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Smoothness of stable holonomies inside center-stable manifolds

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Abstract. Under a suitable bunching condition, we establish that stable holonomies inside center-stable manifolds for $C^{1+\beta}$ diffeomorphisms are uniformly bi-Lipschitz and, in fact, $C^{1+\text{Hölder}}$. This verifies the ergodicity of suitably center-bunched, essentially accessible, partially hyperbolic $C^{1+\beta}$ diffeomorphisms and verifies that the Ledrappier–Young entropy formula holds for $C^{1+\beta}$ diffeomorphisms of compact manifolds.

Key words: holonomies, partial hyperbolicity, ergodicity, entropy
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1. *Introduction*

In this paper we establish that 'fake' stable holonomies inside 'fake' center-stable manifolds for sufficiently bunched $C^{1+\beta}$ diffeomorphismsare uniformly bi-Lipschitz and, in fact, $C^{1+\text{Hölder}}$. This establishes two folklore results in smooth ergodic theory, the primary motivation for this paper. We explain these two folklore results in the remainder of this introduction in order to motivate our main technical result.

1.1. *Ergodicity of partially hyperbolic diffeomorphisms.*

In [**BW1**], Burns and Wilkinson established the ergodicity (and K-property) of partially hyperbolic, center-bunched, essentially accessible, C^2 volume-preserving diffeomorphisms. This extends a number of earlier results, including [**GPS**, **PS2**]. A similar result (with stronger center-bunching conditions) was announced for $C^{1+\delta}$-diffeomorphisms. However, it seems that the bunching condition given in [**BW1**, Theorem 0.3] is possibly too weak for the method of proof. A proof of the technical result needed to establish [**BW1**, Theorem 0.3] was circulated as an unpublished note in [**BW2**]. It seems some of the details of the proof in [**BW2**] are incorrect, specifically [**BW2**, Lemma 1.1]. We replace this lemma with Lemma 3.1 below; however, we note that our Lemma 3.1 requires a stronger bunching condition imposed by condition (6) below. The results of this note replace the main result in [BW2] and we obtain a proof of [BW1, Theorem 0.3] (under a stronger bunching hypothesis).

We refer the reader to [BW1] for definitions and complete arguments. Let M be a compact manifold and, for $\beta > 0$, let $f: M \to M$ be a $C^{1+\beta}$ diffeomorphism. We assume f admits a continuous, (Df)-equivariant partially hyperbolic splitting

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x);$$

in particular, there are continuous functions

$$\mu(x) < \nu(x) < \gamma(x) < \hat{\gamma}(x)^{-1} < \hat{\nu}(x)^{-1} < \hat{\mu}(x)^{-1}$$

with $\hat{\nu}(x), \nu(x) < 1$ such that:

- $\mu(x)\|v\| < \|D_x f v\| < \nu(x)\|v\|$, for all $v \in E^s(x)$;
- $\gamma(x)\|v\| < \|D_x f v\| < \hat{\gamma}(x)^{-1}\|v\|$, for all $v \in E^c(x)$;
- $\hat{\nu}(x)^{-1}\|v\| < \|D_x f v\| < \hat{\mu}(x)^{-1}\|v\|$, for all $v \in E^u(x)$.

THEOREM 1.1. *Let $f: M \to M$ be a volume-preserving, essentially accessible, partially hyperbolic $C^{1+\beta}$ diffeomorphism. Let $\bar{\theta} < \beta$ be such that*

$$\nu(x)\gamma^{-1}(x) < \mu(x)^{\bar{\theta}}, \quad \hat{\nu}(x)\hat{\gamma}^{-1}(x) < \hat{\mu}(x)^{\bar{\theta}}. \tag{1}$$

Assume f satisfies the strong center-bunching hypothesis: there exists $0 < \theta < \bar{\theta}$ such that for all $x \in M$,

$$\nu(x)^{\theta} > \nu(x)^{\beta}\gamma(x)^{-\beta}, \quad \hat{\nu}(x)^{\theta} > \hat{\nu}(x)^{\beta}\hat{\gamma}(x)^{-\beta}, \tag{2}$$

and

$$\max\{\nu(x), \hat{\nu}(x)\}^{\theta} < \gamma(x)\hat{\gamma}(x). \tag{3}$$

Then f is ergodic and has the K-property.

In [BW1, Theorem 0.3], the conclusion of Theorem 1.1 is asserted to hold under the bunching condition

$$\max\{\nu(x), \hat{\nu}(x)\}^{\bar{\theta}} < \gamma(x)\hat{\gamma}(x) \tag{4}$$

where $\bar{\theta}$ satisfies (1). Since $0 < \max\{\nu(x), \hat{\nu}(x)\} < 1$, if $\theta < \bar{\theta}$ we have

$$\max\{\nu(x), \hat{\nu}(x)\}^{\bar{\theta}} < \max\{\nu(x), \hat{\nu}(x)\}^{\theta}$$

and (3) implies (4). In many applications one expects $\gamma(x) < 1$, whence

$$\nu(x) < \nu(x)\gamma(x)^{-1} < \nu(x)^{\beta}\gamma(x)^{-\beta} < 1.$$

In particular, condition (2) is non-trivial even when (1) holds and the bunching condition (3) is strictly stronger than (4). Our proof seems to require a stronger bunching condition in (3) implied by the additional estimate (2).

For justification of Theorem 1.1, we refer the reader to the paragraph preceding [BW1, Theorem 0.3]. See also the discussion in the paragraph preceding ‘Step 2’ on page 467 of [BW1] where Theorem 2.4 below replaces [BW1, Proposition 3.1(vii)].

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We note that if $\bar{\theta}$ satisfies (1) then $\theta = \bar{\theta}\beta$ satisfies (2) and we have the following corollary.

COROLLARY 1.2. *If, in Theorem 1.1, we have*

$$\max\{\nu(x), \hat{\nu}(x)\}^{\bar{\theta}\beta} < \gamma(x)\hat{\gamma}(x)$$

then f is ergodic and has the K-property.

Remark 1.3. Theorem 2.4 below establishes the smoothness of stable holonomies inside center-stable manifolds for a choice of 'globalized' dynamics. In the language of [**BW1**], this establishes the smoothness of holonomy maps by *fake stable manifolds* inside *fake center-stable manifolds*. See the discussion in [**BW1**, Proposition 3.1(vii)]. We note that dynamical coherence and the existence of 'genuine' center manifolds is neither assumed nor required in the proof of [**BW1**].

In the case where the partially hyperbolic diffeomorphism $f: M \to M$ is dynamically coherent, one could likely adapt the proof of Theorem 2.4 to show that the holonomy maps by 'genuine' stable manifolds inside 'genuine' center-stable manifolds are $C^{1+\text{Hölder}}$.

We emphasize that in the case where $f: M \to M$ is dynamically coherent, the center-stable and center manifolds discussed in what follows are not the center-stable and center manifolds for the dynamics of f. In particular, while f might admit compact center manifolds, the 'fake' center manifolds we consider will never be compact.

1.2. *Ledrappier–Young entropy formula.* In two seminal papers [**LY1**, **LY2**], Ledrappier and Young established remarkable results relating the metric entropy of a C^2 diffeomorphism $f: M \to M$ of a compact manifold M, its Lyapunov exponents, and the geometry of conditional measures along unstable manifolds. In [**LY1**], the Sinai–Ruelle–Bowen (SRB) property of measures satisfying the Pesin entropy formula is established for C^2 diffeomorphisms and measures with zero Lyapunov exponents. This extends Ledrappier's result from [**Led**] which established the SRB property for hyperbolic measures invariant under $C^{1+\beta}$ diffeomorphisms satisfying the Pesin entropy formula. In [**LY2**], a more general formula (in terms of Lyapunov exponents and transverse conditional dimensions) for the entropy $h_\mu(f)$ of f with respect to a general ergodic f-invariant probability measure μ is derived.

As remarked in [**LY1**, pp. 526], there is one crucial step in which the C^2 hypothesis rather than the $C^{1+\beta}$ hypothesis on the dynamics is used: establishing the Lipschitzness of unstable holonomies inside center-unstable sets. In [**LY2**], the corresponding estimate is the Lipschitzness of the holonomies along intermediate unstable foliations inside the total unstable manifolds. In the case of hyperbolic measures, the entropy formula from [**LY2**] is known to hold for $C^{1+\beta}$ diffeomorphisms as it is sufficient to establish the Lipschitzness of W^i holonomies inside the W^{i+1} manifolds (corresponding to Lyapunov exponents $\lambda_i > \lambda_{i+1} > 0$) on Pesin sets; this Lipschitzness of holonomies along intermediate unstable manifolds was established in [**BPS**, Appendix]. However, the proof in [**BPS**, Appendix] does not imply Lipschitzness of unstable holonomies inside center-unstable sets which is essential in the proof of the main technical result of [**LY1**]: that the entropy of f is 'carried entirely by the unstable manifolds'; see [**LY1**, Corollary 5.2]. The results of this note establish the Lipschitzness of unstable holonomies inside center-unstable sets which confirms that the results of [**LY1**, **LY2**] hold for $C^{1+\beta}$ diffeomorphisms and invariant measures with zero Lyapunov exponents. See §4 for a brief formulation and justification of the Lipschitzness of unstable holonomies inside center-unstable sets.

To formulate results, fix $\beta > 0$ and let $f: M \to M$ be a $C^{1+\beta}$ diffeomorphism of a compact k-dimensional manifold M. Let μ be an ergodic, f-invariant Borel probability measure. We have the following generalizations of the main results of [**LY1**, **LY2**].

THEOREM 1.4. [**LY2**, Theorem A] *$h_\mu(f)$ satisfies the Pesin entropy formula if and only if μ has the SRB property.*

THEOREM 1.5. [**LY2**, Theorem C′] *For a general ergodic, f-invariant probability measure μ, the entropy formula of [**LY2**, Theorem C′] remains valid.*

1.3. *Outline.* In §2 we present an abstract setup in which the dynamics is assumed to be a perturbation of linear dynamics. Our main result, Theorem 2.2, establishes that stable holonomies are uniformly Lipschitz and, in fact, $C^{1+\text{Hölder}}$ under certain bunching conditions. We formulate our main result, Theorem 2.2, in a sufficiently abstract setting so that it may be applied to a number of settings. We reduce the proof of Theorem 2.2 to the special case of Theorem 2.4. Section 3 is then devoted to the proof of Theorem 2.4. In §4 we briefly formulate and justify the main technical fact needed to establish Theorems 1.4 and 1.5.

2. *Statement of main theorem*

Our main result, Theorem 2.2 below, concerns the smoothness of stable holonomies inside center-stable manifolds for sequences of $C^{1+\beta}$ diffeomorphisms that are assumed to be perturbations of linear maps $L_n: \mathbb{R}^k \to \mathbb{R}^k$ with prescribed hyperbolicity properties. We briefly explain the relationship between this setup and the results outlined above.

Consider a $C^{1+\beta}$ diffeomorphism $f: M \to M$ of a compact manifold and $x \in M$. Using exponential charts $\exp_x: T_xM \to M$ we identify the local dynamics of f near x with a $C^{1+\beta}$ diffeomorphism f_x from a neighborhood of 0 in T_xM to a neighborhood of 0 in $T_{f(x)}M$. Interpolating between f_x and D_xf via a bump function, we extend f_x to a $C^{1+\beta}$ diffeomorphism

$$F_x: T_xM \to T_{f(x)}M$$

which coincides with D_xf outside a neighborhood of the origin. Taking the domain of the bump function sufficiently small, we may further assume $\|F_x - D_xf\|_{C^1}$ and $\|F_x{}^{-1} - (D_xf)^{-1}\|_{C^1}$ are sufficiently small. If $\|F_x - D_xf\|_{C^1}$ and $\|F_x{}^{-1} - (D_xf)^{-1}\|_{C^1}$ are sufficiently small, hyperbolicity properties of Df along the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ induce analogous hyperbolicity properties for the sequence of diffeomorphisms $\{F_{f^n(x)} : n \in \mathbb{Z}\}$. In particular, if f admits a partially hyperbolic splitting then the sequence of maps $\{F_{f^n(x)} : n \in \mathbb{Z}\}$ admits a partially hyperbolic splitting. Moreover, if f satisfies bunching conditions as in (1), (2), and (3) then the sequence of maps $\{F_{f^n(x)} : n \in \mathbb{Z}\}$ satisfy analogous bunching conditions.

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In the setting of non-uniformly hyperbolic dynamics, given a bi-regular point x for the derivative cocycle, one may perform a further sequence of coordinate changes on each $T_{f^n(x)}M$; these are the so-called Lyapunov charts discussed in §4. Relative to these new coordinates one may assume the globalized dynamics $\{F_{f^n(x)} : n \in \mathbb{Z}\}$ is uniformly partially hyperbolic and the hyperbolicity estimates are related to the Lyapunov exponents of the sequence of linear maps $\{D_{f^n(x)} f : n \in \mathbb{Z}\}$.

We formulate our main theorem for dynamics $\{F_{f^n(x)} : n \in \mathbb{Z}\}$ that are perturbations of linear maps, usually thinking of them as globalizations of local dynamics in local coordinates. We establish smoothness of stable holonomies inside center-stable manifolds for these globalized dynamics. If f is a partially hyperbolic diffeomorphism satisfying sufficient bunching conditions then, in the language of [**BW1**, Proposition 3.1], this establishes smoothness of 'fake' stable holonomies inside 'fake' center-stable manifolds associated to an orbit $\{f^n(x) : n \in \mathbb{Z}\}$.

2.1. *Setup.* Fix $k \in \mathbb{N}$ and let $\mathbb{R}^k$ be decomposed into subvector spaces

$$\mathbb{R}^k = \mathbb{R}^s \oplus \mathbb{R}^c \oplus \mathbb{R}^u.$$

For each $n \in \mathbb{Z}$, let $A_n : \mathbb{R}^s \to \mathbb{R}^s$, $B_n : \mathbb{R}^c \to \mathbb{R}^c$, and $C_n : \mathbb{R}^u \to \mathbb{R}^u$ be invertible linear maps and let

$$L_n = \begin{pmatrix} A_n & 0 & 0 \\ 0 & B_n & 0 \\ 0 & 0 & C_n \end{pmatrix}$$

be the associated invertible linear map preserving the decomposition $\mathbb{R}^k = \mathbb{R}^s \oplus \mathbb{R}^c \oplus \mathbb{R}^u$. We assume each component of the decomposition $\mathbb{R}^k = \mathbb{R}^s \oplus \mathbb{R}^c \oplus \mathbb{R}^u$ is non-trivial, though the results can be formulated (with fewer conditions) in the case where $\mathbb{R}^u$ is degenerate.

We assume there are constants

$$-\mu < \eta'_n < \kappa'_n < \gamma'_n \le \hat{\gamma}'_n < \hat{\kappa}'_n < \hat{\eta}'_n < \mu$$

such that for every $n \in \mathbb{Z}$,

(1) $e^{\eta'_n} \le m(A_n) \le \|A_n\| \le e^{\kappa'_n}$,
(2) $e^{\gamma'_n} \le m(B_n) \le \|B_n\| \le e^{\hat{\gamma}'_n}$,
(3) $e^{\hat{\kappa}'_n} \le m(C_n) \le \|C_n\| \le e^{\hat{\eta}'_n}$.

Here $\|\cdot\|$ is the operator norm induced by the standard norm on the corresponding Euclidean spaces and $m(A) := \|A^{-1}\|^{-1}$ denotes the associated conorm of A. Throughout, we will further assume that $\sup\{\kappa'_n\} < 0$. We do not impose any assumptions on the signs† of γ'_n, $\hat{\gamma}'_n$, $\hat{\kappa}'_n$ and $\hat{\eta}'_n$.

We assume, moreover, that

$$\inf\{|\kappa'_n - \gamma'_n|, \hat{\kappa}'_n - \hat{\gamma}'_n\} > 0.$$

† In the context of Theorem 1.1, we may assume that $\inf\{\hat{\kappa}'_n\} > 0$. In the context of Theorems 1.4 and 1.5, given any fixed $\epsilon > 0$, we may assume that $\gamma'_n = -\epsilon < 0 < \epsilon = \hat{\gamma}'_n$ and that $\inf\{\hat{\kappa}'_n\} > 0$. We also note in the case where $\sup\{\hat{\gamma}'_n\} < 0$, our main result, Theorem 2.2 below, should follow from [**BPS**, Appendix], perhaps with stronger bunching conditions.

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Fix some

$$0 < \epsilon_0 \le \inf\{|\kappa_n' - \gamma_n'|, \hat{\kappa}_n' - \hat{\gamma}_n', |\kappa_n'|, 1\}/10.$$

Anticipating perturbing the linear maps L_n below, we set

$$\kappa_n = \kappa_n' + \epsilon_0, \quad \hat{\kappa}_n = \hat{\kappa}_n' - \epsilon_0, \quad \eta_n = \eta_n' - \epsilon_0, \quad \hat{\eta}_n = \hat{\eta}_n' + \epsilon_0,$$

$$\gamma_n = \gamma_n' - \epsilon_0, \quad \hat{\gamma}_n = \hat{\gamma}_n' + \epsilon_0.$$

2.2. *Bunching criteria.* Fix $0 < \beta < 1$, which will be the Hölder regularity of the derivatives of perturbations of L_n below. We assume that $\hat{\gamma}_n' - \gamma_n'$ and ϵ_0 are sufficiently small so that there exists $0 < \bar{\theta} < \beta$ satisfying

$$\sup(\kappa_n - \eta_n\bar{\theta} - \gamma_n) < 0 \quad \text{and} \quad \sup(-\hat{\kappa}_n + \hat{\eta}_n\bar{\theta} + \hat{\gamma}_n) < 0 \tag{5}$$

and $\theta < \bar{\theta}$ with

$$\sup(\kappa_n\beta - \kappa_n\theta - \gamma_n\beta) < 0 \tag{6}$$

and

$$\sup(\hat{\gamma}_n - \gamma_n + \kappa_n\theta) < 0. \tag{7}$$

These are the analogues of (1), (2), and (3) above.

Condition (5) ensures that certain invariant distributions defined below are uniformly $\bar{\theta}$-Hölder. Condition (7) is a standard bunching condition. Note that with $\theta = \bar{\theta}$, (7) is the bunching condition stated in [**BW1**, Theorem 0.3]. Our proof, however, requires a stronger bunching criteria imposed by (6). In particular, we use heavily (6) in our proof of Lemma 3.1 below. Note from (5) that $\theta = \beta\bar{\theta}$ satisfies (6).

2.3. *Family of perturbations.* We introduce the dynamics f_n we study for the remainder as C^1 small perturbations of the linear maps L_n. We begin with some notational conventions used throughout the paper.

2.3.1. *Notational conventions.* We let $\|\cdot\|$ denote the standard Euclidean norm on $\mathbb{R}^k$ and write d for the induced distance. Given a subspace $U \subset \mathbb{R}^k$, we write SU for the unit sphere in U relative to the Euclidean norm $\|\cdot\|$. If $T\colon U \to V$ is linear we write $T_*\colon SU \to SV$ for the induced map. We recall that if $T\colon U \to V$ is a linear isomorphism with $a \le m(T) \le \|T\| \le b$ then T_* is bi-Lipschitz with constants $b^{-1}a$ and ba^{-1}. Finally, if $N \subset \mathbb{R}^k$ is an embedded submanifold we write $SN := STN$ for the unit sphere bundle over N. Given a diffeomorphism $g\colon N_1 \to N_2$, we write $g_*\colon SN_1 \to SN_2$ for the renormalized derivative map

$$g_*(x, v) = \left(g(x), \frac{1}{\|D_x g(v)\|} D_x g(v)\right).$$

In what follows, we consider $C^{1+\beta}$ diffeomorphisms $f\colon \mathbb{R}^k \to \mathbb{R}^k$ with uniform estimates on the $(1+\beta)$-norms: namely, viewing $x \mapsto D_x f$ as a map from $\mathbb{R}^k$ to the space

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of linear maps, we assume that $\sup_{x\in\mathbb{R}^k}\|D_x f\| < \infty$ and that Df is β-Hölder continuous with

$$\text{Höl}^\beta(Df) := \sup_{x\neq y}\left\{\frac{\|D_x f - D_y f\|}{d(x,y)^\beta}\right\} < \infty.$$

Given submanifolds N_1 and N_2 and a diffeomorphism $h\colon N_1 \to N_2$ then, as the linear maps $D_x h$ and $D_y h$ have different domains for $x \neq y \in N_1$, we define the Hölder variation of Dh and h_* as functions between metric spaces: Assuming N_1 has bounded diameter, define the β-Hölder variation of $Dh\colon TN_1 \to TN_2$ to be

$$\text{Höl}^\beta(Dh) := \sup_{(x,v)\neq(y,u)\in SN_1}\left\{\frac{d(Dh(x,v), Dh(y,u))}{d((x,v),(y,u))^\beta}\right\}$$

where, given (x,v) and (y,u) in $T\mathbb{R}^k$, we write

$$d((x,v),(y,u)) = \max\{d(x,y), d(v,u)\}.$$

Similarly define $\text{Höl}^\beta(h_*)$. The $C^{1+\beta}$-norm of h is $\max\{\|h\|_{C^1}, \text{Höl}^\beta(Dh)\}$.

2.3.2. *Families of perturbations.* For the remainder of §2 and throughout §3, we fix $f_n\colon \mathbb{R}^k \to \mathbb{R}^k$ to be a sequence of $C^{1+\beta}$ diffeomorphisms with $f_n(0) = 0$ for each n. Fix $\epsilon' > 0$ sufficiently small satisfying Proposition 2.1 below. We assume there is a $C_0 > 1$ such that for each $n \in \mathbb{Z}$:

(1) $\|f_n - L_n\|_{C^1} \leq \epsilon'$, and $\|f_n{}^{-1} - L_n{}^{-1}\|_{C^1} \leq \epsilon'$;
(2) $\text{Höl}^\beta(Df_n) < C_0$, and $\text{Höl}^\beta(Df_n{}^{-1}) < C_0$.

Note then that for some $C_1 \geq C_0 > 1$ we have:

(3) $\text{Höl}^\beta_{\text{loc}}((f_n)_*) \leq C_1$, and $\text{Höl}^\beta_{\text{loc}}((f_n{}^{-1})_*) \leq C_1$;
(4) $\|(D_x f_n^{\pm1})_*\|_{C^1} \leq C_1$ and $\|D_x f_n^{\pm1}\| \leq C_1$ for every x.

Here, $\text{Höl}^\beta_{\text{loc}}(f_*)$ is the local Hölder variation of $f_*\colon S\mathbb{R}^k \to S\mathbb{R}^k$ defined as

$$\text{Höl}^\beta_{\text{loc}}(f_*) := \sup_{0<d((x,v),(y,u))\leq 1}\left\{\frac{d(f_*(x,v), f_*(y,u))}{d((x,v),(y,u))^\beta}\right\}.$$

Moreover, as it holds in all applications we have in mind, one may assume that $f_n(y) = L_n(y)$ for all y with $\|y\| \geq 1$.

From the graph transform method, given $\epsilon' > 0$ sufficiently small and a sequence f_n of diffeomorphisms as above, we may construct foliations of $\mathbb{R}^k$ by pseudo-stable and pseudo-unstable manifolds. (See [**HPS**, Theorem 5.1], [**BW1**, Proposition 3.1], or [**PS1**, Theorem 3.16] for more details.) To summarize, we have the following proposition.

PROPOSITION 2.1. *There exist $\beta' > \bar\theta$ and $\beta'' > 0$ so that for every sufficiently small $\epsilon' > 0$ and every $C_0 > 1$ as above there is a $\hat C > 0$ such that for every $n \in \mathbb{Z}$, $\star = \{u, c, s, cu, cs\}$, and $x \in \mathbb{R}^k$ there are manifolds $W_n^\star(x)$ containing x with:*

(1) $e^{\hat\kappa_n} d(x,y) \leq d(f_n(x), f_n(y)) \leq e^{\hat\eta_n} d(x,y)$ *for* $y \in W_n^u(x)$;
(2) $e^{\gamma_n} d(x,y) \leq d(f_n(x), f_n(y)) \leq e^{\hat\eta_n} d(x,y)$ *for* $y \in W_n^{cu}(x)$;
(3) $e^{\gamma_n} d(x,y) \leq d(f_n(x), f_n(y)) \leq e^{\hat\gamma_n} d(x,y)$ *for* $y \in W_n^c(x)$;
(4) $e^{\eta_n} d(x,y) \leq d(f_n(x), f_n(y)) \leq e^{\hat\gamma_n} d(x,y)$ *for* $y \in W_n^{cs}(x)$;

(5) *$e^{\eta_n} d(x, y) \le d(f_n(x), f_n(y)) \le e^{\kappa_n} d(x, y)$ for $y \in W^s_n(x)$;*

(6) *$f_n(W^\star_n(x)) = W^\star_{n+1}(f_n(x))$ for every $x \in \mathbb{R}^k$ and $n \in \mathbb{Z}$.*

(7) *If $y \in W^\star_n(x)$ then $W^\star_n(x) = W^\star_n(y)$. In particular, the partition into $W^\star_n$-manifolds foliates $\mathbb{R}^k$; moreover, the partition into W^s_n-manifolds subfoliates each $W^{cs}_n(x)$.*

(8) *Each $W^\star_n(x)$ is the graph of a $C^{1+\text{Hölder}}$ function $G^\star_n(x) \colon \mathbb{R}^\star \to (\mathbb{R}^\star)^\perp$ with $\|D_u G^\star_n(x)\| \le \frac{1}{3}$ for all $u \in \mathbb{R}^\star$ and*

  (a) $\text{Höl}^{\beta}(DG^\star_n(x)) \le \hat{C}$ *for* $\star = s$;

  (b) $\text{Höl}^{\beta'}(DG^\star_n(x)) \le \hat{C}$ *for* $\star = cs, c, cu$;

  (c) $\text{Höl}^{\beta''}(DG^\star_n(x)) \le \hat{C}$ *for* $\star = u$.

  Moreover, the functions $G^\star_n(x)$ depend continuously (in the $C^{1+\text{Hölder}}$ topology) on x.

For a discussion of the $C^{1+\text{Hölder}}$-regularity of individual leaves in property (8) (of Proposition 2.1), see for instance [**PS1**, §6]. Note, in particular, that while the foliation of $\mathbb{R}^k$ into $W^\star_n$-leaves is only continuous, each leaf $W^\star_n(x)$ is a uniformly $C^{1+\text{Hölder}}$-embedded submanifold.

Write $E^\star_n(x) := T_x W^\star_n(x)$. From our choice of $\bar{\theta} > 0$ satisfying (5), it follows (for example, from the C^r-section theorem [**HPS**, p. 30]; see also discussion following [**BW1**]) that the tangent spaces $E^\star_n(x)$ are Hölder continuous with exponent $\bar{\theta}$ and constant uniform in $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ for $\star = cu, cs$, and hence for $\star = c$. For $\star = s, u$, the distributions $E^\star_n(x)$ are Hölder continuous with exponent satisfying analogues of (5). As discussed in [**PS1**, Theorem 6.6], each $W^{cu}_n(x)$ is $C^{1+\beta'}$ whenever $0 < \beta' \le \beta$ satisfies

$$\sup\{\kappa_n - (1+\beta')\gamma_n\} < 0.$$

Since $\eta_n < \gamma_n$, we have

$$\sup\{\kappa_n - (1+\beta')\gamma_n\} < \sup\{\kappa_n - \gamma_n - \beta'\eta_n\}$$

and equation (5) implies each $W^{cu}_n(x)$ is $C^{1+\beta'}$ for some $\beta' > \bar{\theta}$. Similarly, equation (5) implies each $W^{cs}_n(x)$ is $C^{1+\beta'}$ for some $\beta' > \bar{\theta}$; thus each $W^c_n(x)$ is $C^{1+\beta'}$. We also note in the case where $\inf\{\hat{\kappa}_n\} > 0$, we may take $\beta'' = \beta$.

We write

$$W^\star_n(x, R) := \{y \in W^\star_n(x) : d(x, y) < R\}.$$

2.4. *$C^{1+\text{Hölder}}$ holonomies inside center-stable manifolds.* Fix $R > 0$. Fix $p \in \mathbb{R}^k$, $n \in \mathbb{Z}$, and $q \in W^s_n(p, R)$. Let $\hat{D}_1$ and $\hat{D}_2$ be two uniformly $C^{1+\beta'}$ embedded, $\dim(\mathbb{R}^{cu})$-dimensional manifolds without boundary and with $p \in \hat{D}_1$ and $q \in \hat{D}_2$. We assume the diameter of each D_i is less than 1 and that each subspace $T_x \hat{D}_i$ is sufficiently transverse to $\mathbb{R}^s$: Given $v \in T_x \hat{D}_i$, write $v = v^s + v^{cu}$ where $v^s \in \mathbb{R}^s$ and $v^{cu} \in \mathbb{R}^{cu}$; we then assume $\|v^{cu}\| \ge 3\|v^s\|$. Let $D_1 = W^{cs}_n(p) \cap \hat{D}_1$ and $D_2 = W^{cs}_n(q) \cap \hat{D}_2$. Given $x \in D_1$, let $h_{D_1,D_2}(x)$ denote the unique point y in D_2 with $y \in W^s_n(x)$ if such a point exists. Note that the domain and codomain of h_{D_1,D_2} are open subsets of D_1 and D_2. By restriction of domain and codomain we may assume $h_{D_1,D_2} \colon D_1 \to D_2$ is a homeomorphism.

Our main result is the following theorem.

THEOREM 2.2. *The map h_{D_1,D_2} is a $C^{1+\hat{\alpha}}$ diffeomorphism for some $\hat{\alpha} > 0$. Moreover, the $C^{1+\hat{\alpha}}$-norm of h_{D_1,D_2} is uniform across all choices of n, p, q, D_1, and D_2 as above.*

In particular, we have the following corollary.

COROLLARY 2.3. *The map h_{D_1,D_2} is bi-Lipschitz with Lipschitz constants uniform in all choices of n, p, q, D_1, and D_2 as above.*

2.5. *Main theorem.* As discussed below, it is sufficient to prove a special case of Theorem 2.2.

Recall that each $W^c_n(x)$ is a uniformly $C^{1+\beta'}$-embedded manifold and intersects $W^s_n(y)$ for every $y \in W^{cs}_n(x)$. Moreover, each $E^{cu}_n(y)$ is uniformly transverse to both $\mathbb{R}^s$ and $E^s_n(y)$. It suffices to prove Theorem 2.2 for the distinguished family of transversals to W^s_n given by the family of center manifolds. Given $n \in \mathbb{Z}$, $p \in \mathbb{R}^k$, and $q \in W^s_n(p)$, we write $h^s_{p,q,n}\colon W^c_n(p) \to W^c_n(q)$ for the stable holonomy map between center manifolds. More precisely, given $z \in W^c_n(p)$, let

$$h^s_{p,q,n}(z) = W^{cu}_n(q) \cap W^s_n(z).$$

As both $\{W^s_n(x) : x \in W^{cs}_n(p)\}$ and $\{W^c_n(x) : x \in W^{cs}_n(p)\}$ subfoliate $W^{cs}_n(p)$, it follows that $h^s_{p,q,n}(z) \in W^c_n(q)$. Moreover, by the global transverseness of the manifolds, the maps $h^s_{p,q,n}$ have domain all of $W^c_n(p)$ and map onto $W^c_n(q)$.

The main result of this paper is the following theorem.

THEOREM 2.4. *There exist $0 < R_0 < 1$ and $\hat{\alpha} > 0$ with the following property. Let $p \in \mathbb{R}^k$ and $q \in W^s_n(p, R_0)$. Then the holonomy map*

$$h^s_{p,q,n}\colon W^c_n(p, 1) \to W^c_n(q)$$

is a $C^{1+\hat{\alpha}}$-diffeomorphism onto its image. Moreover, the $C^{1+\hat{\alpha}}$-norm of $h^s_{p,q,n}$ is uniform across the choice of p, q and n.

The $R_0 > 0$ for which the theorem holds is given by criteria in §3.1.2 below.

We recall that the composition of finitely many $C^{1+\hat{\alpha}}$ diffeomorphisms is again a $C^{1+\hat{\alpha}}$ diffeomorphism. Since $R_0 > 0$, for any fixed $R > 0$ and any $q \in W^s_n(p, R)$, the holonomy map

$$h^s_{p,q,n}\colon W^c_n(p, 1) \to W^c_n(q)$$

is the composition of finitely many $C^{1+\hat{\alpha}}$-diffeomorphisms and the $C^{1+\hat{\alpha}}$-norm of $h^s_{p,q,n}$ is uniform across the choice of p, q and n.

Taking $\hat{\alpha} < \beta'$ and using that holonomies are uniformly $C^{1+\hat{\alpha}}$, we use Journé's theorem [**Jou**] or related discussions in [**PSW**, §6] to conclude that leaves of the partition $\{W^s_n(x), x \in \mathbb{R}^k\}$ restrict to a $C^{1+\hat{\alpha}}$-foliation inside each $W^{cs}_n(p)$. The smoothness of holonomies for arbitrary transversals in Theorem 2.2 then follows by considering foliation charts. In particular, given arbitrary transversals D_1 and D_2 to $\{W^s_n(x), x \in W^{cs}_n(p)\}$ inside $W^{cs}_n(p)$ as above, it follows that the holonomy map $h^s_{D_1,D_2}$ is uniformly $C^{1+\hat{\alpha}}$ on its domain.

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3. *Proof of Theorem 2.4*

We retain all notation from the previous section. In particular, fix $0 < \theta < \bar{\theta} < \beta$ satisfying (5), (6), and (7).

3.1. *Initial approximations, additional notation, and sequence of approximate holonomies.*

3.1.1. *Initial approximations.* Given $n \in \mathbb{Z}$ and arbitrary $p, q \in \mathbb{R}^k$ with $q \in W^s_n(p, 1)$, we assume there exists a uniformly $C^{1+\beta'}$ initial approximation

$$\pi_{p,q,n} : W^c_n(p, 1) \to W^c_n(q)$$

to the stable holonomy map $h^s_{p,q,n}$ with the following properties. There is a constant $C_2 > 1$ such that for every $n \in \mathbb{Z}$, $p \in \mathbb{R}^k$, and $q \in W^s(p, 1)$ we have:

(1) $d(\pi_{p,q,n}(p), q) \le C_2 d(p, q)$ and $d(p, \pi_{p,q,n}(p)) \le C_2 d(p, q)$;

(2) $d((\pi_{p,q,n})_*(v), v) \le C_2 d(p, q)^{\bar{\theta}}$ for all $v \in SW^c_n(p)$;

(3) $|\|D\pi_{p,q,n}\| - 1| \le C_2 d(p, q)^{\bar{\theta}}$;

(4) if $p' \in W^c_n(p)$ and $q' \in W^s_n(p', 1) \cap W^c_n(q)$ then $\pi_{p,q,n}$ and $\pi_{p',q',n}$ coincide on $W^c_n(p, 1) \cap W^c_n(p', 1)$.

For instance, we may define such a system of approximating maps $\{\pi_{p,q,n}\}$ by linear projection: for $z \in W^c_n(p, 1)$, define $\pi_{p,q,n}(z)$ to be the unique point of intersection of $W^c_n(q)$ and $z + \mathbb{R}^u \oplus \mathbb{R}^s$. One may verify that the above properties hold for this choice of $\pi_{p,q,n}$.

3.1.2. *Additional constants.* Fix $\alpha > 0$ and $0 < \hat{\theta} < \theta < \bar{\theta} < \hat{\beta} < \beta$ for the remainder with

$$\sup\left\{\frac{(1+\alpha)(\hat{\gamma}_n - \gamma_n)}{-\kappa_n}\right\} < \hat{\theta} \tag{8}$$

and

$$\sup\left\{\frac{\kappa_n}{\kappa_n - \gamma_n}\theta\right\} < \hat{\beta}. \tag{9}$$

The existence of such α and $\hat{\theta}$ follow from (7); the existence of such a $\hat{\beta}$ follows from (6). Set $\bar{\kappa} = \sup\{\kappa_n\} < 0$. Set

$$\omega = \sup\{\kappa_n\theta + (1+\alpha)(\hat{\gamma}_n - \gamma_n)\}, \quad \hat{\omega} = \sup\{\kappa_n\hat{\theta} + (1+\alpha)(\hat{\gamma}_n - \gamma_n)\}. \tag{10}$$

We have $\omega < \hat{\omega} < 0$ from the choice of $\hat{\theta} < \theta$ in (8).

For the remainder of §3, fix $0 < \delta < 1$ such that for all $n \in \mathbb{Z}$ we have

$$1 + e^{-(\hat{\gamma}_n - \gamma_n)} C_1 \delta^{\beta - \hat{\beta}} \le e^{\alpha(\hat{\gamma}_n - \gamma_n)}. \tag{11}$$

Such a $\delta > 0$ exists since $\inf\{e^{\alpha(\hat{\gamma}_n - \gamma_n)}\} > 1$.

Given p, q, and n with $q \in W^s_n(p)$, define

$$\rho(p, q, n) := \sup\{d(x, h^s_{p,q,n}(x)) : x \in W^c_n(p, 1)\}.$$

Take $0 < \rho_0 < 1$ such that

$$(3C_2C_1 + 1)^{\bar{\theta}^{-1}} \rho_0 \le \delta \tag{12}$$

where $C_1 \ge 1$ is as in §2.3 and $C_2 \ge 1$ is as above. Fix $0 < R_0 < 1$ so that for all $n \in \mathbb{Z}$, $p \in \mathbb{R}^k$, and $q \in W^s_n(p, R_0)$ we have

$$\rho(p, q, n) \le \rho_0.$$

With this R_0 we establish Theorem 2.4.

3.1.3. *Additional notation.* It is enough to prove Theorem 2.4 in the case where $n = 0$. For the remainder of §3, we fix p and q in $\mathbb{R}^k$ with $q \in W^s_0(p, R_0)$ as in Theorem 2.4. Write $h := h^s_{p,q,0}$.

Given $n, j \in \mathbb{Z}$, write:

- $f_n^{(j)} := \mathrm{id}$, $j = 0$;
- $f_n^{(j)} := f_{n+j-1} \circ \cdots \circ f_n$, $j > 0$;
- $f_n^{(j)} := f_{n+j}{}^{-1} \circ \cdots \circ f_{n-1}{}^{-1}$, $j < 0$;
- for $z \in \mathbb{R}^k$, write $z_n = f_0^{(n)}(z)$;
- write $D_n \subset W^c_n(p_n) := f_0^{(n)}(W^c_0(p, 1))$;
- let $\kappa_n^{(j)} = \begin{cases} \kappa_{n+j-1} + \cdots + \kappa_n, & j > 0, \\ 0, & j = 0, \\ -\kappa_{n+j} - \cdots - \kappa_{n-1}, & j < 0; \end{cases}$
- similarly, define $\hat{\kappa}_n^{(j)}$, $\gamma_n^{(j)}$, and $\hat{\gamma}_n^{(j)}$.

Note that if $x \in W^c_0(p, 1) = D_0$ and $y = h(x) \in W^c_0(q)$ then for all $n \ge 0$ we have

$$d(x_n, y_n) \le e^{\kappa_0^{(n)}} d(x, y) \le e^{\kappa_0^{(n)}} \rho(p, q, 0) < \rho_0.$$

Since $d(x_n, y_n) < 1$, we obtain initial approximations $\pi_{x_n,y_n,n}$ satisfying the properties in §3.1.1. By property (4) of the approximate holonomy maps $\pi_{p_n,q_n,n}$, it follows that the collection of maps $\{\pi_{x_n,y_n,n} : x_n \in D_n\}$ coincide with the restriction of a single approximation which we denote by $\pi_n \colon D_n \to W^c_n(q_n)$ for the remainder. Note that $\pi_n \colon D_n \to W^c_n(q_n)$ has all the properties enumerated in §3.1.

3.1.4. *Approximate holonomies.* For $n \ge 0$, we define $h_n \colon W^c_0(p, 1) \to W^c_0(q)$ to be successive approximations to h given by

$$h_n \colon x \mapsto f_n^{(-n)}(\pi_n(x_n)) = f_n^{(-n)}(\pi_n(f_0^{(n)}(x))).$$

Note that each h_n is a $C^{1+\beta'}$ diffeomorphism onto its image. Although the $(1+\beta')$-norms of the sequence h_n may not be controlled, Theorem 2.2 follows by showing that h_n converges to $h\colon W^c_0(p, 1) \to W^c_0(q)$ in the C^1 topology. We then show $Dh\colon SW^c_0(p, 1) \to TW^c_0(q)$ is Hölder continuous with uniform estimates for some Hölder exponent $0 < \hat{\alpha} < \beta'$.

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3.2. *An auxiliary lemma.* Given $\xi = (x, v)$ and $\zeta = (y, w)$ in $S\mathbb{R}^k$, recall that we write $d(\xi, \zeta) = \sup\{d(x, y), d(v, w)\}$. Given $\xi = (x, v) \in S\mathbb{R}^k$, we write

$$\xi_n = (x_n, v_n) := (f_0^{(n)})_*(\xi) \in S\mathbb{R}^k.$$

Similarly, write $\zeta_n := (y_n, w_n)$.

Recall the parameters $0 < \delta < 1$ satisfying (11) and α and $\hat{\beta}$ satisfying (8).

LEMMA 3.1. *Given $x \in \mathbb{R}^k$, $\xi = (x, v)$, $\zeta = (y, w) \in SW_0^c(x)$, $0 \le r \le \delta$, and $n \ge 0$, suppose that $d(x_n, y_n) \le re^{\kappa_0^{(n)}}$, $d(\xi_n, \zeta_n) \le r^{\bar{\theta}} e^{\kappa_0^{(n)}\theta}$, and for all $0 \le k \le n$ that*

$$d(x_k, y_k) \le \delta.$$

Then, for all $0 \le k \le n$,

$$d(x_k, y_k) \le re^{\kappa_0^{(n)} - \gamma_k^{(n-k)}} \quad \text{and} \quad d(\xi_k, \zeta_k) \le r^{\bar{\theta}} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_k^{(n-k)} - \gamma_k^{(n-k)})}.$$

In particular,

$$d(\xi, \zeta) \le r^{\bar{\theta}} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_0^{(n)} - \gamma_0^{(n)})}.$$

Proof. For the final assertion, note that

$$re^{\kappa_0^{(n)} - \gamma_0^{(n)}} \le r^{\bar{\theta}} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_0^{(n)} - \gamma_0^{(n)})}$$

follows from inequality (5) as $\theta \le ((\kappa_n - \gamma_n)/\eta_n) \le ((\kappa_n - \gamma_n)/\kappa_n)$ holds for all n.

We prove the first two assertions by backwards induction on k starting with $k = n$. We clearly have

$$d(x_{(k-1)}, y_{(k-1)}) \le e^{-\gamma_{k-1}} d(x_k, y_k) \le re^{\kappa_0^{(n)} - \gamma_{k-1}^{(n-(k-1))}}.$$

Moreover, we have

$$\begin{aligned}
d(\xi_{(k-1)}, \zeta_{(k-1)}) &= d((f_k^{(-1)})_*(x_k, v_k), (f_k^{(-1)})_*(y_k, w_k)) \\
&\le d((f_k^{(-1)})_*(x_k, v_k), (f_k^{(-1)})_*(x_k, w_k)) \\
&\quad + d((f_k^{(-1)})_*(x_k, w_k), (f_k^{(-1)})_*(y_k, w_k)) \\
&\le e^{(\hat{\gamma}_{k-1} - \gamma_{k-1})} d(v_k, w_k) + C_1 d(x_k, y_k)^{\beta} \\
&\le e^{(\hat{\gamma}_{k-1} - \gamma_{k-1})} (d(v_k, w_k) + e^{-(\hat{\gamma}_{k-1} - \gamma_{k-1})} C_1 d(x_k, y_k)^{\beta}) \\
&\le e^{(\hat{\gamma}_{k-1} - \gamma_{k-1})} (1 + e^{-(\hat{\gamma}_{k-1} - \gamma_{k-1})} C_1 d(x_k, y_k)^{\beta - \hat{\beta}}) \\
&\quad \cdot \max\{d(x_k, y_k)^{\hat{\beta}}, d(v_k, w_k)\} \\
&\le e^{(1+\alpha)(\hat{\gamma}_{k-1} - \gamma_{k-1})} \max\{r^{\hat{\beta}} e^{\hat{\beta}\kappa_0^{(n)} - \hat{\beta}\gamma_k^{(n-k)}}, r^{\bar{\theta}} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_k^{(n-k)} - \gamma_k^{(n-k)})}\}.
\end{aligned}$$

The last line follows from the induction hypothesis and the choice of $\delta > 0$ in (11).

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From (9) we have

$$\begin{aligned}\hat{\beta}\kappa_0^{(n)} - \hat{\beta}\gamma_k^{(n-k)} &= \hat{\beta}\kappa_0^{(k)} + \hat{\beta}\kappa_k^{(n-k)} - \hat{\beta}\gamma_k^{(n-k)} \\ &\leq \hat{\beta}\kappa_0^{(k)} + \theta\kappa_k^{(n-k)} \\ &\leq \theta\kappa_0^{(n)} \\ &\leq \theta\kappa_0^{(n)} + (1+\alpha)(\hat{\gamma}_k^{(n-k)} - \gamma_k^{(n-k)}).\end{aligned}$$

Hence

$$r^{\hat{\beta}} e^{\hat{\beta}\kappa_0^{(n)} - \hat{\beta}\gamma_k^{(n-k)}} \leq r^{\bar{\theta}} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_k^{(n-k)} - \gamma_k^{(n-k)})}$$

and the result follows. □

3.3. *Step 1: C^0 convergence.* Recall the 'true' holonomies $h^s_{p,q,n}$. We fix $h = h^s_{p,q,0}$ and let h_n be the approximate holonomies in §3.1.4. We have the following lemma.

LEMMA 3.2. *$h_n \to h$ uniformly on $W^c_0(p, 1)$.*

Proof. First (by equivariance of W^s_n-manifolds) we have $f_n^{(-n)} \circ h^s_{p_n,q_n,n} \circ f_0^{(n)} = h^s_{p,q,0}$. For $x \in W^c_0(p, 1)$ we have

$$d(x_n, f_0^{(n)}(h(x))) = d(x_n, h^s_{p_n,q_n,n}(x_n)) \leq e^{\kappa_0^{(n)}} \rho(p, q, 0) \leq e^{\kappa_0^{(n)}}.$$

By property (1) of the maps π_n,

$$\begin{aligned}d(h_n(x), h(x)) &= d(f_0^{(-n)}(\pi_n(x_n)), f_n^{(-n)}(h^s_{p_n,q_n,n}(x_n))) \\ &\leq e^{-\gamma_0^{(n)}} d(\pi_n(x_n), h^s_{p_n,q_n,n}(x_n)) \\ &\leq e^{-\gamma_0^{(n)}} C_2 d(x_n, h^s_{p_n,q_n,n}(x_n)) \\ &\leq C_2 e^{\kappa_0^{(n)} - \gamma_0^{(n)}} \rho_0.\end{aligned}$$

□

3.4. *Step 2: Convergence of the projectivized derivative.* Consider now the projectivized derivatives $(h_n)_*\colon SW^c_0(p, 1) \to SW^c_0(q)$. We show that the sequence $(h_n)_*$ is Cauchy. Set

$$L_1 = C_2 + \sum_{n=0}^{\infty} 3C_2C_1e^{\omega n} \tag{13}$$

where $\omega < 0$ is as in (10).

LEMMA 3.3. *The sequence of maps $(h_n)_*\colon SW^c_0(p, 1) \to SW^c_0(q)$ is uniformly Cauchy. Moreover, defining $h_*\colon SW^c_0(p, 1) \to SW^c_0(q)$ to be the limit $h_* = \lim_{n\to\infty}(h_n)_*$, for $(x, v) \in SW^c_0(p, 1)$ we have*

$$d((x, v), h_*(x, v)) \leq L_1 d(x, h(x))^{\bar{\theta}}.$$

Proof. With $\xi = (x, v)$, let $y = h(x)$ and let $\zeta^n = (y^n, w^n) = (h_n)_*(\xi)$.

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We have

$$d(y^n, y^{n+1}) = d(f_n^{(-n)}(\pi_n(x_n)), f_{n+1}^{(-n-1)}(\pi_{n+1}(x_{n+1})))$$

and, using property (1) of the maps π_n,

$$\begin{aligned} d(\pi_n(x_n), f_{n+1}^{(-1)}(\pi_{n+1}(x_{n+1}))) &\leq d(x_n, \pi_n(x_n)) + (f_{n+1}^{(-1)}(x_{n+1}), f_{n+1}^{(-1)}(\pi_{n+1}(x_{n+1}))) \\ &\leq C_2 e^{\kappa_0^{(n)}} d(x, y) + C_1 C_2 e^{\kappa_0^{(n+1)}} d(x, y) \\ &\leq 2C_1 C_2 e^{\kappa_0^{(n)}} d(x, y). \end{aligned}$$

Similarly,

$$d(\zeta^n, \zeta^{n+1}) = d((f_n^{(-n)})_*(\pi_n)_*(\xi_n), (f_{n+1}^{(-n-1)})_*(\pi_{n+1})_*(\xi_{n+1})) \tag{14}$$

and

$$\begin{aligned} &d((\pi_n)_*(\xi_n), (f_{n+1}^{(-1)})_*(\pi_{n+1})_*(\xi_{n+1})) \\ &\quad \leq d(\xi_n, (\pi_n)_*(\xi_n)) + d((f_{n+1}^{(-1)})_*\xi_{n+1}, (f_{n+1}^{(-1)})_*(\pi_{n+1})_*(\xi_{n+1})). \end{aligned} \tag{15}$$

With $(\pi_{n+1})_*(x_{n+1}, v_{n+1}) = (y', w')$ we have

$$\begin{aligned} &d((f_{n+1}^{(-1)})_*\xi_{n+1}, (f_{n+1}^{(-1)})_*(\pi_{n+1})_*(\xi_{n+1})) \\ &\quad = d((f_{n+1}^{(-1)})_*(x_{n+1}, v_{n+1}), (f_{n+1}^{(-1)})_*(y', w')) \\ &\quad \leq d((f_{n+1}^{(-1)})_*(x_{n+1}, v_{n+1}), (f_{n+1}^{(-1)})_*(x_{n+1}, w')) \\ &\qquad + d((f_{n+1}^{(-1)})_*(x_{n+1}, w'), (f_{n+1}^{(-1)})_*(y', w')) \\ &\quad \leq C_1 d(v_{n+1}, w') + C_1 d(x_{n+1}, y')^{\beta} \\ &\quad \leq C_1 C_2 d(x_{n+1}, y_{n+1})^{\bar\theta} + C_1 (C_2 d(x_{n+1}, y_{n+1}))^{\beta} \\ &\quad \leq C_1 C_2 e^{\kappa_0^{(n+1)}\bar\theta} d(x, y)^{\bar\theta} + C_1 C_2^{\beta} e^{\beta\kappa_0^{(n+1)}} d(x, y)^{\beta} \\ &\quad \leq 2C_1 C_2 e^{\bar\theta\kappa_0^{(n+1)}} d(x, y)^{\bar\theta} \end{aligned} \tag{16}$$

where the first term in (16) follows from property (2) of the approximate holonomies in §3.1.1 and the second term in (16) uses property (1) of the approximate holonomies. Combined with (15) and using property (2) of the approximate holonomies π_n, we then have

$$\begin{aligned} &d((\pi_n)_*(\xi_n), (f_{n+1}^{(-1)})_*(\pi_{n+1})_*(\xi_{n+1})) \\ &\quad \leq C_2 e^{\bar\theta\kappa_0^{(n)}} d(x, y)^{\bar\theta} + 2C_1 C_2 e^{\bar\theta\kappa_0^{(n+1)}} d(x, y)^{\bar\theta} \\ &\quad \leq 3C_2 C_1 e^{\bar\theta\kappa_0^{(n)}} d(x, y)^{\bar\theta}. \end{aligned}$$

Applying Lemma 3.1 (with $r = (3C_1C_2)^{\bar\theta^{-1}} d(x, y)$) to (14) with the choice of ρ_0 satisfying (12), it follows that

$$d((h_n)_*\xi, (h_{n+1})_*\xi) = d(\zeta^n, \zeta^{n+1}) \leq e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat\gamma_0^{(n)} - \gamma_0^{(n)})} 3C_2 C_1 d(x, y)^{\bar\theta}. \tag{17}$$

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By (10),

$$\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_0^{(n)} - \gamma_0^{(n)}) \le n\omega,$$

and it follows that $(h_n)_*$ is uniformly Cauchy on $SW_0^c(p, 1)$.

Moreover, for any $\xi = (x, v) \in SW_0^c(p, 1)$ we have

$$\begin{aligned} d(\xi, h_*(\xi)) &\le d(\xi, (\pi_0)_*\xi) + \sum_{n=0}^{\infty} d((h_n)_*\xi, (h_{n+1})_*\xi) \\ &= d(\xi, (\pi_0)_*\xi) + \sum_{n=0}^{\infty} d(\zeta^n, \zeta^{n+1}) \\ &\le C_2 d(x, y)^{\bar{\theta}} + \sum_{n=0}^{\infty} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_0^{(n)} - \gamma_0^{(n)})} 3C_2C_1 d(x, y)^{\bar{\theta}} \\ &\le C_2 d(x, y)^{\bar{\theta}} + \sum_{n=0}^{\infty} e^{n\omega} 3C_2C_1 d(x, y)^{\bar{\theta}} \end{aligned}$$

where we use properties (1) and (2) of the approximate holonomies π_0 to bound $d(\xi, (\pi_0)_*\xi) \le C_2 d(x, y)^{\bar{\theta}}$. Thus, with L_1 as above, for any $\xi = (x, v) \in SW_0^c(p, 1)$ and with $y = h(x)$ we have

$$d(\xi, h_*(\xi)) \le L_1 d(x, y)^{\bar{\theta}}. \qquad \square$$

Note that the convergence of the projectivized derivative of the stable holonomies in Lemma 3.3 is independent of the choice of $p, q \in \mathbb{R}^k$ or $n \in \mathbb{Z}$ in Theorem 2.2. Thus for all $n \in \mathbb{Z}$, $p' \in \mathbb{R}^k$, and $q' \in W_n^s(p', R_0)$, let $(h^s_{p',q',n})_*$ denote the projectivized derivative of stable holonomies constructed as above. We have for all $\xi' = (x', v') \in SW_n^c(p', 1)$ that

$$d(\xi, (h^s_{p',q',n})_*(\xi)) \le L_1 d(x', h^s_{p',q',n}(x'))^{\bar{\theta}}.$$

Moreover, from the definition of the limit in Lemma 3.3 defining $(h^s_{p',q',n})_*$, we have that $(h^s_{p',q',n})_*$ is a holonomy for the projectivized derivative cocycle:

$$(f_n)_* \circ (h^s_{p',q',n})_* = (h^s_{f_n(p'),f_n(q'),n+1})_* \circ (f_n)_*. \tag{18}$$

Indeed, return to the case $n = 0$, let $h = h^s_{p,q,0}$, and consider $\xi = (x, v) \in SW_0^c(p, 1)$. Then

$$\begin{aligned} (f_0)_* \circ h_*(\xi) &= \lim_{n\to\infty} (f_0)_* \circ (h_n)_*(\xi) \\ &= \lim_{n\to\infty} (f_0)_*(f_n^{(-n)} \circ \pi_n \circ f_0^{(n)})_*(\xi) \\ &= \lim_{n\to\infty} (f_n^{(-(n-1))} \circ \pi_n \circ f_1^{(n-1)})_*(f_0)_*(\xi) \\ &= (h^s_{f_0(p),f_0(q),1})_* \circ (f_0)_*(\xi). \end{aligned}$$

To show that the holonomies are C^1, we next show that each $(h^s_{p',q',n})_*$ coincides with the projectivization of a continuous $Dh^s_{p',q',n} \colon TW_n^c(p', 1) \to TW_n^c(q')$.

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3.5. *Step 3: The sequence of maps Dh_n is uniformly Cauchy.* We return to the notation in Step 2. In particular, we recall our distinguished $p, q \in \mathbb{R}^k$ and the maps h_n approximating $h = h^s_{p,q,0}$.

We first derive two simple distortion estimates. Fix $\xi = (x, v) \in SW^c_0(p, 1)$. With $\xi_n := (f_0^{(n)})_*(\xi)$, let $y = h(x)$, $\zeta = (y, w) := h_*(\xi)$, $\zeta_n = (y_n, w_n) = (f_0^{(n)})_*(\zeta) = (h^s_{p_n,q_n,n})_*(\xi_n)$, and $\hat{\zeta}^n = (h_n)_*(\xi) = (\hat{y}^n, \hat{w}^n)$. Write $\hat{\zeta}^n_i = (f_0^{(i)})_*\hat{\zeta}^n$. Then

$$\hat{\zeta}^n_n := (f_0^{(n)})_*(\hat{\zeta}^n) = (\hat{y}^n_n, \hat{w}^n_n) = (\pi_n)_*(\xi_n).$$

From property (1) of the approximate holonomies π_n,

$$d(y_n, \hat{y}^n_n) \leq C_2 d(x_n, y_n) \leq C_2 e^{\kappa_0^{(n)}} d(x, y) \leq e^{\kappa_0^{(n)}} \delta. \tag{19}$$

From Lemma 3.3, we have

$$d(\xi_n, \zeta_n) \leq L_1 e^{\bar{\theta}\kappa_0^{(n)}} d(x, y)^{\bar{\theta}}.$$

By properties (1) and (2) of the approximate holonomies π_n,

$$d(\xi_n, \hat{\zeta}^n_n) \leq \max\{d(x_n, \hat{y}^n_n), d(v_n, \hat{w}^n_n)\} \leq C_2 e^{\bar{\theta}\kappa_0^{(n)}} d(x, y)^{\bar{\theta}}.$$

Hence,

$$d(\zeta_n, \hat{\zeta}^n_n) \leq (C_2 + L_1) e^{\bar{\theta}\kappa_0^{(n)}}.$$

Let n_0 be such that $(C_2 + L_1) e^{\bar{\theta}\kappa_0^{(n_0)}} \leq e^{\theta\kappa_0^{(n_0)}} \delta^{\bar{\theta}}$ so that for $n \geq n_0$ we have

$$d(\zeta_n, \hat{\zeta}^n_n) \leq e^{\theta\kappa_0^{(n)}} \delta^{\bar{\theta}}. \tag{20}$$

Given $\xi = (x, v) \in T\mathbb{R}^k$, define $\|\xi\| = \|v\|$. For each i, the map $S\mathbb{R}^k \to \mathbb{R}$ given by $\zeta \to \log \|Df_i(\zeta)\|$ is β-Hölder on $S\mathbb{R}^k$ with uniform Hölder constant C_3. Recall $\hat{\theta} < \theta$ satisfying (8) and $\hat{\omega}$ satisfying (10). Let

$$K_1 := \sum_{k=0}^{\infty} C_3 (\delta^{\bar{\theta}} e^{\hat{\omega}k})^{\beta}.$$

LEMMA 3.4. *For all $n \geq n_0$,*

$$\exp(-K_1 e^{\beta(\theta-\hat{\theta})\kappa_0^{(n)}}) \leq \frac{\prod_{i=0}^{n-1} \|Df_i \hat{\zeta}^n_i\|}{\prod_{i=0}^{n-1} \|Df_i \zeta_i\|} \leq \exp(K_1 e^{\beta(\theta-\hat{\theta})\kappa_0^{(n)}}).$$

(Note, in particular, that the middle ratio goes to 1 as $n \to \infty$.)

Proof. Recalling Lemma 3.1 (with $r = \delta$ and estimates (19) and (20)), for $n \geq n_0$ we have

$$\begin{aligned}\left| \log \frac{\prod_{i=0}^{n-1} \|Df_i \hat{\zeta}_i^n\|}{\prod_{i=0}^{n-1} \|Df_i \zeta_i\|} \right| &= \left| \sum_{i=0}^{n-1} \log \|Df_i \hat{\zeta}_i^n\| - \log \|Df_i \zeta_i\| \right| \\ &\leq \sum_{i=0}^{n-1} C_3 d(\hat{\zeta}_i^n, \zeta_i)^\beta \\ &\leq \sum_{i=0}^{n-1} C_3 (\delta^{\bar{\theta}} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_i^{(n-i)} - \gamma_i^{(n-i)})})^\beta \\ &\leq e^{\beta\kappa_0^{(n)}(\theta - \hat{\theta})} K_1\end{aligned}$$

where we use that

$$\begin{aligned}&\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_i^{(n-i)} - \gamma_i^{(n-i)}) \\ &\quad \leq \kappa_0^{(n)}(\theta - \hat{\theta}) + \kappa_i^{(n-i)}\hat{\theta} + (1+\alpha)(\hat{\gamma}_i^{(n-i)} - \gamma_i^{(n-i)}) \\ &\quad \leq \kappa_0^{(n)}(\theta - \hat{\theta}) + \hat{\omega}(n-i)\end{aligned}$$

in the last inequality. □

Similarly, letting

$$K_2 = \exp\left(\sum_{i=1}^{\infty} C_3 L_1^\beta e^{\bar{\kappa}\bar{\theta}\beta i} \right),$$

we have the following lemma.

LEMMA 3.5. *For all $n > 0$,*

$$K_2{}^{-1} \leq \frac{\prod_{i=0}^{n-1} \|Df_i \xi_i\|}{\prod_{i=0}^{n-1} \|Df_i \zeta_i\|} \leq K_2.$$

Proof. From Lemma 3.3 we obtain

$$\begin{aligned}\left| \log \frac{\prod_{i=0}^{n-1} \|Df_i \xi_i\|}{\prod_{i=0}^{n-1} \|Df_i \zeta_i\|} \right| &= \left| \sum_{i=0}^{n-1} \log \|Df_i \xi_i\| - \log \|Df_i \zeta_i\| \right| \\ &\leq \sum_{i=0}^{n-1} C_3 d(\xi_i, \zeta_i)^\beta \\ &\leq \sum_{i=0}^{n-1} C_3 [L_1 d(x_i, y_i)^{\bar{\theta}}]^\beta \\ &\leq \sum_{i=0}^{n-1} C_3 L_1^\beta (e^{\kappa_0^{(i)}\bar{\theta}})^\beta d(x, y)^\beta \\ &\leq \log(K_2).\end{aligned}$$

□

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We now approximate the derivatives $Dh\colon TW_0^c(p) \to TW_0^c(q)$ by the bundle maps

$$\Delta_n\colon TW_0^c(p,1) \to TW_0^c(q)$$

defined as follows: given $n \geq 0$ and $(x,v) \in TW_0^c(p,1)$, let

$$\Delta_n\colon (x,v) \mapsto Df_n^{(-n)}\big((\|Df_0^{(n)}(v)\|(h^s_{p_n,q_n,n})_*((f_0^{(n)})_*(x, \tfrac{v}{\|v\|}))\big).$$

From (18) the projectivization of each Δ_n coincides with $(h^s_{p,q,0})_* = h_*$. With $h = h^s_{p,q,0}$ and $(x,v) \in SW_0^c(0)$, let $h_*(x,v) = (y,w)$. We have

$$\|\Delta_n(x,v)\| = \frac{\|D_x f_0^{(n)}(v)\|}{\|D_y f_0^{(n)}(w)\|}$$

and $\Delta_n(x,v) = (y, \|\Delta_n(x,v)\|w)$. From Lemma 3.5, $\|\Delta_n\|$ is uniformly bounded over the choice of $(x,v) \in SW_0^c(p,1)$ and n.

Given $\xi = (x,v) \in SW_0^c(p)$, we have

$$\|Dh_n(\xi)\| = \|\Delta_n(\xi)\| \cdot \frac{\prod_{i=0}^{n-1} \|Df_i\zeta_i\|}{\prod_{i=0}^{n-1} \|Df_i\hat{\zeta}_i^n\|} \cdot \|D\pi_{p_n,q_n,n}((f_0^{(n)})_*(x,v))\|.$$

It then follows from Lemma 3.4 and property (3) of the approximate holonomies π_n that

$$\|Dh_n(\xi)\| - \|\Delta_n(\xi)\| \to 0$$

uniformly in $\xi \in SW_0^c(p)$. Combined with Lemma 3.3, we thus obtain the following

CLAIM 3.6. $\sup_{\xi\in SW_0^c(p,1)}\{\|Dh_n(\xi) - \Delta_n(\xi)\|\} \to 0$ *as* $n \to \infty$.

It follows that the sequence Dh_n converges uniformly if and only if the sequence Δ_n converges uniformly.

LEMMA 3.7. *The sequence of maps* $\Delta_n\colon SW_0^c(p,1) \to TW_0^c(q)$ *is uniformly Cauchy.*

Proof. Given $\xi = (x,v) \in SW_0^c(p,1)$ with $\xi_n = (x_n, v_n) \in SD_n$, let $\zeta = (y,w) = h_*(\xi) \in SW_0^c(q)$ and $\zeta_n = (y_n, w_n) = (h^s_{p_n,q_n,n})_*(\xi_n)$. Observe that both $\Delta_n(x,v)$ and $\Delta_{n+1}(x,v)$ have footprint y. Then

$$\begin{aligned}
&\|\Delta_n(x,v) - \Delta_{n+1}(x,v)\| \\
&\quad = \frac{\|D_x f_0^{(n)}(v)\|}{\|D_y f_0^{(n)}(w)\|}\|D_{y_n} f_n(w_n)\|^{-1} \\
&\qquad \cdot |\|D_{y_n}(f_n)(w_n)\| - \|D_{x_n}(f_n)(v_n)\|| \\
&\quad \leq K_2C_1(|\|D_{x_n}(f_n)(v_n)\| - \|D_{x_n}(f_n)(w_n)\|| \\
&\qquad + |\|D_{x_n}(f_n)(w_n)\| - \|D_{y_n}(f_n)(w_n)\||) \\
&\quad \leq K_2C_1(C_1L_1e^{\bar{\theta}\kappa_0^{(n)}} + C_0e^{\kappa_0^{(n)}\beta}) \\
&\quad \leq K_2C_12C_1L_1e^{\bar{\theta}\kappa_0^{(n)}} \qquad (21)\\
&\quad \leq K_2C_12C_1L_1e^{\bar{\theta}\bar{\kappa}n}
\end{aligned}$$

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where we use Hölder continuity of $x \mapsto D_x f$ to bound $\|D_{x_n}(f_n)(w_n)\| - \|D_{y_n}(f_n)(w_n)\|$ and Lemma 3.3 to bound

$$\big|\|D_{x_n}(f_n)(v_n)\| - \|D_{x_n}(f_n)(w_n)\|\big| \le C_1\|v_n - w_n\| \le C_1 L_1 d(x_n, y_n)^{\bar{\theta}}. \qquad \square$$

From Claim 3.6 and Lemma 3.7 it follows that the sequence of maps Dh_n converges uniformly. As h_n converges to h we have that $h = h^s_{p,q,n}$ is differentiable and that Dh_n converges to Dh. Furthermore, $\|Dh^s_{p,q,n}\| \le K_2$. This completes the proof of the C^1 properties in Theorem 2.4.

3.6. *Step 4: Hölder continuity of Dh.*

We now show that Dh is Hölder continuous. We begin with the following estimate.

CLAIM 3.8. *There is a $c_0 > 0$ such that if $d(x_k, y_k) \le 1$ for all $0 \le k \le n-1$ and $d((x, v), (y, w)) \le 1$, then*

$$d((f_0^{(n)})_*(x, v), (f_0^{(n)})_*(y, w)) \le e^{c_0 n} d((x, v), (y, w))^{\beta}.$$

Proof. We have

$$\begin{aligned} &d((f_0^{(n)})_*(x, v), (f_0^{(n)})_*(y, w)) \\ &\quad \le d((f_0^{(n)})_*(x, v), (f_0^{(n)})_*(x, w)) + d((f_0^{(n)})_*(x, w), (f_0^{(n)})_*(y, w)) \\ &\quad \le (C_1)^{2n} d(v, w) + d((f_0^{(n)})_*(x, w), (f_0^{(n)})_*(y, w)). \end{aligned}$$

We have

$$d((f_0^{(1)})_*(x, w), (f_0^{(1)})_*(y, w)) \le C_1 d(x, y)^{\beta}.$$

Proceeding inductively, for $k \ge 2$ we claim that

$$d((f_0^{(k)})_*(x, w), (f_0^{(k)})_*(y, w)) \le k C_1^{(1+\beta)k} d(x, y)^{\beta}.$$

Indeed,

$$\begin{aligned} &d((f_0^{(k)})_*(x, w), (f_0^{(k)})_*(y, w)) \\ &\quad = d((D_{x_{k-1}} f_{k-1})_*(D_x f_0^{(k-1)})_*(w), (D_{y_{k-1}} f_{k-1})_*(D_y f_0^{(k-1)})_*(w)) \\ &\quad \le d((D_{x_{k-1}} f_{k-1})_*(D_x f_0^{(k-1)})_*(w), (D_{y_{k-1}} f_{k-1})_*(D_x f_0^{(k-1)})_*(w)) \\ &\qquad + d((D_{y_{k-1}} f_{k-1})_*(D_x f_0^{(k-1)})_*(w), (D_{y_{k-1}} f_{k-1})_*(D_y f_0^{(k-1)})_*(w)) \\ &\quad \le C_1 d(x_{k-1}, y_{k-1})^{\beta} + C_1(k-1) C_1^{(1+\beta)(k-1)} d(x, y)^{\beta} \qquad (22) \\ &\quad \le C_1 (C_1)^{k\beta} d(x, y)^{\beta} + (k-1) C_1^{(1+\beta)k} d(x, y)^{\beta} \\ &\quad \le k C_1^{(1+\beta)k} d(x, y)^{\beta} \end{aligned}$$

where the inductive hypothesis is used in (22). Take $c_0 = 2 \log C_1 + 1$. $\square$

We now show that the maps $h_*\colon SW_0^c(p, 1) \to SW_0^c(q)$ and $Dh\colon SW_0^c(p, 1) \to TW_0^c(q)$ are Hölder continuous.

Fix any $\epsilon_1 > 0$ (to appear only in the proof of Claim 3.10 below).

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- Take $a_0 = \max\{\sup\{(\kappa_n - \hat{\gamma}_n)/\kappa_n\}, \sup\{(\kappa_n - \beta\hat{\gamma}_n - \epsilon_1)/\beta\kappa_n\}, \sup\{(\kappa_n - c_0 - \epsilon_1)/\beta\kappa_n\}\} > 0$. Observe that
$$a_0 \geq \sup\left\{\frac{1 - c_0(\kappa_n)^{-1} - \epsilon_1(\kappa_n)^{-1}}{\beta}\right\} \geq \beta^{-1} > 1.$$
- Set $\bar{\alpha} = \min\{\inf\{(\kappa_n - \gamma_n)/a_0\kappa_n\}, \inf\{(\theta\kappa_n + (1+\alpha)(\hat{\gamma}_n - \gamma_n))/\kappa_n a_0\}, \bar{\theta}/a_0\}$. We have $0 < \bar{\alpha} < \bar{\theta} < 1$.
- Recall our fixed $0 < \rho_0 < 1$ satisfying (12) and $R_0 > 0$ such that $\rho(p, q, n) \leq \rho_0$ for all p and $q \in W^s_n(p, R_0)$.
- Set $\rho_1 = \rho_0^{1/\beta} < \rho_0$.

CLAIM 3.9. *The function $(h^s_{p,q,n})_*$ is $\bar{\alpha}$-Hölder with Hölder constant uniform in all choices of p, $q \in W^s_n(p, R_0)$, and $n \in \mathbb{Z}$.*

Proof. It is enough to consider the case $n = 0$. Fix $p \in \mathbb{R}^k$ and $q \in W^s_0(p, R_0)$.

Given $n \in \mathbb{N}$, set
$$r_n := \rho_1 e^{\kappa_0^{(n)} a_0}.$$
Consider any pair $\xi := (x, v)$ and $\xi' := (x', v')$ in $SW^c_0(p, 1)$. It is enough to consider ξ and ξ' sufficiently close so that for some $1 \leq n$,
$$d(x, x') \leq r_n, \quad d(v, v') \leq r_n,$$
and either
$$d(x, x') \geq r_{n+1} \quad \text{or} \quad d(v, v') \geq r_{n+1}.$$
Let
$$\zeta = (y, w) = h_*(\xi) \quad \text{and} \quad \zeta^n = (y^n, w^n) = (h_n)_*(\xi).$$
Similarly define $\zeta' = (y', w') = h_*(\xi')$ and $\zeta'^n = (y'^n, w'^n) = (h_n)_*(\xi')$. From (17) and (13) we have
$$\begin{aligned}
d(\zeta, \zeta^n) &\leq \sum_{j=n}^{\infty} d(\zeta^j, \zeta^{j+1}) \\
&\leq \sum_{j=n}^{\infty} e^{\kappa_0^{(j)}\theta + (1+\alpha)(\hat{\gamma}_0^{(j)} - \gamma_0^{(j)})} 3C_2C_1 d(x, y)^{\bar{\theta}} \\
&\leq e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_0^{(n)} - \gamma_0^{(n)})} \sum_{j=0}^{\infty} e^{\omega j} 3C_2C_1 d(x, y)^{\bar{\theta}} \\
&\leq e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_0^{(n)} - \gamma_0^{(n)})} L_1 d(x, y)^{\bar{\theta}} \\
&\leq e^{\bar{\alpha} a_0 \kappa_0^{(n)}} L_1 \rho_0^{\bar{\theta}} \\
&= L_1 \rho_1^{\beta\bar{\theta} - \alpha} r_n^{\bar{\alpha}} \\
&\leq L_1 \rho_1^{\beta\bar{\theta} - \bar{\alpha}} e^{\mu\bar{\alpha} a_0} r_{n+1}^{\bar{\alpha}},
\end{aligned}$$

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and similarly,

$$d(\zeta', \zeta'^n) \leq L_1 \rho_1^{\beta\bar{\theta}-\bar{\alpha}} e^{\mu\bar{\alpha}a_0} r_{n+1}^{\bar{\alpha}}.$$

Note that for all $0 \leq k \leq n$ we have

$$d(x_k, x_k') \leq e^{\hat{\gamma}_0^{(k)}} r_n = \rho_1 e^{a_0\kappa_0^{(n)}+\hat{\gamma}_0^{(k)}} \leq \rho_1 e^{\kappa_0^{(k)}} \leq \rho_0 e^{\kappa_0^{(k)}} < \delta < 1.$$

From Claim 3.8 and the choice of a_0, for all $0 \leq k \leq n$ we have

$$d(v_k, v_k') \leq d(\xi_k, \xi_k') \leq e^{c_0 k}(r_n)^\beta = e^{c_0 k} \rho_1^\beta e^{\kappa_0^{(n)} a_0 \beta} \leq \rho_0 e^{\kappa_0^{(n)}}.$$

From properties (1) and (2) of the maps π_n we have

- $d(x_n, y_n^n) \leq C_2 e^{\kappa_0^{(n)}} \rho_0$,
- $d(x_n', y_n'^n) \leq C_2 e^{\kappa_0^{(n)}} \rho_0$,
- $d(v_n, w_n^n) \leq C_2 e^{\kappa_0^{(n)}\bar{\theta}} \rho_0^{\bar{\theta}}$, and
- $d(v_n', w_n'^n) \leq C_2 e^{\kappa_0^{(n)}\bar{\theta}} \rho_0^{\bar{\theta}}$.

Thus

$$d(y_n^n, y_n'^n) \leq (1+2C_2) e^{\kappa_0^{(n)}} \rho_0 < \delta$$

and

$$d(w_n^n, w_n'^n) \leq (\rho_0 e^{\kappa_0^{(n)}} + 2C_2 \rho_0^{\bar{\theta}} e^{\kappa_0^{(n)}\bar{\theta}}) \leq (1+2C_2) e^{\kappa_0^{(n)}\theta} \rho_0^{\bar{\theta}}.$$

From Lemma 3.1 (with $r = (1+2C_2)\rho_0$) we have

$$\begin{aligned} d(y^n, y'^n) &\leq (1+2C_2) e^{\kappa_0^{(n)} - \gamma_0^{(n)}} \rho_0 \\ &\leq (1+2C_2) e^{\bar{\alpha}a_0\kappa_0^{(n)}} \rho_0 \\ &\leq (1+2C_2) e^{\mu\bar{\alpha}a_0} \rho_1^{\beta-\bar{\alpha}} (r_{n+1})^{\bar{\alpha}} \end{aligned}$$

and (from Lemma 3.1 with $r = (1+2C_2)^{\bar{\theta}^{-1}} \rho_0 < \delta$)

$$\begin{aligned} d(w^n, w'^n) &\leq (1+2C_2) \rho_0^{\bar{\theta}} e^{\kappa_0^{(n)}\theta + (1+\alpha)(\hat{\gamma}_0^{(n)} - \gamma_0^{(n)})} \\ &\leq (1+2C_2) \rho_0^{\bar{\theta}} e^{\bar{\alpha}a_0\kappa_0^{(n)}} \\ &\leq (1+2C_2) e^{\mu a_0 \bar{\alpha}} \rho_1^{\bar{\theta}\beta-\bar{\alpha}} (r_{n+1})^{\bar{\alpha}}. \end{aligned}$$

It follows that there is some uniform $K_3 > 0$ so that

$$d(\zeta, \zeta') \leq d(\zeta, \zeta^n) + d(\zeta^n, \zeta'^n) + d(\zeta', \zeta'^n) \leq K_3 r_{n+1}^{\bar{\alpha}}$$

whence $d(\zeta, \zeta') \leq K_3 d(\xi, \xi')^{\bar{\alpha}}$. □

CLAIM 3.10. *The function $SW_n^c(p, 1) \to SW_n^c(q)$ given by*

$$(x, v) \mapsto \|D_x h_{p,q,n}^s(v)\|$$

is $(\bar{\alpha}a_0{}^{-1})$-Hölder with Hölder constant uniform in all choices of $p, q \in W_n^s(p, R_0)$, and $n \in \mathbb{Z}$.

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Proof. Again, it is enough to consider the case $n = 0$. Fix $p \in \mathbb{R}^k$, $q \in W_0^s(p, R_0)$, and set $h = h^s_{p,q,0}$.

We retain the previous notation: given $n \in \mathbb{N}$, set

$$r_n := \rho_1 e^{\kappa_0^{(n)} a_0}$$

and consider $\xi := (x, v)$ and $\xi' := (x', v')$ in $SW_0^c(p, 1)$ with $d(\xi, \xi') \le r_n$ and $d(\xi, \xi') \ge r_{n+1}$. Write $\zeta = (y, w) = h_*(\xi)$ and $\zeta' = (y', w') = h_*(\xi')$.

Recall that for all $(x, v) \in SW_0^c(p)$ we have $\|\Delta_n(x, v)\| - \|D_x h(v)\| \to 0$ as $n \to \infty$. Moreover, from (21) and using that $\bar{\alpha} a_0 \le \bar{\theta}$, we have for some uniform K_4 and K_5 that

$$\big|\|\Delta_n(x, v)\| - \|D_x h(v)\|\big| \le K_4 e^{\bar{\theta}\kappa_0^{(n)}} \le K_5 (r_{n+1})^{\bar{\alpha}} \tag{23}$$

We have that

$$\log(\|\Delta_n(x, v)\|) = \sum_{j=0}^{n-1} \log \|D_{x_j} f_j v_j\| - \log \|D_{y_j} f_j w_j\|.$$

There exists a uniform choice of $L_2 \ge 1$ with

$$\mathrm{Lip}(h|_{W_0^c(p,1)}) := \mathrm{Höl}^1(h|_{W_0^c(p,1)}) \le L_2.$$

Then for all $0 \le j \le n$ we have the following assertions.

- $d(x_j, x_j') \le e^{\hat{\gamma}_0^{(j)}} d(x, x') \le \rho_1 e^{\hat{\gamma}_0^{(j)} + a_0 \kappa_0^{(n)}}$, whence $d(y_j, y_j') \le L_2 \rho_1 e^{\hat{\gamma}_0^{(j)} + a_0 \kappa_0^{(n)}}$.
- By Claim 3.8,
$$d(v_j, v_j') \le e^{c_0 j} d((x, v), (x', v'))^\beta \le e^{c_0 j} \rho_1^\beta e^{a_0 \beta \kappa_0^{(n)}}.$$
- From Claims 3.9 and 3.8,
$$d(w_j, w_j') \le K_3 d(\xi_j, \xi_j')^{\bar{\alpha}} \le K_3 e^{\bar{\alpha} c_0 j} e^{a_0 \bar{\alpha} \beta \kappa_0^{(n)}}.$$

We remark that $v \mapsto \log \|D_x f(v)\|$ is C_1^2-Lipschitz for every x. Then for some uniform choice of K_6, K_7, and K_8 we have

$$\begin{aligned}
&|\log \|\Delta_n(x, v)\| - \log \|\Delta_n(x', v')\||\\
&\le \sum_{j=0}^{n-1} |\log \|D_{x_j} f_j v_j\| - \log \|D_{y_j} f_j w_j\| - \log \|D_{x_j'} f_j v_j'\| + \log \|D_{y_j'} f_j w_j'\||\\
&\le \sum_{j=0}^{n-1} |\log \|D_{x_j} f_j v_j\| - \log \|D_{x_j'} f_j v_j'\|| + \sum_{j=0}^{n-1} |\log \|D_{y_j} f_j w_j\| - \log \|D_{y_j'} f_j w_j'\||\\
&\le \sum_{j=0}^{n-1} C_1^2 d(v_j, v_j') + C_3 d(x_j, x_j')^\beta + C_1^2 d(w_j, w_j') + C_3 d(y_j, y_j')^\beta\\
&\le (K_6 n) \max_j \{(e^{\beta\hat{\gamma}_0^{(j)} + \beta a_0 \kappa_0^{(n)}} + e^{c_0 j} e^{a_0 \beta \kappa_0^{(n)}} + e^{\bar{\alpha} c_0 j} e^{\bar{\alpha} a_0 \beta \kappa_0^{(n)}})\}\\
&\le K_7 \max_j \{(e^{\epsilon_1 n + \beta\hat{\gamma}_0^{(j)} + \beta a_0 \kappa_0^{(n)}} + e^{(\epsilon_1 + c_0) n} e^{a_0 \beta \kappa_0^{(n)}} + e^{\bar{\alpha}(\epsilon_1 + c_0) n} e^{\bar{\alpha} a_0 \beta \kappa_0^{(n)}})\} \qquad (24)\\
&\le K_7 (e^{\kappa_0^{(n)}} + e^{\kappa_0^{(n)}} + e^{\bar{\alpha}\kappa_0^{(n)}})\\
&\le K_8 (r_{n+1})^{\bar{\alpha} a_0^{-1}}
\end{aligned}$$

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where we bound the first term of (24) by

$$\epsilon_1 + \beta\hat{\gamma}_\ell + \beta a_0 \kappa_\ell \le \kappa_\ell$$

when $0 \le \ell < j$ and

$$\epsilon_1 + \beta a_0 \kappa_\ell \le \beta a_0 \kappa_\ell + \epsilon_1 + c_0 \le \kappa_\ell$$

when $j \le \ell < n$. Combined with (23), it follows that

$$(x, v) \mapsto \|D_x h(v)\|$$

is $(\bar{\alpha} a_0{}^{-1})$-Hölder on $SW^c(p, 1)$. □

4. *Lyapunov charts and Ledrappier–Young entropy formula*

For $\beta > 0$, let $f : M \to M$ be a $C^{1+\beta}$ diffeomorphism of a compact k-dimensional manifold M without boundary. Let μ be an ergodic, f-invariant Borel probability measure. We briefly discuss the 'Lipschitz property of unstable manifolds inside center-unstable sets' discussed in §1.2 and justify Theorems 1.4 and 1.5.

4.1. *Lyapunov charts.*

Fix M, $f : M \to M$, and μ as above. Let Λ denote the set of bi-regular points for μ, let $\lambda_1 > \lambda_2 > \cdots > \lambda_p$ denote the Lyapunov exponents, and let $T_x M = \bigoplus_{i=1}^p E^i(x)$ denote Oseledec's splitting for $x \in \Lambda$. Fix a decomposition $\mathbb{R}^k = \bigoplus_{i=1}^p \mathbb{R}_i$ where $\dim \mathbb{R}_i = m_i$ is the dimension of $E_i(x)$ for $x \in \Lambda$. Define the norm $\|\cdot\|'$ on $\mathbb{R}^k$ as follows: writing $v = \sum_{i=1}^p v_i$ where $v_i \in \mathbb{R}_i$ for every $1 \le i \le p$, set $\|v\|' = \max\{\|v_i\|\}$ where $\|v_i\|$ restricts to the standard Euclidean norm on each $\mathbb{R}_i$. Let $\lambda_0 = \max\{|\lambda_1|, |\lambda_p|\}$. We denote by $B(0, r)$ the ball in $\mathbb{R}^k$ of radius r centered at 0 in the norm $\|\cdot\|'$.

Fix a background Riemannian metric and induced distance on M. We have the following standard construction which follows from the construction of a Lyapunov inner product and standard estimates. (See, for example, [**LY1**, Appendix] or [**FHY**, §2].)

PROPOSITION 4.1. *For every sufficiently small $0 < \hat{\epsilon} < 1$ there is a measurable function $\hat{\ell} : \Lambda \to [1, \infty)$ and a measurable family of C^∞ embeddings $\{\hat{\Phi}_x, x \in \Lambda\}$ with the following properties:*

(i) *$\hat{\Phi}_x : B(0, \hat{\ell}(x)^{-1}) \to M$ is a C^∞ diffeomorphism onto a neighborhood of x with $\hat{\Phi}_x(0) = x$;*

(ii) *$D_0\hat{\Phi}_x \mathbb{R}_i = E^i(x)$;*

(iii) *the map $\hat{f}_x : B(0, e^{-\lambda_0 - 3\hat{\epsilon}}\hat{\ell}(x)^{-1}) \to B(0, \hat{\ell}(f(x))^{-1})$ given by*

$$\hat{f}_x(v) = \hat{\Phi}^{-1}_{f(x)} \circ f \circ \hat{\Phi}_x(v)$$

is well defined;

(iv) *$D_0 \hat{f}_x \mathbb{R}_i = \mathbb{R}_i$, and for $v \in \mathbb{R}_i$,*

$$e^{\lambda_i - \hat{\epsilon}}\|v\|' \le \|D_0 \hat{f}_x v\|' \le e^{\lambda_i + \hat{\epsilon}}\|v\|';$$

(v) *$\mathrm{Höl}^\beta(D\hat{f}_x) \le \hat{\epsilon}(\hat{\ell}(x))^\beta$ whence $\mathrm{Lip}(\hat{f}_x - D_0\hat{f}_x) \le \hat{\epsilon}$;*

(vi) *similar statements to (iii), (iv), and (v) hold for f^{-1};*

(vii) *there is a uniform* k_0 *so that* $\mathrm{Lip}(\hat{\Phi}_x) \leq k_0$ *and* $\mathrm{Lip}(\hat{\Phi}_x^{-1}) \leq \hat{\ell}(x)$;

(viii) $e^{-\hat{\epsilon}} \leq \dfrac{\hat{\ell}(f(x))}{\hat{\ell}(x)} \leq e^{\hat{\epsilon}}$.

Given $0 < \hat{\epsilon} < 1$ and corresponding function $\hat{\ell} \colon \Lambda \to [1, \infty)$ as in Proposition 4.1, define new charts $\Phi_x : B(0, 1) \to M$ by rescaling:

$$\Phi_x(v) := \hat{\Phi}_x(\hat{\ell}(x)^{-1} v).$$

We check that with $\epsilon = 4\hat{\epsilon}$ and with $\ell(x) = (\hat{\ell}(x))^2$, for every $x \in \Lambda$ the following hold:

(a) $\Phi_x : B(0, 1) \to M$ is a C^∞ diffeomorphism onto a neighborhood of x with $\Phi_x(0) = x$;

(b) $D_0 \Phi_x \mathbb{R}_i = E^i(x)$;

(c) the map $\tilde{f}_x : B(0, e^{-\lambda_0 - 2\epsilon}) \to B(0, 1)$ given by

$$\tilde{f}_x(v) := \Phi_{f(x)}^{-1} \circ f \circ \Phi_x(v) = \hat{\ell}(f(x))(\hat{f}_x(\hat{\ell}(x)^{-1} v))$$

is well defined;

(d) $D_0 \tilde{f}_x \mathbb{R}_i = \mathbb{R}_i$, and for $v \in \mathbb{R}_i$,

$$e^{\lambda_i - \epsilon} \|v\|' \leq \|D_0 \tilde{f}_x v\|' \leq e^{\lambda_i + \epsilon} \|v\|';$$

(e) $\mathrm{Höl}^\beta(D\tilde{f}_x) \leq \epsilon$, whence $\mathrm{Lip}(\tilde{f}_x - D_0 \tilde{f}_x) \leq \epsilon$;

(f) similar statements to (c), (d), and (e) hold for f^{-1};

(g) there is a uniform k_0 so that $\mathrm{Lip}(\Phi_x) \leq k_0$ and $\mathrm{Lip}(\Phi_x^{-1}) \leq \ell(x)$;

(h) $e^{-\epsilon} \leq \dfrac{\ell(f(x))}{\ell(x)} \leq e^{\epsilon}$.

Indeed, (a), (b), (c), (g), and (h) follow immediately from construction. For (d) and (e), note that for $u \in B(0, 1)$ and $\xi \in \mathbb{R}^k$ with $\|\xi\|' = 1$,

$$D_u \tilde{f}_x(\xi) = \hat{\ell}(f(x)) D_{\hat{\ell}(x)^{-1} u} \hat{f}_x(\hat{\ell}(x)^{-1} \xi) = \frac{\hat{\ell}(f(x))}{\hat{\ell}(x)} D_{\hat{\ell}(x)^{-1} u} \hat{f}_x(\xi),$$

hence

$$D_0 \tilde{f}_x(\xi) = \frac{\hat{\ell}(f(x))}{\hat{\ell}(x)} D_0 \hat{f}_x(\xi)$$

and

$$\begin{aligned}
\|D_u \tilde{f}_x(\xi) - D_v \tilde{f}_x(\xi)\|' &= \frac{\hat{\ell}(f(x))}{\hat{\ell}(x)} \|D_{\hat{\ell}(x)^{-1} u} \hat{f}_x(\xi) - D_{\hat{\ell}(x)^{-1} v} \hat{f}_x(\xi)\|' \\
&\leq \frac{\hat{\ell}(f(x))}{\hat{\ell}(x)} \, \mathrm{Höl}^\beta(D\hat{f}_x) \|\hat{\ell}(x)^{-1} u - \hat{\ell}(x)^{-1} v\|'^\beta \\
&\leq \frac{\hat{\ell}(f(x))}{\hat{\ell}(x)} \hat{\epsilon} \hat{\ell}(x)^\beta (\hat{\ell}(x)^{-1})^\beta \|u - v\|'^\beta \\
&= \frac{\hat{\ell}(f(x))}{\hat{\ell}(x)} \hat{\epsilon} \|u - v\|'^\beta.
\end{aligned}$$

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The family of maps $\{\Phi_x, x \in \Lambda\}$ is called a family of (ϵ, ℓ)-charts. Fix a suitable bump function $\Theta\colon \mathbb{R}^k \to [0,1]$ with $\Theta(v) = 0$ if $\|v\|' \geq e^{-\lambda_0 - 2\epsilon}$ and $\Theta(v) = 1$ if $\|v\|' \leq \frac{1}{2}e^{-\lambda_0-2\epsilon}$. Let

$$F_x = D_0\tilde{f}_x + \Theta \cdot (\tilde{f}_x - D_0\tilde{f}_x)$$

and write $\phi_x = \Theta \cdot (\tilde{f}_x - D_0\tilde{f}_x)$. We have $\mathrm{Lip}(\phi_x) \leq \mathrm{Lip}(\tilde{f}_x - D_0\tilde{f}_x)$, whence $\|\phi_x\|_{C^1} \leq \epsilon$. Taking $\epsilon > 0$ sufficiently small, F_x is a diffeomorphism and

$$F_x^{-1} - (D_0\tilde{f}_x)^{-1} = -(D_0\tilde{f}_x)^{-1} \circ \phi_x \circ (F_x^{-1}).$$

In particular, given $\epsilon' > 0$ (sufficiently small to apply Proposition 2.1), we may take $\epsilon > 0$ sufficiently small so that for every $x \in \Lambda$:

(1) $F_x(u) = \tilde{f}_x(u)$ and $F_x^{-1}(u) = (\tilde{f}_x)^{-1}(u)$ for all u with $\|u\| \leq \frac{1}{2}e^{-2\lambda_0-4\epsilon}$;

(2) $\|F_x - D_0\tilde{f}_x\|_{C^1} < \epsilon'$;

(3) $\|{F_x}^{-1} - (D_0\tilde{f}_x)^{-1}\|_{C^1} < \epsilon'$.

Additionally, there is $C_0 > 0$ so that:

(4) ${\text{Höl}^{\beta}}_{\|\cdot\|'}(DF_x) \leq C_0$;

(5) $\text{Höl}^{\beta}_{\|\cdot\|'}(DF_x^{-1}) \leq C_0$.

Furthermore, taking $\epsilon > 0$ sufficiently small, we can ensure all relevant estimates remain true in the Euclidean norm $\|\cdot\|$.

Given sufficiently small $\epsilon > 0$, fix a family of (ϵ, ℓ)-charts $\{\Phi_x : x \in \Lambda\}$ as above. Let $0 < \delta < 1$ be a reduction factor. (Say $\delta < \frac{1}{4}e^{-3\lambda_0-6\epsilon}$ to adapt the arguments in [**LY1**, equations (2.2) and (2.3)].) For $x \in \Lambda$, let

$$S^{cu}_{\delta,x} := \{y \in \mathbb{R}^k : \|\Phi^{-1}_{f^{-n}(x)} \circ f^{-n} \circ \Phi_x(y)\| < \delta \text{ for all } n \geq 0\}.$$

For $x \in \Lambda$, $\star \in \{s, u, c, su, cu\}$, and $v \in \mathbb{R}^k$, let $\tilde{W}^{\star}_x(v)$ be the corresponding 'fake' manifold through the point v constructed in Proposition 2.1 using the sequence of globalizations $\{F_{f^j(x)}\}$ along the orbit $f^j(x)$.

From the uniformly partially hyperbolic dynamics inside charts we obtain the following claim.

CLAIM 4.2. *For all sufficiently small $\delta > 0$ and almost every x we have $S^{cu}_{\delta,x} \subset \tilde{W}^{cu}_x(0)$.*

From Corollary 2.3, it follows (for sufficiently small $\delta > 0$) that the Lipschitz property of holonomies along unstable manifolds inside the center-unstable sets $S^{cu}_{\delta,x}$ derived in [**LY1**, equation (4.2)] for C^2 maps holds for $C^{1+\beta}$ maps. We similarly obtain that the holonomies along 'fake' W^i manifolds is Lipschitz inside W^{i+1} manifolds for $\lambda_i > \lambda_{i+1} > 0$; this replaces the Lipschitz estimates [**LY2**, Lemma 8.2.5 and equation (8.4)]. Note also that since the coordinate changes intertwining the charts Φ_x and $\hat{\Phi}_x$ are linear we also obtain an analogous Lipschitzness of holonomies relative to the original charts $\hat{\Phi}_x$. In particular, [**LY1**, Proposition 5.1] and [**LY2**, Proposition 11.2] remain valid for $C^{1+\beta}$ diffeomorphisms. It follows that the results of [**LY1**, **LY2**] hold for $C^{1+\beta}$ diffeomorphisms.

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