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IRREDUCIBLE AUTOMORPHISMS OF F_n HAVE NORTH–SOUTH DYNAMICS ON COMPACTIFIED OUTER SPACE

GILBERT LEVITT[1] AND MARTIN LUSTIG[2]

[1]*Laboratoire Émile Picard, UMR CNRS 5580, Université Paul Sabatier, 31062 Toulouse Cedex 4, France* (levitt@picard.ups-tlse.fr)

[2]*Laboratoire de Mathématiques Fondamentales et Appliquées, Université d'Aix-Marseille III, 13397 Marseille Cedex 20, France* (Martin.Lustig@math.u-3mrs.fr)

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Abstract We show that if an automorphism of a non-abelian free group F_n is irreducible with irreducible powers, it acts on the boundary of Culler–Vogtmann's outer space with north–south dynamics: there are two fixed points, one attracting and one repelling, and orbits accumulate only on these points. The main new tool we use is the equivariant assignment of a point $Q(X)$ to any end $X \in \partial F_n$, given an action of F_n on an $\boldsymbol{R}$-tree T with trivial arc stabilizers; this point $Q(X)$ may be in T, or in its metric completion, or in its boundary.

Keywords: free group; automorphism; outer space; tree; pseudo-Anosov; irreducible

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Secondary

1. Introduction

In [**8**], Culler and Vogtmann introduced a space of 'moduli on marked graphs'. This space, CV_n, now also known under the name of *outer space*, is finite dimensional, contractible and, most importantly, its spine admits a discrete co-compact action with finite point stabilizers of $\mathrm{Out}\, F_n$, the group of outer automorphisms of the non-abelian free group F_n (see the survey [**22**]). The emerging picture has a striking analogy with the action of the mapping class group Γ_g on the Teichmüller space $\mathcal{T}_g$—the only really major difference being that $\mathcal{T}_g$ is a manifold, while CV_n is only a simplicial complex (with some faces of certain simplices missing).

In [**2**], Bestvina and Handel, inspired by Thurston's work on surface homeomorphisms, introduced a new basic tool into the theory of free group automorphisms. They developed the notion of *train track maps*, and they used it to prove deep facts (like the Scott Conjecture) about automorphisms of F_n. One of their fundamental contributions is the introduction of *irreducible* automorphisms of F_n, an analogue of Thurston's pseudo-Anosov homeomorphisms of a closed surface S_g of genus g. An automorphism $\alpha \in \mathrm{Aut}\, F_n$

(or its image $\hat{\alpha} \in \operatorname{Out} F_n$) is *irreducible with irreducible powers* (IWIP) if no proper free factor H of F_n is mapped by any positive power of α to a conjugate of H.

One of the crucial innovative contributions of Thurston to surface theory is his boundary $\partial \mathcal{T}_g$, which compactifies $\mathcal{T}_g$ to give topologically a closed ball $\bar{\mathcal{T}}_g$ of dimension $6g - 6$. Using Skora's Theorem [**21**], we can now describe $\partial \mathcal{T}_g$ as the projectivized space $SLF(\pi_1 S_g)$ of small actions of $\pi_1 S_g$ on $\boldsymbol{R}$-trees. In one of the first papers on $\boldsymbol{R}$-trees [**7**], Culler and Morgan have shown that for any finitely generated group G the space $SLF(G)$ is compact.

Now, considering the special case $G = F_n$, and observing that CV_n embeds naturally into $SLF(F_n)$, it is natural to consider the closure $\overline{CV_n}$ of this image. It consists of projective classes $[T]$ of very small actions of F_n on $\boldsymbol{R}$-trees T (see [**22**]). The *Thurston boundary* ∂CV_n of CV_n is defined as the difference $\overline{CV_n} \setminus CV_n$. As in the mapping class group case, the action of $\operatorname{Out} F_n$ on CV_n extends canonically to an action on $\overline{CV_n}$.

It is well known that the action of a pseudo-Anosov automorphism on $\bar{\mathcal{T}}_g$ has *north–south dynamics*: there are precisely two fixed points, both in $\partial \mathcal{T}_g$, one is repelling and the other is attracting, and every other orbit in $\bar{\mathcal{T}}_g$ has the attractor as forward and the repellor as backward limit. The main result of this paper is that the precise analogy is true for IWIP automorphisms of F_n. It answers a question asked in [**4**].

Theorem 1.1. *Let $\alpha \in \operatorname{Aut} F_n$ be irreducible with irreducible powers. Its action on the closure $\overline{CV_n}$ of outer space has north–south dynamics: there exist $[T^+], [T^-] \in \partial CV_n$ such that $\alpha^p([T])$ converges (locally uniformly) to $[T^+]$ as $p \to +\infty$ for all $[T] \neq [T^-]$, and $\alpha^{-p}([T]) \to [T^-]$ for all $[T] \neq [T^+]$.*

It was known beforehand (see [**1, 14, 18, 20**]) that every $\alpha \in \operatorname{Aut} F_n$ has at least one fixed point in $\overline{CV_n}$. It has been shown in [**15, 16**] that IWIP automorphisms have precisely two fixed points, both in ∂CV_n, and in [**4, 13**] that the orbits in the 'interior' CV_n all do converge from the repellor to the attractor. The analogous result for orbits on ∂CV_n, however, turns out to be quite difficult. The main new tool we use is the equivariant assignment of a point $Q(X)$ to any end $X \in \partial F_n$, given an action of F_n on an $\boldsymbol{R}$-tree T with trivial arc stabilizers; this point $Q(X)$ may be in T, or in its metric completion $\bar{T}$, or in the boundary of T (see § 3).

One may wonder about the action of an arbitrary automorphism on ∂CV_n. See [**3, 6**] for dynamics of polynomially growing automorphisms. It is tempting to conjecture that $\operatorname{Out} F_n$ acts on ∂CV_n with *uniformly finite limit sets*: there exists a constant K (depending only on n) such that, for every $\alpha \in \operatorname{Aut} F_n$ and every $[T] \in \partial CV_n$, the sequence $\alpha^p([T])$ has at most K limit points. The analogous statement for the action of Γ_g on $\partial \mathcal{T}_g$ follows easily from Nielsen–Thurston theory (a detailed exposition appears in [**17**]).

In §§ 2 and 3 we consider an arbitrary $\boldsymbol{R}$-tree T with dense orbits and trivial arc stabilizers. We show (corollary 2.3) that there exist F_n-equivariant maps $f : T_0 \to T$, with T_0 a free simplicial F_n-tree, having arbitrarily small backtracking. This allows us in § 3 to associate a point $Q(X)$ to every end $X \in \partial F_n$. These constructions are quite general and may have applications elsewhere.

In § 4 we recall basic facts about train tracks, laminations, trees. In § 5 we prove that there exists a leaf of one of the two α-invariant laminations whose ends X, X' satisfy $Q(X) \neq Q(X')$. In § 6 we deduce the main theorem from this key fact, using a convergence criterion due to Bestvina *et al*. [**4**]. For the convenience of the reader, we provide a proof of this criterion in § 7, with more details than in [**4**].

2. Maps with small backtracking

Let T_0 be a simplicial $\boldsymbol{R}$-tree with a free isometric action of F_n. We denote $\pi : T_0 \to T_0/F_n$ the quotient map and we let $v(T_0)$ be the total length of the graph T_0/F_n.

Let T be an $\boldsymbol{R}$-tree with an isometric action of F_n. In this paper we always assume the action to be very small. We usually assume that the action on T is minimal. But we will also need to consider the metric completion $\bar{T}$ of T, which is minimal if and only if T is simplicial.

In T_0, T and $\bar{T}$, we write $d(P, Q)$ or $|PQ|$ for the distance between two points (length of the segment PQ). We also write $|e|$ for the length of an edge of T_0.

We consider F_n-equivariant maps $f : T_0 \to T$, often requiring that the restriction of f to each edge be isometric, or linear. Note that f is necessarily onto if T is minimal. A segment $PQ \subset T_0$ is *f-backtracking* if $f(P) = f(Q)$.

We say that $f : T_0 \to T$ has the *bounded backtracking property* (BBT) if there exists a constant $C \geqslant 0$ such that the f-image of any segment $PQ \subset T_0$ is contained in the C-neighbourhood of the segment $f(P)f(Q) \subset T$. The smallest such C is the BBT-*constant of f, denoted* $\mathrm{BBT}(f)$. Note that $\mathrm{BBT}(f)$ does not depend on the metric on T_0. Also note that the image of any f-backtracking segment has diameter at most $2\,\mathrm{BBT}(f)$.

We note the following fact (see [**4**, **9**, **11**]).

Lemma 2.1. *Let T be an $\boldsymbol{R}$-tree with a minimal very small action of F_n. Let T_0 be a free simplicial F_n-tree, and $f : T_0 \to T$ an equivariant map isometric on edges. Then f has bounded backtracking, with* $\mathrm{BBT}(f) \leqslant v(T_0)$.

In the rest of this section, and in the next one, we consider a very small F_n-tree T such that some (hence every) orbit is dense. Such a tree has trivial arc stabilizers (as will be recalled in Lemma 4.2).

Proposition 2.2. *Let T be a minimal F_n-tree with dense orbits and trivial arc stabilizers. Given $\varepsilon > 0$, there exists a free simplicial F_n-tree T_0 with $v(T_0) < \varepsilon$, and an equivariant map $f : T_0 \to T$ whose restriction to each edge is isometric.*

Proof. Let $f : T_0 \to T$ be equivariant and isometric on edges. It suffices to show that, given an edge e of T_0, we may replace f by $f' : T_0' \to T$ with $v(T_0') \leqslant v(T_0) - \frac{1}{6}|e|$. By rescaling, we may assume $|e| = 1$.

We first show that *there exists an f-backtracking segment $PQ \subset T_0$ meeting e in a subsegment PR of length $\frac{1}{3}$* (where R is an endpoint of e and P is one of the two points trisecting e). Let M be the midpoint of e. Since orbits in T are dense, there exists a non-trivial $g \in F_n$ such that $d(f(M), f(gM)) \leqslant \frac{1}{6}$. We let P be the point of e located at distance $\frac{1}{6}$ from M on the side of gM, and Q be a point between P and gM such that

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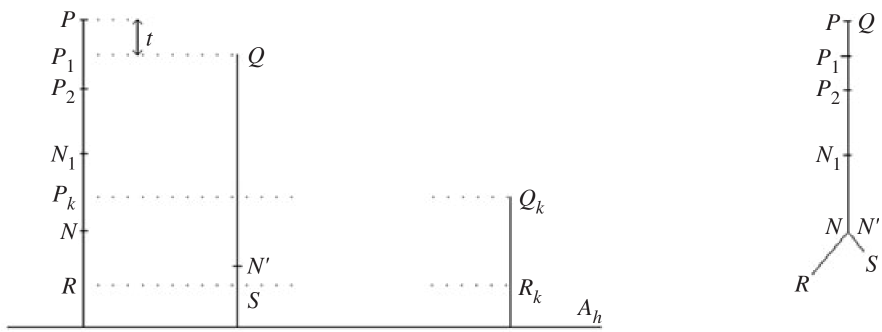

Figure 1.

$f(Q) = f(P)$ (such a Q exists since T is a tree). Note that every interior point x of PR bounds an f-backtracking segment xy with $y \in RQ$.

Consider all segments PQ as above. Since T_0 is a locally finite simplicial tree, we can choose one of minimal length. The easy case is when the interior of RQ does not meet any he, with $h \in F_n$ non-trivial. For then we can redefine f on the orbit of RP and equivariantly fold RP over RQ, obtaining $f' : T_0' \to T$ with $v(T_0') = v(T_0) - \frac{1}{3}$.

Suppose that RQ entirely contains some he. Since every interior point of he bounds an f-backtracking subsegment of PQ, we can replace PQ by a shorter f-backtracking segment, contradicting the choice of PQ.

We therefore reduce to the following situation: RQ intersects some he in a subsegment SQ, and the interior of RS is disjoint from the orbit of e. Minimality of PQ implies $|SQ| \leqslant \frac{1}{3}$ (otherwise there would be a shorter backtracking segment starting at the point of SQ at distance $\frac{1}{3}$ from S). Orient e and he by choosing an orientation of PQ.

First suppose that h maps e to he in an orientation-preserving way (i.e. $hR \neq S$). Then $h(RP)$ does not meet RQ and we can fold. Now suppose that h reverses orientation. Then the interior of PR is disjoint from the translation axis A_h of h, and for every integer $k \geqslant 1$ the intersection of the segment $Ph^k(P)$ with the orbit of e is the union of the two segments PR and $h^k(PR)$.

Note that no interior point Q' of PQ is mapped by f onto $f(P) = f(Q)$. This implies that $f(PR)$ and $f(QS)$ intersect along a non-degenerate arc $f(PN) = f(QN')$.

If $|NR| \geqslant \frac{1}{6}$ (in particular, if $|SQ| \leqslant \frac{1}{6}$), we choose a point $N'' \in RS$ with $f(N'') = f(N)$ and we fold RN over RN'', obtaining T_0' with $v(T_0') = v(T_0) - |NR| \leqslant v(T_0) - \frac{1}{6}$.

Assume therefore $|NR| < \frac{1}{6} < |SQ|$ (see figure 1, showing points in T_0 on the left, and their images by f on the right). Let $t = \frac{1}{3} - |SQ|$ (recall that t is non-negative). Let $P_1 = h^{-1}(Q)$ be the point of PR at distance t from P. Then h maps RP_1 onto SQ in T_0, and $f(RP_1)$ onto $f(SQ)$ in T. In particular, h acts as a translation by t on $f(PN)$. This implies $t > 0$, since otherwise h would fix the non-degenerate segment $f(PN)$. Let $k \geqslant 2$ be the smallest integer such that $kt \geqslant \frac{1}{6}$.

For $1 \leqslant i \leqslant k$, let P_i be the point of PR at distance it from P. We have $f(hP_i) = f(P_{i-1})$ and therefore $f(h^iP_i) = f(Q)$. We let $Q_k = h^kP_k$ and $R_k = h^kR$.

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Then $f(Q_k) = f(h^k P_k) = f(Q)$, and so PQ_k is an f-backtracking segment. As pointed out earlier, the intersection of PQ_k with the orbit of e consists of PR and Q_kR_k. Now $|Q_kR_k| = |P_kR| = \frac{1}{3} - |PP_k| \leqslant \frac{1}{6}$, and therefore $|f(Q)f(R_k)| \leqslant \frac{1}{6}$. If N_1 denotes the midpoint of PR, then $f(N_1)$ separates $f(R)$ from $f(R_k)$. We conclude as above, choosing $N'' \in RR_k$ with $f(N'') = f(N_1)$ and folding RN_1 over RN''. □

Lemma 2.1 and Proposition 2.2 immediately imply the following result.

Corollary 2.3. *Let T be a minimal F_n-tree with dense orbits and trivial arc stabilizers. Given $\varepsilon > 0$, there exists a free simplicial F_n-tree T_0 and an equivariant map $f : T_0 \to T$ with* $\mathrm{BBT}(f) < \varepsilon$.

Remark 2.4. This corollary may be extended to arbitrary actions with trivial arc stabilizers, but not to arbitrary very small actions (see Remark 3.3).

Corollary 2.5. *Let T be as above. Given $P \in T$ and $\varepsilon > 0$, there exists a basis $\{a_1, \dots, a_n\}$ of F_n such that $\sum_{i=1}^{n} d(P, a_iP) < \varepsilon$.*

Proof. Take $f : T_0 \to T$, with $v(T_0) < \varepsilon/2n$. Fix $P_0 \in T_0$ with $f(P_0) = P$, and choose a basis of $\pi_1(T_0/F_n, \pi(P_0))$ represented by loops of length less than $2v(T_0)$. □

This corollary readily extends to points P in the completion $\bar{T}$. Combined with Lemma 2.1, it implies the following.

Remark 2.6. Given $P \in \bar{T}$ and $\varepsilon > 0$, there exists a Cayley tree Γ of F_n and $f : \Gamma \to \bar{T}$ with $\mathrm{BBT}(f) < \varepsilon$ sending the vertex g to gP.

3. The point $Q(X)$

Given an $\boldsymbol{R}$-tree S, we define ∂S as the set of equivalence classes of rays $\rho : [0, +\infty) \to S$, where ρ is an isometric map and two rays are equivalent if their images have infinite intersection. If S is a simplicial tree with a free F_n-action, there is a canonical identification of ∂S with ∂F_n.

Now fix T (as in Proposition 2.2) and $X \in \partial F_n$. We always denote by $f : T_0 \to T$ an equivariant map with T_0 free simplicial. We represent X by a ray ρ in T_0 and we consider $r = f \circ \rho$. We will usually confuse a ray ρ and its image, and similarly r and its image, thus writing $r = f(\rho)$.

We say that X is T-*bounded* if r is bounded in T (this does not depend on the choice of f, as follows from [**11**, § 3] or from the next proof).

If r is not bounded, then clearly it lies in the $\mathrm{BBT}(f)$-neighbourhood of a ray ρ'. This situation was studied in [**11**], with the notation $X = j(\rho')$. Here we shall go in the other direction, defining $Q(X)$ as the point of ∂T corresponding to ρ'.

Now we consider the case when r is bounded.

Proposition 3.1. *Let T be a minimal F_n-tree with dense orbits and trivial arc stabilizers. Suppose $X \in \partial F_n$ is T-bounded. Then there exists a unique point $Q(X) \in \bar{T}$ such that, for any $f : T_0 \to T$ and any ray ρ representing X in T_0, the point $Q(X)$ belongs* *to the closure of* $f(\rho)$ *in* $\bar{T}$*. Furthermore, every* $f(\rho)$ *is contained in the* $2\,\mathrm{BBT}(f)$*-ball centred at* $Q(X)$*, except for an initial part.*

Recall that $\bar{T}$ denotes the metric completion of T. We say that $f(\rho)$ is contained in a set A except for an initial part if $f(\rho(t)) \in A$ for all t larger than some t_0.

Proof. (i) First consider two maps $f : T_0 \to T$ and $f' : T_0' \to T$. We may assume that they are isometric on edges. Let C be the backtracking constant of f. Subdivide T_0' equivariantly so that all edges now have length less than C. Given a vertex v of the subdivided T_0', let $\zeta(v)$ be a point of T_0 such that $f(\zeta(v)) = f'(v)$. We may choose $\zeta(v)$ in an equivariant way, and extend ζ to an equivariant map $\zeta : T_0' \to T_0$ which is linear on each edge. Given an edge vw of T_0', its image by $f \circ \zeta$ is contained in the C-neighbourhood of the segment $f'(v)f'(w)$. This implies that $f \circ \zeta$ is $2C$-close to f' (recall that $|f'(v)f'(w)| \leqslant C$).

Represent $X \in \partial F_n$ by rays ρ, ρ' in T_0, T_0'. Since $\zeta(\rho')$ contains ρ (after truncating if needed), we find that, *except for an initial part,* $r = f(\rho)$ *is contained in the* $2\,\mathrm{BBT}(f)$*-neighbourhood of* $r' = f'(\rho')$. Similarly, r' is contained in the $2\,\mathrm{BBT}(f')$-neighbourhood of r. In particular, boundedness of r depends only on X.

(ii) Now fix f and let $C + \delta > C = \mathrm{BBT}(f)$. We show that *except for an initial part,* r *is entirely contained in a ball of radius* $C + \delta$. Since X is assumed to be T-bounded, we may consider $\eta = \sup_{t>0} d(r(0), r(t))$. Choose t_0 with $d(r(0), r(t_0)) > \eta - \delta$. If $C \geqslant \eta - \delta$, the whole of r is contained in the $(C + \delta)$-ball centred at $r(0)$. If not, let x be the point of the segment $r(0)r(t_0)$ at distance C from $r(t_0)$. For $t \geqslant t_0$ the point x separates $r(0)$ from $r(t)$. Since $d(r(0), x) > \eta - \delta - C$ and $d(r(0), r(t)) \leqslant \eta$ we obtain $d(x, r(t)) \leqslant C + \delta$.

(iii) It is now easy to conclude. Choose a sequence ε_n converging to 0, as well as maps $f_n : T_n \to T$ with $\mathrm{BBT}(f_n) < \varepsilon_n$ (given by Corollary 2.3). Represent X by rays ρ_n in T_n such that $r_n = f_n(\rho_n)$ is contained in an ε_n-ball. As n increases, the distance between these balls goes to 0 and therefore they converge to a unique $Q(X) \in \bar{T}$. For every $r = f(\rho)$ as above, the point $Q(X)$ belongs to the closure of r since r_n is contained in the $2\varepsilon_n$-neighbourhood of r.

To prove the 'furthermore', choose t_0 with $d(Q(X), f(\rho(t_0))) < \mathrm{BBT}(f)$ and suppose there exists $t_1 > t_0$ with $d(Q(X), f(\rho(t_1))) > 2\,\mathrm{BBT}(f)$. For $t \geqslant t_1$ the point located at distance $\mathrm{BBT}(f)$ of $f(\rho(t_1))$ on the segment $Q(X)f(\rho(t_1))$ separates $Q(X)$ from $f(\rho(t))$, a contradiction. □

We can now associate a point $Q(X)$ to every $X \in \partial F_n$. If X is T-bounded, it is the point of $\bar{T}$ provided by Proposition 3.1. If not, then $Q(X)$ is the point of ∂T defined by $r = f(\rho)$, as explained before the statement of Proposition 3.1. It is the unique point of ∂T such that $X = j(Q(X))$ (see [**11**, Lemma 3.5]).

The map $Q : \partial F_n \to \bar{T} \sqcup \partial T$ thus defined is obviously F_n-equivariant. The map Q restricts to a bijection from the set of T-unbounded points of ∂F_n onto ∂T.

Remark 3.2. Suppose T is dual to an arational measured geodesic lamination on a compact hyperbolic surface Σ with geodesic boundary. There is a $\pi_1\Sigma$-equivariant quotient map from the universal covering $\tilde{\Sigma}$ onto T. An infinite geodesic ray in $\tilde{\Sigma}$ maps onto an open segment in T. The corresponding $Q(X)$ is the endpoint of that segment (in ∂T if it has infinite length, in $\bar{T}$ otherwise).

Remark 3.3. Suppose T is a very small simplicial F_n-tree (more generally, a simplicial G-tree where G is a word hyperbolic group and edge stabilizers are quasiconvex (see [**5**])). Then ∂F_n is naturally identified to the disjoint union of ∂T and the union of the boundaries of vertex stabilizers [**5**, Proposition 1.3]. If edge stabilizers are trivial, these boundaries are disjoint and one may define a map Q as above. If $g \in F_n$ fixes an edge, though, one cannot associate a single point of T to $X = \lim_{p\to+\infty} g^p$. This is due to the failure of Corollary 2.3 in this case.

Lemma 3.4. *Let $X \in \partial F_n$ be T-bounded. For any $P \in \bar{T}$, there exists a sequence $g_p \in F_n$ such that $g_p \to X$ and $g_p P \to Q(X)$. Conversely, if $h_p \to X$ and $h_p P$ converges to some point $R \in \bar{T}$, then $R = Q(X)$.*

Proof. Given $\varepsilon > 0$, choose $f : \Gamma \to \bar{T}$ as in Remark 2.6. Long initial subwords of X (in the basis corresponding to Γ) provide elements $g \in F_n$ arbitrarily close to X with $d(gP, Q(X)) < 2\varepsilon$. Conversely, assume $R \neq Q(X)$ and choose f as in Remark 2.6 with $\mathrm{BBT}(f)$ small with respect to $d(R, Q(X))$. Let $h \in F_n$ be the maximal initial subword common to h_p and h_q, for p and q large. The point hP is $2\,\mathrm{BBT}(f)$-close to both R and $Q(X)$, a contradiction. $\square$

It is not true, however, that $h_p P \to Q(X)$ for *every* sequence $h_p \to X$.

Corollary 3.5. *The equivariant map $Q : \partial F_n \to \bar{T} \sqcup \partial T$ is onto.*

Proof. We have to show that every $R \in \bar{T}$ is a $Q(X)$. Choose any $P \in T$ and any sequence h_p such that $h_p P \to R$. Any limit point X of h_p satisfies $Q(X) = R$. $\square$

Remark 3.6. We do not know whether Q is finite-to-one when the action on T is free.

We now consider two distinct points $X, X' \in \partial F_n$. Let $f : T_0 \to T$ be as above. If X, X' are T-bounded and γ is the bi-infinite geodesic joining X to X' in T_0, then $f(\gamma)$ is contained in the $\mathrm{BBT}(f)$-neighbourhood of the segment joining $Q(X)$ to $Q(X')$.

Remark 3.7. In particular, if $Q(X) = Q(X')$, then $f(\gamma)$ is contained in the $\mathrm{BBT}(f)$-ball centred at $Q(X)$.

Given $X, X' \in \partial F_n$, we define $d_T(Q(X), Q(X'))$ as 0 if $X = X'$, as the distance between $Q(X)$ and $Q(X')$ in $\bar{T}$ if both X and X' are T-bounded, and as $+\infty$ in the remaining cases. This gives us a map κ, with values in $[0, +\infty]$, defined on the product of $(\partial F_n)^2$ with the space of F_n-trees with dense orbits and trivial arc stabilizers.

Proposition 3.8. *The map κ is lower semicontinuous.*

Proof. Given $C < d_T(Q(X), Q(X'))$, we have to show that κ remains bigger than C under perturbation. Choose ε small with respect to $d_T(Q(X), Q(X')) - C$. Fix $f : T_0 \to T$ isometric on edges, with $v(T_0) < \varepsilon$. For T' close to T, there exists $f' : T_0 \to T'$, linear on

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edges, such that images of edges have approximately the same length in T and T'. The map f' has Lipschitz constant close to 1 and BBT less than $2v(T_0)$.

Let $\gamma \subset T_0$ be the bi-infinite geodesic joining the ends X and X'. Choose points A, B on γ with $d(f(A), f(B))$ close to $d_T(Q(X), Q(X'))$ (closeness is measured with respect to ε). If a bi-infinite geodesic joining two ends Y, Y' in T_0 contains the segment between A and B, then for T' close enough to T its image by f' has diameter bigger than C, and therefore $d_{T'}(Q(Y), Q(Y')) > C$. □

4. Train tracks, laminations, trees

Let α be an IWIP automorphism of F_n. Let $\Phi : \tau \to \tau$ be a train track map representing α (see [**2**]). We denote by $\tilde{\tau}$ the universal covering of the graph τ. It is equipped with a free action of F_n, and a lift φ of Φ such that $\alpha(g) \circ \varphi = \varphi \circ g$ for $g \in F_n$.

We define the *expanding lamination* Λ as an F_n-invariant collection of geodesic segments in $\tilde{\tau}$, called *leaf segments*, as follows (see [**4**, **13**]). First, a compact segment γ is in Λ if and only if it is contained in $\varphi^k(e)$ for some $k \geqslant 1$ and some edge e. If $\gamma \in \Lambda$, then there exists k_0 such that $\varphi^k(e)$ contains a translate of γ for every edge e and every $k \geqslant k_0$ (this follows from irreducibility of α and its powers). Next, we say that a half-infinite or bi-infinite geodesic is in Λ if every compact subsegment is in Λ. A bi-infinite $\gamma \in \Lambda$ is called a *leaf* of Λ.

We note the following simple facts.

Lemma 4.1.

(1) *If $\gamma \in \Lambda$, then $\varphi_{|\gamma}$ is injective and $\varphi(\gamma) \in \Lambda$.*

(2) *Every $\gamma \in \Lambda$ is contained in a bi-infinite $\gamma' \in \Lambda$.*

(3) *If a bi-infinite γ is in Λ, then the bi-infinite geodesic θ such that $\gamma = [\varphi(\theta)]$ (tightened image) is in Λ; in particular, $\varphi_{|\theta}$ is injective: $\varphi(\theta) = \gamma$.*

(4) *If e, f are edges with a common initial vertex v, there exists a sequence $e = e_0, e_1, \dots, e_p = f$ of distinct edges starting at v such that all edge paths $\bar{e}_i e_{i+1}$ are in Λ (connectedness of the Whitehead graph, see [**4**, **13**]).*

Since the set of ends of $\tilde{\tau}$ is canonically identified with ∂F_n, the set of leaves of Λ may be viewed in a more intrinsic way, as an F_n-invariant collection of unordered pairs $\{X, X'\}$ of distinct elements of ∂F_n. This collection (still denoted Λ) depends only on α (not on τ and Φ), see [**4**].

We define the *support* $s(\Lambda) \subset \partial F_n$ as the set of $X \in \partial F_n$ such that Λ contains some pair $\{X, X'\}$. It is F_n-invariant and α-invariant.

We also consider the *contracting lamination* of α (the expanding lamination of α^{-1}). We usually write Λ^+ for the expanding lamination and Λ^- for the contracting one.

Outer space may be viewed as the space of projective classes of free simplicial actions of F_n on $\boldsymbol{R}$-trees (see [**22**]). Compactified outer space $\overline{CV_n}$ is the space of projective classes of very small actions. We write $[T]$ for the point of $\overline{CV_n}$ determined by a very small $\boldsymbol{R}$-tree T.

Lemma 4.2. *If a very small F_n-tree T has dense orbits, then all arc stabilizers are trivial.*

Proof. This is folklore (compare [**19**, Proposition 1.4], [**1**, proof of Theorem 2.2]). A proof appears in [**10**, Proposition I.10] when T is geometric. We sketch a proof for T not geometric. We first show (with the notations of [**10**]) that for every $x \in T$ the action of $\operatorname{Stab} x$ on $\pi_0(T \setminus \{x\})$ has finitely many orbits. Otherwise, for any N, we may approximate T by a small, minimal, geometric tree T' with a branch point y having at least N $\operatorname{Stab} y$-orbits of directions with cyclic stabilizer (see [**10**]). These directions are contained in the simplicial part of T' by [**10**, Proposition I.10]. But there is a uniform bound for the valence of a vertex in a small minimal graph of groups decomposition of F_n, and therefore N is bounded.

Now let a non-trivial $g \in F_n$ fix a non-degenerate segment $I \subset T$. Branch points of T are dense in I. By the above and [**10**, Corollary III.3], there exists $h \in F_n$ carrying a subsegment $J \subset I$ onto another subsegment of I, in an orientation-preserving way. Then g and $h^{-1}gh$ generate a free group of rank 2 fixing J, a contradiction. □

The automorphism α acts on $\overline{CV_n}$ by precomposing the action with α. Its fixed points $[T^+]$ and $[T^-]$ may be constructed as follows (see [**11**] for a detailed exposition).

Let Φ, $\tilde{\tau}$, φ be as above. The transition matrix of Φ is irreducible, with a Perron–Frobenius eigenvalue $\lambda > 1$. Using an eigenvector associated to λ, one may assign to each segment β of $\tilde{\tau}$ a *Perron–Frobenius length* $\|\beta\|$, in such a way that $\|\varphi(\beta)\| = \lambda\|\beta\|$ if β is a leaf segment of Λ^+ (and $\|\varphi(\beta)\| \leqslant \lambda\|\beta\|$ for arbitrary β).

The $\boldsymbol{R}$-tree T^+ may now be defined as the metric space associated to the pseudo-distance d_+ on $\tilde{\tau}$ giving length $d_+(\beta) = \lim_{p\to+\infty} \|\varphi^p(\beta)\|/\lambda^p$ to a segment β. One constructs T^- similarly, using a train track map associated to α^{-1}.

The length function $\ell_+ : F_n \to \boldsymbol{R}$ of the action of F_n on T^+ satisfies $\ell_+ \circ \alpha = \lambda \ell_+$. If T is a very small F_n-tree, with length function ℓ_T, saying that $\alpha^p([T])$ converges to $[T^+]$ means that there exists a sequence c_p such that $(1/c_p)\ell_T \circ \alpha^p$ converges to ℓ_+.

5. The main argument

Let $\alpha \in \operatorname{Aut} F_n$ be irreducible with irreducible powers. Let Λ^+, Λ^- be the expanding and contracting lamination. The goal of this section is the following statement.

Proposition 5.1. *Let T be a minimal F_n-tree with dense orbits and trivial arc stabilizers. There exists a leaf $\{X, X'\}$ of Λ^+ or of Λ^- such that $Q(X) \neq Q(X')$.*

Recall that $Q(X) = Q(X')$ is possible only if X and X' are T-bounded (because the map $j : \partial T \to \partial F_n$ constructed in [**11**] satisfies $j \circ Q = \mathrm{id}$).

Let $\tilde{\tau}$ be as in § 4. We view it as a metric tree, with each edge having length 1 (we do not use Perron–Frobenius length in this section). The proof of Proposition 5.1 relies on two lemmas.

Lemma 5.2. *Suppose $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^+. Let $Y, Y' \in \partial F_n$ belong to the support $s(\Lambda^+)$. Then the distance in $\bar{T}$ between $Q(\alpha^p(Y))$ and $Q(\alpha^p(Y'))$ tends to 0 as $p \to +\infty$.*

Proof. Let $\gamma \subset \tilde{\tau}$ be the bi-infinite geodesic with ends Y, Y'. The hypothesis that $Y, Y' \in s(\Lambda^+)$ means that the complement of some compact subsegment consists of two half-infinite rays ρ, ρ' contained in Λ^+.

Let e, e' be the first edges of ρ and ρ'. Using the last assertion of Lemma 4.1, we can find a finite sequence $\gamma_1, \dots, \gamma_m$ of edge paths $\gamma_i = e_i e_i'$ of length 2, belonging to Λ^+, with $e_1 = \bar{e}$ and $e_m' = e'$, such that for each i the edges e_i' and e_{i+1} are the same, but not necessarily with the same orientation (thus the union of γ_i and γ_{i+1} is either a tripod or a segment of length 3).

Given ε, choose $f : T_0 \to T$ as in Corollary 2.3, with $\mathrm{BBT}(f) < \varepsilon$. Next choose an equivariant map $\mu : \tilde{\tau} \to T_0$ and note that there exists a constant C such that, if two bi-infinite geodesics in $\tilde{\tau}$ have intersection of length greater than C, then their tightened images in T_0 have non-empty intersection.

For p large, the image of each edge of $\tilde{\tau}$ by φ^p has length greater than C. We complete the proof by showing that the distance between $Q(\alpha^p(Y))$ and $Q(\alpha^p(Y'))$ is at most $2(m+2)\varepsilon$. Let $\lambda_0, \lambda_1, \dots, \lambda_m, \lambda_{m+1} \in \Lambda^+$ be bi-infinite geodesics containing $\rho, \gamma_1, \dots, \gamma_m, \rho'$, respectively. Let $\delta_i \subset T_0$ be the tightened image of $\varphi^p(\lambda_i)$ by μ.

Since λ_i and λ_{i+1} have an edge in common, our choice of p guarantees that δ_i and δ_{i+1} have non-empty intersection. Furthermore, the ends X_i, X_i' of δ_i satisfy $Q(X_i) = Q(X_i')$, and so $f(\delta_i) \subset T$ is contained in the ε-ball centred at $Q(X_i)$ (see Remark 3.7). We conclude that $d(Q(Z), Q(Z')) \leqslant 2(m+2)\varepsilon$, where Z, Z' are the ends determined by $\varphi^p(\rho)$ and $\varphi^p(\rho')$. But the relation $\alpha(g) \circ \varphi = \varphi \circ g$ implies $Z = \alpha^p(Y)$ and $Z' = \alpha^p(Y')$. □

Lemma 5.3. *Suppose $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^-. There exist maps $i_p : \tilde{\tau} \to \bar{T}$ ($p \in \boldsymbol{N}$) such that $i_p \circ \varphi^p$ is F_n-equivariant and $\mathrm{BBT}(i_p) \to 0$ as $p \to +\infty$.*

Proof. Fix X in the support of Λ^-. Also fix an equivariant map π from $\tilde{\tau}$ to a Cayley tree Γ of F_n, obtained by collapsing edges in the lift of a maximal subtree of τ. Choose a base vertex $v \in \Gamma$. Define i_0 as $j_0 \circ \pi$, where $j_0 : \Gamma \to \bar{T}$ is equivariant, linear on edges, and maps v to $Q(X)$. We define i_p analogously as $j_p \circ \pi$, but now j_p is required to map v to $Q(\alpha^{-p}(X))$ and to satisfy the twisted equivariance property $g \circ j_p = j_p \circ \alpha^p(g)$ (which expresses that $i_p \circ \varphi^p$ is equivariant).

We check that $\mathrm{BBT}(i_p) \to 0$. By Lemma 2.1, it suffices to show that the length of $j_p(e)$ tends to 0 for every edge e of Γ. We may assume that e has vertices v and gv, with $g \in F_n$. Now $j_p(v) = Q(\alpha^{-p}(X))$ and

$$j_p(gv) = \alpha^{-p}(g) j_p(v) = \alpha^{-p}(g) Q(\alpha^{-p}(X)) = Q(\alpha^{-p}(gX)).$$

Since X and gX both belong to the support of Λ^-, we conclude by applying Lemma 5.2 to α^{-1}. □

Proof of Proposition 5.1. We argue by way of contradiction, assuming that $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^+ or Λ^-. Let i_p be given by Lemma 5.3. Let e be an edge of $\tilde{\tau}$. If $\gamma \in \Lambda^+$ is a bi-infinite geodesic containing e, then $\varphi^p(\gamma)$ is also a leaf of Λ^+ and therefore its image by i_p is contained in a ball of radius $\mathrm{BBT}(i_p)$. It follows that the diameter of $(i_p \circ \varphi^p)(e)$ is bounded by $2\,\mathrm{BBT}(i_p)$. If u is a conjugacy class in F_n, represented by a loop of length k in τ, its translation length in T is bounded by $2k\,\mathrm{BBT}(i_p)$ for all p (because $i_p \circ \varphi^p$ is equivariant). Thus every u has translation length 0 in T, a contradiction. □

6. Conclusion

In this section we deduce the main theorem from Proposition 5.1 and the following result (proved in the next section).

Proposition 6.1 (cf. Lemmas 3.4 and 3.5.1 of [4]). *Let T be an $\boldsymbol{R}$-tree with a minimal very small action of F_n. Suppose there exist a free simplicial F_n-tree T_0, an equivariant map $f : T_0 \to T$, and a bi-infinite geodesic $\gamma_0 \subset T_0$ representing a leaf of Λ^+ such that $f(\gamma_0)$ has diameter greater than $2\,\mathrm{BBT}(f)$. Then $f(\gamma_0)$ has infinite diameter and there exists a neighbourhood V of $[T]$ in $\overline{CV_n}$ such that $\alpha^p_{|V}$ converges to $[T^+]$ uniformly as $p \to +\infty$.*

We first show that every very small T satisfies the hypothesis of Proposition 6.1, provided we allow γ_0 to be a leaf of Λ^- *or* Λ^+. We distinguish several cases.

If T has dense orbits, it has trivial arc stabilizers by Lemma 4.2. Proposition 5.1 provides $\{X, X'\}$ in $\Lambda^\pm$ with $Q(X) \neq Q(X')$. We choose $f : T_0 \to T$ with $2\,\mathrm{BBT}(f) < d(Q(X), Q(X'))$, using Corollary 2.3, and we let γ_0 be the geodesic joining the ends of T_0 corresponding to X and X'.

Next suppose that orbits are not dense, but T is not simplicial. Then T contains simplicial pieces, as well as non-degenerate subtrees T_v with the property that some subgroup $G_v \subset F_n$ acts on T_v with dense orbits (see [**12**]). In particular, there exists an equivariant collapsing map $\pi : T \to T_m$ where T_m is a very small F_n-tree with dense orbits. Choose X, X' as above, using T_m. By Proposition 6.1, at least one of X, X' (both, in fact) is not T_m-bounded. Thus it is not T-bounded and $f : T_0 \to T$ may be chosen arbitrarily.

The last case is when T is simplicial. It suffices to show that X is T-unbounded whenever $\{X, X'\} \in \Lambda^+$. As pointed out in Remark 3.3, ∂F_n is the disjoint union of ∂T and the boundaries of the vertex stabilizers. A vertex stabilizer H is finitely generated and has infinite index. The proof of Proposition 2.4 in [**4**] shows that X cannot belong to ∂H (consider a half-leaf rather than a whole leaf). Thus X is T-unbounded.

Given any T as in Proposition 6.1, we now know that $\lim_{p\to\infty} \alpha^p([T]) = [T^+]$ or $\lim_{p\to\infty} \alpha^{-p}([T]) = [T^-]$. We also know that $[T^+]$ is an attracting fixed point, because α^p converges to $[T^+]$ uniformly on a neighbourhood of $[T^+]$ as $p \to +\infty$. Similarly, $[T^-]$ is a repelling fixed point. North–south dynamics follow easily.

Indeed, suppose $[T] \neq [T^-]$. The ω-limit set of $[T]$ (the set of limit points of the sequence $\alpha^p([T])$ as $p \to +\infty$) is closed and α-invariant. It must therefore contain $[T^+]$ or $[T^-]$. But it cannot contain the repelling point $[T^-]$, so it contains $[T^+]$. It follows that

$\alpha^p([T])$ tends to $[T^+]$. Similarly, $\alpha^{-p}([T])$ tends to $[T^-]$ if $[T] \neq [T^+]$. This completes the proof of the main theorem.

7. Proof of Proposition 6.1

Let $\varphi : \tilde{\tau} \to \tilde{\tau}$ be as in § 4. Fix an equivariant map $\mu : \tilde{\tau} \to T_0$ and denote by γ the bi-infinite geodesic of $\tilde{\tau}$ such that γ_0 is the tightened image of γ. If $AB \subset \tilde{\tau}$ is a segment, we denote by $d_T(AB)$ the length of the tightened image of AB under $\nu = f \circ \mu$ (i.e. the distance between $\nu(A)$ and $\nu(B)$).

Let A_0, B_0 be points of γ_0 whose images in T have distance bigger than $2\,\mathrm{BBT}(f)$. Let σ be the central subsegment of $f(A_0)f(B_0)$ of length $d(f(A_0), f(B_0)) - 2\,\mathrm{BBT}(f)$. If $A_0'B_0' \subset T_0$ is a segment containing A_0B_0, the segment $f(A_0')f(B_0')$ contains σ.

Since μ has bounded backtracking, we can choose $AB \subset \gamma$ such that $\mu(A')\mu(B')$ contains A_0B_0 for any $A'B' \subset \tilde{\tau}$ containing AB. As a consequence, the tightened image of any such $A'B'$ by ν contains σ. Furthermore, if a segment in $\tilde{\tau}$ contains disjoint subsegments which are translates of AB by elements g_i of F_n, then its tightened image in T contains the $g_i\sigma$ as disjoint subsegments.

Choose m_0 such that $\varphi^{m_0}(e)$ contains a translate of AB for every edge e of $\tilde{\tau}$. If β is any leaf segment contained in Λ^+, then $d_T(\varphi^{m_0}(\beta)) \geqslant |\sigma||\beta|$ (with $|\beta|$ the simplicial length of β in $\tilde{\tau}$, and $|\sigma|$ the length of σ in T).

In particular, $f(\gamma_0)$ has infinite diameter, and for every edge e of $\tilde{\tau}$ the length $d_e(p) = d_T(\varphi^p(e))$ tends to infinity with p.

From now on we equip $\tilde{\tau}$ with a Perron–Frobenius length, characterized by the fact that for every edge path $\beta \subset \Lambda^+$ the image $\varphi(\beta)$ has length $\lambda\|\beta\|$ (where λ is the Perron–Frobenius eigenvalue of the transition matrix of the map Φ).

Lemma 7.1. *There exists a number $c > 0$ such that*

$$\lim_{p\to\infty} \frac{d_T(\varphi^p(\beta))}{\lambda^p\|\beta\|} = c$$

for every edge path $\beta \in \Lambda^+$.

Proof. Let E be the set of edges of a fundamental domain for the action of F_n on $\tilde{\tau}$. For given p, the segment $\varphi^p(\beta)$ is a union of translates of edges of E. Let N_e^p be the number of occurrences of translates of a given $e \in E$. The numbers N_e^p are the components of the image by the pth power of the transition matrix of a fixed non-negative vector (whose components are numbers of occurrences of edges in β). Let $p \to \infty$. By Perron–Frobenius theory, the sequence N_e^p/λ^p has a positive limit, of the form $c_e k_\beta$ with c_e depending only on e and k_β only on β. Since $\|\varphi^p(\beta)\| = \lambda^p\|\beta\| = \sum_{e\in E} N_e^p\|e\|$, we obtain $k_\beta = \|\beta\|$ (up to a normalization). Thus $N_e^p/\lambda^p\|\beta\| \to c_e$ as $p \to \infty$.

Given $\varepsilon > 0$, fix p_0 such that every $d_e(p_0) = d_T(\varphi^{p_0}(e))$ is bigger than $(1/\varepsilon)\,\mathrm{BBT}(\nu)$. Now $\varphi^{p+p_0}(\beta)$ is a union of translates of $\varphi^{p_0}(e)$, with $\varphi^{p_0}(e)$ appearing N_e^p times. Mapping into T by ν, we obtain

$$\sum_{e\in E} N_e^p(d_e(p_0) - 2\,\mathrm{BBT}(\nu)) \leqslant d_T(\varphi^{p+p_0}(\beta)) \leqslant \sum_{e\in E} N_e^p d_e(p_0).$$

Dividing by $\lambda^{p+p_0}\|\beta\|$, we get, as p goes to infinity,

$$(1-2\varepsilon)\sum_{e\in E}\frac{c_e d_e(p_0)}{\lambda^{p_0}} \leqslant \underline{\lim}\,\frac{d_T(\varphi^p(\beta))}{\lambda^p\|\beta\|} \leqslant \overline{\lim}\,\frac{d_T(\varphi^p(\beta))}{\lambda^p\|\beta\|} \leqslant \sum_{e\in E}\frac{c_e d_e(p_0)}{\lambda^{p_0}}$$

for any $\beta \in \Lambda^+$.

Since $d_e(p_0)/\lambda^{p_0}$ is bounded by $\|e\|\,\mathrm{Lip}(\nu)$, we see that $d_T(\varphi^p(\beta))/(\lambda^p\|\beta\|)$ has a limit, which is positive and independent of β. □

We now consider an arbitrary edge path $\beta \subset \tilde\tau$. We let $d_+(\beta)$ be the limit of the non-increasing sequence $\|\varphi^p(\beta)\|/\lambda^p$.

Lemma 7.2. *$d_T(\varphi^p(\beta))/(\lambda^p d_+(\beta))$ tends to c as $p \to \infty$.*

Proof. Write β as a concatenation of k paths $\beta_i \in \Lambda^+$. Then the tightened image $\varphi^p(\beta)$ is the concatenation of the $\varphi^p(\beta_i)$, possibly with cancellation. The total amount of cancellation is $\lambda^p\|\beta\| - \|\varphi^p(\beta)\| \leqslant \lambda^p(\|\beta\| - d_+(\beta))$. Furthermore, if $\varphi^p(\beta_i)$ and $\varphi^p(\beta_{i+1})$ overlap along a distance D, then the cancellation between their tightened images in T is bounded by $D\,\mathrm{Lip}(\nu) + 2\,\mathrm{BBT}(\nu)$. From this we obtain

$$\left|d_T(\varphi^p(\beta)) - \sum_i d_T(\varphi^p(\beta_i))\right| < \lambda^p(\|\beta\| - d_+(\beta))\,\mathrm{Lip}(\nu) + k\,\mathrm{BBT}(\nu).$$

Dividing by $\lambda^p\|\beta\|$ and using Lemma 7.1, we get

$$\overline{\lim_p}\left|\frac{d_T(\varphi^p(\beta))}{\lambda^p\|\beta\|} - c\right| \leqslant \left(1 - \frac{d_+(\beta)}{\|\beta\|}\right)\mathrm{Lip}(\nu).$$

Now apply this inequality to $\varphi^{p_0}(\beta)$,

$$\overline{\lim_p}\left|\frac{d_T(\varphi^{p+p_0}(\beta))}{\lambda^p\|\varphi^{p_0}(\beta)\|} - c\right| \leqslant \left(1 - \frac{d_+(\varphi^{p_0}(\beta))}{\|\varphi^{p_0}(\beta)\|}\right)\mathrm{Lip}(\nu).$$

For p_0 large, the ratio of $\|\varphi^{p_0}(\beta)\|$ to $\lambda^{p_0}d_+(\beta) = d_+(\varphi^{p_0}(\beta))$ is close to 1 and we get the desired result. □

It is now easy to prove Proposition 6.1. For $g \in F_n$, let β be a fundamental domain for the action of g on its translation axis in $\tilde\tau$. Then $d_T(\varphi^p(\beta))$ is the translation length of $\alpha^p(g)$ in T, and $d_+(\beta)$ is the translation length of g in T^+. We thus get $(\ell_T \circ \alpha^p)/(c\lambda^p) \to \ell_+$, and therefore $\alpha^p([T]) \to [T^+]$ in $\overline{CV_n}$.

This convergence is uniform on some neighbourhood of T. Indeed, for T' close to T, we can find $f' : T_0 \to T'$ with $\mathrm{Lip}(f')$ and $\mathrm{BBT}(f')$ bounded (as in the proof of Proposition 3.8). Thus $\mathrm{Lip}(\nu')$ and $\mathrm{BBT}(\nu')$ are uniformly bounded (with $\nu' = f' \circ \mu$). Since $f(\gamma_0)$ has infinite diameter, we can choose the same points A, B for all T' close enough to T, with a positive lower bound for the length of the segment σ. Since all estimates given above depend only on $|\sigma|$, $\mathrm{Lip}(\nu)$, and $\mathrm{BBT}(\nu)$, we have local uniformity.

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