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RESEARCH ARTICLE

Some stochastic comparisons of lower records and lower record spacings

Narayanaswamy Balakrishnan[1], Antonia Castaño-Martínez[2] and Miguel A. Sordo[3]

[1]Department of Mathematics and Statistics, McMaster University, Hamilton, Canada
[2]Department of Statistics and Operations Research, University of Cádiz, Cádiz, Spain
[3]Department of Statistics and Operations Research, University of Cádiz, Cádiz, Spain. E-mail: mangel.sordo@uca.es.

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Abstract
We obtain here sufficient conditions for increasing concave order and location independent more riskier order of lower record values based on stochastic comparisons of minimum order statistics. We further discuss stochastic orderings of lower record spacings. In particular, we show that increasing convex order of adjacent spacings between minimum order statistics is a sufficient condition for increasing convex order of adjacent spacings of their lower records.

1. Introduction

Let X be a continuous random variable with distribution function F, survival function $\bar{F} = 1 - F$ and quantile function $F^{-1}(p) = \inf\{x \in \mathbb{R} \,|\, F(x) \geq p\}, p \in (0,1)$. For $k = 1, \ldots, n$, let $X_{k:n}$ be the kth order statistic from an independent and identically distributed (i.i.d.) random sample $X_1, X_2, \ldots, X_n$, drawn from X. A large body of literature has been devoted to the study of order statistics, which includes numerous characterizations of probability distributions and applications to a wide range of problems, such as statistical estimation, inferential procedures and analysis of censored samples. Order statistics are also important in the context of life testing and reliability models, where they describe lifetimes of k-out-of-n systems. Reviews on theoretical results and applications can be found in the books by Arnold *et al.* [3], David and Nagaraja [23], and the two volume by Balakrishnan and Rao [8,9].

Let $X_1, X_2, \ldots$ be a sequence of independent random variables having the same distribution as X. Then, we say that X_j is a lower record value if it is less than all the previous values of the sequence. Lower record values appear in a natural way in many practical situations. In meteorological analysis, for example, they describe sequences of successive coldest temperatures. In portfolio management, lower record values describe sequences of successive lowest stock market figures, which is of great interest to investors and financial institutions. Similar applications can be given in industrial stress testing, seismology or sporting events. Some important references on records include the books by Ahsanullah [2], Arnold *et al.* [4] and Nevzorov [32]. The indices at which the lower record values occur are given by the record times $\{L_n\}_{n\geq 1}$, where $L_1 = 1$ and

$$L_n = \min\{j : \; j > L_{n-1}, \; X_j < X_{L_{n-1}}\}, \quad n \geq 2.$$

We denote $R_j = X_{L_j}$ as the jth lower record value of the sequence. As the record times of the sequence $\{X_i\}_{i\geq 1}$ are the same as those for the sequence $\{F(X_i)\}_{i\geq 1}$ and that $F(X)$ has a uniform distribution, it is clear that the distribution of L_n, $n \geq 1$, does not depend on F.

It has been observed in the literature (see Charalambides[1], [22]) that R_j can be represented as a minimum order statistic from a sequence of i.i.d. random variables whose sample size is random, that is, $R_j = X_{1:L_j}$. Because $X_{1:n}$ is independent of the event $\{L_j = n\}$, for $n \geq j$ [32] p. 114, the distribution of $R_j = X_{1:L_j}$ can be represented as a countable mixture, mixing the distribution of the jth record time with the distribution of minimum order statistic. One purpose of this paper is to use this mixture representation to provide sufficient conditions for stochastic orderings of records based on stochastic orderings of minimum order statistics. Some other results on stochastic comparisons of record values can be found in Kochar [28,29], Ahmadi and Arghami [1], Khaledi and Shojaei [26], Khaledi *et al.* [27] and Zhao and Balakrishnan [37]. As record values are particular cases of generalized order statistics (GOS), some results on stochastic comparisons of GOS are valid for record values as well (see, e.g., [11,15]).

We first briefly introduce the stochastic orders that are pertinent to the work in this paper.

Definition 1. *Let X and Y be two random variables with respective distribution functions F and G, and let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, be the corresponding survival functions. We say that X is smaller than Y*

(a) in stochastic order, denoted by $X \leq_{\rm st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$, for all x;
(b) in increasing concave order, denoted by $X \leq_{\rm icv} Y$, if

$$\int_{-\infty}^{t} F(x)\,dx \geq \int_{-\infty}^{t} G(x)\,dx, \quad \textit{for all } t;$$

(c) in increasing convex order, denoted by $X \leq_{\rm icx} Y$, if

$$\int_{\rm t}^{+\infty} \bar{F}(x)\,dx \leq \int_{\rm t}^{+\infty} \bar{G}(x)\,dx, \quad \textit{for all } t;$$

(d) in dispersive order, denoted by $X \leq_{\rm disp} Y$, if

$$F^{-1}(p) - F^{-1}(q) \leq G^{-1}(p) - G^{-1}(q), \quad \textit{for all } 0 < q < p < 1;$$

(e) in location independent riskier order, denoted by $X \leq_{\rm lir} Y$, if

$$\int_{-\infty}^{F^{-1}(p)} F(x)\,dx \leq \int_{-\infty}^{G^{-1}(p)} G(x)\,dx, \quad \textit{for all } p \in (0,1).$$

For various properties and applications of these orders, we refer the readers to the books by Shaked and Shanthikumar [33], Belzunce *et al.* [17] and Müller and Stoyan [31]. It is well-known that $X \leq_{\rm st} Y$ implies $X \leq_{\rm icx} Y$ and $X \leq_{\rm icv} Y$ and that $X \leq_{\rm icv} Y$ if and only if $-X \geq_{\rm icx} -Y$. It is also well-known that $X \leq_{\rm disp} Y$ implies $X \leq_{\rm lir} Y$. When the random variables X and Y have a common left endpoint of their supports, then $X \leq_{\rm disp} Y$ implies $X \leq_{\rm st} Y$, and $X \leq_{\rm lir} Y$ implies $X \leq_{\rm st} Y$ (see [36]). It will be useful in the sequel to note (see [25]) that $X \leq_{\rm lir} Y$ if and only if

$$H(u) = \frac{1}{u}\int_0^u (F^{-1}(t) - G^{-1}(t))\,dt \text{ is non-increasing in } u \in (0,1). \tag{1}$$

Using (3.B.6) in Shaked and Shanthikumar [33], it is easy to see that

$$X \leq_{\rm disp} Y \Longleftrightarrow (F^{-1}(p) - X)^+ \leq_{\rm st} (G^{-1}(p) - Y)^+, \quad \text{for all } p \in (0,1), \tag{2}$$

[1] Charalambides, in fact, represents the distribution of the kth upper record value as a countable mixture, mixing the distribution of the kth record time with the distribution of the maximum order statistics.

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where $(x)^+ = \max\{x, 0\}$. Moreover, since $X \leq_{\text{lir}} Y \Leftrightarrow -X \leq_{\text{ew}} -Y$, where $\leq_{\text{ew}}$ denotes the excess wealth order (see Section 3.C.1 of [33]), it follows from Theorem 4.A.43 in [33] that

$$X \leq_{\text{lir}} Y \Longleftrightarrow (F^{-1}(p) - X)^+ \leq_{\text{icx}} (G^{-1}(p) - Y)^+, \quad \text{for all } p \in (0, 1). \tag{3}$$

Using the same argument, it follows from Proposition 3 in Belzunce [13] that when the random variables X and Y have a common right endpoint of their supports, then $X \leq_{\text{lir}} Y$ implies $X \geq_{\text{icv}} Y$. In particular, if u_X and u_Y denote the right endpoints of the supports of X and Y, it follows that

$$X \leq_{\text{lir}} Y \quad \text{and} \quad u_X = u_Y \Longrightarrow E[Y] \leq E[X], \tag{4}$$

a result that will be used in Example 3.

Let $X_1, X_2, \ldots$ be a sequence of independent random variables having the same distribution as X and let $Y_1, Y_2, \ldots$ be another sequence of independent random variables having the same distribution as Y. Let $R_s(X)$ and $R_s(Y)$, $s \geq 1$, be the corresponding sth lower record values of the two sequences. Sufficient conditions on X and Y under which $R_s(X)$ and $R_s(Y)$ are ordered by the orders $\leq_{\text{st}}, \leq_{\text{icv}}, \leq_{\text{disp}}$ and $\leq_{\text{lir}}$ can be immediately derived from well-known results for generalized order statistics and for upper records (see Section 3.7.2 in the book by Belzunce *et al.*, [17], for a review). In particular, it follows from Theorem 4.14 in Balakrishnan *et al.* [11] that $X \leq_{\text{icv}} Y$ implies $R_s(X) \leq_{\text{icv}} R_s(Y)$, and from Theorem 5.2(a) in Belzunce *et al.* [14] that $X \leq_{\text{lir}} Y$ implies $R_s(X) \leq_{\text{lir}} R_s(Y)$ for $s \geq 1$. Note, however, that when the conditions $X \leq_{\text{icv}} Y$ and/or $X \leq_{\text{lir}} Y$ do not hold, the orderings $R_s(X) \leq_{\text{icv}} R_s(Y)$ and/or $R_s(X) \leq_{\text{lir}} R_s(Y)$ are still possible. This motivates us to study new sufficient conditions (weaker than $X \leq_{\text{icv}} Y$ and $X \leq_{\text{lir}} Y$) under which the lower record values $R_s(X)$ and $R_s(Y)$ are ordered for a certain range of values of s. The new sufficient conditions, which are stated in Section 2, are given in terms of comparisons of minimum order statistics. As $X \leq_{\text{icv}} Y$ (respectively, $X \leq_{\text{lir}} Y$) implies $X_{1:n} \leq_{\text{icv}} Y_{1:n}$ (respectively, $X_{1:n} \leq_{\text{lir}} Y_{1:n}$), for $n = 1, 2, \ldots$, and the reverse implications do not hold, our results can be useful for comparing lower records in terms of $\leq_{\text{icv}}$ and $\leq_{\text{lir}}$ when X and Y fail to be ordered. Let us now present an example of two random variables X and Y such that $X \nleq_{\text{icv}} Y$ and $X_{1:2} \leq_{\text{icv}} Y_{1:2}$.

Example 1. *For $i = 1, 2$, let $Z_i \sim W(\alpha_i, \beta_i), \alpha_i > 0, \beta_i > 0$, be two Weibull random variables with survival functions $\overline{F_i}(t) = e^{-(t/\alpha_i)^{\beta_i}}$, $t > 0$, and expectations $E[Z_i] = \alpha_i \Gamma(1 + 1/\beta_i)$. It is well-known (see Table 1.1 in [31]) that*

$$\beta_1 \leq \beta_2 \text{ and } E[Z_1] \leq E[Z_2] \text{ implies } Z_1 \leq_{\text{icv}} Z_2. \tag{5}$$

Consider, in particular, $X \sim W(1, \frac{1}{3})$ and $Y \sim W(2, 1)$. Because

$$\int_0^{100} [1 - e^{-x^{1/3}}]\, dx = 94,95$$

and

$$\int_0^{100} [1 - e^{-x/2}]\, dx = 98,$$

it follows that $X \nleq_{\text{icv}} Y$. Moreover, it is easy to see that $X_{1:2} \sim W(\frac{1}{8}, \frac{1}{3})$ and $Y_{1:2} \sim W(1, 1)$ with expectations $E[X_{1:2}] = \frac{3}{4}$ and $E[Y_{1:2}] = 1$. Thus, it follows from (5) that $X_{1:2} \leq_{\text{icv}} Y_{1:2}$.

Next, we provide an example of two random variables X and Y such that $X \nleq_{\text{lir}} Y$ and $X_{1:2} \leq_{\text{lir}} Y_{1:2}$.

Example 2. *Let X_1 and X_2 be uniform random variables, $X_1 \sim U(\frac{1}{4}, \frac{1}{2})$ and $X_2 \sim U(\frac{1}{2}, 1)$, with distribution functions $F_1(x)$ and $F_2(x)$, respectively. Now, let X be a random variable with distribution function $F(x) = \frac{3}{4} F_1(x) + \frac{1}{4} F_2(x)$ and Y be a standard uniform random variable, $Y \sim U(0, 1)$. Because*

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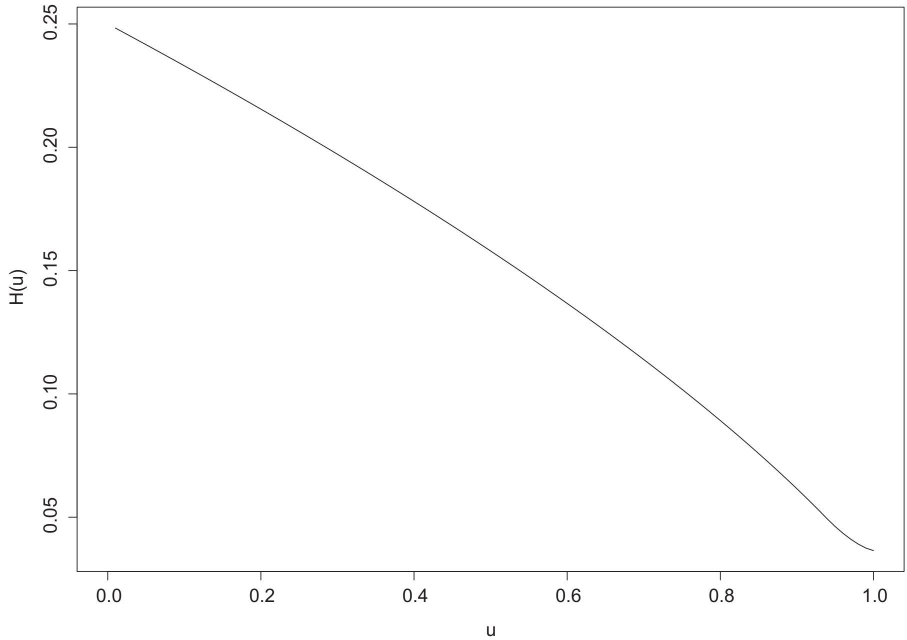

Figure 1. *$H_{1:2}(u)$ as a function of u for Example 3.*

X and Y have a common right endpoint of their supports and $E(X) = \frac{15}{32} < E(Y) = \frac{1}{2}$, it follows from (4) that $X \nleq_{\text{lir}} Y$. The distribution function $F_{1:2}$ of $X_{1:2}$ is

$$F_{1:2}(x) = \begin{cases} 0, & x < 1/4 \\ 1 - (\frac{7}{4} - 3x)^2, & 1/4 \leq x < 1/2 \\ 1 - \frac{1}{4}(1 - x)^2, & 1/2 \leq x < 1 \\ 1, & x \geq 1. \end{cases}$$

Figure 1 shows that the function

$$H_{1:2}(u) = \frac{1}{u} \int_0^u (F_{1:2}^{-1}(t) - G_{1:2}^{-1}(t))\, dt$$

decreases in $u \in (0, 1)$, which implies from (1) that $X_{1:2} \leq_{\text{lir}} Y_{1:2}$.

Another purpose of this paper is to provide sufficient conditions for the comparison of spacings between minimum order statistics drawn from samples of different sizes and for the comparison of spacings of lower records. The time elapsed between two consecutive records is sometimes as important as the record itself and it is related to the variability of the random variable. Given two i.i.d. samples of X of sizes n and $n + k$, $k \geq 1$, the number $\Delta_{1,n,k} = E[X_{1:n} - X_{1:n+k}]$ is a measure of variability of the left-tail of X. As

$$\Delta_{1,n,k} = E[X_{1:n} - X_{1:n+k}] = \int_0^1 F^{-1}(t)\, d(A(t) - B(t)),$$

where $A(t) = 1 - (1 - t)^n$, $B(t) = 1 - (1 - t)^{n+k}$ and $AB^{-1}(t)$ is convex, $\Delta_{1,n,k}$ belongs to the class C_1 of risk measures studied by Sordo [35]. As these measures are consistent with dispersive order of X and Y (Theorem 8 in [35]), a natural question that arises is under what conditions $X_{1:n} - X_{1:n+k}$ and $Y_{1:n} - Y_{1:n+k}$ are stochastically ordered? We address this question in Section 3, where it is also established

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that increasing convex order (respectively, increasing concave order) of adjacent spacings of X and Y is a sufficient condition for increasing convex order (respectively, increasing concave order) of adjacent spacings of their records. Some other works dealing with stochastic comparisons of spacings of GOS and records are due to Hu and Zhuang [24], Belzunce *et al.* [16,18] , Zhao *et al.* [38] and Zhuang and Hu [39]. Stochastic comparisons of relative spacings have also been discussed in the literature by Belzunce *et al.* [19] and Castaño-Martínez *et al.* [21]. Applications of stochastic comparisons of minimum order statistics in welfare theory have been illustrated recently by Castaño-Martínez *et al.* [20].

2. Sufficient conditions for orderings of lower records

In this section, we use the fact that R_s can be represented as a mixture of minima to provide sufficient conditions for the stochastic comparison of lower records. The first result shows that the increasing concave order of $X_{1:n}$ and $Y_{1:n}$ implies the increasing concave order of the lower records $R_s(X)$ and $R_s(Y)$, for $s = n, n+1, \ldots$.

Theorem 1. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. If $X_{1:n} \leq_{\text{icv}} Y_{1:n}$, then $R_s(X) \leq_{\text{icv}} R_s(Y)$, for $s = n, n+1, \ldots$*

Proof. From Lemma 5 in Castaño-Martínez *et al.* [20], if $X_{1:n} \leq_{\text{icv}} Y_{1:n}$, then $X_{1:s} \leq_{\text{icv}} Y_{1:s}$, for $s \geq n$. The result follows from the fact that $R_s(X) = X_{1:L_s}$, where L_s is a random variable with support $\{s, s+1, s+2, \ldots\}$, and the known property that increasing concave order is closed under mixtures (Theorem 4.A.8.b in [33]). □

Remark 1. *By applying Theorem 4 to $-X$ and $-Y$ instead of to X and Y, which transforms lower records to upper records and increasing concave order to increasing convex order, it follows that $X_{n:n} \leq_{\text{icx}} Y_{n:n}$ implies $\tilde{R}_s(X) \leq_{\text{icx}} \tilde{R}_s(Y)$, for $s = n, n+1, \ldots$ where $\tilde{R}_s$ denotes the sth upper record value. This result strengthens Theorem 4.14 in Balakrishnan et al. [11] which is for the case $n = 1$.*

Next, to provide sufficient conditions for comparisons of lower records in location independent riskier order, we need the following lemma.

Lemma 1. *Let $n \geq 1$ and $1 \leq k \leq n$. If $X_{k:n} \leq_{\text{lir}} Y_{k:n}$, then $X_{k:n+r} \leq_{\text{lir}} Y_{k:n+r}$, for all $r \geq 1$.*

Proof. The condition $X_{k:n} \leq_{\text{lir}} Y_{k:n}$ is equivalent (see, e.g., Eq. (6) in [36]) to

$$\int_0^p (F_{k:n}^{-1}(p) - F_{k:n}^{-1}(t))\, dt \leq \int_0^p (G_{k:n}^{-1}(p) - G_{k:n}^{-1}(t))\, dt, \quad p \in (0,1), \tag{6}$$

where $F_{k:n}^{-1}(t)$ and $G_{k:n}^{-1}(t)$ are the quantile functions of $X_{k:n}$ and $Y_{k:n}$, respectively. As

$$F_{k:n}^{-1}(t) = F^{-1}(\beta_{k,n-k+1}^{-1}(t), \quad t \in (0,1),$$

where $\beta_{i,j}$ is Pearson's incomplete beta function (and similarly for $G_{k:n}^{-1}(t)$), we see, by change of variable $x = \beta_{k,n-k+1}^{-1}(t)$, that (6) is equivalent to

$$\begin{aligned}\int_0^{\beta_{k,n-k+1}^{-1}(p)} &(F^{-1}(\beta_{k,n-k+1}^{-1}(p)) - F^{-1}(x))\, d\beta_{k,n-k+1}(x) \\ &\leq \int_0^{\beta_{k,n-k+1}^{-1}(p)} (G^{-1}(\beta_{k,n-k+1}^{-1}(p)) - G^{-1}(x))\, d\beta_{k,n-k+1}(x), \ p \in (0,1).\end{aligned}$$

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This shows that $X_{k:n} \le_{\rm lir} Y_{k:n}$ is equivalent to

$$\int_0^p (F^{-1}(p) - F^{-1}(x))\, d\beta_{k,n-k+1}(x) \le \int_0^p (G^{-1}(p) - G^{-1}(x))\, d\beta_{k,n-k+1}(x),\ p \in (0,1) \tag{7}$$

or, equivalently,

$$\int_0^p (G^{-1}(p) - G^{-1}(x) - F^{-1}(p) + F^{-1}(x))\, d\beta_{k,n-k+1}(x) \ge 0, \quad p \in (0,1).$$

Given $r = 1, 2, \ldots$, the function

$$h(x) = \begin{cases} \dfrac{(n+r)!(n-k)!}{n!(n+r-k)!}(1-x)^r, & t \in (0,p], \\ 0, & t \in (p,1), \end{cases}$$

is non-negative and decreasing. It then follows from Lemma 7.1(b) in Chapter 4 of Barlow and Proschan [12] that

$$\int_0^p (G^{-1}(p) - G^{-1}(x) - F^{-1}(p) + F^{-1}(x))h(x)\, d\beta_{k,n-k+1}(x) \ge 0, \quad p \in (0,1).$$

As $(d\beta_{k,n+r-k+1}/d\beta_{k,n-k+1})(x) = h(x)$, we obtain

$$\int_0^p (F^{-1}(p) - F^{-1}(x))\, d\beta_{k,n+r-k+1}(x) \le \int_0^p (G^{-1}(p) - G^{-1}(x))\, d\beta_{k,n+r-k+1}(x), \quad \text{for all } p \in (0,1),$$

which, using (7) again, is the same as $X_{k:n+r} \le_{\rm lir} Y_{k:n+r}$. □

The next result shows that location independent riskier order of $X_{1:n}$ and $Y_{1:n}$ is a sufficient condition for location independent riskier order of lower records $R_s(X)$ and $R_s(Y)$, for $s = n, n+1, \ldots$. Observe, however, that as location independent riskier order is not, in general, closed under mixtures, we cannot use the same argument as in Theorem 4 for establishing this result.

Theorem 2. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. If $X_{1:n} \le_{\rm lir} Y_{1:n}$, then $R_s(X) \le_{\rm lir} R_s(Y)$, for $s = n, n+1, \ldots$*

Proof. Let $F_{1:n}(x)$ be the distribution function of $X_{1:n}$. Then, we can express

$$F_{1:n+r}(x) = h_r(F_{1:n}(x)), \quad r = 1, 2, \ldots,$$

where $h_r(t) = 1 - (1-t)^{(n+r)/n}$, $0 \le t \le 1$, is an increasing concave function such that $h_r : [0,1] \to [0,1]$, $h_r(0) = 0$ and $h_r(1) = 1$. Now, let $s = n, n+1, \ldots$. As $R_s(X) = X_{1:L_s}$, where L_s is a random

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variable with support $\{s, s+1, s+2, \ldots\}$, the distribution function $F_{R_s}(x)$ of $R_s(X)$ can be expressed as

$$F_{R_s}(x) = \sum_{r=0}^{\infty} F_{1:s+r}(x)P[L_s = s+r] = \sum_{r=0}^{\infty} h_r(F_{1:s}(x))P[L_s = s+r] = H_s(F_{1:s}(x)), \quad (8)$$

where

$$H_s(t) = \sum_{r=0}^{\infty} h_r(t)P[L_s = s+r], \quad s = n, n+1, \ldots, \quad (9)$$

is an increasing concave function such that $H_s : [0,1] \to [0,1]$, $H_s(0) = 0$ and $H_s(1) = 1$. Moreover, since $X_{1:n} \leq_{\text{lir}} Y_{1:n}$, it follows from Lemma 6 that $X_{1:s} \leq_{\text{lir}} Y_{1:s}$, for $s = n, n+1, \ldots$. Now, by applying Theorem 2.1(ii) in Shaked *et al.* [34], it follows that

$$\int_{-\infty}^{F_{1:s}^{-1}(p)} H_s(F_{1:s}(x))\,dx \leq \int_{-\infty}^{G_{1:s}^{-1}(p)} H_s(G_{1:s}(x))\,dx, \quad p \in (0,1), \ s = n, n+1, \ldots,$$

which can be rewritten as

$$\int_{-\infty}^{F_{1:s}^{-1}(H_s^{-1}(p))} H_s(F_{1:s}(x))\,dx \leq \int_{-\infty}^{G_{1:s}^{-1}(H_s^{-1}(p))} H_s(G_{1:s}(x))\,dx, \quad p \in (0,1), s = n, n+1, \ldots.$$

Using (8), this is equivalent to

$$\int_{-\infty}^{F_{R_s}^{-1}(p)} F_{R_s}(x)\,dx \leq \int_{-\infty}^{G_{R_s}^{-1}(p)} G_{R_s}(x)\,dx, \quad \forall p \in (0,1), \ s = n, n+1, \ldots,$$

which implies $R_s(X) \leq_{\text{lir}} R_s(Y)$, $s = n, n+1, \ldots$, as required. □

Remark 2. *In particular, by taking $n = 1$ in Theorem 7, we have that $X \leq_{\text{lir}} Y$ implies $R_s(X) \leq_{\text{lir}} R_s(Y)$, for $s \geq 1$. This also follows, by using the same argument as in Remark 5, from Theorem 5.2.(a) in Belzunce et al. [14].*

3. Increasing convex ordering of spacings of partial minima and lower records

In this section, we provide sufficient conditions for stochastic comparisons of spacings of partial minima and lower records. For establishing the main results, we need the following lemma.

Lemma 2. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F. Then, for $1 \leq j < m$ and $p \in (0,1)$,*

$$[F^{-1}(p) - X_{1:m} \mid X_{1:j} = F^{-1}(p)] \stackrel{d}{=} (F^{-1}(p) - X_{1:m-j})^+,$$

where $\stackrel{d}{=}$ denotes "equal in distribution".

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Proof. Given $y \geq 0$, we have

$$\begin{aligned} P(F^{-1}(p) - X_{1:m} \leq y \mid X_{1:j} = F^{-1}(p)) &= P(X_{1:m} \geq F^{-1}(p) - y \mid X_{1:j} = F^{-1}(p)) \\ &= (\bar{F}(F^{-1}(p) - y))^{m-j} \\ &= P((F^{-1}(p) - X_{1:m-j})^{+} \leq y), \end{aligned}$$

where the second equality follows from the fact that

$$P(X_{1:m} > x \mid X_{1:j} = F^{-1}(p)) = \begin{cases} (\bar{F}(x))^{m-j}, & x < F^{-1}(p) \\ 0, & x \geq F^{-1}(p). \end{cases}$$

□

Now, we show that if X and Y are ordered in dispersive order, then $X_{1:n} - X_{1:n+k}$ and $Y_{1:n} - Y_{1:n+k}$ are ordered in the usual stochastic order.

Theorem 3. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. If $X \leq_{\text{disp}} Y$, then*

$$X_{1:n} - X_{1:n+k} \leq_{\text{st}} Y_{1:n} - Y_{1:n+k}, \quad \textit{for } n \geq 1 \textit{ and } k \geq 1.$$

Proof. It is easy to see that $X \leq_{\text{disp}} Y$ holds if and only if $X_{1:k} \leq_{\text{disp}} Y_{1:k}$, for $k = 1, 2, \ldots$. From (2), it follows that

$$(F^{-1}(p) - X_{1:k})^{+} \leq_{\text{st}} (G^{-1}(p) - Y_{1:k})^{+}, \quad \text{for all } p \in (0,1) \text{ and } k \geq 1,$$

where we have used the fact that $F_{1:k}^{-1}(p) = F^{-1}(1 - (1-p)^{1/k})$. Let $y > 0$, $n \geq 1$ and $k \geq 1$. Then, by using Lemma 9, we see that

$$\begin{aligned} P(X_{1:n} - X_{1:n+k} \leq y) &= \int_0^1 P(F^{-1}(p) - X_{1:n+k} \leq y \mid X_{1:n} = F^{-1}(p))\, d\beta_{1,n}(p) \\ &= \int_0^1 P((F^{-1}(p) - X_{1:k})^{+} \leq y)\, d\beta_{1,n}(p) \\ &\geq \int_0^1 P((G^{-1}(p) - Y_{1:k})^{+} \leq y)\, d\beta_{1,n}(p) \\ &= P(Y_{1:n} - Y_{1:n+k} \leq y), \end{aligned}$$

where the inequality follows from (2). This proves the required result. □

Remark 3. *For random variables with equal left-end support points, the dispersive order implies the usual stochastic order (see Theorem 3.B.13 in [33]). Therefore, a natural question that arises is whether the assumption $X \leq_{\text{disp}} Y$ in Theorem 10 can be replaced by $X \leq_{\text{st}} Y$. In general, the answer is no. To see this, let us consider two Power random variables $X \sim \text{Pow}(1)$ and $Y \sim \text{Pow}(2)$, with respective survival functions $\bar{F}(x) = 1 - x$ and $\bar{G}(x) = 1 - x^2$, $x \in (0,1)$. Then, $X \leq_{\text{st}} Y$ but $X \nleq_{\text{disp}} Y$ (see Theorem 3.B.14 in [33]). Straightforward calculations show that*

$$E[X_{1:1} - X_{1:2}] = \frac{1}{6} > E[Y_{1:1} - Y_{1:2}] = \frac{2}{15}. \tag{10}$$

Therefore, $X_{1:1} - X_{1:2} \nleq_{\text{st}} Y_{1:1} - Y_{1:2}$.

Next, we provide sufficient conditions for increasing convex ordering of spacings of partial minima and lower records. The following theorem shows that location independent riskier order of $X_{1:n}$ and $Y_{1:n}$

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is a sufficient condition for increasing convex order of the spacings $X_{1:j} - X_{1:k}$ and $Y_{1:j} - Y_{1:k}$, whenever $k - j \geq n$.

Theorem 4. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. Furthermore, let $n \geq 1$. If $X_{1:n} \leq_{\text{lir}} Y_{1:n}$, then*

$$X_{1:j} - X_{1:k} \leq_{\text{icx}} Y_{1:j} - Y_{1:k}, \quad \textit{for } j \geq 1 \textit{ and } k - j \geq n.$$

Proof. Let $1 \leq j < k$. Using Lemma 9, we can write

$$\begin{aligned} P(X_{1:j} - X_{1:k} > y) &= \int_0^1 P(F^{-1}(p) - X_{1:k} > y \mid X_{1:j} = F^{-1}(p))\, d\beta_{1,j}(p) \\ &= \int_0^1 P[(F^{-1}(p) - X_{1:k-j})^+ > y]\, d\beta_{1,j}(p) \\ &= \int_0^1 \bar{F}_{(F^{-1}(p)-X_{1:k-j})^+}(y)\, d\beta_{1,j}(p), \quad y > 0. \end{aligned}$$

The assumption $X_{1:n} \leq_{\text{lir}} Y_{1:n}$ implies, via Lemma 6, that $X_{1:r} \leq_{\text{lir}} Y_{1:r}$, for $r \geq n$. By (3), this is equivalent to

$$(F_{1:r}^{-1}(p) - X_{1:r})^+ \leq_{\text{icx}} (G_{1:r}^{-1}(p) - Y_{1:r})^+, \quad \text{for all } p \in (0,1).$$

Upon using the fact that $F_{1:r}^{-1}(p) = F^{-1}(1-(1-p)^{1/r})$, we see that $X_{1:r} \leq_{\text{lir}} Y_{1:r}$ is equivalent to $(F^{-1}(p) - X_{1:r})^+ \leq_{\text{icx}} (G^{-1}(p) - Y_{1:r})^+$, for all $p \in (0,1)$ and $r \geq n$. This implies that

$$\int_s^{+\infty} \bar{F}_{(F^{-1}(p)-X_{1:r})^+}(y)\, dy \leq \int_s^{+\infty} \bar{G}_{(G^{-1}(p)-Y_{1:r})^+}(y)\, dy, \quad \text{for all } s > 0.$$

Therefore,

$$\int_0^1 \int_s^{+\infty} \bar{F}_{(F^{-1}(p)-X_{1:k-j})^+}(y)\, dy\, d\beta_{1,j}(p) \leq \int_0^1 \int_s^{+\infty} \bar{G}_{(G^{-1}(p)-Y_{1:k-j})^+}(y)\, dy\, d\beta_{1,j}(p),$$

for all $s > 0$ and $p \in (0,1)$, which implies $X_{1:j} - X_{1:k} \leq_{\text{icx}} Y_{1:j} - Y_{1:k}$, for $k - j \geq n$. □

Remark 4. *For random variables with equal left-end support points, the location independent riskier order implies the usual stochastic order (see Theorem 6 in [36]). It is natural to wonder whether the assumption $X_{1:n} \leq_{\text{lir}} Y_{1:n}$ in Theorem 12 can be replaced by $X_{1:n} \leq_{\text{st}} Y_{1:n}$. The same counterexample as in Remark 11 can be used to show that the answer is no. Recall that $X \sim$ Pow(1) and $Y \sim$ Pow(2). Clearly $X_{1:n} \leq_{\text{st}} Y_{1:n}$. However, since $X_{1:n}$ and $Y_{1:n}$ have the same finite support, $X_{1:n} \not\leq_{\text{lir}} Y_{1:n}$ (see Corollary 7 in [36]). Moreover, it follows from (10) that $X_{1:1} - X_{1:2} \not\leq_{\text{icx}} Y_{1:1} - Y_{1:2}$.*

The following theorem shows that increasing convex order (respectively, the increasing concave order) of adjacent spacings between minimum order statistics is a sufficient condition for increasing convex order (respectively, the increasing concave order) of adjacent spacings of their records.

Theorem 5. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. If*

$$X_{1:n} - X_{1:n+1} \leq_{\text{icx}} Y_{1:n} - Y_{1:n+1}, \quad \textit{for all } n \geq i, \tag{11}$$

then

$$R_j(X) - R_{j+1}(X) \leq_{\text{icx}} R_j(Y) - R_{j+1}(Y), \quad \textit{for all } j \geq i. \tag{12}$$

The result remains true if $\leq_{\text{icx}}$ is replaced by $\leq_{\text{icv}}$ in both (11) and (12).

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Proof. First, we prove the result for the increasing convex order. Let us rewrite (11) and (12) in a more convenient form. Let $n \geq i$ and denote by $\bar{F}_{X^*_{1:n}}(y)$ the survival function of $X_{1:n} - X_{1:n+1}$. Then, condition (11) is equivalent to

$$\int_s^\infty (\bar{G}_{Y^*_{1:n}}(y) - \bar{F}_{X^*_{1:n}}(y))\, dy \geq 0, \quad \text{for all } s. \tag{13}$$

Using Lemma 9, we see that

$$\begin{aligned}
\bar{F}_{X^*_{1:n}}(y) &= \int_0^1 P(F^{-1}(p) - X_{1:n+1} > y \mid X_{1:n} = F^{-1}(p)) n(1-p)^{n-1}\, dp \\
&= \int_0^1 P((F^{-1}(p) - X)^+ > y) n(1-p)^{n-1}\, dp \\
&= \int_0^1 F(F^{-1}(p) - y) n(1-p)^{n-1}\, dp, \quad y \geq 0.
\end{aligned} \tag{14}$$

Similarly,

$$\bar{G}_{Y^*_{1:n}}(y) = \int_0^1 G(G^{-1}(p) - y) n(1-p)^{n-1}\, dp, \quad y \geq 0. \tag{15}$$

Substituting (14) and (15) in (13) and changing the order of integration by Fubini's theorem, we see that (11) is equivalent to

$$I(s,n) \geq 0, \quad \text{for all } n \geq i, \text{ for all } s, \tag{16}$$

where we have denoted

$$I(s,n) = \int_0^1 \left(\int_s^\infty (G(G^{-1}(p) - y) - F(F^{-1}(p) - y))\, dy \right) (1-p)^{n-1}\, dp.$$

Now, let $j \geq i$ and denote by $\bar{F}_{R^*_j}(y)$ the survival function of $R_j(X) - R_{j+1}(X)$. Then, condition (12) is equivalent to

$$\int_s^\infty (\bar{G}_{R^*_j}(y) - \bar{F}_{R^*_j}(y))\, dy \geq 0, \quad \text{for all } s. \tag{17}$$

Now, upong using the fact that $[R_{j+1}(X) \mid R_j(X) = F^{-1}(p)] \stackrel{d}{=} [X \mid X < F^{-1}(p)]$, we have

$$\begin{aligned}
\bar{F}_{R^*_j}(y) &= \int_0^1 P(F^{-1}(p) - R_{j+1}(X) > y \mid R_j(X) = F^{-1}(p))\, d(1 - \gamma_j(-\log(p))) \\
&= \int_0^1 \frac{F(F^{-1}(p) - y)}{p}\, d(1 - \gamma_j(-\log(p))), \quad y \geq 0,
\end{aligned} \tag{18}$$

where we have used the fact that the distribution function of R_j is given by

$$F_{R_j}(x) = 1 - \gamma_j(-\log(F(x))), \quad x \in \mathbb{R},\ j \geq 1,$$

with

$$\gamma_j(x) = \int_0^x \frac{t^{j-1}}{(j-1)!} e^{-t}\, dt, \quad x \geq 0;$$

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see Arnold *et al.* [4]. As $R_j(U) = U_{1:L_j}$, where $U \sim U(0,1)$, the density of $R_j(U)$ can be expressed as

$$\begin{aligned} f_{R_j(U)}(p) &= d(1-\gamma_j(-\log(p))) \\ &= \sum_{m\geq j} f_{U_{1:m}}(p)P(L_j=m) \\ &= \sum_{m\geq j} m(1-p)^{m-1}P(L_j=m), \end{aligned}$$

and so

$$\frac{d(1-\gamma_j(-\log(p)))}{p} = \sum_{m\geq j}\sum_{r\geq m} m(1-p)^{r-1}P(L_j=m).$$

Substituting this expression in (18) and then changing sums to integral (justified by dominated convergence theorem), we have

$$\bar{F}_{R_j^*}(y) = \sum_{m\geq j}\sum_{r\geq m} mP(L_j=m)\int_0^1 F(F^{-1}(p)-y)(1-p)^{r-1}\,dp. \tag{19}$$

Similarly, we have

$$\bar{G}_{R_j^*}(y) = \sum_{m\geq j}\sum_{r\geq m} mP(L_j=m)\int_0^1 G(G^{-1}(p)-y)(1-p)^{r-1}\,dp. \tag{20}$$

Now, upon substituting (19) and (20) in (17) and changing the order of integration by Fubini's theorem, we see that (12) is equivalent to

$$\sum_{m\geq j}\sum_{r\geq m} mP(L_j=m)I(s,r) \geq 0, \quad \text{for all } s. \tag{21}$$

As $j \geq i$, it is clear that (16) implies (21), which proves the required result for the increasing convex order.

To prove that the result remains true if $\leq_{\text{icx}}$ is replaced by $\leq_{\text{icv}}$ in both (11) and (12), note that

$$X_{1:n} - X_{1:n+1} \leq_{\text{icv}} Y_{1:n} - Y_{1:n+1}, \quad \text{for all } n \geq i,$$

is equivalent to

$$\int_{-\infty}^{s} (\bar{G}_{Y_{1:n}^*}(y) - \bar{F}_{X_{1:n}^*}(y))\,dy \geq 0, \quad \text{for all } s,$$

whereas

$$R_j(X) - R_{j+1}(X) \leq_{\text{icv}} R_j(Y) - R_{j+1}(Y), \quad \text{for all } j \geq i,$$

is equivalent to

$$\int_{-\infty}^{s} (\bar{G}_{R_j^*}(y) - \bar{F}_{R_j^*}(y))\,dy \geq 0, \quad \text{for all } s.$$

The proof for the increasing concave order follows upong replacing the integral between s and ∞ by the integral between $-\infty$ and s along the proof for the increasing convex order. □

Remark 5. *In particular, from Theorems 12 (with $n = 1$) and 14 (with $i = 1$), we see that $X \leq_{\text{lir}} Y$ implies $R_j(X) - R_{j+1}(X) \leq_{\text{icx}} R_j(Y) - R_{j+1}(Y)$, for $j \geq 1$. This also follows by applying Corollary 4.1 in Belzunce et al. [18] to $-X$ and $-Y$ instead of to X and Y, which transforms upper records to lower records and excess wealth order to location independent riskier order.*

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Using arguments similar to those used in the proofs of Theorems 12 and 14, we can obtain the following result, for which we omit the proof for the sake of conciseness.

Theorem 6. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. Let $j \geq 1$. If $R_j(X) \leq_{\text{lir}} R_j(Y)$, then*

$$R_s(X) - R_m(X) \leq_{\text{icx}} R_s(Y) - R_m(Y), \quad \text{for } s \geq 1 \text{ and } m - s \geq j.$$

Remark 6. *We use the same counterexample as in Remarks 11 and 13 to show that the assumption $R_j(X) \leq_{\text{lir}} R_j(Y)$ cannot be replaced, in general, by $R_j(X) \leq_{\text{st}} R_j(Y)$. It is well-known that if $V \sim \text{Pow}(\alpha)$, with $\alpha > 0$, their corresponding lower records can be expressed as $R_n(V) = \prod_{i=1}^{n} V_i$, with $\{V_i\}_{i\geq 1}$ being independent random variables with the same distribution as V (see [4]). Then, $E[R_n(V)] = (E[V])^n = (\alpha/(\alpha+1))^n$ and*

$$E[R_1(V) - R_2(V)] = \frac{\alpha}{(\alpha+1)^2}.$$

Now, let $X \sim \text{Pow}(1)$, $Y \sim \text{Pow}(2)$ and $j = 1$. It is then easy to check that $R_1(X) \leq_{\text{st}} R_1(Y)$, but

$$E[R_1(X) - R_2(X)] = \tfrac{1}{4} > E[R_1(Y) - R_2(Y)] = \tfrac{2}{9},$$

which implies that $R_1(X) - R_2(X) \nleq_{\text{icx}} R_1(Y) - R_2(Y)$.

4. Further remarks

In the proof of Theorem 7, we have used the property that the distribution function of $R_s(X)$ can be expressed as an increasing and concave distortion of the distribution function of $X_{1:s}$. This fact can also be utilized to establish ordering conditions for lower record values in terms of some other stochastic orderings. For illustrating this point, for example, we present the following results.

Definition 2. *Let X and Y be two non-negative random variables with respective distribution functions F and G, and let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively, be their survival functions. We say that X is smaller than Y*

(a) in total time on test transform order, denoted by $X \leq_{\text{ttt}} Y$, if

$$\int_0^{F^{-1}(p)} \bar{F}(x)\,dx \leq \int_0^{G^{-1}(p)} \overline{G}(x)\,dx, \quad \text{for all } p \in (0,1);$$

(b) in quantile mean inactivity time order, denoted by $X \leq_{\text{qmit}} Y$, if

$$\frac{\int_0^{F^{-1}(p)} F(x)\,dx}{\int_0^{G^{-1}(p)} G(x)\,dx} \quad \text{decreases in } p \in (0,1).$$

For references on these orders, see Kochar *et al.* [30] and Arriaza *et al.* [6]. We then have the following result.

Theorem 7. *Let $X_1, X_2, \ldots$ be non-negative i.i.d. as X with continuous distribution function F. Similarly, let $Y_1, Y_2, \ldots$ be non-negative i.i.d. as Y with distribution function G. Furthermore, let $n \geq 1$.*

(a) If $X_{1:n} \leq_{\text{ttt}} Y_{1:n}$, then $R_s(X) \leq_{\text{ttt}} R_s(Y)$, for $s = n, n+1, \ldots$;
(b) If $X_{1:n} \leq_{\text{qmit}} Y_{1:n}$, then $R_s(X) \leq_{\text{qmit}} R_s(Y)$, for $s = n, n+1, \ldots$.

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Proof. First, we observe that $\bar{F}_{1:s}(x) = h_s^*(\bar{F}_{1:n}(x))$, with $h_s^*(t) = t^{s/n}$ being an increasing convex function for $s \geq n$, such that $h_s^* : [0,1] \to [0,1]$, $h_s^*(0) = 0$ and $h_s^*(1) = 1$. Then, using Theorem 1 and Remark 2 in [5], we have $X_{1:s} \leq_{\text{ttt}} Y_{1:s}$ and $X_{1:s} \leq_{\text{qmit}} Y_{1:s}$, for $s \geq n$, respectively.

On the other hand, let $s \geq n$. It then follows from (8) that

$$\bar{F}_{R_s}(x) = H_s^*(\bar{F}_{1:s}(x)),$$

with $H_s^*(t) = 1 - H_s(1-t)$ (where H_s is as in (9)) is an increasing convex function such that $H_s^* : [0,1] \to [0,1]$, $H_s^*(0) = 0$ and $H_s^*(1) = 1$. Then, the required results follow from Theorem 1 and Remark 2 of Arriaza and Sordo [5]. □

A further observation regarding kth lower records is the following. In the continuous case, kth lower record values obtained from $X \sim F$, denoted by $R_j^{(k)}(X)$ (or $R_j^{(k)}(F)$), are distributed exactly as a sequence of ordinary lower records from the distribution $F_{k:k}(x) = (F(x))^k$, $R_j(F_{k:k})$. Using similar arguments as those used in the proofs of Lemma 6, Remark 8 and Theorem 16, we can then state the following results.

Theorem 8. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. If $X_{k:k} \leq_{\text{lir}} Y_{k:k}$, then $R_j^{(k')}(X) \leq_{\text{lir}} R_j^{(k')}(Y)$, for $j \geq 1$ and $1 \leq k' \leq k$.*

Theorem 9. *Let $X_1, X_2, \ldots$ be i.i.d. as X with continuous distribution function F, and $Y_1, Y_2, \ldots$ be i.i.d. as Y with continuous distribution function G. Let $j \geq 1$ and $k \geq 1$. If $R_j^{(k)}(X) \leq_{\text{lir}} R_j^{(k)}(Y)$, then*

$$R_s^{(k')}(X) - R_m^{(k')}(X) \leq_{\text{icx}} R_s^{(k')}(Y) - R_m^{(k')}(Y), \quad \textit{for } s \geq 1,\ m - s \geq j,\ 1 \leq k' \leq k.$$

One more remark before ending this work. Balakrishnan and Mi [7] and Balakrishnan *et al.* [10] have shown that the ordering satisfied by the two underlying distributions implies the ordering of the maximum likelihood estimates of the parameters arising from samples from the two distributions. A similar question can be asked in the present context about the estimation of distributional parameters based on lower record values having been observed from the two distributions. This is an open problem that we plan to consider as our future work.

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