Existence of travelling-wave solutions representing domain wall motion in a thin ferromagnetic nanowire

Ross G. Lund
Department of Mathematical Sciences,
New Jersey Institute of Technology, University Heights,
Newark, NJ 07102, USA

J. M. Robbins and **Valeriy Slastikov**
University of Bristol School of Mathematics, University Walk,
Bristol BS8 1TW, UK (`valeriy.slastikov@bristol.ac.uk`)

We study the dynamics of a domain wall under the influence of applied magnetic fields in a one-dimensional ferromagnetic nanowire, governed by the Landau–Lifshitz–Gilbert equation. Existence of travelling-wave solutions close to two known static solutions is proven using implicit-function-theorem-type arguments.

1. Introduction

The subject of domain wall motion in ferromagnetic nanowires has attracted a great deal of interest in recent years, from both the physics and applied mathematics communities. This is, in part, due to proposed magnetic storage media such as the so-called racetrack memory [16]. This device makes use of the fact that, in a thin ferromagnetic nanowire, the magnetic domains (regions of uniform magnetization, which are separated by small transition layers called *domain walls*) prefer to align along the wire axis (in either direction), thus providing a two-state system which can be used to represent information. In order to read or write information, the domains are propagated through the wire by application of a magnetic field or an electric current.

In order to understand the workings of such a device, the primary problem to study is the dynamics of a single domain wall separating two domains of opposite magnetization, under the influence of an applied magnetic field. The magnetization dynamics is modelled by the Landau–Lifshitz–Gilbert (LLG) equation [4,11].

An ansatz method that has been successful in approximately describing one-dimensional domain wall motion was introduced by Schryer and Walker [17], gen-

eralized by Malozemoff and Slonczewski [13], and applied to various cases by a number of authors (see, for example, [1, 12, 15, 19–24]). This approach consists of looking for approximate solutions belonging to a three-parameter family of profiles, in which the parameters represent translations, rotations and re-scalings of a suitably chosen static profile. The time dependence of the parameters is determined so as to satisfy the equations of motion as closely as possible. A recent asymptotic analysis [6] provides a systematic foundation for this approach.

The phenomenology of solution behaviours in this problem is reasonably well understood by applying the approximate methods discussed above. Exact solutions are obtained only in some special cases [5, 17], including a travelling-wave (TW) solution known as the Walker solution. For values of the driving field (that is, the component of the applied field along the wire axis) up to a certain critical value, which depends on other physical parameters (such as the transverse applied field components and the anisotropy coefficients of the nanowire), the velocity and width of the domain wall are asymptotically constant in time: the behaviour is apparently that of a TW. For fields above the critical value, oscillatory motion sets in: the domain wall precesses about the wire axis, and its velocity and width oscillate in time. This phenomenon is known in the literature as Walker breakdown, due to its discovery in the specific case studied by Schryer and Walker [17]. An approximate description of these transitions in the general one-dimensional problem, including predictions of the critical fields, has been provided elsewhere [6, 7].

Notwithstanding the success of these approximate methods, rigorous results concerning domain wall motion in nanowires are currently few [2, 3]. In this article, we analyse TW solutions of the LLG equation in thin nanowires. We consider nanowires of biaxial anisotropy, with easy axis along the wire, subject to a uniform magnetic field with components along and transverse to the wire, and prove that TW solutions exist provided that the magnetic field component along the wire is sufficiently small. The proof is an application of the implicit-function theorem.

1.1. Domain wall motion in a thin nanowire: statement of the problem

We work in the continuum-mechanical theory of micromagnetics (see, for example, [8]). The equation governing magnetization dynamics is the LLG equation. We study the following dimensionless form:

$$\hat{\boldsymbol{M}}_t + \alpha \hat{\boldsymbol{M}} \times \hat{\boldsymbol{M}}_t = \hat{\boldsymbol{M}} \times \boldsymbol{H}(\hat{\boldsymbol{M}}), \tag{1.1}$$

where $\hat{\boldsymbol{M}} \in S^2$ is the magnetization, $\alpha > 0$ is the Gilbert damping constant (typical value in the range 0.01–0.2), and $\boldsymbol{H}(\hat{\boldsymbol{M}})$ is the effective magnetic field. We study the LLG equation in a *thin* cylindrical magnetic nanowire, $\Omega \subset \mathbb{R}^3$; 'thin' meaning that the radius of the wire is small in comparison to both the length of the wire and the magnetic exchange length of the material. We work in scaled units such that the exchange length is equal to 1, and take the length of the wire to be infinite. We use Cartesian coordinates on Ω: x for the coordinate along the wire axis, and y, z for the coordinates on the cross-section.

In the thin-wire regime it is known that the micromagnetic energy Γ-converges to a one-dimensional energy [10, 18], given in dimensionless form by

$$E(\hat{\boldsymbol{M}}) = \frac{1}{2} \int_{\mathbb{R}} |\hat{\boldsymbol{M}}_x|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} (1 - (\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{x}})^2 + K_2 (\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{y}})^2) \, \mathrm{d}x, \tag{1.2}$$

where $K_2 \geqslant 0$ is the effective hard-axis anisotropy coefficient and $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ are the usual Cartesian basis vectors. The magnetization vector depends only on the axial coordinate, $x \in \mathbb{R}$, and time, $t \in \mathbb{R}^+$, i.e. $\hat{\boldsymbol{M}} = \hat{\boldsymbol{M}}(x,t)$. The effective field is given by

$$\boldsymbol{H}(\hat{\boldsymbol{M}}) = -\frac{\delta E}{\delta \hat{\boldsymbol{M}}} + \boldsymbol{H}_a = \hat{\boldsymbol{M}}_{xx} + (\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{x}})\hat{\boldsymbol{x}} - K_2(\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{y}})\hat{\boldsymbol{y}} + \boldsymbol{H}_a, \qquad (1.3)$$

where $\boldsymbol{H}_a = H_1\hat{\boldsymbol{x}} + H_2\hat{\boldsymbol{y}} + H_3\hat{\boldsymbol{z}} \in \mathbb{R}^3$ is a uniform applied magnetic field. For convenience, we denote the parameters of the system, namely the hard-axis anisotropy K_2 and applied magnetic field $\boldsymbol{H}_a$, collectively by

$$\Lambda = (H_1, H_2, H_3, K_2). \qquad (1.4)$$

We note that the effective field may also be written as

$$\boldsymbol{H}(\hat{\boldsymbol{M}}) = \hat{\boldsymbol{M}}_{xx} - \nabla_{\hat{\boldsymbol{M}}} U^\Lambda(\hat{\boldsymbol{M}}), \qquad (1.5)$$

where the potential U^Λ is given by

$$U^\Lambda(\hat{\boldsymbol{M}}) = \tfrac{1}{2}(1 - (\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{x}})^2 + K_2(\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{y}})^2 - 2\boldsymbol{H}_a \cdot \hat{\boldsymbol{M}}). \qquad (1.6)$$

As we are interested in domain wall dynamics, we impose the boundary conditions

$$\lim_{x \to \pm\infty} \hat{\boldsymbol{M}} = \hat{\boldsymbol{m}}^\Lambda_\pm, \qquad (1.7)$$

where $\hat{\boldsymbol{m}}^\Lambda_\pm$ correspond to distinct local minima of U^Λ whose projections along $\hat{\boldsymbol{x}}$ have opposite signs. For definiteness, we take

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{m}}^\Lambda_+ > 0, \qquad \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{m}}^\Lambda_- < 0, \qquad (1.8)$$

which corresponds to 'tail-to-tail' domains ('head-to-head' domains may be treated similarly).

It will be convenient to introduce polar coordinates $(\psi(\xi), \beta(\xi))$ for the magnetization, writing

$$\hat{\boldsymbol{m}}(\psi, \beta) = (\sin\psi \cos\beta, \cos\psi, \sin\psi \sin\beta), \qquad (1.9)$$

and letting $(\psi^\Lambda_\pm, \beta^\Lambda_\pm)$ denote the polar coordinates of the boundary values $\hat{\boldsymbol{m}}^\Lambda_\pm$. We note that we have taken the polar axis along the hard axis $\hat{\boldsymbol{y}}$ rather than, as is often done, the easy axis $\hat{\boldsymbol{x}}$. This is because the profiles of interest do not take values near the hard axis, which is energetically costly, so that their representation in terms of (ψ, β) avoids the coordinate singularities at $\psi = 0$ and $\psi = \pi$.

1.2. Overview

The remainder of the paper is structured as follows: In § 2 we derive the system of ordinary differential equations (ODEs) satisfied by TWs and identify static (zero velocity) solutions for two subsets of parameters in which the component of the applied field along the wire vanishes, namely (i) $K_2 > 0$, $\boldsymbol{H}_a = 0$, corresponding to biaxial anisotropy and vanishing applied field; and (ii) $K_2 = H_1 = 0 < H_2^2 + H_3^2 < 1$, corresponding to uniaxial anisotropy and non-vanishing transverse applied field. In § 3 we reformulate the TW equations as a map between Banach spaces whose zeros

correspond to solutions of the TW equations. In §4 we state and prove our main results, which establish existence of TW solutions to the LLG equation (1.1) for values of the parameters in neighbourhoods of the subsets (i) and (ii).

2. Travelling waves

We look for TW solutions of the LLG equation (1.1) of the form

$$\hat{\boldsymbol{M}}(x,t) = \hat{\boldsymbol{m}}(x - Vt), \tag{2.1}$$

where V denotes the velocity of the TW. Substituting (2.1) into (1.1) and imposing the boundary conditions (1.7), we obtain the TW equation

$$V\hat{\boldsymbol{m}}' + \alpha V \hat{\boldsymbol{m}} \times \hat{\boldsymbol{m}}' + \hat{\boldsymbol{m}} \times \boldsymbol{H}(\hat{\boldsymbol{m}}) = 0, \quad \lim_{\xi \to \pm\infty} \hat{\boldsymbol{m}}(\xi) = \hat{\boldsymbol{m}}_{\pm}^{\Lambda}, \tag{2.2}$$

a system of second-order ODEs, where we have introduced the travelling coordinate $\xi := x - Vt$. We note that its solutions are determined up to translation; that is, if $\hat{\boldsymbol{m}}(\xi)$ satisfies (2.2), so does $\hat{\boldsymbol{m}}(\xi - \xi_0)$.

2.1. Static solutions

Given a solution $\hat{\boldsymbol{m}}$ of the TW equation (2.2), one can derive the following identity for the velocity [7]:

$$V = \frac{U^{\Lambda}(\hat{\boldsymbol{m}}_{-}^{\Lambda}) - U^{\Lambda}(\hat{\boldsymbol{m}}_{+}^{\Lambda})}{\alpha \int_{\mathbb{R}} \hat{\boldsymbol{m}}' \cdot \hat{\boldsymbol{m}}'}. \tag{2.3}$$

(Formally, (2.3) is obtained by noting that the TW equation (2.2) implies that $\alpha V \hat{\boldsymbol{m}}' \cdot \hat{\boldsymbol{m}}'$ is equal to $(\frac{1}{2}\hat{\boldsymbol{m}}' \cdot \hat{\boldsymbol{m}}' - U^{\Lambda}(\hat{\boldsymbol{m}}))'$.) Therefore, if the stable domain orientations $\hat{\boldsymbol{m}}_{+}^{\Lambda}$ and $\hat{\boldsymbol{m}}_{-}^{\Lambda}$ have same energy, then the velocity vanishes – equivalently, there may exist a static solution of the TW equation. We always have that $U^{\Lambda}(\hat{\boldsymbol{m}}_{-}^{\Lambda}) = U^{\Lambda}(\hat{\boldsymbol{m}}_{+}^{\Lambda})$ whenever $H_1 = 0$, i.e. the component of the applied field along the wire vanishes, since in this case, the potential $U^{\Lambda}(\hat{\boldsymbol{m}})$ is symmetric under the reflection $(m_1, m_2, m_3) \to (-m_1, m_2, m_3)$.

We can find explicit static solutions in two parameter regimes with $H_1 = 0$. The first is the case of vanishing transverse applied field, which we denote by

$$\Lambda_W = (0, 0, 0, K_2), \quad K_2 > 0. \tag{2.4}$$

Allowing $H_1 \neq 0$, this encompasses the regime of the Walker solution. We note that, in this regime, we have restricted to positive K_2 since the case in which $K_2 = H_2 = H_3 = 0$ is degenerate and our results do not apply there, though an explicit dynamic solution to (1.1) is available that is not a simple TW [5].

The static profile $\hat{\boldsymbol{m}}_W$, for $\Lambda = \Lambda_W$, is given by the usual Bloch domain wall,

$$\hat{\boldsymbol{m}}_W(\xi) = (\tanh \xi, 0, \operatorname{sech} \xi). \tag{2.5}$$

The polar representation of the static profile (2.5) is given by

$$\psi_W(\xi) = \frac{\pi}{2}, \qquad \beta_W(\xi) = 2 \tan^{-1}(e^{-\xi}). \tag{2.6}$$

It is easily shown that $\hat{\boldsymbol{m}}_W$ corresponds to a minimizer of the micromagnetic energy (1.2). A second minimizer, $(\tanh\xi, 0, -\operatorname{sech}\xi)$, may be obtained by reflection through the intermediate axis $\hat{\boldsymbol{z}}$. There exist another pair of solutions of (2.2) given by $(\tanh\xi, \pm\operatorname{sech}\xi, 0)$, which correspond to critical points of the energy, in which the domain wall is aligned along the hard axis $\hat{\boldsymbol{y}}$. As they are energetically unstable, we do not consider them further.

The second regime corresponds to vanishing hard-axis anisotropy and non-vanishing transverse field, and is denoted by

$$\Lambda_T = (0, H_2, H_3, 0), \quad 0 < H_2^2 + H_3^2 < 1. \tag{2.7}$$

In view of the rotational symmetry about the easy axis, we may assume that $H_2 = 0$. In terms of the polar coordinates (ψ, β), the static profile is given by the solution of the first-order equations

$$\psi_T = \frac{\pi}{2}, \qquad \beta_T' = H_3 - \sin\beta_T, \tag{2.8}$$

with boundary conditions

$$\lim_{\xi\to-\infty} \beta_T = \pi - \sin^{-1} H_3, \qquad \lim_{\xi\to\infty} \beta_T = \sin^{-1} H_3. \tag{2.9}$$

One may explicitly compute β_T (see [7]), but the expression is rather long and in any case will not be required in what follows. We will only need the fact that $\sin\beta_T \geqslant H_3$ so that β_T is strictly decreasing.

3. TW equations as a map between Banach spaces

Let Λ_* denote either of the static parameters Λ_W or Λ_T, and let (ψ_*, β_*) denote the corresponding static solutions. We introduce reference profiles $(\psi^\Lambda, \beta^\Lambda)$ that coincide with (ψ_*, β_*) for $\Lambda = \Lambda_*$ as follows:

$$\left.\begin{aligned} \psi^\Lambda(\xi) &:= \psi_*(\xi) + \Theta(\xi)(\psi_+^\Lambda - \psi_+^{\Lambda_*}) + \Theta(-\xi)(\psi_-^\Lambda - \psi_-^{\Lambda_*}), \\ \beta^\Lambda(\xi) &:= \beta_*(\xi) + \Theta(\xi)(\beta_+^\Lambda - \beta_+^{\Lambda_*}) + \Theta(-\xi)(\beta_-^\Lambda - \beta_-^{\Lambda_*}). \end{aligned}\right\} \tag{3.1}$$

Here, $\Theta(\xi)$ is a switching function with the following properties: (i) $\Theta(\xi) = 0$ for $\xi < 0$, (ii) $\Theta(\xi) = 1$ for $\xi > \xi_0$ for some $\xi_0 > 0$, and (iii) on the interval $(0, \infty)$, $1 - \Theta$ belongs to $H^2((0, \infty))$. It is then straightforward to check that $\lim_{\xi\to\pm\infty}(\psi^\Lambda(\xi), \beta^\Lambda(\xi)) = (\psi_\pm^\Lambda, \beta_\pm^\Lambda)$ and that $\psi^{\Lambda\prime\prime}, \beta^{\Lambda\prime\prime} \in L^2(\mathbb{R})$.

We consider profiles with polar-coordinate representation (a, b) of the form

$$a = \psi^\Lambda + u, \qquad b = \beta^\Lambda + w, \tag{3.2}$$

where $u, w \in H^2(\mathbb{R})$. These satisfy the boundary conditions (1.7) and include all profiles sufficiently close to the reference profile $(\psi^\Lambda, \beta^\Lambda)$. Substituting $\hat{\boldsymbol{m}}(a, b) := (\sin a \cos b, \cos a, \sin a \sin b)$ into the TW equation (2.2), we obtain a second-order system for u and w in the form $G_1 = G_2 = 0$, where

$$\left.\begin{aligned} G_1(u, w, V; \Lambda) &:= \sin a\, b'' + 2\cos a\, a' b' + V a' + \alpha V \sin a\, b' - F_1^\Lambda(a, b), \\ G_2(u, w, V; \Lambda) &:= a'' - \tfrac{1}{2}\sin 2a\, {b'}^2 + \alpha V a' - V \sin a\, b' - F_2^\Lambda(a, b), \end{aligned}\right\} \tag{3.3}$$

and

$$
\left.\begin{aligned}
F_1^\Lambda(a,b) &:= -\hat{\boldsymbol{p}} \cdot \nabla_{\hat{\boldsymbol{m}}} U^\Lambda(\hat{\boldsymbol{m}}(a,b)), \\
F_2^\Lambda(a,b) &:= \hat{\boldsymbol{n}} \cdot \nabla_{\hat{\boldsymbol{m}}} U^\Lambda(\hat{\boldsymbol{m}}(a,b)).
\end{aligned}\right\}
\tag{3.4}
$$

Here, $\hat{\boldsymbol{n}} := (\cos a \cos b, -\sin a, \cos a \sin b)$ and $\hat{\boldsymbol{p}} := \hat{\boldsymbol{m}} \times \hat{\boldsymbol{n}}$ form an orthogonal basis for the plane normal to $\hat{\boldsymbol{m}}$. The boundary conditions (1.7) imply that

$$
F_1^\Lambda(\psi_\pm^\Lambda, \beta_\pm^\Lambda) = F_2^\Lambda(\psi_\pm^\Lambda, \beta_\pm^\Lambda) = 0,
\tag{3.5}
$$

which expresses the vanishing of torques on $\hat{\boldsymbol{m}}(\xi)$ as ξ approaches $\pm\infty$.

It is straightforward to show that a', b', a'' and b'' belong to $L^2(\mathbb{R})$ while a, b, a', b' belong to $L^\infty(\mathbb{R})$. It follows that the terms in G_1 and G_2 involving derivatives of a and b belong to $L^2(\mathbb{R})$. From (3.1), it is straightforward to show that F_1^Λ and F_2^Λ also belong to $L^2(\mathbb{R})$. For example, to show that F_1^Λ is square-integrable on $(0, \infty)$, we note that for $\xi > 0$, (a, b) can be expressed as $(\psi_+^\Lambda + s, \beta_+^\Lambda + t)$, where

$$
s = (\psi_* - \psi_+^{\Lambda*}) - (1 - \theta)(\psi_+^\Lambda - \psi_+^{\Lambda*}) + u, \qquad t = (\beta_* - \beta_+^{\Lambda*}) - (1 - \theta)(\beta_+^\Lambda - \beta_+^{\Lambda*}) + w.
\tag{3.6}
$$

It is evident that $s, t \in L^2((0, \infty))$. Therefore, since $F_1^\Lambda(\psi_+^\Lambda, \beta_+^\Lambda,) = 0$, it follows from the mean-value theorem and boundedness of derivatives of F_1^Λ that $F_1^\Lambda(a, b) = F_1^\Lambda(\psi_+^\Lambda + s(\xi), \beta_+^\Lambda + t(\xi))$ is bounded by $|s(\xi)| + |t(\xi)|$ up to a multiplicative constant. A similar argument, with $(\psi_+^\Lambda, \beta_+^\Lambda)$ replaced by $(\psi_-^\Lambda, \beta_-^\Lambda)$, shows that F_1^Λ is square-integrable on $(-\infty, 0)$, and likewise for F_2^Λ.

Finally, just as solutions $\hat{\boldsymbol{m}}$ of (2.2) are determined up to translations, so too are solutions (u, w) of $G_1 = G_2 = 0$. This degeneracy can be lifted by choosing the translate which is closest to the static solution $\hat{\boldsymbol{m}}_*$ in the $L^2(\mathbb{R})$-norm. This is equivalent to the condition $g = 0$, where

$$
g(u, w, V; \Lambda) := \langle b, \beta_*' \rangle_{L^2(\mathbb{R})}.
\tag{3.7}
$$

With these considerations, we can formulate the TW equation in terms of a map between Banach spaces. We define

$$
X = H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times \mathbb{R}, \qquad Y = \mathbb{R}^4 = \{\Lambda\}, \qquad Z = L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{R}, \tag{3.8}
$$

and

$$
\boldsymbol{G}\colon X \times Y \to Z; (u, w, V; \Lambda) \mapsto (G_1, G_2, g)(u, w, V; \Lambda).
\tag{3.9}
$$

Then $\hat{\boldsymbol{m}}(a, b)$ satisfies (2.2) if and only if $\boldsymbol{G}(u, w, V; \Lambda) = 0$. In order to prove existence of solutions by applying the implicit-function theorem, we shall need the following result.

PROPOSITION 3.1. *$\boldsymbol{G}$ is continuously Fréchet differentiable.*

Proof. The arguments are standard, but because of the large number of terms contained in the Fréchet derivative of $\boldsymbol{G}$, we restrict the discussion to a few representative terms.

The functional derivative with respect to w of the first term in G_1, namely $T := \sin a b''$, is the linear operator $B\colon H^2(\mathbb{R}) \to L^2(\mathbb{R}); \phi \mapsto \sin a \phi''$. This is clearly bounded, and depends continuously on u, w and Λ. The functional derivative of T with respect to u is the linear operator $A\colon \chi \mapsto b'' \cos a \chi$. We note that $b'' \in L^2(\mathbb{R})$

while $\chi, \cos a \in L^\infty(\mathbb{R})$, so that $b'' \cos a \chi \in L^2(\mathbb{R})$. Also, $\|\chi\|_{L^\infty}$ is bounded by $\|\chi\|_{H^2}$. It follows that B is a bounded map from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and is continuous in its arguments. The functional derivatives of the remaining terms with respect to u and w are treated similarly; in general, they can be expressed as linear maps $f \mapsto h$, where h is given by the product of an L^2-function and an L^∞-function, so that $h \in L^2(\mathbb{R})$. Moreover, the L^2-norm of h can be bounded by the H^2-norm of f.

Establishing continuous Fréchet differentiability with respect to Λ requires the vanishing-torque condition (3.5). Consider, for example, the term F_1^Λ as given by (3.4). It is necessary to show that $D_\Lambda F_1^\Lambda(a,b)$ belongs to $L^2(\mathbb{R})$ and that, as an element of $L^2(\mathbb{R})$, $D_\Lambda F_1^\Lambda(a,b)$ depends continuously on u, w and Λ. We have that

$$D_\Lambda F_1^\Lambda = \left(\frac{\partial F_1^\Lambda}{\partial a} D_\Lambda a + \frac{\partial F_1^\Lambda}{\partial b} D_\Lambda b + \frac{\partial F_1^\Lambda}{\partial \Lambda} \right)\Bigg|_{a,b,\Lambda}, \tag{3.10}$$

where the notation $f|_x$ indicates the evaluation of function f at the point x. Consider first the interval $\xi \in (0,\infty)$. From (3.6), $(a,b) = (\psi_+^\Lambda + s, \beta_+^\Lambda + t)$, where s, t, $D_\Lambda s$ and $D_\Lambda t$ all belong to $L^2((0,\infty))$. Substituting into (3.10), we may write that

$$D_\Lambda F_1^\Lambda = Q + \left(\frac{\partial F_1^\Lambda}{\partial a} D_\Lambda s + \frac{\partial F_1^\Lambda}{\partial b} D_\Lambda t \right)\Bigg|_{a,b}, \tag{3.11}$$

where

$$Q = \left(\frac{\partial F_1^\Lambda}{\partial a} D_\Lambda \psi_+^\Lambda + \frac{\partial F_1^\Lambda}{\partial b} D_\Lambda \beta_+^\Lambda + \frac{\partial F_1^\Lambda}{\partial \Lambda} \right)\Bigg|_{a,b}. \tag{3.12}$$

Since $\partial F_1^\Lambda / \partial a$ and $\partial F_1^\Lambda / \partial b$ are bounded, the terms

$$(\partial F_1^\Lambda / \partial a) D_\Lambda s \quad \text{and} \quad (\partial F_1^\Lambda / \partial b) D_\Lambda t$$

belong to $L^2((0,\infty))$. As for Q, we note that

$$Q|_{\psi_+^\Lambda, \beta_+^\Lambda, \Lambda} = \left(\frac{\partial F_1^\Lambda}{\partial a} D_\Lambda \psi_+^\Lambda + \frac{\partial F_1^\Lambda}{\partial b} D_\Lambda \beta_+^\Lambda + \frac{\partial F_1^\Lambda}{\partial \Lambda} \right)\Bigg|_{\psi_+^\Lambda, \beta_+^\Lambda, \Lambda} = 0, \tag{3.13}$$

which follows from differentiating the vanishing-torque condition (3.5) with respect to Λ. Using the mean-value theorem, we may write that

$$\frac{\partial F_1^\Lambda}{\partial a}(a,b) = \frac{\partial F_1^\Lambda}{\partial a}(\psi_+^\Lambda, \beta_+^\Lambda) + \frac{\partial^2 F_1^\Lambda}{\partial a^2}(\tilde{a},\tilde{b})s + \frac{\partial^2 F_1^\Lambda}{\partial a \partial b}(\hat{a},\hat{b},\Lambda)t, \tag{3.14}$$

where $\psi_+^\Lambda \leqslant \tilde{a}, \hat{a} \leqslant \psi_+^\Lambda + s$ and $\beta_+^\Lambda \leqslant \tilde{b}, \hat{b} \leqslant \beta_+^\Lambda + t$. Using similar expressions for the remaining terms in Q, we may write that

$$Q = \left(\frac{\partial^2 F_1^\Lambda}{\partial a^2} D_\Lambda \psi_+^\Lambda + \frac{\partial^2 F_1^\Lambda}{\partial a \partial b} D_\Lambda \beta_+^\Lambda + \frac{\partial^2 F_1^\Lambda}{\partial a \partial \Lambda} \right)\Bigg|_{\tilde{a},\tilde{b}} s$$
$$+ \left(\frac{\partial^2 F_1^\Lambda}{\partial a \partial b} D_\Lambda \psi_+^\Lambda + \frac{\partial^2 F_1^\Lambda}{\partial b^2} D_\Lambda \beta_+^\Lambda + \frac{\partial^2 F_1^\Lambda}{\partial b \partial \Lambda} \right)\Bigg|_{\hat{a},\hat{b}} t. \tag{3.15}$$

As s and t belong to $L^2((0,\infty))$ and their coefficients in (3.15) are bounded, it follows that $Q \in L^2((0,\infty))$.

An analogous argument shows that for $\xi < 0$, $D_\Lambda F_1^\Lambda(a,b) \in L^2((-\infty, 0))$. It follows that $D_\Lambda F_1^\Lambda(a,b) \in L^2(\mathbb{R})$, and is continuous with respect to u, w, Λ. One may apply similar arguments to all other terms, and the claim follows. $\qquad\square$

4. Existence of TWs

The existence of TWs is given by theorem 4.1 in the anisotropy-dominated case, and by theorem 4.5 in the transverse-field dominated case. Properties of certain one-dimensional Schrödinger operators relevant to the analysis are proved in lemmas 4.3, 4.4 and 4.6.

THEOREM 4.1 (anisotropy-dominated TWs). *For all parameters*

$$\Lambda = (H_1, H_2, H_3, K_2)$$

sufficiently close to Λ_W, there exists a solution $\hat{\boldsymbol{m}}(\xi; \Lambda)$ of the TW equation (2.2) with velocity $V(\Lambda)$. Both $\hat{\boldsymbol{m}}$ and V are continuously differentiable in Λ, and for $\Lambda = \Lambda_W$, $\hat{\boldsymbol{m}}$ coincides with the static profile $\hat{\boldsymbol{m}}_W$.

Proof. We recall the implicit-function theorem (see, for example, [9]): given Banach spaces X, Y, Z and a continuously Fréchet differentiable map $\boldsymbol{G} \colon X \times Y \to Z$ such that $\boldsymbol{G}(x_0, y_0) = 0$, if the linear operator $D_x \boldsymbol{G}(x_0, y_0) \colon X \to Z$ has bounded inverse, then there exist open balls $B_X = B_r(x_0) \subset X$ and $B_Y = B_s(y_0) \subset Y$ such that for all $y \in B_Y$, there exists a unique $x \in B_X$ satisfying $\boldsymbol{G}(x, y) = 0$. Moreover, the implicit function $f \colon B_Y \to B_X$ defined by $\boldsymbol{G}(f(y), y) = 0$ is continuously differentiable on B_Y.

We take X, Y, Z as in (3.8) and $\boldsymbol{G}$ as in (3.9), with $\Lambda_* = \Lambda_W = (0, 0, 0, K_2)$. From proposition 3.1, $\boldsymbol{G}$ is continuously Fréchet differentiable. We let $x_0 := (u_0, w_0, V_0) = (0, 0, 0)$ and $y_0 = \Lambda_W$, so that $\boldsymbol{G}(x_0, y_0) = 0$. We let

$$\mathcal{D}_W := D_{(u,w,V)}\boldsymbol{G}(0, 0, 0; \Lambda_W) \colon X \to Z. \tag{4.1}$$

Then for $(f_1, f_2, \mu) \in X = H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times \mathbb{R}$, we have that

$$\mathcal{D}(f_1, f_2, \mu) = (-Lf_2 + \alpha\mu\beta_W', -(L + K_2)f_1 - \mu\beta_W', \langle \beta_W', f_2 \rangle), \tag{4.2}$$

where L is the Schrödinger operator given by

$$L := -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + W(\xi), \qquad W(\xi) = \cos 2\beta_W = \frac{\beta_W'''}{\beta_W'} = 1 - 2\operatorname{sech}^2 \xi. \tag{4.3}$$

Clearly,

$$L\beta_W' = 0, \tag{4.4}$$

so that $\ker L$ is spanned by β_W'. (Note that the kernel of a Schrödinger operator is at most one dimensional, since its eigenfunctions are necessarily non-degenerate.)

First, we show that $\mathcal{D}_W$ is bounded away from zero. We write $f_j = f_j^\perp + c_j\beta_W'$, where $f_j^\perp$ is orthogonal to β_W'. From (4.2) and (4.4) it follows that

$$\|\mathcal{D}_W(f_1, f_2, \mu)\|_Z = \|(L+K_2)f_1^\perp\|_{L^2}^2 + \|Lf_2^\perp\|_{L^2}^2 + ((K_2c_1 + \mu)^2 + \alpha^2\mu^2 + c_2^2)\|\beta_W'\|_{L^2}^2. \tag{4.5}$$

For $\alpha \neq 0$, the quantity $(K_2 c_1 + \mu)^2 + \alpha^2 \mu^2$, regarded as a quadratic form in c_1 and μ, is positive definite, so that there exists $C > 0$ such that

$$\|\beta'_W\|_{L^2}((K_2 c_1 + \mu)^2 + c_2^2 + \alpha^2 \mu^2) \geqslant C(\|c_1 \beta'_W\|_{H^2}^2 + \|c_2 \beta'_W\|_{H^2}^2 + \mu^2).$$

From (4.3), it is clear that (i) W is smooth, (ii) $\lim_{\xi \to \pm\infty} W(\xi) = 1$, and (iii) $\lim_{\xi \to \pm\infty} W''(\xi) = 0$. In lemma 4.3, it is shown that if the potential W in a one-dimensional Schrödinger operator has these properties, then it is bounded away from zero on the orthogonal complement of its kernel, which for L is the subspace orthogonal to β'_W. In lemma 4.4 it is shown that $L + K_2$ has trivial kernel for $K_2 > 0$. Together with lemma 4.3, this implies that $L + K_2$ is bounded away from zero. Therefore, there exists $C' > 0$ such that

$$\|(L + K_2) f_1^\perp\|_{L^2}^2 \geqslant C'\|f_1^\perp\|_{H^2}^2, \qquad \|L f_2^\perp\|_{L^2}^2 \geqslant C'\|f_2^\perp\|_{H^2}^2.$$

Taking $C'' = \min(C, C')$, we obtain

$$\|\mathcal{D}_W(f_1, f_2, \mu)\|_Z \geqslant C''\|(f_1, f_2, \mu)\|_X. \tag{4.6}$$

Next, we establish that $\mathcal{D}_W \colon X \to Z$ is onto. The fact that $\mathcal{D}_W$ is bounded away from zero implies that $\mathcal{D}_W$ has closed range $R(\mathcal{D}_W)$, so it suffices to show that $R(\mathcal{D}_W)^\perp = 0$.

Suppose that $(\phi_1, \phi_2, r) \in R(\mathcal{D}_W)^\perp \subset Z = L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times \mathbb{R}$, so that

$$\langle (\phi_1, \phi_2, r), \mathcal{D}_W(f_1, f_2, \mu) \rangle_Z = 0 \quad \text{for all } (f_1, f_2, \mu) \in X. \tag{4.7}$$

Writing out the terms in (4.7), we get that

$$-\langle \phi_1, L f_2 - \alpha\mu\beta'_W \rangle_{L^2} - \langle \phi_2, (L + K_2) f_1 + \mu\beta'_W \rangle_{L^2} + r\langle \beta'_W, f_2 \rangle_{L^2} = 0. \tag{4.8}$$

Taking $f_2 = 0$, $\mu = 0$, we have for all $f_1 \in H^2(\mathbb{R})$ that

$$\langle \phi_2, (L + K_2) f_1 \rangle_{L^2} = 0.$$

Regarding L as an unbounded operator on $L^2(\mathbb{R})$ with domain $D(L)$, the preceding equation implies that $\phi_2 \in D(L^\dagger)$. Since L is self-adjoint, it follows that

$$0 = \langle \phi_2, (L + K_2) f_1 \rangle_{L^2} = \langle f_1, (L^\dagger + K_2)\phi_2 \rangle_{L^2} = \langle f_1, (L + K_2)\phi_2 \rangle_{L^2}.$$

Thus, $(L + K_2)\phi_2 = 0$. Lemma 4.4 implies that $L + K_2$ has trivial kernel, so that $\phi_2 = 0$.

Next, we take $f_1 = 0$, $\mu = 0$ in (4.8) to obtain that

$$\langle \phi_1, L f_2 \rangle_{L^2} = r\langle \beta'_W, f_2 \rangle_{L^2}$$

for all $f_2 \in H^2(\mathbb{R})$. Arguing as above, it follows that $\phi_1 \in D(L^\dagger)$, so that we have $\langle L\phi_1, f_2 \rangle_{L^2} = r\langle \beta'_W, f_2 \rangle_{L^2}$ for all $f_2 \in H^2(\mathbb{R})$, which implies that

$$L\phi_1 = r\beta'_W.$$

Taking inner products with β'_W, we get that $r = 0$ and that $\phi_1 \in \ker L = \text{span}\{\beta'_W\}$. Finally, taking $f_1 = f_2 = 0$ in (4.8), we see that

$$\langle \phi_1, \beta'_W \rangle_{L^2} = 0, \tag{4.9}$$

which implies that $\phi_1 = 0$.

It follows that $\mathcal{D}_W$ is onto, and therefore invertible. Since $\mathcal{D}_W$ is bounded away from zero, $\mathcal{D}_W^{-1}$ is bounded. The assertions in theorem 4.1 then follow from the implicit-function theorem. $\qquad\square$

REMARK 4.2. The potential W in (4.3) is a particular case of the modified Pöschl–Teller potential [14], whose spectral properties are known rather explicitly. We shall not make use of these explicit results, however, but instead give a self-contained analysis in lemmas 4.3 and 4.4. This has the advantage of carrying over to the case of transverse-field-dominated TWs, for which corresponding explicit results for the Schrödinger operator that appear there are not available.

LEMMA 4.3. *Let $L = -\mathrm{d}^2/\mathrm{d}\xi^2 + W(\xi)\colon H^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a Schrödinger operator. If* (i) *$W \in C^2(\mathbb{R})$,* (ii) *W has positive limiting values $W_\pm$ as $\xi \to \pm\infty$, and* (iii) *$W''(\xi) \to 0$ as $\xi \to \pm\infty$, then L is bounded away from zero on the orthogonal complement of $\ker L$. That is, there exists $C > 0$ such that*

$$\|Lu\|_{L^2(\mathbb{R})} \geqslant C\|u\|_{H^2(\mathbb{R})} \quad \text{for all } u \in \ker(L)^\perp.$$

Proof. Assuming to the contrary that there exists a normalized sequence $u_n \in \ker(L)^\perp \cap H^2(\mathbb{R})$ such that $Lu_n \to 0$ strongly in $L^2(\mathbb{R})$, we can then extract a weakly convergent subsequence (not relabelling), $u_n \rightharpoonup u$ in $H^2(\mathbb{R})$. As L is bounded, it follows that $Lu_n \rightharpoonup Lu$ weakly in $L^2(\mathbb{R})$. As the weak and strong limits must coincide, it follows that $Lu = 0$, i.e. $u \in \ker L$. Since $u \in (\ker L)^\perp$ by assumption, we conclude that $u = 0$, so that u_n converges weakly to 0 in $H^2(\mathbb{R})$.

Integrating by parts, we can write that

$$\|Lu_n\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} (|u_n''|^2 + 2W|u_n'|^2 + (W^2 - W'')|u_n|^2).$$

Taking $c = \min(W_+, W_-) > 0$, we may write that

$$\|Lu_n\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} (|u_n''|^2 + 2c|u_n'|^2 + c^2|u_n|^2) - J_n, \tag{4.10}$$

where

$$J_n := \int_{\mathbb{R}} 2(c - W)|u_n'|^2 + (c^2 - W^2 + W'')|u_n|^2.$$

Given $\varepsilon > 0$, we can choose $\ell > 0$ such that $2(c - W) > -\varepsilon$ and $c^2 - W^2 + W'' > -\varepsilon$ for $|\xi| > \ell$. Then

$$J_n \geqslant -\varepsilon + \int_{-\ell}^{\ell} 2(c - W)|u_n'|^2 + (c^2 - W^2 + W'')|u_n|^2,$$

since $\|u_n\|_{L^2}^2 + \|u_n'\|_{L^2}^2 \leqslant 1$. By the Rellich–Kondrachov compact embedding theorem, the fact that u_n converges weakly to 0 in $H^2(\mathbb{R})$ implies that u_n and u_n' converge strongly to 0 in $L^2((-\ell, \ell))$. Therefore,

$$\lim_{n\to\infty} \int_{-\ell}^{\ell} 2(c - W)|u_n'|^2 + (c^2 - W^2 + W'')|u_n|^2 = 0.$$

It follows that $\lim_{n\to\infty} J_n \geqslant -\varepsilon$, and, since ε is arbitrary, that $\lim_{n\to\infty} J_n \leqslant 0$. From (4.10), we conclude that

$$\lim_{n\to\infty} \|Lu_n\|^2_{L^2(\mathbb{R})} \geqslant \lim_{n\to\infty} \int_{\mathbb{R}} (|u''_n|^2 + 2c|u'_n|^2 + c^2|u_n|^2) \geqslant \min\{1, 2c, c^2\} > 0. \quad (4.11)$$

But this contradicts the fact that $Lu_n \to 0$. $\qquad\square$

LEMMA 4.4. *Let L be the Schrödinger operator given by (4.3). Then*

$$\langle \phi, (L + K_2)\phi \rangle_{L^2(\mathbb{R})} \geqslant K_2 \|\phi\|^2_{L_2}.$$

Proof. We take $\phi = u\beta'_W$. Since $|\beta'_W| > 0$, it follows that $u \in H^2_{\text{loc}}(\mathbb{R})$. Then

$$L\phi = -u''\beta'_W - 2u'\beta''_W,$$

and hence

$$\langle \phi, L\phi \rangle_{L^2} = -\int_{\mathbb{R}} u(u''\beta'^2_W + 2u'\beta'_W\beta''_W) = -\int_{\mathbb{R}} u(u'\beta'^2_W)' = \int_{\mathbb{R}} u'^2\beta'^2_W \geqslant 0,$$

where we have used the fact, easily checked, that $\beta'^2_W u u'$ vanishes at $\xi = \pm\infty$. Therefore, $\langle \phi, (L + K_2)\phi \rangle_{L^2(\mathbb{R})} \geqslant K_2 \|\phi\|^2_{L_2}$. $\qquad\square$

THEOREM 4.5 (transverse-field-dominated TWs). *For all parameters*

$$\Lambda = (H_1, H_2, H_3, K_2)$$

sufficiently close to Λ_T, there exists a solution $\hat{\boldsymbol{m}}(\xi; \Lambda)$ of the TW equation (2.2) with velocity $V(\Lambda)$. Both $\hat{\boldsymbol{m}}$ and V are continuously differentiable in Λ, and, for $\Lambda = \Lambda_T$, $\hat{\boldsymbol{m}}$ coincides with the static profile $\hat{\boldsymbol{m}}_T$.

Proof. When Λ_W is replaced by Λ_T, $\mathcal{D}_T := D_{(u,w,V)}\boldsymbol{G}(0,0,0;\Lambda_T)$ is given by

$$\mathcal{D}_T(f_1, f_2, \mu) = (-Mf_2 + \alpha\mu\beta'_T, -Nf_1 - \mu\beta'_T, \langle\beta'_T, f_2\rangle). \quad (4.12)$$

Here, $\beta_T(\xi) \in H^2(\mathbb{R})$ satisfies (see (2.8) and (2.9))

$$\beta'_T = H_3 - \sin\beta_T, \qquad \lim_{\xi\to-\infty}\beta_T = \pi - \sin^{-1}(H_3), \qquad \lim_{\xi\to\infty}\beta_T = \sin^{-1}(H_3), \quad (4.13)$$

and M, N are Schrödinger operators given by

$$M := -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{\beta'''_T}{\beta'_T}, \qquad N := -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{(\cos\beta_T)''}{\cos\beta_T} + 1. \quad (4.14)$$

Clearly,

$$M\beta'_T = 0, \quad (4.15)$$

so that β'_T spans $\ker M$.

Comparing (4.2) and (4.12), we see that $\mathcal{D}_T$ is obtained from $\mathcal{D}_W$ by making the replacements $\beta'_W \to \beta'_T$, $L \to M$ and $L + K_2 \to N$. The proof of theorem 4.1 carries over directly provided we show that M is bounded away from zero on $\ker(M)^\perp$, and that N is bounded away from zero.

Firstly, lemma 4.3 implies directly that $M = -\mathrm{d}^2/\mathrm{d}\xi^2 + W$ is bounded away from zero on $\ker(M)^\perp$. Secondly, with W taken to be $(\cos\beta_T)''/\cos\beta_T + 1$, the fact

that $N = -\mathrm{d}^2/\mathrm{d}\xi^2 + W$ is bounded away from zero follows from lemma 4.3 (as (4.13) implies that W is smooth, $\lim_{\xi \to \pm\infty} W(\xi) = 1$, and $\lim_{\xi \to \pm\infty} W''(\xi) = 0$), combined with lemma 4.6 below, which implies that N has trivial kernel. □

LEMMA 4.6. *Let N be the Schrödinger operator given by (4.14). Then*

$$\langle \phi, N\phi \rangle_{L^2(\mathbb{R})} \geqslant H_3^2 \|\phi\|_{L_2}^2.$$

Proof. From (4.2) and (4.12), we get that

$$N = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \cos 2\beta_T + 3H_3 \sin \beta_T - H_3^2, \quad (\beta_T')'' = (\cos 2\beta_T + H_3 \sin \beta_T)\beta_T'.$$

Therefore,

$$N\beta_T' = H_3(2\sin \beta_T - H_3)\beta_T'.$$

We let $\phi = u\beta_T'$. Since $|\beta_T'| < 0$, it follows that $u \in H_{\mathrm{loc}}^2(\mathbb{R})$. A calculation gives

$$N\phi = -u''\beta_T' - 2u'\beta_T'' + H_3(2\sin \beta_T - H_3)\phi,$$

and hence

$$
\begin{aligned}
\langle \phi, N\phi \rangle_{L^2(\mathbb{R})} &= -\int_{\mathbb{R}} u(u''\beta_T'^2 + 2u'\beta_T'\beta_T'') + H_3 \int_{\mathbb{R}} (2\sin \beta_T - H_3)\phi^2 \\
&= \int_{\mathbb{R}} u'^2 \beta_T'^2 + H_3 \int_{\mathbb{R}} (2\sin \beta_T - H_3)\phi^2 \\
&\geqslant H_3^2 \|\phi\|_{L^2}^2,
\end{aligned}
$$

where in the second equality we have used the fact (easily checked) that $\beta_T'^2 uu'$ vanishes at $\xi = \pm\infty$, while in the last equality we have $\sin \beta_T \geqslant H_3$. □

5. Conclusion

We have proven existence of TW solutions to the LLG equation in a thin ferromagnetic nanowire. This was accomplished by virtue of the implicit-function theorem, which provides local existence and uniqueness (up to translation) of TW profiles close to static energy minimizers satisfying tail-to-tail boundary conditions (thus representing propagating domain walls), with velocities in a neighbourhood of 0. We proved the existence of solutions of this type for values of the physical parameters close to two ranges in the parameter space: the first with zero applied field $\boldsymbol{H}_a = 0$ and non-zero transverse anisotropy $K_2 > 0$, and the second with non-vanishing transverse applied field $\boldsymbol{H}_a = H_2\hat{\boldsymbol{y}} + H_3\hat{\boldsymbol{z}}$, $0 < H_2^2 + H_3^2 < 1$, and vanishing hard axis anisotropy $K_2 = 0$.

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