J. Plasma Physics (2002), *vol.* 67, *parts* 2&3, *pp.* 163–173. © 2002 Cambridge University Press 163
DOI: 10.1017/S0022377801001556 Printed in the United Kingdom

Asymptotic theory of a collision-dominated space-charge sheath with a velocity-dependent ion mobility

M. S. BENILOV[1] and R. N. FRANKLIN[2]

[1]Departmento de Física, Universidade da Madeira Largo do Município, 9000 Funchal, Portugal

[2]Oxford Research Unit, The Open University Boars Hill, Oxford OX1 5HR, England

(Received 9 April 2001 and in revised form 19 September 2001)

Abstract. The method of matched asymptotic approximations is used to examine an ordering of the physical quantities in a collisional plasma–collisional sheath model that has not been previously explored, namely in the constant ion mean free path model with high electric field. The asymptotic theory for a collision-dominated space-charge sheath at a negative surface is developed for the case when the ions interact with neutral particles as rigid spheres while the electric field is high both in the sheath and in the quasineutral plasma. The ion mobility is inversely proportional to the square root of the electric field in such a case. The ratio λ_D/L of the Debye length at the center of the quasineutral plasma to a characteristic dimension of the plasma is considered as a small asymptotic parameter. The general structure of the asymptotic solution obtained is similar to that in the case of low electric field (or in the framework of the model of constant collision frequency), considered by previous workers; however, the scalings are different. In particular, the thickness of the sheath is found to have the order $\lambda_D^{4/5}L^{1/5}$, while the respective order in the case of low electric field is $\lambda_D^{2/3}L^{1/3}$.

1. Introduction

The asymptotic theory of a collision-dominated space-charge sheath at a negative surface in a weakly ionized plasma was first described in the classical papers [1, 2] for the cases of large and moderate surface potentials, respectively; refined treatments have been given in [3, 4]. The theory in the above-cited works was developed with reference to electric probes in high-pressure plasmas. In [5], a treatment similar to that in [2] was applied to the problem of distribution of parameters in a cross section of a DC glow discharge.

Designate by L a length scale characterizing the region of quasineutral plasma. (Typically, L can be set equal to the probe radius or to the radius of the glow discharge tube.) If the surface potential is moderate, which is the case that will be considered in this work, the space-charge sheath has no internal structure and can be characterized by a single length scale, which has the meaning of a scale of sheath thickness and will be designated δ. The continuum asymptotic approach employed in the above papers applies in the case

$$\lambda_i \ll \delta \ll L, \tag{1}$$

where λ_i is the mean free path of ions in the gas of neutral particles. This case will be considered also in this work.

If the energy gained in the electric field in a collision-dominated plasma over an ion mean free path is much smaller than the mean thermal energy of the neutral particles,

$$eE\lambda_i \ll kT_a, \tag{2}$$

then the diffusion coefficient and the mobility of the ions may be considered as given constants, regardless of the model of the ion–atom interaction being considered. (Here T_a is the temperature of neutral particles; it is assumed for simplicity that the masses of the ions and of the neutral particles are comparable; the temperature and density of the neutral particles are assumed to be constant.) This case is usually referred to as the case of low electric field; see, e.g., [6]. If condition (2) is not satisfied, the dependence of the diffusion coefficient and mobility of the ions on the local electric field should be taken into account, an exception being the case when the interaction of the ions with neutral particles can be approximated by the model of constant collision frequency and the ion mobility is constant. Note that the case when the inequality (2) is reversed is usually referred to as the case of high electric field; see, e.g., [6].

The electric field in the quasineutral region and (provided that the surface potential is moderate) in the space-charge sheath may be estimated as $E = O(kT_e/el)$, where T_e is the electron temperature and l is the local length scale ($l = L$ in the quasineutral region and $l = \delta$ in the sheath). It follows that the inequality (2) amounts to

$$\lambda_i \ll l\frac{T_a}{Te}. \tag{3}$$

If T_a is comparable to T_e, the inequality (3) for a collision-dominated plasma ($\lambda_i \ll l$) is always satisfied. However, in many situations of practical interest, $T_a \ll T_e$; in particular, under conditions of a glow discharge, the ratio T_a/T_e may be of the order of 10^{-2}. Assuming a collision-dominated sheath ($\lambda_i \ll \delta$), one can distinguish the following three limiting cases in such situations:

$$\lambda_i \ll \delta\frac{T_a}{T_e}; \qquad \delta\frac{T_a}{T_e} \simeq \lambda_i \ll L\frac{T_a}{T_e}, \delta; \qquad L\frac{T_a}{T_e} \simeq \lambda_i \ll \delta. \tag{4}$$

In the first case, the inequality (3) holds both in the quasineutral region and in the sheath, and the diffusion coefficient and the mobility of the ions may be considered as given constants. In the second case, the diffusion coefficient and the mobility of the ions may be considered as given constants in the quasineutral plasma; the dependence of these coefficients on the local electric field should be taken into account in the sheath. In the third case, the dependence of the ion transport coefficients on the local electric field must be taken into account both in the sheath and in the quasineutral region. Obviously, the third case can be realized only if $\delta/L \gg T_a/T_e$.

The treatment in [2, 4, 5] refers to the first case. It is the third case that will be (asymptotically) analyzed in this work. For definiteness, the theory will be developed with reference to a planar column of a DC glow discharge. Apart from being of independent interest, such an analysis can provide a useful check on computations similar to [7], just as [5] did for the computations given in [8]. The results obtained are complementary to those with different ordering of the physical quantities. The case $\delta \ll L \ll \lambda_i$ was considered in [9] and $\delta \ll \lambda_i \ll L$ more recently in [10], where in both cases the powerful technique of matched asymptotic approximations shows that the Bohm criterion applies and that there is necessarily a structure of plasma–transition layer–collisionless sheath. In the present case, just as in [5], the structure is collisional plasma–collisional sheath, with plasma and sheath joining smoothly and the Bohm criterion has no relevance. We conclude, as did Riemann and Meyer [11], that there is no such thing as a collisionally modified Bohm criterion.

The use of the technique of matched asymptotic approximations went through an initial phase in the 1960s, of which [1, 2, 5, 9] are examples, but it is only now being used regularly in the analysis of different and complex plasma situations. Recent examples include the variation of ion temperature [12], dusty plasmas [13], fusion plasmas [14], electronegative plasmas [15], and flowing plasmas [16].

2. The model

We consider a plane collision-dominated column of a low-pressure DC glow discharge. The column is confined by two (planar) walls and has width $2L$. In accord with the estimates of the preceding section, the average ionic energy is of the order of $kT_e\lambda_i/L$ in the quasineutral region and of the order of $kT_e\lambda_i/\delta$ in the sheath. In either case, this energy is much smaller than kT_e and the flux of ions caused by diffusion is much smaller than the (drift) flux caused by the electric field. The distribution of parameters in a cross-section of the column is governed by the system of equations including the continuity equation for the ions, the equation describing their drift, the Boltzmann distribution for the electron density, and Poisson's equation:

$$\frac{d}{dx}(n_i v_i) = -\alpha n_e. \qquad v_i = \mu_i \frac{d\phi}{dx}, \tag{5}$$

$$n_e = n_{e0} \exp\left(\frac{e\phi}{kT_e}\right), \qquad \epsilon_0 \frac{d^2\phi}{dx^2} = e(n_e - n_i). \tag{6}$$

Here the x axis is directed from the wall into the plasma, n_i and n_e are the number densities of the ions and the electrons, v_i is the mean velocity of the motion of the ions in the direction to the wall, α is the ionization rate (the eigenvalue), ϕ is the electrostatic potential, and μ_i is the ion mobility. For definiteness, one can set the value of the potential at the plane of symmetry equal to zero; then n_{e0} is the electron density at the plane of symmetry.

It will be assumed that the ions interact with the atoms as rigid spheres (the model of constant mean free path). Furthermore, we restrict consideration to the case when the third inequality in (4) is satisfied in a somewhat stronger form: $LT_a/T_e \ll \lambda_i \ll \delta$. In this case, the high-electric field regime occurs in the whole calculation domain, i.e., both in the quasineutral plasma and in the space-charge sheath. The mobility of the ions in this regime in the framework of the model of rigid spheres is inversely proportional to the square root of the electric field, and one can write

$$v_i = A\sqrt{\frac{d\phi}{dx}}, \tag{7}$$

where A is a given positive constant. Note that this constant is equal, to the accuracy of a factor, to $\sqrt{e\lambda_i/m_i}$, where λ_i is the (constant) mean free path of the ions in the gas of neutral atoms and m_i is the mass of an ion. This can be seen, e.g., from equation (12) of [17]; a similar result can be obtained from equation (5-2-23) of [6]. Note that the above-mentioned factor depends on the ratio of the mass of an ion to the mass of a neutral particle and is of order unity if the two masses are comparable.

Boundary conditions for the above equations are as follows. Since the net electric current density to an (insulating) wall is zero, one can write (see, e.g., [18])

$$x = 0: \qquad \tfrac{1}{4} n_e \bar{C}_e = n_i v_i. \tag{8}$$

Here $\bar{C} = (8kT_e/\pi m_e)^{1/2}$ is the electron mean thermal speed (m_e is the mass of the electron). At the axis of symmetry of the discharge, one can write

$$x = L: \qquad \phi = 0, \qquad v_i = 0, \qquad \frac{dn_i}{dx} = 0. \tag{9}$$

We introduce dimensionless variables

$$X = \frac{x}{L}, \qquad N = \frac{n_i}{n_{e0}}, \qquad \Phi = \frac{e\phi}{kT_e}. \tag{10}$$

The problem assumes the form

$$\frac{d}{dX}\left(N\sqrt{\frac{d\Phi}{dX}}\right) = -Ze^{\Phi}, \qquad \varepsilon \frac{d^2\Phi}{dX^2} = e^{\Phi} - N, \tag{11}$$

$$X = 0: \qquad Be^{\Phi} = N\sqrt{\frac{d\Phi}{dX}}, \tag{12}$$

$$X = 1: \qquad \Phi = 0, \qquad \frac{d\Phi}{dX} = 0, \qquad \frac{dN}{dX} = 0, \tag{13}$$

where

$$\varepsilon = \left(\frac{\lambda_D}{L}\right)^2, \qquad Z = \frac{\alpha L}{A}\sqrt{\frac{eL}{kT_e}}, \qquad B = \frac{\bar{C}_e}{4\Lambda}\sqrt{\frac{eL}{kT_e}}. \tag{14}$$

Here $\lambda_D = \sqrt{\varepsilon_0 kT_e/n_{e0}e^2}$ is the Debye length at the center of the quasineutral plasma.

The stated problem involves two dimensionless control parameters: ε and B. ε is the squared ratio of the Debye length to the characteristic dimension, and is usually much smaller than unity. B is equal, to the accuracy of a factor, to $\sqrt{m_i L/m_e \lambda_i}$; since $m_i/m_e \gg 1$ and (in the conditions considered in this work) $L \gg \lambda_i$, $B \gg 1$. It is the aim of this work to develop an asymptotic solution in which ε is considered as a small parameter and B as a large parameter. Use will be made of the method of matched asymptotic expansions (see, e.g., [19–23]).

Analysis shows that the asymptotic structure of the solution depends on the relative orders of magnitude of B and $\varepsilon^{-1/5}$. In the next section, a solution will be obtained for the case $B = O(\varepsilon^{-1/5})$. As is shown in Sec. 4, a solution describing the space-charge sheath in this limiting case applies also to the case $B \gg \varepsilon^{-1/5}$. Results for the case $1 \ll B \ll \varepsilon^{-1/5}$ are given in the Appendix, but this ordering requires smaller Debye lengths than usually occur.

3. Asymptotic treatment

We seek an asymptotic expansion of the (dimensionless) eigenvalue in the form

$$Z(\varepsilon, B) = Z_1 + \dots \quad . \tag{15}$$

The straightforward (outer) expansion of the unknown functions has the form

$$N(X;\varepsilon, B) = N_1(X) + \dots, \qquad \Phi(X;\varepsilon, B) = \Phi_1(X) + \dots, \tag{16}$$

and applies at $0 < X \leqslant 1$. Substituting this expansion into (11) and (13), expanding and retaining leading terms, one arrives at

$$\frac{d}{dX}\left(N_1\sqrt{\frac{d\Phi_1}{dX}}\right) = -Z_1 e^{\Phi_1}, \qquad N_1 = e^{\Phi_1}. \tag{17}$$

$$X = 1: \qquad \Phi_1 = 0, \qquad \frac{dN_1}{dX} = 0. \tag{18}$$

Taking into account the second equation in (17), one can see that the second boundary condition in (13) is a consequence of the last one, and may therefore be dropped.

The boundary condition at $X = 0$ should be formulated is such a way as to make possible matching with an inner expansion. By analogy with the treatment for the case of low field [2, 4, 5], one can expect that this condition is (see also the discussion in [24])

$$X = 0: \qquad N_1 = 0. \tag{19}$$

Now the problem (17)–(19) is closed and may be solved. Eliminating the potential from (17) gives

$$\frac{d}{dX}\left(\sqrt{N_1\frac{dN_1}{dX}}\right) = -Z_1 N_1. \tag{20}$$

This equation may be transformed to a first-order equation:

$$N_1 Y^{1/2}\frac{dY}{dN_1} + Y^{3/2} + 2Z_1 N_1^{3/2} = 0, \tag{21}$$

where $Y = dN_1/dX$. The solution to this equation, subject to the boundary condition $Y|_{N_1=1} = 0$, is

$$Y = Z_1^{2/3}(N_1^{-3/2} - N_1^{3/2})^{2/3}. \tag{22}$$

Applying the boundary condition (19), one finds

$$X = \int_0^{N_1} \frac{N_1\, dN_1}{Z_1^{2/3}(1 - N_1^3)^{2/3}}. \tag{23}$$

Furthermore the first boundary condition (18) now gives

$$Z_1 = (2\pi)^{3/2} 3^{-9/4}. \tag{24}$$

Thus, the first term of the straightforward expansion has been found. Its asymptotic behavior at $X \to 0$ may be found to be

$$N_1 = Z_1^{1/3}\sqrt{2X} + \dots, \qquad \Phi_1 = \tfrac{1}{2}\ln X + \ln(\sqrt{2}Z_1^{1/3}) + \dots \quad . \tag{25}$$

Since the function Φ_1 has a singularity at the point $X = 0$, the straightforward expansion (16) does not apply in the vicinity of this point. In order to describe a solution in this vicinity, one should consider another (inner) expansion. The form of this expansion may be found out as follows. The term on the left-hand side of the second equation in (11) increases proportionally to $1/X^2$ as $X \to 0$, while the terms on the right-hand side decrease proportionally to $\sqrt{X}$. It follows that at $X = O(\varepsilon^{2/5})$, the term on the left-hand side of the second equation in (11), which has been dropped in the above analysis, becomes comparable to the terms on the right-hand side, and therefore the above analysis breaks down. Thus, one should introduce an inner expansion associated with the variable $X_2 = X/\varepsilon^{2/5}$ and applicable at $X_2 \geqslant 0$. (This implies that the scale of the sheath region is $\varepsilon^{2/5}L$, i.e., $\lambda_D^{4/5}L^{1/5}$.) Taking (25) into account, one should seek this expansion in the form

$$N(X;\varepsilon,B) = \varepsilon^{1/5}N_2(X_2;B_1) + \dots, \qquad \Phi(X;\varepsilon,B) = \tfrac{1}{5}\ln\varepsilon + \Phi_2(X_2;B_1) + \dots, \tag{26}$$

where $B_1 = \varepsilon^{1/5}B = O(1)$.

To a first approximation, (11) assumes the form

$$\frac{d}{dX_2}\left(N_2\sqrt{\frac{d\Phi_2}{dX_2}}\right) = 0, \qquad \frac{d^2\Phi_2}{dX_2^2} = e^{\Phi_2} - N_2. \tag{27}$$

A boundary condition at the surface is supplied by (12):

$$X_2 = 0: \qquad B_1 e^{\Phi_2} = N_2\sqrt{\frac{d\Phi_2}{dX_2}}. \tag{28}$$

Boundary conditions at $X_2 \to \infty$ are obtained by matching with the outer expansion, and read

$$N_2 = Z_1^{1/3}\sqrt{2X_2} + \dots, \tag{29}$$

$$\Phi_2 = \tfrac{1}{2}\ln X_2 + \ln(\sqrt{2}Z_1^{1/3}) + \dots \quad . \tag{30}$$

Integrating the first equation in (27) and making use of the boundary conditions (29) and (30), one finds

$$N_2 = Z_1^{1/3}\left(\frac{d\Phi_2}{dX_2}\right)^{-1/2}. \tag{31}$$

Substituting this expression into the second equation in (27), one obtains an equation for the potential:

$$\frac{d^2\Phi_2}{dX_2^2} = e^{\Phi_2} - Z_1^{1/3}\left(\frac{d\Phi_2}{dX_2}\right)^{-1/2}. \tag{32}$$

The boundary condition at $X_2 \to \infty$ is supplied by (30); the boundary condition at the wall follows from (28):

$$X_2 = 0: \qquad \Phi_2 = \ln\frac{Z_1^{1/3}}{B_1}. \tag{33}$$

Now the boundary-value problem for the function Φ_2 is closed and may be solved. Since no analytical solution exists, numerical calculations are necessary. Before applying a numerical treatment, it is convenient to introduce new variables:

$$X_3 = (2Z_1^{2/3})^{1/5}(X_2 + 2C), \qquad \Phi_3 = \Phi_2 - \tfrac{2}{5}\ln(2Z_1^{2/3}), \tag{34}$$

where C is a constant that will be specified at a later stage. Equation (32) assumes

https://doi.org/10.1017/S0022377801001556 Published online by Cambridge University Press

the form

$$\frac{d^2\Phi_3}{dX_3^2} = e^{\Phi_3} - \left(2\frac{d\Phi_3}{dX_3}\right)^{-1/2}. \tag{35}$$

In order to derive a convenient initial condition for numerical calculations, we find higher-order terms in the expansion (30), thus giving results analogous to those for the case of a constant collision frequency given in the appendix of [5] and in [25]. We shall seek the asymptotic expansion of the function $\Phi_3(X_3)$ at $X_3 \to \infty$ in the form

$$\Phi_3 = \tfrac{1}{2}\ln X_3 + \frac{C_1}{X_3^{p_1}} + \frac{C_2}{X_3^{p_2}} + \dots, \tag{36}$$

where $C_1, p_1, C_2, p_2, \dots$ are constants to be determined, $1 < p_1 < p_2 < \dots$. Note that (35) is invariant with respect to a shift of the independent variable, and such a shift results, in a general case, in appearance of a term in $1/X_3$ in this expansion. We assume that the constant C is chosen in such a way that this term is absent (i.e., $p_1 > 1$).

Substituting the expansion (36) into (35), one readily finds that $p_1 = \frac{5}{2}$, $p_2 = 5$, $p_3 = \frac{15}{2}$, etc. Note that the analogous quantities in [5, 25] were 3, 6, 9, etc. This difference stems from the fact that the exponent of the second term on the right-hand side of (35) is $-\frac{1}{2}$ rather than -1, as in [5, 25].

After considerable manipulation one finds that

$$C_1 = \frac{1}{3}, \qquad C_2 = -\frac{281}{288}, \qquad C_3 = \frac{82\,733}{33\,696}. \tag{37}$$

Using (36) and (37), one can find, to any desired accuracy, values of Φ_3 and of $d\Phi_3/dX_3$ at large enough X_3. After that, one can numerically integrate (35) in the direction of decreasing X_3. The integration stops when Φ_3 reaches the value $\ln(Z_1^{1/15}/2^{2/5}B_1)$, which determines the value of X_3 at the wall and hence the constant C in (34).

Within this approach, the sheath is described by the same curve $\Phi_3(X_3)$ for all B_1, the effect of B_1 manifesting itself only through the position of the wall. This generic solution is given in Fig. 1, along with the generic solutions for $d\Phi_3/dX_3$, $N_3 = (2\,d\Phi_3/dX_3)^{-1/2}$ and $N_{e3} = \exp(\Phi_3)$. At large distances from the wall, the charged particle densities from (25) are proportional to $X^{1/2}$, and this is seen to be so in Fig. 1. In other words, the behavior for large X_3 matches the 'plasma edge' behavior of the plasma solution given by (25). Note that Fig. 1 is analogous to Fig. 2 of [25] and has the same general properties. It conforms also to the results of numerical treatment [7].

4. Discussion

The asymptotic structure of the solution obtained is similar to that in the case of a low electric field [2, 4, 5]: there is a region of the quasineutral plasma, which occupies the bulk of the tube and is described by the asymptotic expansion (16), and an asymptotically thin space-charge sheath, which is adjacent to the wall and is described by the expansion (26), the charged-particle density and the electric field in the sheath being, respectively, much lower and much higher than those in the quasineutral plasma. On the other hand, it follows from (25) that the density of the charged particles in the quasineutral plasma approaches the wall with an

https://doi.org/10.1017/S0022377801001556 Published online by Cambridge University Press

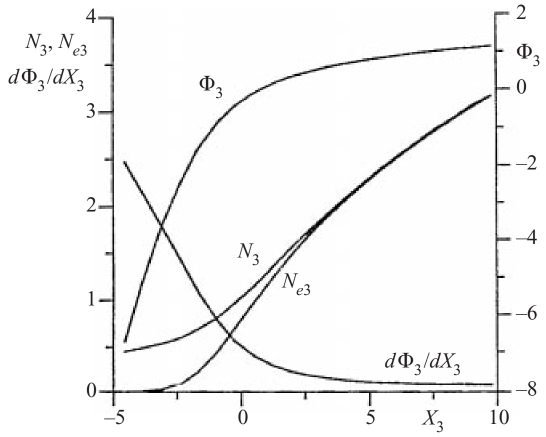

Figure 1. Generic solution describing the space-charge sheath. Φ_3 is the (normalized) potential, $d\Phi_3/dX_3$ is the electric field, N_3 is the ion density, and N_{e3} is the electron density. The wall is located at the point where $\Phi_3 = \ln(Z_1^{1/5}/2^{2/5}B_1)$ i.e. its position in terms of X_3 varies with the mass ratio of ions and electrons and the ratio of the plasma dimension to the ion mean free path. The densities tend to a common value for large X_3, with a parabolic increase with distance into the plasma. The electric field goes to zero in the plasma and increases linearly near the wall.

infinite derivative. This asymptotic behavior is different from that in the case of a low electric field, and results in different scalings of parameters in the sheath (see the paragraph after (25)): while the sheath thickness, the charged-particle density, and the electric field in the sheath in the case of low field are of the orders of, respectively, $\varepsilon^{1/3}$, $\varepsilon^{1/3}$, and $\varepsilon^{-1/3}$ [2, 4, 5], in the case considered in this work, the respective orders are $\varepsilon^{2/5}$, $\varepsilon^{1/5}$, and $\varepsilon^{-2/5}$. Note that the dimensional scale of sheath thickness is $\delta = \lambda_D^{2/3} L^{1/3}$ in the case of low electric field and $\delta = \lambda_D^{4/5} L^{1/5}$ in the case considered in this work.

The third term of the expansion (30) is C/X_2. This term will generate a term of the order of $\varepsilon^{2/5}$ in the outer expansion. It follows that the second term of the outer asymptotic expansion (16) is of the order of $\varepsilon^{2/5}$.

The expansion of the dimensionless eigenvalue is

$$Z(\varepsilon, B) = (2\pi)^{3/2} 3^{-9/4} + O(\varepsilon^{2/5}). \tag{38}$$

A second approximation may be found from analysis of the second term of the outer expansion; cf. the calculation of the current–voltage characteristic of an electrostatic probe [4]. Alternatively, the eigenvalue in the second approximation can be determined as follows. Replacing the independent variable in the outer expansion by the variable $\tilde{X} = X + 2\varepsilon^{2/5}C$, one can eliminate from this expansion a term of the order of $\varepsilon^{2/5}$. The resulting expansion breaks down at $\tilde{X} = 0$ or, which is the same, at $X = -2\varepsilon^{2/5}C$. It follows that the region of the quasineutral plasma, while having thickness L in the first approximation, has thickness $L(1 + 2\varepsilon^{2/5}C)$ in the second approximation. Hence, in order to find a second-approximation formula for the dimensional eigenvalue, it is sufficient to write the respective formula in the first approximation,

$$\alpha = \frac{(2\pi)^{3/2} A}{3^{9/4} L} \sqrt{\frac{kT_e}{eL}}, \tag{39}$$

and to replace L in the obtained relationship by $L(1 + 2\varepsilon^{2/5}C)$.

The relationship (39) is the plasma balance equation relating the volume generation rate to the wall loss rate [26]. This is equivalent to the result obtained by Schottky [27] for the constant-mobility case, namely, $\alpha = (\pi^2/L^2)\mu_i kT_e/e$.

The above asymptotic analysis refers to the case $LT_a/T_e \ll \lambda_i \ll \delta$, when the electric field is high and (7) is applicable both in the region of quasineutral plasma and in the sheath. If a less restrictive inequality holds, $\delta T_a/T_e \ll \lambda_i \ll \delta$, the electric field in the region of quasineutral plasma is not necessarily high, and the solution (23), (24) is, generally speaking, invalid. However, the electric field is still high and (7) is applicable in the sheath; hence the solution obtained above for the sheath remains applicable, the difference being that Z_1 is no longer given by (24). Similarly, the sheath solution remains applicable to a cylindrical (rather than planar) discharge column.

The above results refer to the case $B = O(\varepsilon^{-1/5})$. In the case $B \gg \varepsilon^{-1/5}$, the space-charge sheath becomes non-uniform and includes the ion–electron layer, which is an outer section of the sheath where the ion and electron densities are comparable, and the ion layer, where the electron density is much smaller than the ion density. The ion–electron layer has an asymptotic structure similar to that of the (uniform) space-charge sheath considered above. In particular, the ionization in the ion–electron layer is a minor effect due to the thinness of the layer. The ion layer is not necessarily asymptotically thin; however, the ionization is still a minor effect in it due to the very low values of the electron density. Since the only approximation made when writing (27) was the neglect of the ionization, one may conclude that the above-obtained sheath solution remains applicable also in the case $B \gg \varepsilon^{-1/5}$, when the sheath is non-uniform.

Results for the case $1 \ll B \ll \varepsilon^{-1/5}$ follow from the analysis given in the Appendix. In particular, it follows that the plasma remains quasineutral right up to the wall, i.e., there is no space-charge sheath in this case. The eigenvalue to a first approximation equals $(2\pi)^{3/2}3^{-9/4}$. It follows that the formula $Z(\varepsilon, B) = (2\pi)^{3/2}3^{-9/4}$ is to a first approximation uniformly valid in the whole range $1 \ll B \leqslant O(\varepsilon^{-1/5})$.

Acknowledgements

This work was performed within activities of Project 32411/99 of the program POCTI of Fundação para a Ciência e a Tecnologia and FEDER.

Appendix. Solution for the case $O(1) \leqslant B < 0(\varepsilon^{-1/5})$

The aim of this appendix is to present, a first-approximation asymptotic solution for the limiting case $\varepsilon \to 0$, $B \to \infty$, provided that $B\varepsilon^{1/5} \to 0$. The simplest way to do this is to first treat the limiting case $\varepsilon \to 0$, B fixed and then set in the obtained solution $B \to \infty$.

Equations (15)–(18) in the limiting case $\varepsilon \to 0$, B fixed remain valid, the difference being that $Z_1 = Z_1(B)$, $N_1 = N_1(X; B)$, and $\Phi_1 = \Phi_1(X; B)$ in this case. The boundary condition for the straightforward expansion at $X = 0$ can be obtained from (12), and reads

$$X = 0: \quad \frac{d\Phi_1}{dX} = B^2. \tag{A 1}$$

https://doi.org/10.1017/S0022377801001556 Published online by Cambridge University Press

Equations (20)–(22) remain valid. The solution (23) is replaced by

$$X = \int_{N_{1w}}^{N_1} \frac{N_1\, dN_1}{Z_1^{2/3}(1 - N_1^3)^{2/3}}, \tag{A 2}$$

where $N_{1w} = N_1(0; B)$ is defined by the equation

$$N_{1w}^3 = \frac{Z_1}{B^3 + Z_1}. \tag{A 3}$$

The eigenvalue $Z_1 = Z_1(B)$ is defined by the equation

$$Z_1^{2/3} = \int_{N_{1w}}^{1} \frac{N_1\, dN_1}{(1 - N_1^3)^{2/3}}, \tag{A 4}$$

which should be solved jointly with (A3).

Thus, a first-approximation straightforward asymptotic solution for the limiting case $\varepsilon \to 0$, B fixed has been completed. This solution has no singularity at the surface, and therefore remains applicable in the whole calculation domain. In other words, the plasma remains quasineutral in the whole discharge vessel right up to the walls.

Now one can set $B \to \infty$ in the obtained solution. The expansion of the eigenvalue Z_1 may be found to be

$$Z_1(B) = \frac{(2\pi)^{3/2}}{3^{9/4}} \left(1 - \frac{3}{4B^2} + \dots\right). \tag{A 5}$$

Furthermore, one finds

$$N_{1w} = O\left(\frac{1}{B}\right), \qquad \frac{d^2\Phi_1}{dX^2}(0; B) = O(B^4). \tag{A 6}$$

It follows that the term on the left-hand side of the second equation in (11) remains small at the wall as compared with the terms on the right-hand side, provided that $\varepsilon B^5 \ll 1$. In other words, the above analysis performed for the limiting case $\varepsilon \to 0$, B fixed remains applicable, (at least to a first approximation) to the case $\varepsilon \to 0$, $B \to \infty$, provided that B tends to infinity slower than $\varepsilon^{-1/5}$.

References

[1] Su, C. H. and Lam, S. H., *Phys. Fluids* **6**, 1479 (1963).
[2] Cohen, I. M., *Phys. Fluids* **6**, 1492 (1963).
[3] Bush, W. B. and Fendell, F. E., *J. Plasma Phys.* **4**, 317 (1970).
[4] Benilov, M. S., *Fluid Dyn.* **17**, 773 (1982).
[5] Blank, J. L., *Phys. Fluids* **11**, 1686 (1968).
[6] Mason, E. A. and McDaniel, E. W., *Transport Properties of Ions in Gases*. New York: Wiley, 1988.
[7] Franklin, R. N. and Snell, J., *Phys. Plasmas* **8**, 643 (2001).
[8] Franklin, R. N. and Snell, J., *Phys. Plasmas* **7**, 3077 (2000).
[9] Franklin, R. N. and Ockendon, J. R., *J. Plasma Phys.* **4**, 371 (1970).
[10] Riemann, K. U., *Phys. Plasmas* **4**, 4158 (1997).
[11] Riemann, K. U. and Meyer, P., *Phys. Plasmas* **3**, 4751 (1996).
[12] Benilov, M. S., *Phys. Plasmas* **7**, 135 (2000).
[13] Benilov, M. S. and Shukla, P. K., *Phys. Rev.* **E63**, 016410-1 (2000).
[14] Baritello, V., Porcelli, F. and Subba, F., *Phys. Rev.* **E60**, 4733 (1999).

[15] Franklin, R. N., Daniels, P. G. and Snell, J., *J. Phys. D: Appl. Phys.* **26**, 1638 (1993).
[16] Goswani, M. and Ramachandran, H., *Phys. Plasmas* **6**, 4522 (1999).
[17] Benilov, M. S., *Phys. Plasmas* **7**, 4403 (2000).
[18] Franklin, R. N., *Plasma Phenomena in Gas Discharges*. Oxford: Clarendon Press, 1976.
[19] Van Dyke, M., *Perturbation Methods in Fluid Mechanics*. New York: Academic Press, 1964.
[20] Cole, J. D., *Perturbation Methods in Applied Mathematics*. Waltham, MA: Blaisdell, 1968.
[21] Nayfeh, A. H., *Perturbation Methods*. New York: Wiley, 1973.
[22] Nayfeh, A. H., *Introduction to Perturbation Techniques*. New York: Wiley, 1981.
[23] Nayfeh, A. H., *Problems in Perturbation*. New York: Wiley, 1985.
[24] Benilov, M. S. and Naidis, G. V., *Phys. Rev.* **E57**, 2230 (1998).
[25] Franklin, R. N. and Snell, J., *J. Phys. D: Appl. Phys.* **30**, L15 (1997).
[26] Tonks, L. and Langmuir, I., *Phys. Rev.* **34**, 876 (1929).
[27] Schottky, W., *Phys. Z.* **25**, 635 (1924).