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ON CONTRA-CLASSICAL VARIANTS OF NELSON LOGIC N4 AND ITS CLASSICAL EXTENSION

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Abstract. In two recent articles, Norihiro Kamide introduces unusual variants of Nelson's paraconsistent logic and its classical extension. Kamide's systems, **IP** and **CP**, are unusual insofar as double negations in these logics behave as intuitionistic and classical negations, respectively. In this article we present Hilbert-style axiomatizations of both **IP** and **CP**. The axiom system for **IP** is shown to be sound and complete with respect to a four-valued Kripke semantics, and the axiom system for **CP** is characterized by four-valued truth tables. Moreover, we note some properties of **IP** and **CP**, and emphasize that these logics are unusual also because they are contra-classical and inconsistent but nontrivial. We point out that Kamide's approach exemplifies a general method for obtaining contra-classical logics, and we briefly speculate about a linguistic application of Kamide's logics.

§1. Introduction.

1.1. Background and aim. In his recent articles [10, 11], Norihiro Kamide introduces variants of Nelson's paraconsistent logic **N4** (cf. [22, 12, 13]) and its classical extension $\mathbf{B}_4^{\rightarrow}$ (cf. [15]). As the titles of the two articles suggest, Kamide introduces sequent systems in which double negations behave as intuitionistic and classical negations, respectively. However, the given semantics, shown to be sound and complete with respect to the Gentzen-style systems, are the so-called bivaluational semantics, which are not the most intuitive ones.

Based on these, we establish that the system **CP**, the variant of $\mathbf{B}_4^{\rightarrow}$, is a four-valued logic and offer a four-valued Kripke semantics for **IP**, the variant of **N4**. We also observe that the four-valued algebra that is sound and complete with respect to **CP** is functionally complete, which implies the Post completeness of **CP**, and offer some reflections on the unusual unary operation. Moreover, we observe that both systems are contra-classical logics, and point out that there is a general method to obtain contra-classical logics out of a certain family of nonclassical logics. Although Kamide's systems are rather unusual, they might perhaps be applied to model negative concord in certain natural languages.

1.2. Preliminaries. The language $\mathcal{L}$ consists of a finite set $\{\sim, \wedge, \vee, \rightarrow\}$ of propositional connectives and a countable set Prop of propositional variables which we denote

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by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

§2. Proof systems. We first review the Gentzen-style sequent systems introduced by Kamide in [10, 11]. We then present a Hilbert-style system. The equivalence of the two systems will be established in §3.3.

***2.1. Gentzen-style system for* IP.** First, we introduce the Gentzen-style system for **IP**. A sequent is an expression of the form $\Gamma \Rightarrow A$ or $\Gamma \Rightarrow$, where $A \in \mathsf{Form}$ and Γ is a finite subset of Form. We write $B \Rightarrow A$ instead of $\{B\} \Rightarrow A$ and $\Gamma, \Sigma \Rightarrow A$ instead of $\Gamma \cup \Sigma \Rightarrow A$.

DEFINITION 2.1 (Kamide). *The initial sequents of $\mathcal{G}$**IP** are of the following form, for any atomic formula p,*

$$p \Rightarrow p \qquad\qquad {\sim}p \Rightarrow {\sim}p.$$

*The structural inference rules of $\mathcal{G}$**IP** are of the form*:

$$\frac{\Gamma \Rightarrow A \quad A, \Sigma \Rightarrow C}{\Gamma, \Sigma \Rightarrow C}\ (\text{cut}) \qquad \frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ (\text{we-left}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (\text{we-right}).$$

*The pure logical inference rules of $\mathcal{G}$**IP** are of the form*:

$$\frac{A, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge\text{left1}) \qquad \frac{B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ (\wedge\text{left2}) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ (\wedge\text{right})$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ (\vee\text{right1}) \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ (\vee\text{right2})$$

$$\frac{\Gamma \Rightarrow A \quad B, \Sigma \Rightarrow C}{A \to B, \Gamma, \Sigma \Rightarrow C}\ (\to \text{left}) \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\to \text{right}).$$

*The ${\sim}$-combined logical inference rules of $\mathcal{G}$**IP** are of the form*:

$$\frac{\Gamma \Rightarrow A}{{\sim}{\sim}A, \Gamma \Rightarrow}\ ({\sim}{\sim}\text{left}) \qquad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow {\sim}{\sim}A}\ ({\sim}{\sim}\text{right})$$

$$\frac{{\sim}A, \Gamma \Rightarrow C \quad {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \wedge B), \Gamma \Rightarrow C}\ ({\sim}\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow {\sim}A}{\Gamma \Rightarrow {\sim}(A \wedge B)}\ ({\sim}\wedge\text{right1}) \qquad \frac{\Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \wedge B)}\ ({\sim}\wedge\text{right2})$$

$$\frac{{\sim}A, \Gamma \Rightarrow C}{{\sim}(A \vee B), \Gamma \Rightarrow C}\ ({\sim}\vee\text{left1}) \qquad \frac{{\sim}B, \Gamma \Rightarrow C}{{\sim}(A \vee B), \Gamma \Rightarrow C}\ ({\sim}\vee\text{left2}) \qquad \frac{\Gamma \Rightarrow {\sim}A \quad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \vee B)}\ ({\sim}\vee\text{right})$$

$$\frac{A, \Gamma \Rightarrow C}{{\sim}(A \to B), \Gamma \Rightarrow C}\ ({\sim}{\to}\text{left1}) \qquad \frac{{\sim}B, \Gamma \Rightarrow C}{{\sim}(A \to B), \Gamma \Rightarrow C}\ ({\sim}{\to}\text{left2}) \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \to B)}\ ({\sim}{\to}\text{right}).$$

REMARK 2.2. Note that if we replace (${\sim}{\sim}$left) and (${\sim}{\sim}$right) by the following rules, we obtain a Gentzen-style system for **N4**.

$$\frac{A, \Gamma \Rightarrow C}{{\sim}{\sim}A, \Gamma \Rightarrow C} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow {\sim}{\sim}A}.$$

***2.2. Gentzen-style system for* CP.** Second, we introduce the Gentzen-style system for **CP**. A sequent is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite subsets of Form. We write $B \Rightarrow A$ instead of $\{B\} \Rightarrow \{A\}$ and $\Gamma, \Sigma \Rightarrow \Delta, \Theta$ instead of $\Gamma \cup \Sigma \Rightarrow \Delta \cup \Theta$.

DEFINITION 2.3 (Kamide). *The initial sequents of* $\mathcal{G}$**CP** *are of the following form, for any atomic formula* p,

$$p \Rightarrow p \qquad \sim p \Rightarrow \sim p.$$

The structural inference rules of $\mathcal{G}$**CP** *are of the form*:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}) \qquad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\text{we-right}).$$

The pure logical inference rules of $\mathcal{G}$**CP** *are of the form*:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{right})$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Sigma \Rightarrow \Pi}{A \to B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to \text{left}) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ (\to \text{right}).$$

The $\sim$*-combined logical inference rules of* $\mathcal{G}$CP *are of the form*:

$$\frac{\Gamma \Rightarrow \Delta, A}{\sim\sim A, \Gamma \Rightarrow \Delta}\ (\sim\sim\text{left}) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\sim A}\ (\sim\sim\text{right})$$

$$\frac{\sim A, \Gamma \Rightarrow \Delta \quad \sim B, \Gamma \Rightarrow \Delta}{\sim(A \wedge B), \Gamma \Rightarrow \Delta}\ (\sim\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim A, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \wedge B)}\ (\sim\wedge\text{right})$$

$$\frac{\sim A, \sim B, \Gamma \Rightarrow \Delta}{\sim(A \vee B), \Gamma \Rightarrow \Delta}\ (\sim\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim A \quad \Gamma \Rightarrow \Delta, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \vee B)}\ (\sim\vee\text{right})$$

$$\frac{A, \sim B, \Gamma \Rightarrow \Delta}{\sim(A \to B), \Gamma \Rightarrow \Delta}\ (\sim \to \text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \to B)}\ (\sim \to \text{right}).$$

REMARK 2.4. Here is a brief summary of the results established by Kamide in [11]. Kamide proves some theorems regarding syntactical embeddings between $\mathcal{G}$**CP** and Gentzen's sequent calculus **LK** for classical propositional logic, which are used to prove cut-elimination and decidability for $\mathcal{G}$**CP**. Moreover, he introduces bivaluational semantics for $\mathcal{G}$**CP** and proves some theorems regarding semantical embeddings between $\mathcal{G}$**CP** and **LK**. And by both syntactical and semantical embeddings, Kamide proves that the sequent calculus is sound and complete with respect to the bivaluational semantics. Similarly, Kamide proves several theorems for syntactical embeddings between $\mathcal{G}$**IP** and Gentzen's sequent calculus **LJ** for intuitionistic propositional logic and uses these results to prove cut-elimination, decidability, and paraconsistency for $\mathcal{G}$**IP**. A sound and complete Kripke semantics for **IP** is used in proving theorems for semantically embedding **IP** into **LJ** and vice versa. The article concludes with Glivenko and Gödel–Gentzen translation theorems for **CP** and **IP**.

***2.3. Hilbert-style system for* IP.** We now introduce a Hilbert-style system for **IP**.

DEFINITION 2.5. *The system* $\mathcal{H}$**IP** *consists of the following axiom schemata and a rule of inference, where* $A \leftrightarrow B$ *abbreviates* $(A \to B) \wedge (B \to A)$.

(Ax1)	$A \to (B \to A)$		
(Ax2)	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	(Ax9)	$(A \to {\sim\sim}A) \to {\sim\sim}A$
(Ax3)	$(A \wedge B) \to A$	(Ax10)	$(A \wedge {\sim\sim}A) \to B$
(Ax4)	$(A \wedge B) \to B$	(Ax11)	${\sim}(A \wedge B) \leftrightarrow ({\sim}A \vee {\sim}B)$
(Ax5)	$(C \to A) \to ((C \to B) \to (C \to (A \wedge B)))$	(Ax12)	${\sim}(A \vee B) \leftrightarrow ({\sim}A \wedge {\sim}B)$
(Ax6)	$A \to (A \vee B)$	(Ax13)	${\sim}(A \to B) \leftrightarrow (A \wedge {\sim}B)$
(Ax7)	$B \to (A \vee B)$	(MP)	$\frac{A \quad A \to B}{B}$
(Ax8)	$(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$		

Finally, we write $\Gamma \vdash_{\mathbf{hi}} A$ *iff there is a sequence of formulas* $\langle B_1, \dots, B_n, A\rangle$ $(n \geq 0)$*, called* a derivation*, such that every formula in the sequence either* (*i*) *belongs to* Γ; (*ii*) *is an axiom of* **IP**; (*iii*) *is obtained by* (*MP*) *from formulas preceding it in the sequence. As usual, we write* $\Gamma, A_1, \dots, A_n \vdash_{\mathbf{hi}} B$ *for* $\Gamma \cup \{A_1, \dots, A_n\} \vdash_{\mathbf{hi}} B$.

REMARK 2.6. If we replace (Ax9) and (Ax10) by ${\sim\sim}A \leftrightarrow A$, then we obtain an axiomatization for **N4**.

PROPOSITION 2.7 (Deduction theorem). *For any* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$*,* $\Gamma, A \vdash_{\mathbf{hi}} B$ *iff* $\Gamma \vdash_{\mathbf{hi}} A \to B$.

Proof. The left-to-right direction can be proved in the usual manner in the presence of axioms (Ax1) and (Ax2), and (MP) the sole rule of inference. For the other direction, we use (MP). □

***2.4. Hilbert-style system for* CP.** We now introduce a Hilbert-style system for **CP** as a variant of $\mathcal{H}$**IP**.

DEFINITION 2.8. *The system* $\mathcal{H}$**CP** *is obtained by replacing (Ax9) by the following in the system* $\mathcal{H}$**IP**:

$$A \vee {\sim\sim}A. \tag{Ax9'}$$

Finally, we write $\Gamma \vdash_{\mathbf{hc}} A$ *iff there is a sequence of formulas* $\langle B_1, \dots, B_n, A\rangle$ $(n \geq 0)$*, called* a derivation*, such that every formula in the sequence either* (*i*) *belongs to* Γ; (*ii*) *is an axiom of* **CP**; (*iii*) *is obtained by* (*MP*) *from formulas preceding it in the sequence. As usual, we write* $\Gamma, A_1, \dots, A_n \vdash_{\mathbf{hc}} B$ *for* $\Gamma \cup \{A_1, \dots, A_n\} \vdash_{\mathbf{hc}} B$.

REMARK 2.9. Note that Peirce's law, $((A \to B) \to A) \to A$, is derivable in view of (Ax9') and (Ax10) together with positive intuitionistic logic. Indeed, by (Ax10) we obtain ${\sim\sim}A \to (A \to B)$, and this together with (Ax9') implies that $A \vee (A \to B)$. Here we use the following thesis of positive intuitionistic logic.

$$(A \vee B) \to ((B \to C) \to (A \vee C)) \tag{$*$}$$

Recall then that we have both $A \to (((A \to B) \to A) \to A)$ and $(A \to B) \to (((A \to B) \to A) \to A)$ as theses of positive intuitionistic logic. Thus, by another application of $(*)$, we obtain Peirce's law.

§3. Semantics, soundness, and completeness.

3.1. Semantics. We first introduce the semantics for **IP**, and then turn to the semantics for **CP** by considering a special case.

DEFINITION 3.1. *An* **IP***-model for the language* $\mathcal{L}$ *is a triple* $\langle W, \leq, V\rangle$*, where* W *is a nonempty set* (*of states*); $\leq$ *is a partial order on* W; *and* $V : W \times \mathsf{Prop} \longrightarrow$

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$\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ *is an assignment of truth values to state-variable pairs with the condition that* $i \in V(w_1, p)$ *and* $w_1 \leq w_2$ *only if* $i \in V(w_2, p)$ *for all* $p \in \mathsf{Prop}$, *all* $w_1, w_2 \in W$ *and* $i \in \{0, 1\}$. *Valuations V are then extended to interpretations I to state-formula pairs by the following conditions*:

- $I(w, p) = V(w, p)$,
- $1 \in I(w, \sim A)$ *iff* $0 \in I(w, A)$,
- $0 \in I(w, \sim A)$ *iff for all* $x \in W$ *such that* $w \leq x$: $1 \notin I(x, A)$,
- $1 \in I(w, A \wedge B)$ *iff* $1 \in I(w, A)$ *and* $1 \in I(w, B)$,
- $0 \in I(w, A \wedge B)$ *iff* $0 \in I(w, A)$ *or* $0 \in I(w, B)$,
- $1 \in I(w, A \vee B)$ *iff* $1 \in I(w, A)$ *or* $1 \in I(w, B)$,
- $0 \in I(w, A \vee B)$ *iff* $0 \in I(w, A)$ *and* $0 \in I(w, B)$,
- $1 \in I(w, A \to B)$ *iff for all* $x \in W$ *such that* $w \leq x$: $1 \notin I(x, A)$ *or* $1 \in I(x, B)$,
- $0 \in I(w, A \to B)$ *iff* $1 \in I(w, A)$ *and* $0 \in I(w, B)$.

Finally, semantic consequence is now defined as follows:

$\Sigma \models_{\mathbf{IP}} A$ *iff for all* **IP**-*models* $\langle W, \leq, I\rangle$, *and for all* $w \in W$: $1 \in I(w, A)$ *if* $1 \in I(w, B)$ *for all* $B \in \Sigma$.

REMARK 3.2. If we replace the falsity condition for the negation by the following clause, then we obtain the semantics for **N4**:

$$0 \in I(w, \sim A) \text{ iff } 1 \in I(w, A).$$

Note also that the following holds for all formulas $A \in \mathsf{Form}$, all $w_1, w_2 \in W$ and $i \in \{0, 1\}$:

$$i \in V(w_1, A) \text{ and } w_1 \leq w_2 \text{ only if } i \in V(w_2, A).$$

Now, by considering the special case of **IP**-models in which W is a singleton, we obtain the following four-valued semantics. Note that **t**, **b**, **n** and **f** in the following definition correspond to $\{1\}$, $\{0, 1\}$, $\emptyset$ and $\{0\}$, respectively, in the above definition.

DEFINITION 3.3. *A* **CP**-valuation *is a function from the set* Form *to the set* $\mathcal{V}$ *of truth values, induced by the following matrix* $\langle\langle \mathcal{V}, \mathcal{O}\rangle, \mathcal{D}\rangle$, *where*

- $\mathcal{V} = \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$,
- $\mathcal{D} = \{\mathbf{t}, \mathbf{b}\}$,
- $\mathcal{O}$ *consists of the following truth functions*:

x	$\sim x$
t	**n**
b	**t**
n	**f**
f	**b**

$x \wedge y$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**b**	**b**	**f**	**f**
n	**n**	**f**	**n**	**f**
f	**f**	**f**	**f**	**f**

$x \vee y$	**t**	**b**	**n**	**f**
t	**t**	**t**	**t**	**t**
b	**t**	**b**	**t**	**b**
n	**t**	**t**	**n**	**n**
f	**t**	**b**	**n**	**f**

$x \to y$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**t**	**t**	**t**	**t**
f	**t**	**t**	**t**	**t**

Based on this, we define the semantic consequence relation in the usual manner: $\Gamma \models_{\mathbf{CP}} A$ *iff for all* **CP**-*valuations* v, *if* $v(B) \in \mathcal{D}$ *for all* $B \in \Gamma$ *then* $v(A) \in \mathcal{D}$.

REMARK 3.4. First, if we replace the above truth table for $\sim$ by that for the usual de Morgan negation in **FDE**, then we obtain the matrix semantics for $\mathbf{B}_4^{\to}$. Second, note that $\sim\sim$ is one of the sixteen classical negations (cf. [6, §2]) which corresponds to the Boolean complement operation from an algebraic perspective. Third, $\sim$ here rotates the "diamond" the other way around compared to Paul Ruet's unary operation ↻, introduced in [18], which has the following truth table:

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x	$\circlearrowleft x$
t	**b**
b	**f**
n	**t**
f	**n**

Finally, note that in [8] Lloyd Humberstone introduces "demi-negation" which satisfies, among other things, that two application of demi-negation are equivalent to classical negation or intuitionistic negation.

3.2. Soundness and completeness. We now establish the soundness and completeness of $\mathcal{H}$**IP** and $\mathcal{H}$**CP** with respect to the semantics presented in the previous section. We take the strategy to first prove the results for $\mathcal{H}$**IP**, and the result for $\mathcal{H}$**CP** will be a corollary by considering a special case.

THEOREM 3.5 (Soundness). *For any* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \vdash_{\mathbf{hi}} A$ *then* $\Gamma \models_{\mathbf{IP}} A$.

Proof. By a straightforward verification that each instance of each axiom schema always takes a designated value, and that (MP) preserves designated values. □

We now turn to completeness. First, we introduce some standard terminologies.

DEFINITION 3.6. *A set of formulas,* Σ, *is* deductively closed *if the following holds*:

$$\text{if } \Sigma \vdash A \text{ then } A \in \Sigma.$$

And Σ *is* prime *if the following holds*:

$$\text{if } A \vee B \in \Sigma \text{ then } A \in \Sigma \text{ or } B \in \Sigma.$$

Σ *is* prime deductively closed (*pdc*) *if it is both. Finally,* Σ *is* nontrivial *if* $A \notin \Sigma$ *for some* A.

Then the following lemmas are well-known, and thus we will omit the details of the proofs.

LEMMA 3.7 (Lindenbaum). *If* $\Sigma \nvdash A$ *then there is a pdc set,* Δ, *such that* $\Sigma \subseteq \Delta$ *and* $\Delta \nvdash A$.

LEMMA 3.8. *If* Σ *is pdc and* $A \to B \notin \Sigma$, *there is a pdc set* Δ *such that* $\Sigma \subseteq \Delta$, $A \in \Delta$ *and* $B \notin \Delta$.

We are now ready to prove the completeness.

THEOREM 3.9 (Completeness). *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \models_{\mathbf{IP}} A$ *then* $\Gamma \vdash_{\mathbf{hi}} A$.

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash_{\mathbf{hi}} A$. Then by Lemma 3.7, there is a $\Pi \supseteq \Gamma$ such that Π is pdc and $A \notin \Pi$. Define the interpretation $\mathfrak{A} = \langle X, \leq, I \rangle$, where $X = \{\Delta : \Delta \text{ is a nontrivial pdc set}\}$, $\Delta \leq \Sigma$ iff $\Delta \subseteq \Sigma$ and I is defined thus. For every state Σ and propositional parameter, p:

$$1 \in I(\Sigma, p) \text{ iff } p \in \Sigma \text{ and } 0 \in I(\Sigma, p) \text{ iff } {\sim} p \in \Sigma.$$

We show that this condition holds for any arbitrary formula, B:

$$1 \in I(\Sigma, B) \text{ iff } B \in \Sigma \text{ and } 0 \in I(\Sigma, B) \text{ iff } {\sim} B \in \Sigma. \quad (*)$$

It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models_{\mathbf{IP}} A$. The proof of $(*)$ is by a simultaneous induction on the complexity of B with respect to the positive and the negative clause.

For negation: We begin with the positive clause.

$$\begin{aligned} 1 \in I(\Sigma, \sim C) &\text{ iff } 0 \in I(\Sigma, C) \\ &\text{ iff } \sim C \in \Sigma && \text{IH} \end{aligned}$$

The negative clause is also straightforward.

$$\begin{aligned} 0 \in I(\Sigma, \sim C) &\text{ iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, 1 \notin I(\Delta, C) \\ &\text{ iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, C \notin \Delta && \text{IH} \\ &\text{ iff } \sim\sim C \in \Sigma \end{aligned}$$

For the last equivalence, assume $\sim\sim C \in \Sigma$ and for reductio that $C \in \Delta_0$ for some Δ_0 such that $\Sigma \subseteq \Delta_0$. Then by $\Sigma \subseteq \Delta_0$ and $\sim\sim C \in \Sigma$, we obtain $\sim\sim C \in \Delta_0$. Therefore, we have $\Delta_0 \vdash C \wedge \sim\sim C$, and by (Ax10), we obtain that Δ_0 is trivial which is a contradiction. For the other direction, suppose that for all Δ s.t. $\Sigma \subseteq \Delta$, $C \notin \Delta$. Then, by Lemma 3.8, it follows that $C \rightarrow \sim\sim C \in \Sigma$. Thus, in view of (Ax9), we obtain $\sim\sim C \in \Sigma$.

For disjunction: We begin with the positive clause.

$$\begin{aligned} 1 \in I(\Sigma, C \vee D) &\text{ iff } 1 \in I(\Sigma, C) \text{ or } 1 \in I(\Sigma, D) \\ &\text{ iff } C \in \Sigma \text{ or } D \in \Sigma && \text{IH} \\ &\text{ iff } C \vee D \in \Sigma && \Sigma \text{ is a prime theory} \end{aligned}$$

The negative clause is also straightforward.

$$\begin{aligned} 0 \in I(\Sigma, C \vee D) &\text{ iff } 0 \in I(\Sigma, C) \text{ and } 0 \in I(\Sigma, D) \\ &\text{ iff } \sim C \in \Sigma \text{ and } \sim D \in \Sigma && \text{IH} \\ &\text{ iff } \sim C \wedge \sim D \in \Sigma && \Sigma \text{ is a theory} \\ &\text{ iff } \sim(C \vee D) \in \Sigma && \text{(Ax12)} \end{aligned}$$

For conjunction: Similar to the case for disjunction, and thus we leave the details to the reader.

For implication: We begin with the positive clause.

$$\begin{aligned} 1 \in I(\Sigma, C \rightarrow D) &\text{ iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, \text{ if } 1 \in I(\Delta, C) \text{ then } 1 \in I(\Delta, D) \\ &\text{ iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, \text{ if } C \in \Delta \text{ then } D \in \Delta && \text{IH} \\ &\text{ iff } C \rightarrow D \in \Sigma \end{aligned}$$

For the last equivalence, assume $C \rightarrow D \in \Sigma$ and $C \in \Delta$ for any Δ such that $\Sigma \subseteq \Delta$. Then by $\Sigma \subseteq \Delta$ and $C \rightarrow D \in \Sigma$, we obtain $C \rightarrow D \in \Delta$. Therefore, we have $\Delta \vdash C \rightarrow D$, so by (MP), we obtain $\Delta \vdash D$, i.e., $D \in \Delta$, as desired. On the other hand, suppose $C \rightarrow D \notin \Sigma$. Then by Lemma 3.8, there is a $\Sigma' \supseteq \Sigma$ such that $C \in \Sigma'$, $D \notin \Sigma'$ and Σ' is pdc. Furthermore, nontriviality of Σ' is obvious by $D \notin \Sigma'$.

As for the negative clause, it is straightforward.

$$\begin{aligned} 0 \in I(\Sigma, C \rightarrow D) &\text{ iff } 1 \in I(\Sigma, C) \text{ and } 0 \in I(\Sigma, D) \\ &\text{ iff } C \in \Sigma \text{ and } \sim D \in \Sigma && \text{IH} \\ &\text{ iff } C \wedge \sim D \in \Sigma && \Sigma \text{ is a theory} \\ &\text{ iff } \sim(C \rightarrow D) \in \Sigma && \text{(Ax13)} \end{aligned}$$

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Thus, we obtain the desired result. □

The soundness and completeness for **CP** follows immediately by considering a special case for **IP**.

THEOREM 3.10 (Soundness and completeness). *For any* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\mathbf{hc}} A$ *iff* $\Gamma \models_{\mathbf{CP}} A$.

Proof. For soundness, just note that every one-element model validates (Ax9'). For completeness, note first that the presence of (Ax9') makes the partial order on the canonical model trivial. More specifically, for two nontrivial pdcs Σ and Δ, we obtain the following:

$$\Sigma \subseteq \Delta \text{ only if } \Delta \subseteq \Sigma.$$

Indeed, suppose for reductio that $\Sigma \subseteq \Delta$ and that for some A_0, $A_0 \in \Delta$ but $A_0 \notin \Sigma$. Then, by Remark 2.9, we have $A_0 \vee (A_0 \to B) \in \Sigma$ for arbitrary B. In view of $A_0 \notin \Sigma$ and that Σ is prime, we obtain $(A_0 \to B) \in \Sigma$. This together with $\Sigma \subseteq \Delta$ implies $(A_0 \to B) \in \Delta$, and with $A_0 \in \Delta$, we obtain $B \in \Delta$. But since B is arbitrary, Δ will be trivial and this contradicts the assumption that Δ is nontrivial.

We can then consider the submodel of the canonical model with $X = \{\Pi\}$ where $\Pi \supseteq \Gamma$ such that Π is pdc and $A \notin \Pi$, obtained in view of Lemma 3.7. This completes the proof. □

3.3. Equivalence of Gentzen and Hilbert systems. By making use of the above completeness results, we establish the equivalence of the Gentzen and Hilbert systems.

PROPOSITION 3.11. *For any finite set* $(\Gamma \cup \{A\}) \subseteq \mathsf{Form}$, *if* $\Gamma \vdash_{\mathbf{hi}} A$ *then* $\mathcal{G}\mathbf{IP} \vdash \Gamma \Rightarrow A$.

Proof. The proof is by induction on the length of derivations in $\mathcal{H}\mathbf{IP}$. □

PROPOSITION 3.12. *For any finite set* $(\Gamma \cup \{A\}) \subseteq \mathsf{Form}$, *if* $\mathcal{G}\mathbf{IP} \vdash \Gamma \Rightarrow A$ *then* $\Gamma \vdash_{\mathbf{hi}} A$.

Proof. Let $\tau(\Gamma \Rightarrow \Delta) = \bigwedge \Gamma \to \bigvee \Delta$, where $\bigwedge \varnothing = (p \to p)$ and $\bigvee \varnothing = (p \wedge {\sim}{\sim} p)$ for some fixed $p \in \mathsf{Prop}$. We first note that $\mathcal{G}\mathbf{IP} \vdash \Gamma \Rightarrow A$ iff $\mathcal{G}\mathbf{IP} \vdash \varnothing \Rightarrow \tau(\Gamma \Rightarrow A)$. In view of the completeness of $\mathcal{H}\mathbf{IP}$ w.r.t. the semantics from Definition 3.1, it is enough to show that for every sequent rule $\frac{S_1 \ldots S_n}{S}$ of $\mathcal{G}\mathbf{IP}$, we have $\{\tau(S_1), \ldots, \tau(S_n)\} \models_{\mathbf{IP}} \tau(S)$. For axiomatic sequents, this is obvious, and since (cut) is eliminable, (cut) need not be considered. Here we present two of the remaining cases.

- Case (${\sim}{\sim}$left): Suppose $1 \in I(w, \tau(\Gamma \Rightarrow A))$, $1 \in I(w, {\sim}{\sim} A)$, and $1 \in I(w, \bigwedge \Gamma)$. Then: $1 \in I(w, {\sim}{\sim} A)$ iff $0 \in I(w, {\sim} A)$ iff for all $x \in W$ such that $w \leq x$: $1 \notin I(x, A)$. Since $w \leq w$, we have $1 \in I(w, \bigwedge \Gamma)$ and $1 \notin I(w, A)$, which contradicts $1 \in I(w, \bigwedge \Gamma \to A)$. Thus, $1 \in I(w, \tau({\sim}{\sim} A, \Gamma \Rightarrow))$.
- Case (${\sim}{\sim}$right): We reason by contraposition. Suppose that $1 \notin I(w, \tau(\Gamma \Rightarrow {\sim}{\sim} A))$. Then $1 \in I(w, \bigwedge \Gamma)$ and $1 \notin I(w, {\sim}{\sim} A)$. But then $1 \notin I(w, {\sim}{\sim} A)$ iff $0 \notin I(w, {\sim} A)$ iff there exists $x \in W$ with $w \leq x$ and $1 \in I(x, A)$. By heredity, $1 \in I(x, \bigwedge \Gamma)$, which contradicts $1 \in I(w, \tau(A, \Gamma \Rightarrow))$.

This completes the proof. □

PROPOSITION 3.13. *For any finite set* $(\Gamma \cup \{A\}) \subseteq \mathsf{Form}$, *if* $\Gamma \vdash_{\mathbf{hc}} A$ *then* $\mathcal{G}\mathbf{CP} \vdash \Gamma \Rightarrow A$.

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Proof. The proof is by induction on the length of derivations in $\mathcal{H}\mathbf{CP}$. □

PROPOSITION 3.14. *For any finite set* $(\Gamma \cup \{A\}) \subseteq \mathsf{Form}$, *if* $\mathcal{G}\mathbf{CP} \vdash \Gamma \Rightarrow A$ *then* $\Gamma \vdash_{\mathbf{hc}} A$.

Proof. Note that $\mathcal{G}\mathbf{CP} \vdash \Gamma \Rightarrow A$ iff $\mathcal{G}\mathbf{CP} \vdash \varnothing \Rightarrow \tau(\Gamma \Rightarrow A)$. By the completeness of $\mathcal{H}\mathbf{CP}$ w.r.t. its truth table semantics from Definition 3.3, it is enough to show that for every sequent rule $\frac{S_1 \ldots S_n}{S}$ of $\mathcal{G}\mathbf{CP}$, we have $\{\tau(S_1), \ldots, \tau(S_n)\} \models_{\mathbf{CP}} \tau(S)$. Since (cut) is eliminable, (cut) need not be considered. For the axiomatic sequents the proof is obvious, and for the remaining sequent rules of $\mathcal{G}\mathbf{CP}$ one may use the truth tables. □

§4. Technical reflections.

Here are a few more results related to **CP** and **IP**.

4.1. Inconsistency. We first observe that there is a contradictory pair of formulas being provable in both **CP** and **IP**. To this end, the following lemma is useful.

LEMMA 4.1. *The following formulas are derivable in* **IP**.

$$(A \to B) \to ({\sim}{\sim}B \to {\sim}{\sim}A) \quad (1)$$

$${\sim}{\sim}(A \vee B) \to ({\sim}{\sim}A \wedge {\sim}{\sim}B) \quad (2)$$

$$A \to {\sim}{\sim}{\sim}{\sim}A \quad (3)$$

Proof. We only need to recall the axioms (Ax9) and (Ax10). The details are left to interested readers. □

PROPOSITION 4.2. *The following formulas are derivable in* **IP**, *and therefore in* **CP** *as well.*

$${\sim}{\sim}(A \to A) \to B \quad (4)$$

$${\sim}{\sim}{\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A) \quad (5)$$

$${\sim}{\sim}{\sim}{\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A) \quad (6)$$

Proof. For (4), just note that we have $((A \to A) \wedge {\sim}{\sim}(A \to A)) \to B$ as an instance of (Ax10). For (5), the proof runs as follows:

1	${\sim}(A \wedge {\sim}{\sim}A) \leftrightarrow ({\sim}A \vee {\sim}{\sim}{\sim}A)$	[(Ax11)]
2	${\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A) \leftrightarrow {\sim}{\sim}({\sim}A \vee {\sim}{\sim}{\sim}A)$	[1, (1)]
3	${\sim}{\sim}({\sim}A \vee {\sim}{\sim}{\sim}A) \leftrightarrow ({\sim}{\sim}{\sim}A \wedge {\sim}{\sim}{\sim}{\sim}{\sim}A)$	[2, (2)]
4	$({\sim}{\sim}{\sim}A \wedge {\sim}{\sim}{\sim}{\sim}{\sim}A) \to {\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A)$	[(Ax10), taking ${\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A)$ for B]
5	${\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A) \to {\sim}{\sim}{\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A)$	[2, 3, 4]
6	${\sim}{\sim}{\sim}{\sim}{\sim}(A \wedge {\sim}{\sim}A)$	[5, (Ax9)]

Finally, for (6), note first that we obtain ${\sim}{\sim}(A \wedge {\sim}{\sim}A)$ by making use of (Ax9') and (Ax10). Therefore, in view of (3), we obtain the desired result. □

REMARK 4.3. (4) implies that **CP** is not a subsystem of classical logic, but is orthogonal to it like systems of connexive logic (cf. [24]).[1] Moreover, (5) and (6) show that **CP** and **IP** are inconsistent.[2] Note finally that we have a more simple instance in the case of **CP**,

[1] We here assume that $\sim$ is a negation. More on this is discussed in §5.1.

[2] An anonymous reviewer correctly pointed out to us that we have both ${\sim}(A \wedge B)$ and ${\sim}{\sim}(A \wedge B)$ as theses if we have that ${\sim}A$ is a thesis and that B is like a bottom, namely that $B \Rightarrow$ is derivable

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namely $\sim(A \wedge \sim\sim A)$ and $\sim\sim(A \wedge \sim\sim A)$. The latter is already provable in **IP**, as we observed above, and the former is immediate in view of (Ax11) and (Ax9').

4.2. Functional completeness. We now turn to show that the matrix that characterizes **CP** is functionally complete as a corollary of a general characterization of functional completeness. To this end, we first introduce some related notions.

DEFINITION 4.4 (Functional completeness). *An algebra* $\mathfrak{A} = \langle A, f_1, \ldots, f_n \rangle$*, is said to be* functionally complete *provided that every finitary function* $f : A^m \to A$ *is definable by superpositions of the functions* $f_1, \ldots, f_n$ *alone. A matrix* $\langle \mathfrak{A}, \mathcal{D} \rangle$ *is* functionally complete *if* $\mathfrak{A}$ *is functionally complete.*

DEFINITION 4.5 (Definitional completeness). *A logic* **L** *is* definitionally complete *if there exists a functionally complete matrix that is strongly adequate for L.*

For the characterization of the functional completeness, the following theorem of Jerzy Słupecki is useful. In order to state the result, we need the following definition.

DEFINITION 4.6. *Let* $\mathfrak{A} = \langle A, f_1, \ldots, f_n \rangle$ *be an algebra, and* f *be a binary operation defined in* $\mathfrak{A}$*. Then,* f *is* unary reducible *iff for some unary operation g definable in* $\mathfrak{A}$*,* $f(x, y) = g(x)$ *for all* $x, y \in A$ *or* $f(x, y) = g(y)$ *for all* $x, y \in A$*. And* f *is* essentially binary *if* f *is not unary reducible.*

THEOREM 4.7 (Słupecki, [19]). $\mathfrak{A} = \langle \langle \mathcal{V}, f_1, \ldots, f_n \rangle, \mathcal{D} \rangle$ ($\sharp\mathcal{V} \geq 3$) *is functionally complete iff in* $\langle \mathcal{V}, f_1, \ldots, f_n \rangle$

(i) *all unary functions on* $\mathcal{V}$ *are definable and*
(ii) *at least one surjective and essentially binary function on* $\mathcal{V}$ *is definable.*

Based on this elegant characterization by Słupecki, the desired result is obtained as follows. In case of expansions of the algebra related to **FDE**, we can simplify even further.

THEOREM 4.8. *Given any expansion* $\mathcal{F}$ *of the algebra* $\langle \mathcal{V}, \wedge, \vee \rangle$ *the following are equivalent*:

(i) $\mathcal{F}$ *is functionally complete.*
(ii) *All of the* δ_a*s as well as* C_a*s* ($a \in \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$) *of the following tables are definable.*

x	$\delta_{\mathbf{t}}(x)$	$\delta_{\mathbf{b}}(x)$	$\delta_{\mathbf{n}}(x)$	$\delta_{\mathbf{f}}(x)$	$\mathsf{C}_a(x)$
t	**t**	**f**	**f**	**f**	a
b	**f**	**t**	**f**	**f**	a
n	**f**	**f**	**t**	**f**	a
f	**f**	**f**	**f**	**t**	a

REMARK 4.9. The above result is essentially the theorem proved in [17, Theorem 3.6], but with the simplification that we can replace $\langle \mathcal{V}, \sim, \wedge, \vee \rangle$ (where $\sim$ here is the de Morgan negation) by $\langle \mathcal{V}, \wedge, \vee \rangle$. This is possible since if we have the eight unary functions, then we can define the de Morgan negation as follows:

$$(\delta_{\mathbf{t}}(x) \wedge \mathsf{C}_{\mathbf{f}}) \vee (\delta_{\mathbf{b}}(x) \wedge \mathsf{C}_{\mathbf{b}}) \vee (\delta_{\mathbf{n}}(x) \wedge \mathsf{C}_{\mathbf{n}}) \vee (\delta_{\mathbf{f}}(x) \wedge \mathsf{C}_{\mathbf{t}}).$$

THEOREM 4.10. **CP** *is definitionally complete.*

in $\mathcal{G}$**IP**. Although it is rather easy to find a bottom, note that it is not so obvious to find a thesis of the form $\sim A$, as we have a rather lengthy derivation above.

Proof. In view of the above theorem, it suffices to prove that all of the δ_as as well as C_as ($a \in \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$) are definable in $\langle \mathcal{V}, \sim, \wedge, \vee, \rightarrow \rangle$. And this can be done as follows.

x	$\neg\neg(\sim x \vee \sim\sim x)$	$\neg(\sim x \rightarrow \sim\sim x)$	$\neg(\sim\sim x \rightarrow \sim x)$	$\neg\neg(\sim x \vee \sim\sim x)$	$x \vee \sim\sim x$	$\sim(x \wedge \sim\sim x)$	$\sim(x \vee \sim\sim x)$	$x \wedge \sim\sim x$
t	**t**	**f**	**f**	**f**	**t**	**b**	**n**	**f**
b	**f**	**t**	**f**	**f**	**t**	**b**	**n**	**f**
n	**f**	**f**	**t**	**f**	**t**	**b**	**n**	**f**
f	**f**	**f**	**f**	**t**	**t**	**b**	**n**	**f**

Note here that $\neg x$ is defined as $x \rightarrow (x \wedge \sim\sim x)$ and has the following truth table.

x	$\neg x$
t	**f**
b	**f**
n	**t**
f	**t**

This completes the proof. □

REMARK 4.11. If the functional completeness of **CP** is the only concern, then by a result established by Arnon Avron in [3, Theorem 3.10], it suffices to show that de Morgan negation, $\mathsf{C}_{\mathbf{b}}$, and $\mathsf{C}_{\mathbf{n}}$ are definable in $\langle \mathcal{V}, \sim, \wedge, \vee, \rightarrow \rangle$.[3]

4.3. Post completeness. Finally, we note the Post completeness of **CP**.

DEFINITION 4.12. *The logic* **L** *is* Post complete *iff for every formula* A *such that* $\nvdash A$, *the extension of* **L** *by* A *becomes trivial, i.e.,* $\vdash_{\mathbf{L} \cup \{A\}} B$ *for any* B.

THEOREM 4.13 (Tokarz, [20]). *Definitionally complete logics are Post complete.*

In view of Theorems 4.10 and 4.13, we obtain the following result.

COROLLARY 4.14. **CP** *is Post complete.*

REMARK 4.15. Given its Post completeness, we obtain that the addition of any formulas which are not valid in **CP** will make the system trivial. In particular, the following formulas can not be added without falling into triviality:

- $\sim\sim A \rightarrow A$
- $A \rightarrow \sim\sim A$
- $(A \wedge \sim A) \rightarrow B$
- $A \vee \sim A$

§5. Philosophical reflections.

5.1. Is $\sim$ a negation? The pressing question is probably the following: is $\sim$ a negation at all? Note first that a very modest requirement for a unary connective to deserve the classification as a negation can be found in [14, 2, 25]: a unary connective $\neg$ is a negation in a logic **L** if there exist **L**-formulas A and B such that in **L**, $A \nvdash \neg A$ and $\neg B \nvdash B$. This condition is clearly satisfied by Kamide's $\sim$.

Can we say something even more? We believe we can. To this end, let us first recall the truth and falsity conditions for $\sim$ from Definition 3.1.[4]

3 We would like to thank an anonymous reviewer for pointing this out to us.

4 In the following discussion, we will not relativize the conditions with respect to worlds/states for the sake of simplicity.

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- $1 \in v(\sim A)$ iff $0 \in v(A)$
- $0 \in v(\sim A)$ iff $1 \notin v(A)$

This shows that there is a sense in which $\sim$ *is* a negation: the truth condition is exactly the one for the negation in **FDE**. However, there is another sense in which $\sim$ is *not* a negation: the falsity condition is more like that for "affirmative" operators. In the end, one's answer heavily depends on the account of negation one is ready to accept. In what follows, we classify how one may answer the question.

One of the related questions that is crucial is: what is negation? We believe that the following answer, offered in [7], will be almost universally accepted:

> Negation is in the first place a phenomenon of semantical opposition.

Of course, the next question will be to ask what a semantical opposition is. In the present context in which the two-valued relational (or Dunn) semantics is available, it seems that there are at least two kinds of semantical opposition: one between truth and falsity, and the other between truth and untruth.[5]

5.1.1. Semantic opposition in terms of truth and falsity. If one follows the idea that semantic opposition should be understood in terms of truth and falsity, then there seem to be three choices:

- require that $\sim A$ is true iff A is false;
- require that $\sim A$ is false iff A is true;
- require that $\sim A$ is true iff A is false, and that $\sim A$ is false iff A is true.

The first choice is considered, for example, by Arnon Avron in [4] who regards the non-classical negation as representing "the idea of falsehood within the language" ([4, p. 160]). The second choice seems to be not considered explicitly in the literature, but it also has some intuitive appeal: if A is true and negation is about semantic opposition between truth and falsity, then it seems to be quite reasonable to require that $\sim A$ is false, and vice versa. The third choice, combining the first and the second options, boils down to the account of negation understood as the flip-flopping operator between truth and falsity as in **FDE**.

Among these three choices, only the first choice justifies to claim that the unary operation $\sim$ in **CP** is a negation. But the point to be emphasized here is that there is a way to make sense of the rather unusual unary operation as a negation.

5.1.2. Semantic opposition in terms of truth and untruth. If one follows the idea that semantic opposition should be understood in terms of truth and untruth, then there seems to be only one way to go:

- require that $\sim A$ is true iff A is untrue.

But this path will not lead us to claim that the unary operation $\sim$ in **CP** or **IP** is a negation. Note, of course, that in view of Remark 3.4, $\sim\sim$ will be counted as negation in **CP**, as the title of [10] claims.

In sum, we observed that there are accounts of negation, including the one discussed by Avron, that count the unary operation $\sim$ of **IP** and **CP** as a negation.

5.2. Contra-classicality. Contra-classical logics are, roughly speaking, those that are orthogonal to classical logic. More specifically, following Humberstone, a logic is contra-

[5] To be sure, there are more kinds of opposition, namely those between falsity and nonfalsity, between nonfalsity and untruth, between truth and nonfalsity, and between falsity and untruth.

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classical "just in case not everything provable in the logic is provable in classical logic" ([9, p. 438]).[6] There are a few contra-classical logics in the literature, such as connexive logics and Abelian logics, but one might have the impression that contra-classical logics are quite remote from the other classical and nonclassical logics.[7] This is, however, not necessarily the case in a certain context.

Indeed, let us recall the semantics for **IP** and **CP**. Then, the only deviation from the more familiar systems is the falsity condition for the unary operation $\sim$. In fact, this can be generalized in the following manner. In the context of expansions of the four-valued logic of Belnap and Dunn, if we take the relational semantics due to Dunn, then some tweaking on the falsity conditions of connectives will lead us to contra-classical logics. Here are a couple of examples.

EXAMPLE 5.1. *First, let us deal with the conditional, and for the sake of simplicity assume the following truth condition*:

- *$A \rightarrow B$ is true iff A is not true or B is true.*

Now, the most well-studied falsity condition is probably the following:

- *$A \rightarrow B$ is false iff A is true and B is false.*

However, if we replace this condition by the following condition, then we obtain so-called connexive logics:

- *$A \rightarrow B$ is false iff A is not true or B is false.*

The resulting expansion of Belnap–Dunn logic is the system **MC** *from* [*24*]. *Too see the contra-classicality, we may observe that both* $(A \wedge \sim A) \rightarrow A$ *and its negation is valid/derivable in* **MC**.[8]

Note finally that this idea from [*23*] *can be applied to many other contexts, not restricted to the above simple case. Indeed, the conditional maybe constructive, or even relevant* (*cf.* [*23, 16*]).

EXAMPLE 5.2. *Second, let us deal with disjunction. Then the standard truth and falsity conditions are as follows:*

- *$A \vee B$ is true iff A is true or B is true,*
- *$A \vee B$ is false iff A is false and B is false.*

Now, in the literature of bilattice logics, there is a connective $\oplus$, called informational join, which shares the truth condition with the usual disjunction, but has the following falsity condition:

- *$A \oplus B$ is false iff A is false or B is false.*

That is, the falsity condition for $\oplus$ is exactly the same as that of the more standard conjunction. As is well-known, this difference in the falsity condition is a result of some bilattice theoretic considerations, not something introduced in an ad hoc *manner.*

The resulting expansion of Belnap–Dunn logic obtained by adding informational meet as well as the standard, "weak" (*not the connexive!*) *conditional is known as* $\mathbf{BL}_{\supset}$ *due to*

[6] For a much more detailed discussion on the definition of contra-classical logics, see [9].
[7] Note here that there are some contra-classical modal logics discussed by Humberstone in [9].
[8] Note of course that $\sim$ here is the more usual de Morgan negation.

Arieli and Avron [*1*]. *And to see the contra-classicality, we may observe that both* $(A \supset A) \oplus \sim(A \supset A)$ *and its negation is valid/derivable in* $\mathbf{BL}_{\supset}$.

So, these are instances in which the falsity conditions for the conditional and disjunction are involved. Repeating the earlier remark, our observations in this article show that we can also take a somewhat unusual falsity condition for negation to obtain more contra-classical logics.

In sum, we hope to have established that contra-classicality is not too far away if we are already in the realm of Belnap–Dunn logic.

5.3. Applications. One may wonder whether the simulation of intuitionistic negation and classical negation as a double paraconsistent negation is just a formal possibility (or even curiosity), or whether treating double negation as negation could have some interesting applications.[9] It is well-known that multiple negation in natural language often comes with a combination of different kinds of negation, as, for instance, in "not unhappy" or "not impossible", where the strongly (predicate term) negating prefixes "un-" and " im-" occur in the scope of the weaker sentential negation "not". In [7] it is pointed out that whilst such examples of multiple negation in natural language require a more subtle analysis of the idea of *duplex negatio affirmat*, the phenomenon of negative concord exemplifies the principle *duplex negatio negat*. In [5, p. 4], the phenomenon of negative concord is introduced by explaining that in negative concord sentences, "single negative meanings are expressed by two or more negative words." Negative concord can be found in certain varieties of English, for example in lyrics ("I shot the sheriff, but I didn't shoot no deputy", Bob Marley; "We don't need no education", Pink Floyd). Some natural languages, such as Classical Latin, seem not to display negative concord; some languages, such as German, seem to have preserved it only in a number of dialects, whereas in lyrics it seems to work ("Keine Macht für Niemand", Ton Steine Scherben). In other natural languages, such as Italian, negative concord is often obligatory.

Van der Wouden [21], distinguishes between four kinds of multiple negation in natural language. In [26, p. 57 f.] they are presented as follows:

- Double Negation: Two negative elements cancel each other and yield an affirmative.
- Weakening Negation: One negative element weakens the negation of another negative element. The result is somewhere between a positive and a negative.
- Negative Concord: two or more negative elements yield one negation in the semantics.
- Emphatic Negation: One negative element enforces another negative element. The result is stronger than it would be the case with just the second negative element.

According to that classification, Kamide's negations as double negations express negative concord.

§6. Concluding remarks. In [10, 11], Norihiro Kamide exploits the explicit distinction between support of truth and support of falsity conditions in the semantics on Nelson's paraconsistent logic **N4** to simulate intuitionistic negation by double negation in a variant **IP** of **N4**, and classical negation in a variant **CP** of $\mathbf{B}_4^{\rightarrow}$. We have presented both systems, **IP** and **CP**, as Hilbert-style axiom systems, and, in particular, we have presented **CP** as a

9 Note that Humberstone discusses a motivation very briefly in [8, p. 2].

four-valued logic. Although at first sight Kamide's approach may appear to be disturbingly unusual, the fact that it exemplifies a general method for obtaining contra-classical systems might turn out to be useful in future investigations of contra-classicality. Moreover, systems such as **CP** and **IP** might turn out to be useful for analyzing the phenomenon of negative concord.

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