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Co-actions of groups

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Let $f : G \to H$ be a fixed homomorphism and $p' : G * H \to G$ and $p'' : G * H \to H$ the two projections of the free product. Then a co-action relative to f is a homomorphism $s : G \to G * H$ such that $p's = \text{id}$ and $p''s = f$. We study this notion and investigate the following questions. What restrictions does s place on the structure of the group G? What form does s take in special cases? When does s induce a co-multiplication on H? What is the relation between associativity of s and associativity of the induced co-multiplication m on H? What are the properties of the operation of $\text{Hom}(H, B)$ on $\text{Hom}(G, B)$ induced by $s : G \to G * H$? In addition, we give several diverse examples of co-actions in the last section.

1. Introduction

Let $f : X \to Y$ be a map of topological spaces and C_f the mapping cone of f. Let ΣX be the suspension of X and $p' : C_f \vee \Sigma X \to C_f$ and $p'' : C_f \vee \Sigma X \to \Sigma X$ the two projections of the wedge. Then there is a homotopy co-action of ΣX on C_f which is given by a map $s : C_f \to C_f \vee \Sigma X$ such that $p's$ is homotopic to the identity map of C_f and $p''s$ is homotopic to the collapsing map $C_f \to \Sigma X$. Geometrically, s is obtained by identifying the 'equator' of the cone $CX \subseteq C_f$ to the base point. This homotopy co-action is a basic concept of homotopy theory and has proved to be an extremely useful tool (see [3, ch. 11, 14], [7, ch. 2]). In the case Y is a point, the co-action becomes the canonical co-multiplication $m : \Sigma X \to \Sigma X \vee \Sigma X$ of the suspension ΣX.

The notion of a co-multiplication for groups has been considered by several authors [1, 2, 4]. For a group G, this consists of a homomorphism $m : G \to G * G$, the free product of G with itself, whose composition with each of the two projections is the identity homomorphism id of G. In the present work we transfer the notion of homotopy co-action from the homotopy category of spaces to the category of groups, thereby obtaining a generalization of the notion of a group with a co-multiplication. More precisely, let $f : G \to H$ be a fixed homomorphism and $p' : G * H \to G$ and $p'' : G * H \to H$ the two projections. Then a co-action of H on G relative to f is a homomorphism $s : G \to G * H$ such that $p's = \text{id}$ and $p''s = f$ (definition 4.5). This is the central notion which we study in this paper.

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We now briefly summarize the contents of the paper. Section 2 introduces our terminology and notation in group theory. In § 3, after stating some known results on co-multiplications, we prove that a stable subgroup for a co-multiplication is a free factor (theorem 3.7). We begin the study of co-actions in § 4. We show a co-action $s : G \to G * H$ rel f gives a free product decomposition of G into a free subgroup and a free product of subgroups of $K = \text{kernel} f$ on which s is determined (proposition 4.8). We then obtain a complete description of s on all finite subgroups of G (corollary 4.10). In § 5 we give necessary and sufficient conditions for a co-action s to induce a co-multiplication m on H (proposition 5.3). For such co-actions, we study the relation between associativity of s and associativity of m (proposition 5.11). We then obtain in theorem 5.15 a canonical set of generators of G in the case when the co-action s is associative. We consider in § 6 homomorphisms $f : G \to H$, called free homomorphisms, such that G and H are free and f is compatible with certain bases of G and H (definition 6.1). We investigate the right action of $\text{Hom}(H, B)$ on $\text{Hom}(G, B)$ induced by a co-action s rel f, for any group B. For certain co-actions s, we prove that the pre-images of $i^* : \text{Hom}(G, B) \to \text{Hom}(K, B)$ are precisely the orbits of $\text{Hom}(G, B)$ under the action of $\text{Hom}(H, B)$, where i is the inclusion of K in G (theorem 6.7, remark 6.8). The final section deals with several diverse examples of co-actions. These are intended to illustrate our results.

2. Preliminaries

In this section we introduce our conventions regarding group theory and fix our notation. All groups will be written multiplicatively. If G is a group, then $1 \in G$ is the unit or identity of G. If $g \in G$, we usually denote the inverse g^{-1} by $\bar{g}$. The commutator $[g, h]$ of elements $g, h \in G$ is given by $[g, h] = \bar{g}\bar{h}gh$. If $g \in G$ and $H \subseteq G$ is a subgroup, then the conjugate subgroup $\bar{g}Hg$ is denoted H^g. For subgroups $H, K \subseteq G$, we denote the smallest subgroup containing H and K by HK. If $S \subseteq G$ is a subset of the group G, then S^G is the normal closure of S in G, i.e. the smallest normal subgroup containing S.

For groups G and H, the free product $G * H$ is defined in the usual way. An element $\xi \in G * H$ can be written

$$\xi = g_1 h_1 \cdots g_n h_n,$$

where $g_i \in G$ and $h_i \in H$. We call ξ reduced if $g_2, \dots, g_n \neq 1$ and $h_1, \dots, h_{n-1} \neq 1$. We frequently write $\xi = g_1' h_1'' \dots g_n' h_n''$, especially in the case $G = H$, where g_i' signifies that g_i' is in the first factor of the free product and h_i'' signifies that h_i'' is in the second factor. For $g \in G$, $h \in H$ and $k \in K$, we denote by g', h'' and k''' the obvious elements in the triple free product $G * H * K$. If G is a group, then a subgroup $H \subseteq G$ is called a free factor of G if there exists a subgroup $K \subseteq G$ such that $G = H * K$.

The identity homomorphism of G is denoted $\text{id} : G \to G$. For a free product $G * H$, there are projection homomorphisms $p_G : G * H \to G$ and $p_H : G * H \to H$ and injection homomorphisms $i_G : G \to G * H$ and $i_H : H \to G * H$. When $G = H$, we write $p' = p_G$, $p'' = p_H$, $i' = i_G$ and $i'' = i_H$. Homomorphisms $f : G \to L$ and

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$g : H \to M$ induce a homomorphism $f * g : G * H \to L * M$ in the standard way. If $L = M$, then we obtain a canonical homomorphism $(f, g) : G * H \to L$.

We will frequently work with free groups F and consider bases of F. If X is a basis of F, then we write $F = \langle X \rangle$ to indicate that F is generated by the basis X. Let G be a group, H a set with distinguished element e and $f : G \to H$ a surjection such that $f(1) = e$. Then a section of f is a function $\sigma : H \to G$ such that $f\sigma = \mathrm{id} : H \to H$ and $\sigma(e) = 1$. Since f is onto, a section σ always exists. If H is a group and $f : G \to H$ is an epimorphism, we still refer to a section as a function $\sigma : H \to G$ such that $f\sigma = \mathrm{id}$ and $\sigma(1) = 1$. If σ is a homomorphism, we call it a section homomorphism. If $f : G \to H$ is an epimorphism of groups and H is free, there is a section homomorphism $\sigma : H \to G$.

3. Co-multiplications

We begin this section by recalling some known results on co-multiplications. Let H be a group and $p', p'' : H * H \to H$ the two projections. We define $E_H \subseteq H * H$ to be the *equalizer* of p' and p''. Thus if $\xi = g_1' h_1'' \cdots g_n' h_n'' \in H * H$, then $\xi \in E_H$ if and only if $g_1 \cdots g_n = h_1 \cdots h_n$. For each $h \in H$, set $\xi_h = h' h'' \in E_H$ and let $X_H = \{\xi_h \mid h \in H,\ h \neq 1\}$.

THEOREM 3.1 (see theorem 1.4 of [2], proposition 3.1 of [1]). *The group E_H is a free group with basis X_H.*

REMARK 3.2. In proposition 3.1 of [1], there is an algorithm to express $\xi = g_1' h_1'' \cdots g_n' h_n''$ in E_H in terms of the basis X_H. If

$$\delta_1 = g_1, \qquad \delta_2 = \bar{h}_1 g_1, \qquad \delta_3 = \bar{h}_1 g_1 g_2, \ldots, \delta_{2n-1} = \bar{h}_{n-1} \ldots \bar{h}_1 g_1 \ldots g_n,$$

then $\xi = \xi_{\delta_1} \bar{\xi}_{\delta_2} \xi_{\delta_3} \bar{\xi}_{\delta_4} \cdots \xi_{\delta_{2n-1}}$.

DEFINITION 3.3. A homomorphism $m : H \to H * H$ is called a *co-multiplication* if $p'm = p''m = \mathrm{id} : H \to H$, where $p', p'' : H * H \to H$ are the two projections. We call m *associative* if $(m * \mathrm{id})m = (\mathrm{id} * m)m : H \to H * H * H$.

For any group H, there is a homomorphism $\pi_H : E_H \to H$ defined by $\pi_H = p'|E_H = p''|E_H$. If m is a co-multiplication of H, then m induces a homomorphism (also called m) $H \to E_H$, which is a section homomorphism of π_H. Conversely, a section homomorphism of π_H determines a co-multiplication of H. We shall often not distinguish the co-multiplication $H \to H * H$ from the section homomorphism $H \to E_H$. Moreover, if m is a co-multiplication of H, then $m : H \to E_H$ is a monomorphism and so H is a free group by theorem 3.1.

DEFINITION 3.4. If $m : H \to H * H$ is a co-multiplication, then the set

$$D_m = \{h \mid h \in H,\ h \neq 1,\ mh = h'h''\} \subseteq H$$

is called the *diagonal set* of m.

THEOREM 3.5 (see corollary 3.12 of [4], corollary 4.6 of [1]). *Let m be a co-multiplication of H. Then m is associative $\Longleftrightarrow$ D_m is a basis of H.*

This concludes our summary of known results on co-multiplications which we shall need. The remainder of this section is devoted to a new result on co-multiplications which is needed in later sections.

Let H be a group with co-multiplication m and let $A \subseteq H$ be a subgroup.

DEFINITION 3.6. We say that A is *left stable* (with respect to m) if $m(A) \subseteq A * H$. A similar definition holds for right stable.

THEOREM 3.7. *Let m be a co-multiplication of H such that $A \subseteq H$ is left stable. Then A is a free factor of H.*

Proof. Let $\rho : H \to H/A$ be the natural projection onto the set of left co-sets of A defined by $\rho(h) = hA$. Since ρ is onto, we choose a section $\sigma : H/A \to H$ of ρ. We set $\hat{h} = \sigma\rho(h)$ for each $h \in H$ and note that the set of all $\hat{h}$ is a complete set of co-set representatives of H modulo A. The basis X_H of the equalizer E_H (theorem 3.1) can be written as the disjoint union $X_A \cup Y' \cup Z$, where $X_A = \{\xi_w \mid w \in A,\ w \neq 1\}$, $Y' = \{\xi_k \mid k \notin A,\ k \neq \hat{k}\}$ and $Z = \{\xi_{\hat{k}} \mid \hat{k} \neq 1\}$. By an elementary transformation, we obtain that $X_A \cup Y \cup Z$ is also a basis of E_H, where $Y = \{\bar{\xi}_k \xi_{\hat{k}} \mid k \neq \hat{k},\ k \notin A\}$.

Now let $w \in A$ and write $mw = g_1'h_1'' \cdots g_n'h_n''$, where $g_i \in A$ and $h_i \in H$. Let $\delta_1, \delta_2, \ldots, \delta_{2n-1}$ be the sequence of elements of H defined in remark 3.2. Note that each $\delta_i = \eta_i\gamma_i$, where $\eta_i \in H$ and $\gamma_i \in A$. Also, $\delta_1 = \gamma_1$ and $\eta_{2k} = \eta_{2k+1}$, so that $\hat{\delta}_{2k} = \hat{\delta}_{2k+1}$. Thus

$$
\begin{aligned}
mw &= \xi_{\delta_1}\bar{\xi}_{\delta_2}\xi_{\delta_3}\cdots\xi_{\delta_{2n-1}} \\
&= \xi_{\gamma_1}\prod_{k=1}^{n-1}(\bar{\xi}_{\delta_{2k}}\xi_{\delta_{2k+1}}) \\
&= \xi_{\gamma_1}\prod_{k=1}^{n-1}(\bar{\xi}_{\delta_{2k}}\xi_{\hat{\delta}_{2k}})(\bar{\xi}_{\delta_{2k+1}}\xi_{\hat{\delta}_{2k+1}})^{-1}.
\end{aligned}
$$

If $\hat{\delta}_i = 1$, then $\bar{\xi}_{\delta_i}\xi_{\hat{\delta}_i} = \bar{\xi}_{\delta_i}$, and so either $\bar{\xi}_{\delta_i}\xi_{\hat{\delta}_i} = 1$ or $\bar{\xi}_{\delta_i}\xi_{\hat{\delta}_i} \in \bar{X}_A$. If $\hat{\delta}_i \neq 1$, then either $\bar{\xi}_{\delta_i}\xi_{\hat{\delta}_i} = 1$ or $\bar{\xi}_{\delta_i}\xi_{\hat{\delta}_i} \in Y$. Since $\xi_{\gamma_1} = 1$ or $\xi_{\gamma_1} \in X_A$, we have that mw lies in the subgroup of E_H generated by the subset $X_A \cup Y$ of the basis $X_A \cup Y \cup Z$ of E_H. It follows that $m(A)$ is a free factor of $m(H)$ [5, exercise 32, p. 117]. Since $m : H \to m(H)$ is an isomorphism, A is a free factor of H. □

REMARK 3.8. Let $m : H \to H * H$ be a co-multiplication and $A_m \subseteq H$ the equalizer of $(m * \mathrm{id})m$ and $(\mathrm{id} * m)m$. We proved in [1, theorem 4.4(2)] that A_m is left and right stable under m, and then showed that A_m is a free factor of H with basis D_m. We now see that this latter result is a special case of theorem 3.7.

COROLLARY 3.9. *Let $A \subseteq H$ be a subgroup, $j : A \to H$ be the inclusion and $m' : A \to A * A$ a co-multiplication. Then m' extends to a co-multiplication of $H \iff H$ is free and A is a free factor of H.*

4. Co-actions

In this section $f : G \to H$ will be a fixed homomorphism. We let $K = \operatorname{kernel} f$, $I = \operatorname{image} f$ and denote the inclusions $i : K \to G$ and $j : I \to H$. For every $g \in G$, we denote $\eta_g = gf(g) \in G * H$, which is sometimes written $g'f(g)''$.

Definition 4.1. The *equalizer* of $fp_G : G * H \to H$ and $p_H : G * H \to H$ is denoted $\mathcal{E}_f \subseteq G * H$. The *semi-equalizer* E_f is the subgroup of $G * H$ generated by $\eta_g = g' f(g)''$, for all $g \in G$.

Clearly, $E_f \subseteq \mathcal{E}_f$, but they are not equal in general (see remark 4.7(iii) and lemma 4.4). We introduce some notation next.

Let $p_f = p_G|_{E_f} : E_f \to G$ and $\pi_f = p_G|_{\mathcal{E}_f} : \mathcal{E}_f \to G$. Note that $i_G : G \to G * H$ carries K to E_f and so induces homomorphisms $i_f : K \to E_f$ and $\iota_f : K \to \mathcal{E}_f$. Also, $f * \mathrm{id} : G * H \to H * H$ carries E_f to E_I since $(f * \mathrm{id})(\eta_g) = \xi_{f(g)} \in E_I$. We let $\nu = (f * \mathrm{id})|_{E_f} : E_f \to E_I$. Now $f : G \to H$ is onto I and so determines a surjection $f' : G \to I$. Let $\phi : I \to G$ be a section of f'. If $g \in G$, we write $\hat{g} = \phi f(g)$. Then ϕ determines a section homomorphism $\sigma : E_I \to E_f$ of ν by setting $\sigma(\xi_{f(g)}) = \eta_{\hat{g}}$. Our next few results deal with E_f.

Proposition 4.2. *$E_f = i_f(K) * \sigma(E_I)$, and so E_f is isomorphic to $K * E_I$.*

Proof. If $k \in K$, note that $\eta_k = k' = i_f(k)$. For each $g \in G$, there exists a $k_g \in K$ such that $g = k_g \hat{g}$. In particular, a generator η_g of E_f can be written $\eta_g = k'_g \hat{g} f(\hat{g}) = k'_g \eta_{\hat{g}}$. Thus every non-trivial element of E_f can be written as a product

$$\pi_1 k'_1 \pi_2 k'_2 \cdots \pi_n k'_n, \tag{$*$}$$

with $n \geqslant 1$, where (a) $k_i \in K$ and $k_i \neq 1$ for $i = 1, \ldots, n-1$ and (b) $\pi_i \in \sigma(E_I)$, $\pi_i \neq 1$ for $i = 2, \ldots, n$ and π_i is a product of factors $\eta_{\hat{g}_{i,j}}^{\epsilon_{i,j}}$, with $\epsilon_{i,j} \neq 0$, $\hat{g}_{i,j} \neq 1$ and $\hat{g}_{i,j} \neq \hat{g}_{i,j+1}$. Thus cancellation cannot occur in the terms of $(*)$, and so $E_f = i_f(K) * \sigma(E_I)$. □

We note that $i_f(K)$ is a canonical free factor of E_f, but the other factor depends on the choice of section ϕ.

Corollary 4.3. *E_f is a free group $\Longleftrightarrow$ K is a free group.*

Lemma 4.4. *If $f : G \to H$ is onto, then $\mathcal{E}_f = E_f$.*

Proof. An element $c \in \mathcal{E}_f$ can be written

$$c = g'_1 h''_1 g'_2 h''_2 \cdots g'_n h''_n,$$

with $g_i \in G$ and $h_i \in H$, where $f(g_1 \cdots g_n) = h_1 \cdots h_n$. Since f is onto, $h_i = f(x_i)$, for some $x_i \in G$. We apply the method given in remark 3.2 to define elements of G: $\delta_1 = g_1$, $\delta_2 = \bar{x}_1 g_1$, $\delta_3 = \bar{x}_1 g_1 g_2, \ldots, \delta_{2n-1} = \bar{x}_{n-1} \cdots \bar{x}_1 g_1 \cdots g_n$. Then $c = \eta_{\delta_1} \bar{\eta}_{\delta_2} \eta_{\delta_3} \bar{\eta}_{\delta_4} \cdots \eta_{\delta_{2n-1}}$, and so $c \in E_f$. □

We now give the main definition of the paper.

Definition 4.5. Let $f : G \to H$ be a homomorphism. A homomorphism $s : G \to G * H$ is called a *right co-action rel f* if $p_G s = \mathrm{id}$ and $p_H s = f$.

There is clearly a definition of left co-action. However, we shall usually consider right co-actions and call them co-actions.

Note that a co-action $s : G \to G * H$ factors through $\mathcal{E}_f \subseteq G * H$, and we also call this homomorphism $s : G \to \mathcal{E}_f$. The next lemma is then obvious.

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LEMMA 4.6. *$s : G \to G * H$ is a co-action rel $f \iff s : G \to \mathcal{E}_f$ is a section homomorphism of $\pi_f : \mathcal{E}_f \to G$.*

REMARK 4.7.

(i) If $G = H$ and $f = \mathrm{id}$, then a co-action $s : G \to G*G$ is just a co-multiplication of G and $\mathcal{E}_f = E_G = E_f$.

(ii) If $f : G \to H$ is onto, then $s : G \to G * H$ is a co-action rel f if and only if $s : G \to E_f$ is a section homomorphism of $p_f : E_f \to G$.

(iii) In examples 7.2 and 7.4 we show that $s(G)$ need not be contained in E_f and hence $\mathcal{E}_f \neq E_f$.

(iv) If $m : H \to H * H$ is a co-multiplication and $A \subseteq H$ is left stable (definition 3.6), then $m|_A : A \to A * H$ is a right co-action rel the inclusion $A \to H$.

The following proposition and its corollaries show that it is only the free part of G on which a co-action is non-trivial. Apart from its independent interest, this will justify considering co-actions on free groups.

In the case that s is a co-action and f is onto, $s(G) \subseteq E_f$ and $E_f \approx K * E_I$ (proposition 4.2). By the Kurosh theorem [6, theorem 5.1], we then obtain an isomorphism of $s(G)$ with the free product of a free group and a free product of conjugates of certain subgroups of K. The next result, which also uses the Kurosh theorem, extends this by not requiring that f be onto and by giving a precise description of s on the non-free factor.

PROPOSITION 4.8. *If $s : G \to G * H$ is a co-action rel f, then $G = L * K_1$ for subgroups L and K_1 of G such that L is a free group, $K_1 \subseteq K = \mathrm{kernel}\, f$ and K_1 is isomorphic to $\mathop{*}_{j \in J} U_j$ for subgroups $U_j \subseteq K$. Furthermore, for every $j \in J$, there exists $w_j \in \ker p_G$ such that $s(u) = \bar{w}_j u w_j$, for every $u \in U_j$.*

Proof. We apply the Kurosh theorem [6, theorem 5.1, p. 219] to the subgroup $s(G) \subseteq G * H$ (which is isomorphic to G) and obtain

$$G = L * (\mathop{*}_{j \in J} U_j) * (\mathop{*}_{i \in A} V_i),$$

where L is free and $s(U_j)$ (respectively, $s(V_i)$) is conjugate in $G*H$ to a subgroup of G (respectively, H). Thus $p_H s(U_j) = p_G s(V_i) = 1$ and $f = p_H s$ (respectively, $\mathrm{id} = p_G s$) gives $U_j \subseteq K$ (respectively, $V_i = 1$). Finally, if $u \in U_j$, $s(u) = \bar{y}_j w y_j$ for some $w \in U_j$. Then $u = p_G(s(u)) = (p_G(\bar{y}_j)) w (p_G(y_j))$, and so $w = (p_G(y_j)) u (p_G(\bar{y}_j))$. Thus $s(u) = \bar{w}_j u w_j$, where $w_j = (p_G(\bar{y}_j)) y_j \in \ker p_G$. □

COROLLARY 4.9. *K is free $\iff$ G is free. In particular, if $f : G \to H$ is one-to-one, then G is free.*

We next show how a co-action s behaves on finite subgroups of G.

COROLLARY 4.10. *Let $s : G \to G * H$ be a co-action rel $f : G \to H$ and $T \subseteq G$ a finite subgroup. Then $T \subseteq K$ and there is an element $v \in \ker p_G$ such that $s(t) = \bar{v} t v$, for all $t \in T$.*

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Proof. Since $s(T) \subseteq G * H$ is a finite subgroup, by [5, p. 194, exercise 12], $s(T)$ is conjugate to a subgroup of G or to a subgroup of H. The argument in the proof of proposition 4.8 shows that $s(T) = \bar{a}Ua$, where $U \subseteq K$, and that, for $t \in T$, $s(t) = \bar{a}ta$, with $a \in \ker p_G$. □

COROLLARY 4.11. *Let G be a finite group. If $s : G \to G*H$ is a co-action rel f, then f is the trivial homomorphism and there exists $v \in \ker p_G$ such that $s(g) = \bar{v}gv$, for $g \in G$. Moreover, there is a one-to-one correspondence between co-actions of G and elements of $\ker p_G$.*

COROLLARY 4.12. *If $s : G \to G * H$ is a co-action rel $f : G \to H$ and L is as in proposition 4.8, then there exists a co-action $\bar{s} : L \to L * H$ rel $f \mid L : L \to H$ such that the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{s} & G * H \\ {\scriptstyle p_L}\downarrow & & \downarrow{\scriptstyle p_L * \mathrm{id}} \\ L & \xrightarrow{\bar{s}} & L * H \end{array}$$

Proof. The proof is an easy exercise and hence omitted. □

5. Co-multiplications induced by co-actions

Let $s : G \to G * H$ be a co-action rel f.

DEFINITION 5.1. A co-multiplication $m : H \to H*H$ is *induced by* s if the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{s} & G * H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f * \mathrm{id}} \\ H & \xrightarrow{m} & H * H \end{array}$$

In this section we determine when a co-action induces a co-multiplication. In that case, we then study the relationship between properties of the co-action and of the co-multiplication.

DEFINITION 5.2. Let $s : G \to G * H$ be a co-action rel f. We say that s is *inductive* if $s(K) \subseteq \ker(f * \mathrm{id})$, where $f * \mathrm{id} : G * H \to H * H$.

We now give necessary and sufficient conditions for s to induce a co-multiplication on H.

PROPOSITION 5.3. *Let $s : G \to G * H$ be a co-action rel f and H be a free group. Then we have the following.*

(i) *If f is onto, then s induces a co-multiplication on $H \iff s$ is inductive.*

(ii) *For any $f : G \to H$, s induces a co-multiplication on $H \iff s$ is inductive and $I = f(G)$ is a free factor of H.*

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Proof.

(i) $\Longleftarrow$: If s is inductive, then $(f * \mathrm{id})(sK) = 1$, and so $(f * \mathrm{id})s : G \to H * H$ induces a homomorphism $m : H \to H * H$ such that the diagram in definition 5.1 commutes. Since f is onto, m is a co-multiplication.

$\Longrightarrow$: Conversely, if there is a co-multiplication $m : H \to H * H$ such that the diagram in definition 5.1 commutes, then $s(K) \subseteq \ker(f * \mathrm{id})$. Thus s is inductive.

(ii) $\Longleftarrow$: Apply (i) to the homomorphism $f' : G \to I$ to conclude that there is a co-multiplication m_I on I such that $m_I f' = (f * \mathrm{id})s$. But $H = I * J$, for some subgroup J. Since H is free, so is J, and hence there is a co-multiplication $m_J : J \to J * J$. Then m_I and m_J determine a co-multiplication m on H with the desired properties.

$\Longrightarrow$: If s induces m, then $(f * \mathrm{id})s(K) = 1$, so s is inductive. Also, H is free by § 3. Finally, $m(f(G)) \subseteq (f * \mathrm{id})(G * H) = f(G) * H$. Therefore, I is left stable with respect to m. By theorem 3.7, I is a free factor of H. □

Remark 5.4.

(i) In the proof of the second part of proposition 5.3, we wrote $H = I * J$, where H is free. If Z is a basis for J, then we may assume that the co-multiplication m of H satisfies $mz = z'z''$ for all $z \in Z$. Unless otherwise stated, we will assume that m is so defined.

(ii) Let $s : G \to G * H$ be a co-action rel f and $\bar{s} : L \to L * H$ the associated co-action rel $f|L$ given in corollary 4.12. Then it is not difficult to show that s induces a co-multiplication $m : H \to H * H$ if and only if $\bar{s}$ induces the co-multiplication $m : H \to H * H$.

(iii) In example 7.2 we show that a co-action need not induce a co-multiplication.

We next use proposition 5.3 to extend theorem 3.7 from co-multiplications to certain co-actions.

Definition 5.5. Let $s : G \to G * H$ be a co-action rel f and $A \subseteq G$ a subgroup. We say that A is *stable* with respect to s if $s(A) \subseteq A * H$.

Proposition 5.6. *If $A \subseteq G$ is stable with respect to a co-action $s : G \to G * H$ rel f, f is one-to-one and I is a free factor of H, then A is a free factor of G.*

Proof. Since I is a free factor of H, there is a projection $p_I : H \to I$. The homomorphism $s' : G \to G * I$ given by

$$G \xrightarrow{s} G * H \xrightarrow{\mathrm{id} * p_I} G * I$$

is then a co-action rel $f' = p_I f$. Since $f' : G \to I$ is an isomorphism, s' is inductive and so, by proposition 5.3, s' induces a co-multiplication m on I,

$$\begin{array}{ccc} G & \xrightarrow{s'} & G * H \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f' * \mathrm{id}} \\ I & \xrightarrow{m} & I * I \end{array}$$

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Then $m(f'A) = (f' * \mathrm{id})s'A \subseteq (f' * \mathrm{id})(\mathrm{id} * p_I)(A * H) \subseteq f'A * I$. Thus $f'A$ is left stable with respect to m. By theorem 3.7, $f'(A)$ is a free factor of $I = f'(G)$. Therefore, A is a free factor of G. □

In example 7.3, we show that if f is not one-to-one, then A need not be a free factor.

We next turn to a consideration of associativity for a co-action.

DEFINITION 5.7. Let $s : G \to G * H$ be a co-action rel f which induces a co-multiplication $m : H \to H * H$. We say that (s, m) is *associative* or, more briefly, that s is *associative* if the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{s} & G * H \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s*\mathrm{id}} \\ G * H & \xrightarrow{\mathrm{id} * m} & G * H * H \end{array}$$

More generally, the *associator* A_s of s is the equalizer of $(s * \mathrm{id})s$ and $(\mathrm{id} * m)s : G \to G * H * H$.

REMARK 5.8.

(i) If $G = H$ and $f = \mathrm{id}$, then $s = m$, $A_s = A_m$ and associativity of the co-action s is just associativity of the co-multiplication m (definition 3.3 and remark 3.8).

(ii) $f(A_s) \subseteq A_m$.

(iii) s is associative $\iff A_s = G$.

DEFINITION 5.9. For $i = 1, 2$, let $s_i : G_i \to G_i * H$ be a co-action rel $f_i : G_i \to H$. A homomorphism $\phi : G_1 \to G_2$ is a *co-action homomorphism* $\phi : (G_1, s_1) \to (G_2, s_2)$ if $f_1 = f_2\phi$ and the following diagram commutes:

$$\begin{array}{ccc} G_1 & \xrightarrow{s_1} & G_1 * H \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi*\mathrm{id}} \\ G_2 & \xrightarrow{s_2} & G_2 * H \end{array}$$

For example, in corollary 4.12, $p_L : (G, s) \to (L, \bar{s})$ is a co-action homomorphism. The proof of the following proposition is then clear.

PROPOSITION 5.10. *If $s : G \to G * H$ is a co-action rel f and $m : H \to H * H$ is a co-multiplication, then we have the following.*

(i) *m is induced by $s \iff f : (G, s) \to (H, m)$ is a co-action homomorphism.*

(ii) *(s, m) is associative $\iff s : (G, s) \to (G * H, \mathrm{id} * m)$ is a co-action homomorphism.*

PROPOSITION 5.11. *Let $s : G \to G * H$ be a co-action rel f which induces a co-multiplication $m : H \to H * H$ that satisfies the condition in remark 5.4(i).*

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(i) *If s is associative, then m is associative.*

(ii) *If f is one-to-one, then s is associative if and only if m is associative.*

Proof.

(i) By remark 5.8(ii), $I = f(G) \subseteq A_m$. Since $mz = z'z''$ for $z \in Z$, a basis of J, it follows that $J \subseteq A_m$. But $H = I * J$, and so $A_m = H$. Thus m is associative.

(ii) We assume f is one-to-one and m is associative. Then

$$(\mathrm{id} * m)mf = (m * \mathrm{id})mf,$$
$$(\mathrm{id} * m)(f * \mathrm{id})s = (m * \mathrm{id})(f * \mathrm{id})s,$$
$$(f * \mathrm{id} * \mathrm{id})(\mathrm{id} * m)s = (f * \mathrm{id} * \mathrm{id})(s * \mathrm{id})s.$$

Since $f * \mathrm{id} * \mathrm{id}$ is one-to-one, s is associative. □

REMARK 5.12. In examples 7.5 and 7.6 we give examples of non-associative co-actions. In example 7.7 we give an example in which m is associative but s is not.

For the last theorem of this section we introduce the following definition.

DEFINITION 5.13. Let $s : G \to G * H$ be a co-action rel f. An element $g \in G$ is called *s-characteristic* if $s(g) = h_1'' g' h_2''$ for some $h_1, h_2 \in H$.

Note that $f(g) = h_1 h_2$.

The proof of the next lemma is then clear.

LEMMA 5.14. *Let the co-action s rel f induce a co-multiplication m on H and let $g \in G$ be an s-characteristic element. If $s(g) = h_1'' g' h_2''$, then $\bar{h}_1, h_2 \in D_m \cup \{1\}$.*

Then we have the following theorem, whose proof is a modification of [4, theorem 3.10].

THEOREM 5.15. *Let the co-action s rel f induce a co-multiplication m on H. Then (s, m) is associative if and only if G is generated by the set of s-characteristic elements.*

Proof. If the s-characteristic elements generate G, then (s, m) is associative by lemma 5.14. For the converse, let $g \in G$ and write $s(g) = \prod_{i=1}^{n} g_i' h_i''$ in reduced form. The number, between $2n - 2$ and $2n$ inclusive, of non-trivial factors in $s(g)$ is denoted $|g|$. We prove by induction on $|g|$ that g is in the subgroup generated by the s-characteristic elements. If $|g| \leqslant 2$, the result is clear, and so we assume $|g| \geqslant 3$.

CASE 1 ($g \neq 1$). Our hypothesis gives $g_1'(m(h_1))'g_2' \cdots = s(g_1)h_1''' \cdots$. Comparing the two expressions up to the first occurrence of a triple prime term, we obtain that $s(g_1)$ must equal $g_1' h''$ for some $h \in H$. Clearly, $h = f(g_1)$ and so g_1 is s-characteristic. But $|\bar{g}_1 g| < |g|$, and thus, by induction, g is in the subgroup generated by s-characteristic elements.

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Case 2 ($g = 1$). Then we have $m(h_1)'g_2' \cdots = h_1'''s(g_2)\cdots$. Comparing the two expressions up to the first occurrence of a single prime term, we get $m(h_1) = h_1''h_1'$ and $s(g_2) = h_1''g_2'h''$, where h may be trivial. As in case 1, g is in the subgroup generated by s-characteristic elements.

This completes the induction. □

Corollary 5.16. *Let the co-action s rel f induce a co-multiplication m on H. Then the associator A_s is generated by all elements in A_s which are s-characteristic. Consequently, A_s is stable with respect to s.*

Corollary 5.17. *Let m be an associative co-multiplication on H. Then $A \subseteq H$ is a stable subgroup if and only if A has a basis of elements of the form c or $\bar{d}e$, where c, d and e belong to D_m.*

Proof. In this case, $s = m \mid A : A \to A * H$ is a co-action relative to the inclusion and (s, m) is obviously associative. Thus A has a set of generators of the required form and, by eliminating redundancies, a basis of such elements. □

6. Operations in exact Hom sequences

In this section we study the exact sequence of homomorphism sets obtained by applying the functor $\mathrm{Hom}(-, B)$ to a certain sequence of groups and homomorphisms. We show that the existence of a co-action yields more structure in the exact sequence and hence more information regarding exactness. The motivation for this section comes from a study in topology of the Puppe sequence of homotopy sets of a co-fibration [7, ch. 2].

We put restrictions on the homomorphism $f : G \to H$. We consider inductive co-actions $s : G \to G * H$ rel f, and so H must be free. We will also assume that G is free. Although there are non-trivial co-actions $s : G \to G * H$ when G is not free (see example 7.8), the results 4.8, 4.9, 4.12 and 5.4(ii) provide strong reasons for studying co-actions in the case when G is free. Thus, for a homomorphism $f : G \to H$ with kernel K and image I, we introduce the following definition.

Definition 6.1. The homomorphism $f : G \to H$ is called *free* if there are disjoint sets X, Y, Y', Z such that G is free with basis $X \cup Y$, I is free with basis Y', H is free with basis $Y' \cup Z$, $f|_Y : Y \to Y'$ is a bijection and $f(x) = 1$, for all $x \in X$.

Thus if we set $K_0 = \langle X \rangle$, then $K = K_0^G$ and $G = K_0 * \langle Y \rangle$. We also identify Y and Y' under f and write Y for Y'. We set $J = \langle Z \rangle$ and so $H = I * J$. Furthermore, $f : G = K_0 * I \to H = I * J$ can be regarded as $f = i_I p_I$. We also set

$$i_0 = i_{K_0} : K_0 \to G = K_0 * I \quad \text{and} \quad k = p_J : H = I * J \to J.$$

Lemma 6.2. *If f is a homomorphism of free groups of finite rank, then f is free homomorphism $\iff$ I is a free factor of H.*

Proof. The proof follows immediately from [5, theorem 3.3]. □

In this section we assume f is a free homomorphism.

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Definition 6.3. The sequence of groups and homomorphisms

$$1 \to K_0 \xrightarrow{i_0} G \xrightarrow{f} H \xrightarrow{k} J \to 1$$

is called the *co-fibre sequence* of the free homomorphism f.

Note that i_0 is one-to-one and k is onto, but that the co-fibre sequence is not exact. However, the kernel of each homomorphism is the normal closure of the image of the previous homomorphism.

For groups A and B, let $\mathrm{Hom}(A,B)$ denote the set of homomorphisms $A \to B$. Then the constant homomorphism which carries A to $1 \in B$ is a distinguished element of $\mathrm{Hom}(A,B)$. A homomorphism $g : A' \to A$ induces $g^* : \mathrm{Hom}(A,B) \to \mathrm{Hom}(A',B)$ defined by $g^*(a) = ag$.

Definition 6.4. For any free homomorphism $f : G \to H$ and group B, the co-fibre sequence of f yields the following sequence,

$$1 \to \mathrm{Hom}(J,B) \xrightarrow{k^*} \mathrm{Hom}(H,B) \xrightarrow{f^*} \mathrm{Hom}(G,B) \xrightarrow{i_0^*} \mathrm{Hom}(K_0,B) \to 1,$$

which is called the *Puppe sequence* of the co-fibre sequence of f.

Proposition 6.5. *The Puppe sequence of the co-fibre sequence of f is an exact sequence of based sets and maps.*

Now we consider a co-action $s : G \to G * H$ rel f. Then s induces a right action of the set $\mathrm{Hom}(H,B)$ on the set $\mathrm{Hom}(G,B)$, which is defined as follows. Let $\alpha \in \mathrm{Hom}(G,B)$ and $\beta \in \mathrm{Hom}(H,B)$ and set $\alpha \cdot \beta \in \mathrm{Hom}(G,B)$ equal to the composition

$$G \xrightarrow{s} G * H \xrightarrow{(\alpha,\beta)} B.$$

Definition 6.6. Let $f : G \to H$ be a free homomorphism and $s : G \to G * H$ a co-action rel f. Then s is called *special* if

(i) $s(k_0) = k_0'$ for every $k_0 \in K_0$, and

(ii) $X' \cup Y' \cup sY \cup Z''$ is a basis for $G * H$.

Here, $X' = i_G(X)$, $Y' = i_G(Y)$ and $Z'' = i_H(Z)$.

We note that every special co-action is inductive. For if s is special and $k \in K$, then k is a product of elements of the form $\bar{x}k_0x$, with $x \in G$ and $k_0 \in K_0$. But $s(\bar{x}k_0x) = (s\bar{x})k_0'(sx)$, and so $s(\bar{x}k_0x)$ is in the kernel of $f * \mathrm{id}$. However, not every inductive co-action is special, as example 7.3 shows.

Theorem 6.7. *Let $f : G \to H$ be a free homomorphism, $s : G \to G*H$ a special co-action rel f and $i_0^* : \mathrm{Hom}(G,B) \to \mathrm{Hom}(K_0,B)$ the map induced by $i_0 : K_0 \to G$. For $\alpha, \alpha' \in \mathrm{Hom}(G,B)$, $i_0^*(\alpha) = i_0^*(\alpha')$ if and only if there exists a $\beta \in \mathrm{Hom}(H,B)$ such that $\alpha' = \alpha \cdot \beta$. Furthermore, if f is onto, then β is unique.*

Proof. Clearly, $i_0^*(\alpha \cdot \beta) = i_0^*(\alpha)$. Now suppose $i_0^*(\alpha) = i_0^*(\alpha')$ and consider P the push-out of $i_0 : K_0 \to G$ and $i_0 : K_0 \to G$ with inclusions $j_1, j_2 : G \to P$. Then

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P is just the free product of G with itself with amalgamated subgroup K_0. Since $si_0 = i_G i_0$, there is a homomorphism $\theta : P \to G*H$ such that $\theta j_1 = s$ and $\theta j_2 = i_G$. Because X is a basis of K_0 and $X \cup Y$ is a basis of G, then $W = X' \cup Y' \cup Y''$ is a basis of P. Thus $\theta|_W$ is given by $\theta(x') = x'$, $\theta(y') = sy$ and $\theta(y'') = y'$, for $x \in X$ and $y \in Y$. Hence $\theta(W) = X' \cup sY \cup Y'$, a subset of the basis $X' \cup Y' \cup sY \cup Z''$ of $G*H$ (definition 6.6). Therefore, there is a homomorphism $\mu : G*H \to P$ such that $\mu\theta = \mathrm{id}$. Note that if f is onto, Z is empty, and so $\mu = \theta^{-1}$ is an isomorphism. Now let $\alpha, \alpha' \in \mathrm{Hom}(G, B)$ with $i_0^*(\alpha) = i_0^*(\alpha')$. Then α and α' determine $\alpha|\alpha' : P \to B$ such that $(\alpha'|\alpha)j_1 = \alpha'$ and $(\alpha'|\alpha)j_2 = \alpha$. We define $\beta : H \to B$ as the composition

$$H \xrightarrow{i_H} G * H \xrightarrow{\mu} P \xrightarrow{\alpha'|\alpha} B.$$

Then a straightforward argument yields $\alpha \cdot \beta = \alpha'$.

Now assume the homomorphism f is onto. Then θ is an isomorphism and so $\theta^* : \mathrm{Hom}(G * H, B) \to \mathrm{Hom}(P, B)$ is a bijection. Suppose $\beta, \gamma \in \mathrm{Hom}(H, B)$ and $\alpha \cdot \beta = \alpha \cdot \gamma$. Then $(\alpha, \beta)s = (\alpha, \gamma)s$. Thus

$$\theta^*(\alpha, \beta)j_1 = \theta^*(\alpha, \gamma)j_1 \quad \text{and} \quad \theta^*(\alpha, \beta)j_2 = \theta^*(\alpha, \gamma)j_2.$$

Therefore, $\theta^*(\alpha, \beta) = \theta^*(\alpha, \gamma)$ and so $(\alpha, \beta) = (\alpha, \gamma)$. Hence $\beta = \gamma$. □

REMARK 6.8. If $i : K \to G$ is the inclusion, then theorem 6.7 holds, with i replacing i_0. This is so because K is the normal closure of K_0 in G and hence $i_0^*(\alpha) = i_0^*(\alpha')$ if and only if $i^*(\alpha) = i^*(\alpha')$.

Now let $f : G \to H$ be free and $s : G \to G * H$ an inductive co-action rel f. By proposition 5.3, s induces a co-multiplication m on H. Then m determines a binary operation, denoted '+', on the set $\mathrm{Hom}(H, B)$ in the usual way.

PROPOSITION 6.9. *Let $s : G \to G * H$ be an inductive co-action rel a free homomorphism $f : G \to H$. If $\alpha, \alpha' \in \mathrm{Hom}(H, B)$, then $f^*(\alpha) = f^*(\alpha')$ if and only if there is a unique $\gamma \in \mathrm{Hom}(J, B)$ such that $\alpha' = k^*\gamma + \alpha$, where $k = p_J : H \to J$.*

Proof. Consider $t = (k * \mathrm{id})m$ defined as the following composition,

$$H \xrightarrow{m} H * H \xrightarrow{k*\mathrm{id}} J * H,$$

which is then a left co-action rel k. Note that $H = I * J = \langle Y \rangle * \langle Z \rangle$, and let $j = i_I : I \to H$. Then we have the co-fibre sequence of k,

$$1 \to I \xrightarrow{j} H \xrightarrow{k} J \to 1 \to 1,$$

where k is a free homomorphism. One easily shows that t is special and then applies theorem 6.7 (for left co-actions) to complete the proof. □

REMARK 6.10. If $s : G \to G * H$ is a special co-action rel a free homomorphism f, then theorem 6.7 and proposition 6.9 are applicable to s and give additional information on the exactness of the Puppe sequence. In fact, theorem 6.7 and proposition 6.9 are the group-theoretic analogues of proposition 2.48 of [7].

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7. Examples

We give examples to show the necessity of the hypotheses of some of the previous results and to illustrate the possibilities. We let $\langle a_1, \dots, a_k\rangle$ denote the free group with basis $\{a_1, \dots, a_k\}$.

EXAMPLE 7.1. A co-action with H not free. Let $\boldsymbol{Z}$ be the infinite cyclic group generated by x and let α be the non-trivial element in the two-element group $\boldsymbol{Z}_2$. Let $f : \boldsymbol{Z} \to \boldsymbol{Z}_2$ be defined by $f(x) = \alpha$. Then a co-action $s : \boldsymbol{Z} \to \boldsymbol{Z} * \boldsymbol{Z}_2$ can be defined by

$$s(x) = x^{n_1}\alpha x^{n_2}\alpha\cdots\alpha x^{n_k},$$

where n_i are integers such that $n_2, \dots, n_{k-1} \neq 0$, $\sum n_i = 1$ and $k-1$ (the number of occurrences of α) is odd. This, in fact, determines all co-actions $s : \boldsymbol{Z} \to \boldsymbol{Z} * \boldsymbol{Z}_2$ rel f.

EXAMPLE 7.2. A co-action that does not induce a co-multiplication. Let $G = \langle y\rangle$ be the free group on y and $H = \langle z, u\rangle$ the free group on z and u. Define $f : G \to H$ by $f(y) = z^2$. Define a co-action $s : G \to G * H$ rel f by

$$sy = y'^2 z'' u'' \bar{y}' \bar{u}'' z''.$$

Then s does not induce a co-multiplication on H. For, if it did, the co-multiplication would carry z^2 to $z'^4 z'' u'' \bar{z}'^2 \bar{u}'' z''$, which is not possible since the latter term is not a square. Note that f is not free and $s(G) \not\subseteq E_f$.

EXAMPLE 7.3. An inductive co-action which is not special (also a counterexample to extending proposition 5.6). $G = \langle x, y\rangle$, $H = \langle y\rangle$ and $f : G \to H$ is the free homomorphism given by $f(x) = 1$ and $f(y) = y$. Define $s : G \to G * H$ by

$$sx = \bar{y}'' x' y'' \quad \text{and} \quad sy = y' y''.$$

Then s is clearly inductive but not special. Let $M \subseteq G$ be the subgroup generated by x^2. Then M is stable under s, but M is not a free factor of G.

EXAMPLE 7.4. A co-action with $s(G) \not\subseteq E_f$. Let $G = \langle x, y\rangle$, $H = \langle y, z\rangle$ and $f : G \to H$ be the free homomorphism defined by $f(x) = 1$ and $f(y) = y$. Define $s : G \to G * H$ by

$$\begin{aligned} sx &= x', \\ sy &= \bar{x}' \bar{z}'' x' z'' y' y'' \\ &= [x', z''] y' y''. \end{aligned}$$

EXAMPLE 7.5. A co-action which is not associative in the case f is one-to-one. $G = \langle y\rangle$, $H = \langle y, z\rangle$ and $f : G \to H$ is given by $f(y) = y$. Define $s : G \to G * H$ by

$$sy = y'^2 y'' z'' \bar{y}' \bar{z}''.$$

Then it is easily seen that s is special (hence inductive). The induced co-multiplication m on H satisfies

$$my = y'^2 y'' z'' \bar{y}' \bar{z}'' \quad \text{and} \quad mz = z' z'',$$

and, by theorem 3.5, is seen to be non-associative. By proposition 5.11, s is not associative.

EXAMPLE 7.6. A co-action which is not associative in the case f is onto. Let $G = \langle x, y_1, y_2, y_3\rangle$, $H = \langle y_1, y_2, y_3\rangle$, $f(x) = 1$ and $f(y_i) = y_i$, $i = 1, 2, 3$. Define $s : G \to G * H$ by

$$\begin{aligned} sx &= x', \\ sy_1 &= y_1' x' y_1'' \bar{x}', \\ sy_2 &= y_2' y_2'', \\ sy_3 &= y_3' y_3'' [\bar{y}_1', [\bar{y}_1', y_2'']]. \end{aligned}$$

Then s is special since x', y_1', y_2', y_3', sy_1, sy_2, sy_3 are a basis of $G * H$. The induced co-multiplication $m : H \to H * H$ is given by

$$my_i = y_i' y_i'', \quad i = 1, 2, \quad \text{and} \quad my_3 = y_3' y_3'' [\bar{y}_1', [\bar{y}_1', y_2'']].$$

By [1, example 3.7(4)], m is not associative. By proposition 5.11, s is not associative.

EXAMPLE 7.7. m associative does not imply s associative. Let $G = \langle x, y_1, y_2\rangle$, $H = \langle y_1, y_2\rangle$, $f(x) = 1$ and $f(y_i) = y_i$, $i = 1, 2$. Define $s : G \to G * H$ by

$$\begin{aligned} sx &= x', \\ sy_1 &= y_1' y_1'' [x', y_2''], \\ sy_2 &= y_2' y_2''. \end{aligned}$$

Then s is special and the induced co-multiplication $m : H \to H * H$ satisfies $my_i = y_i' y_i''$, $i = 1, 2$. Thus m is associative, but a simple computation shows that s is not associative.

EXAMPLE 7.8. A co-action with G not free. Let $\boldsymbol{Z} = \langle x\rangle$ and $\boldsymbol{Z}_2$ be the two-element group with generator α. Let $\nu : \boldsymbol{Z} \to \boldsymbol{Z}_2$ be defined by $\nu(x) = \alpha$. Let $G = \boldsymbol{Z} * \boldsymbol{Z}_2$, $H = \boldsymbol{Z}_2$ and let $f : G = \boldsymbol{Z} * \boldsymbol{Z}_2 \to H = \boldsymbol{Z}_2$ be the homomorphism which is ν on $\boldsymbol{Z}$ and trivial on $\boldsymbol{Z}_2$. Then a co-action $s : G \to G * H$ rel f is defined by

$$s(x) = x' \alpha'' \alpha''' \bar{\alpha}'' \quad \text{and} \quad s(\alpha) = \bar{\alpha}''' \alpha'' \alpha'''.$$

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