_{\theta_s \omega}) \end{aligned}$$

$$\begin{aligned} &= \delta_{\bar{Y}(s,\omega)}(A_{\theta_s\omega}) \\ &= \mu(s, \omega)(A). \end{aligned}$$

Thus, μ is a weak random periodic probability measure of φ .

For $A \in \mathcal{F} \otimes \mathcal{B}(\bar{E})$, define

$$\tilde{\mu}(A) := \int_{\Omega} \frac{1}{T\omega} \int_0^{T\omega} \mu(s, \omega)(A) ds P(d\omega).$$

Then, by using (3.3) and following the same argument of the proof of Theorem 3.2, we can complete the proof. ■

4 Examples

In this section, we use examples to illustrate the weak random periodic phenomena of dynamical systems induced by random and stochastic differential equations.

First, we investigate the periodic behavior of RDEs of type (2.2) by virtue of weak random periodic solutions.

Example 4.1 Let $X(\omega)$ be a positive random variable and $a_k, b_k \in \mathbb{R}, 1 \leq k \leq N$, for some $N \in \mathbb{N}$. Consider the following RDE:

$$(4.1) \quad \frac{d^2x(t)}{dt^2} = \sum_{k=1}^N [a_k \sin(kX(\omega)(t+s)) + b_k \cos(kX(\omega)(t+s))],$$

where $s \in \mathbb{R}$. Note that equation (4.1) is equivalent to

$$(4.2) \quad \begin{cases} dx_1(t) = x_2(t) dt, \\ dx_2(t) = \left\{ \sum_{k=1}^N [a_k \sin(kX(\omega)(t+s)) + b_k \cos(kX(\omega)(t+s))] \right\} dt. \end{cases}$$

Denote by ν the distribution of $X(\omega)$. Define

$$V = \left\{ (x, y) : x \in (0, \infty), y \in \left[0, \frac{2\pi}{x} \right] \right\}.$$

We equip $(V, \mathcal{B}(V))$ with the probability measure P_V :

$$P_V(A) = \int_0^\infty \int_0^\infty \frac{x 1_A(x, y)}{2\pi} dy \nu(dx), \quad A \in \mathcal{B}(V).$$

Define

$$g_{x,y}(t) = \sum_{k=1}^N [a_k \sin(kx(t+y)) + b_k \cos(kx(t+y))], \quad t \in \mathbb{R}, (x, y) \in V,$$

and

$$\Omega = \{g_{x,y} : (x, y) \in V\}.$$

Set $J : V \mapsto \Omega, J(x, y) = g_{x,y}$. Define

$$\mathcal{F} = J(\mathcal{B}(V)), \quad P = P_V \circ J^{-1},$$

and

$$\theta_t \omega(s) = \omega(t + s), \quad \omega \in \Omega, s, t \in \mathbb{R}.$$

Then, $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system and equation (4.2) is equivalent to the following RDE:

$$(4.3) \quad \begin{cases} dx_1(t) = x_2(t)dt, \\ dx_2(t) = \omega(t)dt. \end{cases}$$

The random dynamical system $\Psi : \mathbb{R} \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ induced by equation (4.3) is given by

$$\begin{aligned} & \Psi(t, g_{x,y})(x_1, x_2) \\ &= \left(x_1 + x_2 t - \sum_{k=1}^N \frac{a_k [\sin(kx(t+y)) - kxt \cos(kxy) - \sin(kxy)]}{k^2 x^2} \right. \\ & \quad \left. - \sum_{k=1}^N \frac{b_k [\cos(kx(t+y)) + kxt \sin(kxy) - \cos(kxy)]}{k^2 x^2}, \right. \\ & \quad \left. x_2 + \sum_{k=1}^N \frac{-a_k [\cos(kx(t+y)) - \cos(kxy)] + b_k [\sin(kx(t+y)) - \sin(kxy)]}{kx} \right), \end{aligned}$$

where $t \in \mathbb{R}$, $(x, y) \in V$, and $(x_1, x_2) \in \mathbb{R}^2$. Fix a $\mathcal{B}(\mathbb{R})$ -measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$. For $t \in \mathbb{R}$ and $(x, y) \in V$, define

$$(4.4) \quad \begin{aligned} & Y(t, g_{x,y}) \\ &= \left(h(x) - \sum_{k=1}^N \frac{a_k \sin(kx(t+y)) + b_k \cos(kx(t+y))}{k^2 x^2}, \right. \\ & \quad \left. \sum_{k=1}^N \frac{-a_k \cos(kx(t+y)) + b_k \sin(kx(t+y))}{kx} \right) \end{aligned}$$

and

$$Tg_{x,y} = \frac{2\pi}{x}.$$

Then, (Y, T) is a weak random periodic solution of Ψ . Further, by Theorem 3.2, we know that Ψ has an invariant probability measure.

Remark 4.2 Theorem 3.2 shows that, if a random dynamical system has a weak random periodic solution, then it has an invariant probability measure induced by this solution. On the other hand, Example 4.1 shows that different weak random periodic solutions can be obtained for the same random dynamical system through choosing different functions h in (4.4). It is interesting to consider how weak random periodic solutions affect the ergodicity of random dynamical systems.

Next, we consider a system of RDEs driven by periodic multiplicative noises.

Example 4.3 Suppose that $d \geq 1$. Denote by $C(\mathbb{R}; \mathbb{R}^d)$ and $C^1(\mathbb{R}; \mathbb{R}^d)$ the spaces of all continuous and continuously differentiable \mathbb{R}^d -valued functions on \mathbb{R} , respectively. We equip $C(\mathbb{R}; \mathbb{R}^d)$ with topology of locally uniform convergence. Define $\Omega = C^1(\mathbb{R}; \mathbb{R}^d)$ and

$$\mathcal{F} = \{A \cap \Omega : A \in \mathcal{B}(C(\mathbb{R}; \mathbb{R}^d))\}.$$

For $\omega = (\omega_1, \dots, \omega_d) \in \Omega$ and $s, t \in \mathbb{R}$, set $(\theta_t \omega)(s) = \omega(t + s) - \omega(t)$.

We choose $\omega^1, \omega^2, \dots \in \Omega$ with periods $T_1 < T_2 < \dots$, respectively, and $a_1, a_2, \dots \in (0, \infty)$ satisfying $\sum_{n=1}^\infty a_n = 1$. Define

$$\Omega_n := \{\theta_t \omega^n : 0 \leq t < T_n\}, \quad n \in \mathbb{N}.$$

Denote by \mathcal{L} the Lebesgue measure on \mathbb{R} . We define a probability measure P on (Ω, \mathcal{F}) by

$$P\left(\Omega \setminus \bigcup_{n=1}^\infty \Omega_n\right) = 0,$$

and

$$P(\{\theta_t \omega_n : t \in A\}) = \frac{a_n \mathcal{L}(A)}{T_n}, \quad \forall A \in \mathcal{B}([0, T_n]), n \in \mathbb{N}.$$

Set

$$T\omega = T_n, \quad \forall \omega \in \Omega_n, n \in \mathbb{N},$$

and

$$T\omega = 1, \quad \forall \omega \notin \bigcup_{n=1}^\infty \Omega_n.$$

Then, $\{\theta_t\}$ are P -measure preserving and $T : \Omega \rightarrow (0, \infty)$ is a measurable map such that for almost all $\omega \in \Omega$,

$$\omega(s + T\omega) = \omega(s), \quad \forall s \in \mathbb{R}.$$

Let A be a $d \times d$ hyperbolic matrix and $\sigma = (\sigma_{ij})_{1 \leq i, j \leq d}$ with $\sigma_{ij} : \mathbb{R}^d \rightarrow \mathbb{R}$ being Lipschitz-continuous and satisfying $\sigma_{ij}(x) = o(|x|)$ as $|x| \rightarrow \infty$. Consider the RDE

$$(4.5) \quad dx(t) = Ax(t)dt + \sigma(x(t))d\omega(t).$$

By [1, Theorem 22.1], we know that (4.5) has a T_n -periodic solution for each ω^n , which is denoted by $x(t, \omega^n)$. For $s, t \in \mathbb{R}$, define

$$Y(t, \theta_s \omega^n) = x(t + s, \omega^n).$$

Then, (Y, T) is a weak random periodic solution of the random dynamical system induced by equation (4.5). Further, by Theorem 3.2, we know that this random dynamical system has an invariant probability measure.

The third example is concerned with a random dynamical system induced by stochastic differential equations (SDEs), which is an extension of the example given by Zhao and Zheng (see [25, Section 2]).

Example 4.4 Let $\Omega := C(\mathbb{R}; \mathbb{R})$ and $\{\omega(t)\}_{t \in \mathbb{R}}$ be a one-dimensional two-sided Brownian motion on the path space $(\Omega, \mathcal{B}(\Omega), P)$ with θ being the shift operator $(\theta_t \omega)(s) = \omega(t + s) - \omega(t)$ for $s, t \in \mathbb{R}$. We define an equivalence relation \sim on Ω by $\omega \sim \omega'$ if and only if there exists $t \in \mathbb{R}$ such that $\omega' = \theta_t \omega$. For $\omega \in \Omega$, denote by $[\omega]$ the equivalence class of ω . Let $\Omega' := \Omega / \sim$ be the quotient space of Ω , $\mathcal{B}(\Omega') := \{\cup_{\omega \in F} [\omega] : F \in \mathcal{B}(\Omega)\}$, and P' be the induced image measure of P on $(\Omega', \mathcal{B}(\Omega'))$.

Before stating the example, we present a proposition. To the best of our knowledge, this is a novel result in the literature, which is of independent interest.

Proposition 4.5 $(\Omega', \mathcal{B}(\Omega'), P')$ is isomorphic (mod 0) to $[0, 1]$ with the Lebesgue measure.

Proof We equip Ω with the locally uniform metric:

$$d(\omega_1, \omega_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{-n \leq t \leq n} (|\omega_1(t) - \omega_2(t)| \wedge 1), \quad \omega_1, \omega_2 \in \Omega.$$

Then, Ω is a Polish space. By [20, pp. 14 and 24], we know that Ω is a Lebesgue space (also called standard probability space or Lebesgue–Rohlin probability space). Note that $(\Omega, \mathcal{B}(\Omega))$ is countably generated (cf. [16, Sections 1 and 2] for the definition of countably generated). Let $\{O_n, n \in \mathbb{N}\}$ be a sequence of open subsets of Ω which generates $\mathcal{B}(\Omega)$ and separates Ω . Define $O'_n := \cup_{\omega \in O_n} [\omega]$, $n \in \mathbb{N}$. Then, $\{O'_n, n \in \mathbb{N}\}$ generates $\mathcal{B}(\Omega')$ and separates Ω' . Hence, $(\Omega', \mathcal{B}(\Omega'))$ is also countably generated. Further, by [4, Theorem 3.2], we conclude that $(\Omega', \mathcal{B}(\Omega'), P')$ is a Lebesgue space.

It is known that any Lebesgue space is isomorphic (mod 0) to an interval with Lebesgue measure, a finite or countable set of atoms, or a combination (disjoint union) of both (see [20, p. 20]). Then, to show that $(\Omega', \mathcal{B}(\Omega'), P')$ is isomorphic (mod 0) to $[0, 1]$ with the Lebesgue measure, it suffices to prove that the probability space $(\Omega', \mathcal{B}(\Omega'), P')$ has no atom, equivalently, $P([\omega]) = 0$ for almost all $\omega \in \Omega$. Note that

$$[\omega] = \{\theta_t \omega : t \in [0, \infty)\} \cup \{\theta_t \omega : t \in (-\infty, 0]\}.$$

By symmetry, it suffices to show that for almost all $\omega \in \Omega$,

$$P^*(\{\theta_t \omega|_{[0, \infty)} : t \in [0, \infty)\}) = 0,$$

where P^* is the restriction of P on $C[0, \infty)$.

Define

$$A = \left\{ \omega \in C[0, \infty) : \limsup_{t \rightarrow \infty} \omega(t) = \infty \text{ and } \liminf_{t \rightarrow \infty} \omega(t) = -\infty \right\}.$$

It is well known that $P^*(A) = 1$. Fix an $\omega_0 \in A$. We choose $0 < t_1 < t_2 < \infty$ such that $\omega_0(t_1) = \omega_0(t_2) = 0$ and

$$m_1 := \max_{0 \leq u \leq t_1} \omega_0(u) > 0, \quad m_2 := \max_{t_1 \leq u \leq t_2} \omega_0(u) > 0.$$

Then, there exists $\varepsilon > 0$ such that

$$m_1 = \max_{\varepsilon \leq u \leq t_1 + \varepsilon} \omega_0(u), \quad m_2 = \max_{t_1 + \varepsilon \leq u \leq t_2 + \varepsilon} \omega_0(u).$$

Define

$$B = \left\{ \omega \in C[0, \infty) : \left[\max_{t_1 \leq u \leq t_2} \omega(u) \right] - \left[\max_{0 \leq u \leq t_1} \omega(u) \right] = m_2 - m_1 \right\}.$$

Then,

$$(4.6) \quad \{\theta_t \omega_0 : t \in [0, \varepsilon)\} \subset B.$$

For $\omega \in C[0, \infty)$, we have

$$\begin{aligned} & \left[\max_{t_1 \leq u \leq t_2} \omega(u) \right] - \left[\max_{0 \leq u \leq t_1} \omega(u) \right] \\ &= \left[\max_{t_1 \leq u \leq t_2} \{\omega(u) - \omega(t_1)\} \right] - \left[\max_{0 \leq u \leq t_1} \{\omega(u) - \omega(t_1)\} \right] \\ &= \left[\max_{0 \leq v \leq t_2 - t_1} \{\omega(t_1 + v) - \omega(t_1)\} \right] - \left[\max_{0 \leq v \leq t_1} \{\omega(t_1 - v) - \omega(t_1)\} \right] \\ &:= M_2 - M_1. \end{aligned}$$

It is known that M_2 and M_1 are two independent continuous random variables such that (cf. [12, p. 96])

$$P^*(M_2 \in dx) = \frac{2}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{x^2}{2(t_2 - t_1)}} dx, \quad P^*(M_1 \in dx) = \frac{2}{\sqrt{2\pi t_1}} e^{-\frac{x^2}{2t_1}} dx; \quad x > 0.$$

Then, $P^*(B) = 0$, which together with (4.6) implies that

$$P^*(\{\theta_t \omega_0 : t \in [0, \varepsilon)\}) = 0.$$

Applying the similar argument, we can show that for any $s \geq 0$, there exists $\varepsilon_s > 0$ such that

$$P^*(\{\theta_t(\theta_s \omega_0) : t \in [0, \varepsilon_s)\}) = 0,$$

which implies that

$$(4.7) \quad P^*(\{\theta_t \omega_0 : t \in [s, s + \varepsilon_s)\}) = 0, \quad \forall s \geq 0.$$

Define

$$C = \sup\{c : P^*(\{\theta_t \omega_0 : t \in [0, c)\}) = 0\}.$$

Then, by (4.7), we obtain that $C = \infty$. Hence,

$$P^*(\{\theta_t \omega_0 : t \in [0, \infty)\}) = 0.$$

Since $\omega_0 \in A$ is arbitrary, we conclude that for almost all $\omega \in \Omega$,

$$P^*(\{\theta_t \omega|_{[0, \infty)} : t \in [0, \infty)\}) = 0.$$

Therefore, $(\Omega', \mathcal{B}(\Omega'), P')$ is isomorphic (mod 0) to $[0, 1]$ with the Lebesgue measure. ■

We consider the SDE

$$(4.8) \quad \begin{cases} dx(t) = \{x(t) - y(t) - x(t)[x^2(t) + y^2(t)]\}dt + x(t) \circ d\omega(t), \\ dy(t) = \{x(t) + y(t) - y(t)[x^2(t) + y^2(t)]\}dt + y(t) \circ d\omega(t), \end{cases}$$

where $\circ d\omega(t)$ denotes the Stratonovich stochastic integral. Using polar coordinates

$$x = \rho \cos(2\pi\alpha), \quad y = \rho \sin(2\pi\alpha),$$

we can transform equation (4.8) on \mathbb{R}^2 to the following equation on $[0, \infty) \times [0, \infty)$:

$$(4.9) \quad \begin{cases} d\rho(t) = [\rho(t) - \rho^3(t)]dt + \rho(t) \circ d\omega(t), \\ d\alpha(t) = \frac{1}{2\pi} dt. \end{cases}$$

Equation (4.9) has a unique closed form solution as follows:

$$\rho(t, \alpha_0, \rho_0, \omega) = \frac{\rho_0 e^{t+\omega(t)}}{(1 + 2\rho_0^2 \int_0^t e^{2s+2\omega(s)} ds)^{\frac{1}{2}}}, \quad \alpha(t, \alpha_0, \rho_0, \omega) = \alpha_0 + \frac{t}{2\pi}.$$

One can check that

$$\rho^*(\omega) = \left(2 \int_{-\infty}^0 e^{2s+2\omega(s)} ds \right)^{-\frac{1}{2}}$$

is the stationary solution of the first equation of (4.9), i.e.,

$$\rho(t, \alpha_0, \rho^*(\omega), \omega) = \rho^*(\theta_t \omega).$$

By Proposition 4.5, we can choose a surjective measurable map $T' : \Omega' \rightarrow (0, \infty)$. Define $T\omega := T'[\omega]$ for $\omega \in \Omega$. Then, T is a measurable map on Ω . Define

$$\Psi^*(t, \omega)(\alpha_0, \rho_0) = \left(\left(\alpha_0 + \frac{t}{T\omega} \right) \bmod 1, \rho(t, \alpha_0, \rho_0, \omega) \right).$$

We find that

$$\Psi^*(0, \omega)(\alpha, \rho) = (\alpha, \rho), \quad \Psi^*(t+s, \omega) = \Psi^*(t, \theta_s \omega) \Psi^*(s, \omega), \quad \forall (\alpha, \rho) \in [0, 1) \times [0, \infty), s, t \in \mathbb{R}.$$

Hence, $\Psi^*(t, \omega) = (\Psi_1^*(t, \omega), \Psi_2^*(t, \omega))$ defines a random dynamical system on the cylinder $[0, 1) \times [0, \infty)$. Next, we transform the random dynamical system Ψ^* back to \mathbb{R}^2 . For $(x, y) \in \mathbb{R}^2$, $x = \rho \cos(2\pi\alpha)$, $y = \rho \sin(2\pi\alpha)$, define

$$\Psi(t, \omega)(x, y) = (\Psi_2^*(t, \omega)(\alpha, \rho) \cdot \cos[2\pi\Psi_1^*(t, \omega)(\alpha, \rho)], \Psi_2^*(t, \omega)(\alpha, \rho) \cdot \sin[2\pi\Psi_1^*(t, \omega)(\alpha, \rho)]).$$

Now, we investigate weak random periodic solutions of the random dynamical system Ψ . Fix an $\alpha_0 \in [0, 1)$ and define

$$Y(t, \omega) = \left(\rho^*(\theta_t \omega) \cos \left[2\pi\alpha_0 + \frac{2\pi t}{T\omega} \right], \rho^*(\theta_t \omega) \sin \left[2\pi\alpha_0 + \frac{2\pi t}{T\omega} \right] \right).$$

Then, we have

$$\begin{aligned} & \Psi(T\omega, \theta_{-T\omega} \omega) Y_0(\theta_{-T\omega} \omega) \\ &= \Psi(T\omega, \theta_{-T\omega} \omega) (\rho^*(\theta_{-T\omega} \omega) \cos(2\pi\alpha_0), \rho^*(\theta_{-T\omega} \omega) \sin(2\pi\alpha_0)) \\ &= (\Psi_2^*(T\omega, \theta_{-T\omega} \omega) \cdot \cos[2\pi\Psi_1^*(T\omega, \theta_{-T\omega} \omega)(\alpha_0, \rho^*(\theta_{-T\omega} \omega))], \\ & \quad \Psi_2^*(T\omega, \theta_{-T\omega} \omega) \cdot \sin[2\pi\Psi_1^*(T\omega, \theta_{-T\omega} \omega)(\alpha_0, \rho^*(\theta_{-T\omega} \omega))]) \\ &= \left(\rho(T\omega, \alpha_0, \rho^*(\theta_{-T\omega} \omega), \theta_{-T\omega} \omega) \cdot \cos \left[2\pi \left(\alpha_0 + \frac{T\omega}{T(\theta_{-T\omega} \omega)} \right) \bmod 1 \right], \right. \end{aligned}$$

$$\begin{aligned} & \rho(T\omega, \alpha_0, \rho^*(\theta_{-T\omega}\omega), \theta_{-T\omega}\omega) \cdot \sin \left[2\pi \left(\left(\alpha_0 + \frac{T\omega}{T(\theta_{-T\omega}\omega)} \right) \bmod 1 \right) \right] \\ &= (\rho^*(\omega) \cos(2\pi\alpha_0), \rho^*(\omega) \sin(2\pi\alpha_0)) \\ &= Y_0(\omega), \end{aligned}$$

which implies that (2.4) holds. Therefore, by Proposition 2.2, we find that (Y, T) is a weak random periodic solution of Ψ . Further, by Theorem 3.2, we conclude that Ψ has an invariant probability measure.

Remark 4.6 By Proposition 4.5, we know that there are different choices of the random period map T' . Hence, Example 4.7 implies that the weak random periodic solution of a random dynamical system is not necessarily a random periodic solution defined as in [7, 25].

The last example is related to SDEs in random environments.

Example 4.7 Let Z be an \mathbb{N} -valued random variable on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$. Suppose that $d \geq 1$, $\Omega_2 := C(\mathbb{R}; \mathbb{R}^d)$, and $\{\omega_2(t)\}_{t \in \mathbb{R}}$ is a d -dimensional two-sided Brownian motion on the path space $(\Omega_2, \mathcal{B}(\Omega_2), P_2)$ with θ_2 being the shift operator $(\theta_{2,t}\omega_2)(s) = \omega_2(t+s) - \omega_2(t)$ for $s, t \in \mathbb{R}$. Let $m \in \mathbb{N}$, $T_n > 0$, $b_{n,i} : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$, $\sigma_{n,ij} : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq d$, $n \in \mathbb{N}$. Suppose

$$b_{n,i}(t + T_n, x) = b_{n,i}(t, x), \quad \sigma_{n,ij}(t + T_n, x) = \sigma_{n,ij}(t, x), \quad t \in \mathbb{R}, x \in \mathbb{R}^m, n \in \mathbb{N}.$$

Assume that, for each $n \in \mathbb{N}$, the stochastic semiflow φ_n induced by the following SDE has a random periodic solution $\{Y_n(t, \omega_2)\}$ with period T_n :

$$dx_n(t) = b_n(t, x_n(t))dt + \sigma_n(t, x_n(t))d\omega_2(t).$$

We refer the reader to [5–8] for some concrete examples of random periodic solutions of nonautonomous SDEs.

Let $(\Omega, \mathcal{F}, P) = (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{B}(\Omega_2), P_2)$ and $\theta_t(\omega_1, \omega_2) = (\omega_1, \theta_{2,t}(\omega_2))$. We consider the following SDE in random environment:

$$dx(t) = b_Z(t, x(t))dt + \sigma_Z(t, x(t))d\omega_2(t),$$

which induces a stochastic semiflow φ :

$$\varphi(t, s, \omega_1, \omega_2) = \varphi_{Z(\omega_1)}(t, s, \omega_2), \quad (\omega_1, \omega_2) \in \Omega, s \leq t.$$

Define

$$T(\omega_1, \omega_2) = T_{Z(\omega_1)}, \quad Y(t, \omega_1, \omega_2) = Y_{Z(\omega_1)}(t, \omega_2), \quad (\omega_1, \omega_2) \in \Omega, t \in \mathbb{R}.$$

Then, (Y, T) is a weak random periodic solution of φ , which has a weak-invariant probability measure by Theorem 3.4.

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