EQUIVARIANT COMPACTIFICATIONS OF TWO-DIMENSIONAL ALGEBRAIC GROUPS

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Abstract   We classify generically transitive actions of semi-direct products $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ on $\mathbb{P}^2$. Motivated by the program to study the distribution of rational points on del Pezzo surfaces (Manin's conjecture), we determine all (possibly singular) del Pezzo surfaces that are equivariant compactifications of homogeneous spaces for semi-direct products $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$.

1. Introduction

In this note, we are concerned with the classification of algebraic surfaces that are equivariant compactifications of two-dimensional connected linear algebraic groups. Over an algebraically closed field K of characteristic 0, any such group is isomorphic to the torus $\mathbb{G}_\mathrm{m}^2$, the additive group $\mathbb{G}_\mathrm{a}^2$ or a semi-direct product $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$.

Here, varieties admitting an action of a connected linear algebraic group G with an open dense orbit are called *equivariant compactifications of homogeneous spaces for G*. If the stabilizer of a point in the open dense orbit is trivial, then we simply say that the variety is an *equivariant compactification of G*.

Equivariant compactifications of tori are widely studied in toric geometry. The classification of equivariant compactification of additive groups $\mathbb{G}_\mathrm{a}^n$ was initiated by Hassett and Tschinkel [**18**]. Here, we start the classification of equivariant compactifications of semi-direct products $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$. We focus on del Pezzo surfaces (possibly with rational double points) having such a structure.

This has arithmetic motivations. Namely, the distribution of rational points on Fano varieties over number fields is predicted by Manin's conjecture [**3**], giving a precise asymptotic formula for the number of rational points of bounded height. Using methods of harmonic analysis, it has been proved for toric varieties [**4**], for equivariant compactifications of $\mathbb{G}_\mathrm{a}^n$ [**9**], and recently for certain equivariant compactifications of $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ [**22**].

Furthermore, Manin's conjecture is studied systematically in dimension 2, where Fano varieties are del Pezzo surfaces, primarily using universal torsors combined with various analytic techniques (see [**7**, Chapter 2] for an overview). In the version stated in [**5**], Manin's conjecture is expected to hold for any del Pezzo surface whose singularities are rational double points (i.e. canonical); different behaviour occurs if one allows other singularities (see [**5**, Example 5.1.1]).

Therefore, it is important to know which del Pezzo surfaces with at most rational double points are equivariant compactifications, so they may be covered by the results from harmonic analysis. It turns out that this depends only on the *type* of a del Pezzo surface (which can be expressed by its degree, the types of its singularities in the **ADE** classification and the number of its lines, where the latter is relevant only in a few cases).

Toric del Pezzo surfaces are easily identified (see, for example, [**13**, Figure 1]). Del Pezzo surfaces that are equivariant compactifications of $\mathbb{G}_\mathrm{a}^2$ were classified in [**14**]. This leaves the classification of those del Pezzo surfaces that are equivariant compactifications of semi-direct products $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$, which is the main theorem of this paper.

Theorem 1.1. *A del Pezzo surface S, possibly singular with rational double points, is an equivariant compactification of some semi-direct product $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ if and only if it has one of the following types.*

- *Degree $\geqslant 7$: all types.*

- *Degree 6: types $\boldsymbol{A}_2 + \boldsymbol{A}_1$, $\boldsymbol{A}_2$, $2\boldsymbol{A}_1$, $\boldsymbol{A}_1$ (with three or four lines).*

- *Degree 5: types $\boldsymbol{A}_3$, $\boldsymbol{A}_2 + \boldsymbol{A}_1$, $\boldsymbol{A}_2$.*

- *Degree 4: types $\boldsymbol{A}_3 + 2\boldsymbol{A}_1$, $\boldsymbol{D}_4$, $\boldsymbol{A}_3 + \boldsymbol{A}_1$.*

Additionally, precisely the following types are equivariant compactifications of a homogeneous space for some semi-direct product $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$.

- *Degree 5: type $\boldsymbol{A}_4$.*

- *Degree 4: type $\boldsymbol{D}_5$, $\boldsymbol{A}_4$.*

- *Degree 3: type $\boldsymbol{E}_6$, $\boldsymbol{A}_5 + \boldsymbol{A}_1$.*

Theorem 1.1 is visualized diagrammatically in Figure 1. Note that, as remarked in [**18**, § 2], if a variety can be given the structure of a toric variety, this structure is unique up to equivalence (see Definition 2.2). This may, however, fail for other algebraic groups. For example, even $\mathbb{P}^n$ has infinitely many different structures as an equivariant compactification of $\mathbb{G}_\mathrm{a}^n$ for $n \geqslant 6$ [**18**, Example 3.6]. We consider the corresponding problem for each semi-direct product $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$. In the case where $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ is not the direct product $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$, we show that up to equivalence $\mathbb{P}^2$ admits precisely two different struc-

tures as an equivariant compactification of $\mathbb{G}_a \rtimes \mathbb{G}_m$ (see Theorem 3.3). We also prove that it admits infinitely many different structures as an equivariant compactification of a homogeneous space for each $\mathbb{G}_a \rtimes \mathbb{G}_m$.

Note that a related result is proved in [**1**, §6]. There, however, only the classification of those equivariant compactifications of homogeneous spaces ('almost homogeneous' in their terminology) having Picard number 1 is considered, while our techniques allow us to identify the equivariant compactifications of $\mathbb{G}_a \rtimes \mathbb{G}_m$. Moreover, in §4 we also give results towards classifying the possible actions that may occur for the surfaces listed in Theorem 1.1; for example, we show which stabilizers may arise.

The layout of this paper is as follows. In §2 we gather various facts on algebraic group actions and on equivariant compactifications of homogeneous spaces. In §3 we classify the different structures that $\mathbb{P}^2$ admits as an equivariant compactification of a homogeneous space for each semi-direct product $\mathbb{G}_a \rtimes \mathbb{G}_m$. Finally, in §4, we consider del Pezzo surfaces and prove Theorem 1.1. Throughout this paper we work over an algebraically closed field K of characteristic 0, and all algebraic groups are linear.

2. Generalities on algebraic groups

2.1. Actions of algebraic groups

We begin by collecting various results on actions of (always linear) algebraic groups on varieties.

Definition 2.1. Let G be a connected algebraic group and let X be a proper normal variety. If X admits an action of G that is *generically transitive* (i.e. transitive on some dense open subset), we say that X is an *equivariant compactification of a homogeneous space for G*. If, moreover, the action is also *generically free* (i.e. free on some dense open subset), then we say that X is an *equivariant compactification of G*.

For motivation with this terminology, suppose that X is an equivariant compactification of a homogeneous space for G and let H be the stabilizer of a general point (i.e. a point in the open dense orbit). Then, X contains an open subset isomorphic to the homogeneous space G/H, and the action of G on X extends the natural action of G on G/H. If, moreover, H is reductive, then the quotient G/H is affine (see [**20**, Theorem 1.1]), and so the complement of G/H in X is a divisor [**16**, Corollaire 21.12.7], which we call the *boundary* of the action. As an example, note that a toric variety is by definition an equivariant compactification of an algebraic torus. As algebraic tori are commutative, however, every homogeneous space for a torus is in fact itself a torus; in particular, every equivariant compactification of a homogeneous space for an algebraic torus is also a toric variety. To obtain homogeneous spaces that are not themselves algebraic groups, one needs to consider non-commutative groups; we present many such examples in §4.

We are interested in classifying generically transitive actions up to the following notion of equivalence.

Definition 2.2. Let G be an algebraic group acting on varieties X_1 and X_2. An equivalence of (left) G-actions is then a commutative diagram

$$
\begin{array}{ccc}
G \times X_1 & \xrightarrow{(\alpha,j)} & G \times X_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\quad j \quad} & X_2
\end{array}
$$

where $\alpha\colon G \to G$ is an automorphism and $j\colon X_1 \to X_2$ is an isomorphism.

Note that, in order to classify generically transitive actions up to equivalence, we need only consider *left* actions. Indeed, if G acts on the right on a variety X via $(x,g) \mapsto xg$, then we obtain a left action of G on X defined by $(g,x) \mapsto xg^{-1}$. This left action is obviously generically transitive (or generically free) if and only if the original action is. Throughout this paper we therefore assume that all groups act on the left.

Recall that given an action of an algebraic group G on a variety X and a line bundle L on X, a G-linearization of L is a fibrewise linear action of G on L that respects the action of G on X (see [20, Chapter 1] and [15, Chapter 7]).

Lemma 2.3. *Let G be a connected algebraic group such that $\mathrm{Pic}(G) = 0$, and suppose that G acts on some normal variety X. Then, every line bundle on X admits a G-linearization.*

In particular, for any $n \in \mathbb{N}$, every projective representation $G \to \mathrm{PGL}_n$ admits a lift to a representation $G \to \mathrm{GL}_n$, i.e. there exists a homomorphism $G \to \mathrm{GL}_n$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
G & \longrightarrow & \mathrm{GL}_n \\
 & \searrow & \downarrow \\
 & & \mathrm{PGL}_n
\end{array}
$$

Proof. By [15, Theorem 7.2], as G is connected, we have an exact sequence

$$
\mathrm{Pic}^G(X) \to \mathrm{Pic}(X) \to \mathrm{Pic}(G),
$$

where $\mathrm{Pic}^G(X)$ denotes the group of isomorphism classes of G-linearized line bundles on X. As $\mathrm{Pic}(G) = 0$, the map $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ is surjective, and hence every line bundle on X admits a G-linearization.

To prove the second part of the lemma, note that a projective representation $G \to \mathrm{PGL}_n$ gives rise to an action of G on $\mathbb{P}^{n-1}$. By the first part of the lemma, the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ admits a G-linearization. Therefore, we obtain an action on the n-dimensional vector space $H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$, which is the required lift to a representation $G \to \mathrm{GL}_n$. $\qquad\square$

The algebraic groups of primary interest in this paper (namely, $\mathbb{G}_a, \mathbb{G}_m$ and semi-direct products $\mathbb{G}_a \rtimes \mathbb{G}_m$) all have trivial Picard groups [15, Remark 7.3]. Note also that,

in general, the choice of linearization is not unique if G admits non-trivial characters (see [**15**, (7.3)]).

Next, we obtain a criterion to help determine whether certain morphisms to projective space are equivariant.

Lemma 2.4. *Let X be a normal variety together with the action of an algebraic group G. Let L be a line bundle on X that is generated by its global sections such that $W = H^0(X, L)$ is finite dimensional and that admits a G-linearization. Let $V \subset W$ be a base-point free linear series. Then, if $\varphi \colon X \to \mathbb{P}(V)$ denotes the associated morphism, the following are equivalent.*

(1) *$V \subset W$ is invariant under the action of G.*

(2) *The composed morphism $X \to \mathbb{P}(V) \subset \mathbb{P}(W)$ is G-equivariant.*

(3) *$\mathbb{P}(V) \subset \mathbb{P}(W)$ is invariant under the action of G.*

Proof. The proof that (1) implies (2) can be found in [**15**, §7.3]. To show that (2) implies (3), first note that if we let $Y = \varphi(X)$, then $\mathbb{P}(V)$ is the only linear subspace of $\mathbb{P}(W)$ of dimension $n = \dim V + 1$ that contains Y. Indeed, choose a basis $s_0, \ldots, s_n$ for V and suppose that $H \subset \mathbb{P}(W)$ is another such subspace. Then, $H \cap \mathbb{P}(V)$ is a linear subspace of dimension at most $n - 1$ containing Y, which implies that there is a linear relation between $s_0, \ldots, s_n$, giving a contradiction. Therefore, as the G-equivariance of φ implies that $g\mathbb{P}(V)$ contains Y for all $g \in G$, we see that $g\mathbb{P}(V) = \mathbb{P}(V)$, i.e. $\mathbb{P}(V) \subset \mathbb{P}(W)$ is invariant under the action of G. This proves (3).

Finally, we show that (3) implies (1). The fact that $\mathbb{P}(V) \subset \mathbb{P}(W)$ is invariant under the action of G implies that for any line $E \subset V$ we have $gE \subset V$ for all $g \in G$. Applying this to the line spanned by each $s \in V$, we deduce that $gs \in V$ for all $g \in G$, which proves (1). □

Note that, as (3) in Lemma 2.4 is independent of the choice of G-linearization on L, we see that (1) is also independent of the choice of G-linearization. We next consider how the property of being an equivariant compactification of a homogeneous space behaves with respect to birational morphisms.

Lemma 2.5. *Let G be a connected algebraic group and let X be an equivariant compactification of a homogeneous space for G. Let $\pi \colon \tilde{X} \to X$ be the blow-up of X at a subvariety $V \subset X$ that is invariant under the action of G. Then, $\tilde{X}$ is an equivariant compactification of a homogeneous space for G in such a way that π is a G-equivariant morphism.*

Proof. From the universal property of blow-ups [**17**, Corollary II.7.15], we obtain a morphism $G \times \tilde{X} \to \tilde{X}$. It is easy to see that this gives the required action (see the proof of [**14**, Lemma 3]). □

Lemma 2.6. *Let G be a connected algebraic group and let X be a smooth equivariant compactification of a homogeneous space for G. Let $\pi \colon X \to Y$ be a birational*

morphism to a normal projective variety Y. Then, Y is an equivariant compactification of a homogeneous space for G in such a way that π is a G-equivariant morphism.

Proof. For equivariant compactifications of G, see [**22**, Proposition 1.3]. The exact same proof works for equivariant compactifications of homogeneous spaces for G, as the fact that the stabilizer of a general point is trivial is not used in the proof. $\qquad\square$

Combining these results we obtain the following.

Proposition 2.7. *Let G be a connected algebraic group, let S be a singular projective normal surface and let $\pi\colon \tilde{S} \to S$ be a minimal desingularization. Then, S is an equivariant compactification of a homogeneous space for G if and only if $\tilde{S}$ is, in which case π is a G-equivariant morphism.*

Proof. The proof of this lemma is essentially the same as the proof of [**14**, Lemma 4]. The fact that S is normal implies that the singular locus consists of a finite set of singularities. As G is connected, each of these singularities must be fixed under the action of G. Since the map π is given by successively blowing up these singularities, on applying Lemmas 2.5 and 2.6 we deduce the result. $\qquad\square$

Note that as the G-equivariant morphisms in Lemmas 2.5, 2.6 and Proposition 2.7 are birational, they preserve the order of the stabilizer of each point in the open dense orbit.

2.2. Semidirect products $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$

We now turn our attention to semi-direct products of $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$. Note that one may write all such groups in a fairly simple way. Namely, a semi-direct product $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m}$ is given by a homomorphism $\mathbb{G}_\mathrm{m} \to \mathrm{Aut}(\mathbb{G}_\mathrm{a}) = \mathbb{G}_\mathrm{m}$. Since homomorphisms $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ are given by $t \mapsto t^d$ for any integer d, any such semi-direct product has the form $G_d = \mathbb{G}_\mathrm{a} \rtimes_{\phi_d} \mathbb{G}_\mathrm{m}$ with $\phi_d(t)(b) = t^d b$. The group law on G_d is given by

$$(b, t) \cdot (b', t') = (b + t^d b', tt').$$

We keep this notation throughout this paper. Note that we have obvious isomorphisms $G_d \cong G_{-d}$ and $G_0 \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$.

Later, we will require some information about stabilizers of generically transitive actions. For this it is useful to know which finite subgroups can occur.

Lemma 2.8. *Any finite subgroup of G_d is conjugate to one of the form*

$$\mu_n \to G_d, \qquad \zeta \mapsto (0, \zeta),$$

for some $n \in \mathbb{N}$. Such a subgroup is normal if and only if $n \mid d$, in which case $G_d/\mu_n \cong G_{d/n}$.

Proof. Let $H \subset G_d$ be a finite subgroup. Restricting the exact sequence

$$0 \to \mathbb{G}_\mathrm{a} \to G_d \to \mathbb{G}_\mathrm{m} \to 1$$

to H, we see that H injects into $\mathbb{G}_{\mathrm{m}}$. Indeed, $H \cap \mathbb{G}_{\mathrm{a}} = 0$, since $\mathbb{G}_{\mathrm{a}}$ has no non-trivial finite subgroups as K has characteristic 0. Therefore, there exists $n \in \mathbb{N}$ such that $H \cong \mu_n$ as an algebraic group; in particular, H is cyclic and generated by a semi-simple element. Such an element is conjugate to one in the maximal torus $T = \{(0,t) \colon t \in \mathbb{G}_{\mathrm{m}}\}$ by [**6**, Theorem III.10.6]. This completes the proof of the first part of the lemma.

A simple calculation shows that μ_n is not normal if $n \nmid d$. If $n \mid d$, then the map

$$G_d \to G_{d/n}, \qquad (b,t) \mapsto (b, t^n)$$

has kernel μ_n and gives the required isomorphism. $\qquad\square$

Note that it follows from Lemma 2.8 that if we wish to classify generically transitive actions of G_d on a certain surface S for every $d \in \mathbb{Z}$, we may reduce to the case where the action is *faithful*. Indeed, as G_d and S have the same dimension, the stabilizer of a general point is finite, and hence the kernel of the action is a finite normal subgroup. Quotienting out we obtain a faithful generically transitive action of $G_{d/n}$ on S for some $n \mid d$.

3. Actions on the projective plane

We now classify the generically transitive actions of G_d on $\mathbb{P}^2$. We begin with a lemma on three-dimensional representations of $\mathbb{G}_{\mathrm{a}}$.

Lemma 3.1. *Let $f \colon \mathbb{G}_{\mathrm{a}} \to \mathrm{GL}_3$ be a faithful representation whose image consists only of upper triangular matrices. There then exist $\alpha_1, \alpha_2, \alpha_3 \in K$ not all 0 such that*

$$f(b) = \begin{pmatrix} 1 & \alpha_1 b & \alpha_2 b + \dfrac{\alpha_1 \alpha_3}{2} b^2 \\ 0 & 1 & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. By assumption, we have

$$f(b) = \begin{pmatrix} f_{1,1}(b) & f_{1,2}(b) & f_{1,3}(b) \\ 0 & f_{2,2}(b) & f_{2,3}(b) \\ 0 & 0 & f_{3,3}(b) \end{pmatrix},$$

where all the $f_{i,j}(b)$ are polynomial expressions in b. For this to define an action we must have that

$$f_{i,i}(b) \cdot f_{i,i}(b') = f_{i,i}(b + b') \tag{3.1}$$

for $i = 1, 2, 3$, i.e. each $f_{i,i}$ defines a homomorphism $f_{i,i} \colon \mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{m}}$. Such a homomorphism must be trivial; hence, we have $f_{i,i}(b) = 1$ for each $b \in \mathbb{G}_{\mathrm{a}}$ and $i = 1, 2, 3$.

Next, differentiating the map f gives an injection of Lie algebras $\mathrm{d}f \colon \mathfrak{g} \to \mathfrak{gl}_3$, where $\mathfrak{g}$ denotes the Lie algebra of $\mathbb{G}_{\mathrm{a}}$. The morphism $\mathrm{d}f$ sends a generator of $\mathfrak{g}$ to a nilpotent matrix

$$\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \alpha_3 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha_1, \alpha_2, \alpha_3 \in K$, at least one of which is non-zero. On exponentiating this map, we obtain the result. □

The following lemma is the key step in the classification of the generically transitive actions on $\mathbb{P}^2$ up to equivalence. It is used later in our study of such actions on generalized del Pezzo surfaces.

Lemma 3.2. *Let $d \in \mathbb{Z}$ and let $\rho\colon G_d \to \mathrm{PGL}_3$ be a faithful representation whose image consists of only upper triangular matrices. There then exist an element $g \in G_d$ and $k_1, k_2 \in \mathbb{Z}$ not both 0 and $\alpha_1, \alpha_2, \alpha_3 \in K$ not all 0 such that*

$$\rho(g^{-1}(b,t)g) = \begin{pmatrix} t^{k_1} & \alpha_1 b t^{k_2} & \alpha_2 b + \dfrac{\alpha_1 \alpha_3}{2} b^2 \\ 0 & t^{k_2} & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, the following four conditions must hold:

- $\alpha_1 = 0$ or $k_1 = k_2 + d$,

- $\alpha_2 = 0$ or $k_1 = d$,

- $\alpha_3 = 0$ or $k_2 = d$,

- $\alpha_1 \alpha_2 \alpha_3 = 0$.

Proof. Let $U = \{(b,1)\colon b \in \mathbb{G}_a\}$ denote the normal subgroup of G_d isomorphic to $\mathbb{G}_a$, and let T denote the maximal torus $T = \{(0,t)\colon t \in \mathbb{G}_m\}$. The first step of the proof is to analyse the behaviour of ρ when restricted to U and T. Note that, by Lemma 2.3, there exists a lift of ρ to a faithful representation $f\colon G_d \to \mathrm{GL}_3$ that takes the form

$$\begin{pmatrix} f_{1,1}(b,t) & f_{1,2}(b,t) & f_{1,3}(b,t) \\ 0 & f_{2,2}(b,t) & f_{2,3}(b,t) \\ 0 & 0 & f_{3,3}(b,t) \end{pmatrix},$$

where all the $f_{i,j}(b,t)$ are polynomial expressions in b, t, t^{-1}. For this to define an action, the following relations must hold:

$$f_{i,i}(b,t) \cdot f_{i,i}(b',t') = f_{i,i}(b + t^d b', tt') \tag{3.2}$$

for $i = 1, 2, 3$. Applying Lemma 3.1 we see that $f_{i,i}(b,0) = 1$. Therefore, it follows from (3.2) that each $f_{i,i}$ defines a homomorphism $f_{i,i}\colon T \to \mathbb{G}_m$, so we must have $f_{i,i}(b,t) = f_{i,i}(0,t) = t^{k_i}$ for some $k_i \in \mathbb{Z}$ and $i = 1, 2, 3$. Note that we may obviously choose the lift f so that $k_3 = 0$. Moreover, we claim that at least one of k_1 and k_2 is non-zero. Indeed, otherwise, f restricted to T would give a map $T \to \mathrm{GL}_3$ whose image is unipotent. As $T \cong \mathbb{G}_m$, such a map must be trivial, which contradicts the fact that f is faithful.

Next, we find a maximal torus of G_d that has diagonal image under f. Let $D_3 \subset \mathrm{GL}_3$ denote the subgroup of diagonal matrices and let $H = D_3 \cap f(G_d)$, which is a closed algebraic subgroup of both D_3 and $f(G_d)$. Since one of the k_i is non-zero, we see that H is not finite. Thus, if we let H^0 denote the connected component of the identity of H, it follows that H^0 is an algebraic torus of dimension 1, as it is a connected one-dimensional algebraic subgroup of $D_3 \cong \mathbb{G}_\mathrm{m}^3$. So, H^0 defines a maximal torus in $f(G_d)$, and, pulling back via f, we obtain a maximal torus in G_d with diagonal image. However, as any two maximal tori are conjugate (see, for example, [**6**, Theorem III.10.6]), there exists an element $g \in G_d$ such that $f(g^{-1}Tg)$ consists of diagonal matrices. Moreover, by the above we may assume that $f(g^{-1}(0,t)g) = \mathrm{diag}(t^{k_1}, t^{k_2}, 1)$.

Next, note that the map $b \mapsto f(g^{-1}(b,1)g)$ is a faithful representation of $\mathbb{G}_\mathrm{a}$ that consists of upper triangular matrices. Hence, applying Lemma 3.1 and using the fact that $f(g^{-1}(b,t)g) = f(g^{-1}(b,1)g)f(g^{-1}(0,t)g)$, we see that there exist $\alpha_1, \alpha_2, \alpha_3 \in K$ not all 0 such that $f(g^{-1}(b,t)g)$ is given by

$$
\begin{pmatrix}
t^{k_1} & \alpha_1 b t^{k_2} & \alpha_2 b + \dfrac{\alpha_1 \alpha_3}{2} b^2 \\
0 & t^{k_2} & \alpha_3 b \\
0 & 0 & 1
\end{pmatrix}.
$$

One can check that this defines a homomorphism if and only if

$$
\alpha_1(t^{k_1} - t^{d+k_2}) = \alpha_2(t^{k_1} - t^d) = \alpha_3(t^{k_2} - t^d) = 0 \tag{3.3}
$$

for all $t \in K^*$. This gives the list of conditions in the lemma. To finish the proof, it suffices to note that if $\alpha_1 \alpha_2 \alpha_3 \neq 0$, then (3.3) implies that $k_1 = k_2 = d = 0$, which does not give a faithful representation. $\qquad\square$

We are now ready to classify the faithful generically transitive actions of G_d on $\mathbb{P}^2$. We first define the actions that we are interested in. Let $d \in \mathbb{Z}$ and let $k \in \mathbb{Z} \setminus 0$. We define a generically transitive action of G_d on $\mathbb{P}^2$ by

$$
\tau_{d,k}(b,t) =
\begin{pmatrix}
t^k & 0 & 0 \\
0 & t^d & b \\
0 & 0 & 1
\end{pmatrix}.
$$

The following facts are easy to check. We use the coordinates $(x{:}y{:}z)$ on $\mathbb{P}^2$.

- The stabilizer of a general point has order $|k|$.

- The representation is faithful if and only if $\gcd(|k|, |d|) = 1$.

- The boundary divisor consists of the two lines $\{x = 0\}$ and $\{z = 0\}$.

- If $k \neq d$, the only fixed points are $(1{:}0{:}0)$ and $(0{:}1{:}0)$. If $k = d$, then the fixed points are exactly the points on the line $\{z = 0\}$.

Note that $\tau_{d,k}$ is not equivalent to $\tau_{d,k'}$ for any $|k| \neq |k'|$, as the stabilizers of a general point are different in each case. Also, $\tau_{d,k}$ is not equivalent to $\tau_{d,-k}$ for $d \neq 0$, as these have inequivalent action on the line $\{z = 0\}$. One easily sees, however, that $\tau_{0,k}$ is equivalent to $\tau_{0,-k}$ on applying the automorphism $(b,t) \mapsto (b,t^{-1})$ of $G_0 = \mathbb{G}_a \times \mathbb{G}_m$. We also have another faithful generically transitive action of G_d on $\mathbb{P}^2$ given by

$$\rho_d(b,t) = \begin{pmatrix} t^{2d} & bt^d & b^2/2 \\ 0 & t^d & b \\ 0 & 0 & 1 \end{pmatrix}$$

for any $d \neq 0$. Here again it is easy to check the following.

- The stabilizer of a general point has order $2|d|$.

- The boundary divisor consists of the line $\{z = 0\}$ and the conic $\{y^2 = 2xz\}$.

- The only fixed point is $(1{:}0{:}0)$.

Note that the boundary divisor for ρ_d does not have strict normal crossings, as the conic lies tangent to the line. Also, it is easy to see that ρ_d is not equivalent to $\tau_{d,k}$ for any $kd \neq 0$, as there is no automorphism of $\mathbb{P}^2$ that swaps a line and a conic. Our main theorem in this section is that any faithful generically transitive action of G_d of $\mathbb{P}^2$ is of the above form, up to equivalence.

Theorem 3.3. *Let $d \neq 0$. Any faithful generically transitive action of G_d on $\mathbb{P}^2$ is equivalent to either $\tau_{d,k}$ for a unique $k \neq 0$ with $\gcd(|k|,|d|) = 1$ or ρ_d. Any faithful generically transitive action of G_0 on $\mathbb{P}^2$ is equivalent to $\tau_{0,1}$.*

Proof. The action of G_d on $\mathbb{P}^2$ gives rise to a faithful representation $\rho \colon G_d \to \mathrm{PGL}_3$. As G_d is solvable, it follows from the Lie–Kochin theorem [**6**, Corollary III.10.5] (applied to a lift of ρ obtained via Lemma 2.3) that we may conjugate by an element of PGL_3 to obtain an equivalent action whose image consists of only upper triangular matrices. This corresponds to the fact that the action on $\mathbb{P}^2$ leaves $(1{:}0{:}0)$ and $\{z = 0\}$ invariant. Therefore, applying Lemma 3.2, we see that up to equivalence $\rho(b,t)$ takes the form

$$\begin{pmatrix} t^{k_1} & \alpha_1 bt^{k_2} & \alpha_2 b + \dfrac{\alpha_1 \alpha_3}{2} b^2 \\ 0 & t^{k_2} & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

We now proceed by considering the various possibilities on the α_i given by Lemma 3.2. First, if $\alpha_1 \alpha_2 \neq 0$, then we see that $k_2 = 0$, $k_1 = d$ and $\alpha_3 = 0$. This action is not generically transitive for any d; indeed, it preserves the lines $y = \lambda z$ for any $\lambda \in K$.

Next, consider the case where $\alpha_1 \alpha_3 \neq 0$ and, hence, $\alpha_2 = 0$. Lemma 3.2 then implies that $k_1 = 2d$ and $k_2 = d$ and, therefore, $d \neq 0$. We claim that this action is equivalent to ρ_d. Indeed, the conic $\{\alpha_1 y^2 = 2\alpha_3 xz\}$ is invariant under the action. The automorphism

of $\mathbb{P}^2$ given by $x \mapsto \alpha_3 x / \alpha_1$ moves this conic to the conic $\{y^2 = 2xz\}$ and gives rise to an equivalent action given by

$$\begin{pmatrix} t^{2d} & \alpha_3 b t^d & (\alpha_3 b)^2/2 \\ 0 & t^d & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

On performing the automorphism $(b, t) \mapsto (b/\alpha_3, t)$ of G_d, which rescales b, we obtain ρ_d. Thus, we may assume that $\alpha_1 \alpha_3 = 0$ and that ρ takes the form

$$\begin{pmatrix} t^{k_1} & \alpha_1 b t^{k_2} & \alpha_2 b \\ 0 & t^{k_2} & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\alpha_2 \alpha_3 \neq 0$, then Lemma 3.2 tells us that $k_1 = k_2 = d$ and $\alpha_1 = 0$. Clearly, this action is not faithful unless $d = 1$, in which case it gives

$$\begin{pmatrix} t & 0 & \alpha_2 b \\ 0 & t & \alpha_3 b \\ 0 & 0 & 1 \end{pmatrix}.$$

The boundary here consists of the lines $\{z = 0\}$ and $\{\alpha_2 y = \alpha_3 x\}$. Hence, as before, we may perform an automorphism of $\mathbb{P}^2$ that moves the line $\{\alpha_2 y = \alpha_3 x\}$ to the line $\{x = 0\}$ to obtain an action equivalent to $\tau_{1,1}$.

Thus, we have reduced to the case where $\alpha_1 \alpha_2 = \alpha_1 \alpha_3 = \alpha_2 \alpha_3 = 0$. In particular, only one of the α_i can be non-zero, and we may even assume that $\alpha_i = 1$, since applying the automorphism $(b, t) \mapsto (b/\alpha_i, t)$ of G_d gives an equivalent action. This leaves the three cases

$$\begin{pmatrix} t^k & 0 & 0 \\ 0 & t^d & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} t^d & 0 & b \\ 0 & t^k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} t^{d+k} & b t^k & 0 \\ 0 & t^k & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first action is $\tau_{d,k}$, by definition, whereas the second action is seen to be equivalent to $\tau_{d,k}$ on performing the automorphism of $\mathbb{P}^2$ that swaps x and y. As for the third one, we note that in PGL_3 we have

$$\begin{pmatrix} t^{d+k} & b t^k & 0 \\ 0 & t^k & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} t^d & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-k} \end{pmatrix},$$

which is easily seen to be equivalent to $\tau_{d,-k}$ on performing the automorphism of $\mathbb{P}^2$ that swaps y and z. $\qquad \square$

4. Actions on generalized del Pezzo surfaces

4.1. Recap on del Pezzo surfaces

We now recall various facts that we need on del Pezzo surfaces, which can be found, for example, in [**10, 11, 19**]. As before, we work over an algebraically closed field K of characteristic 0.

A *generalized del Pezzo surface* $\tilde{S}$ is a non-singular projective surface whose anticanonical class $-K_{\tilde{S}}$ is big and nef. A normal projective surface S with ample anticanonical class $-K_S$ is called an *ordinary del Pezzo surface* if it is non-singular and a *singular del Pezzo surface* if its singularities are rational double points. Ordinary del Pezzo surfaces and minimal desingularizations of singular del Pezzo surfaces are generalized del Pezzo surfaces, and, conversely, every generalized del Pezzo surface arises in this way (see [**10**, Proposition 0.6]).

The *degree* of a generalized del Pezzo surface $\tilde{S}$ is the self-intersection number $(-K_{\tilde{S}}, -K_{\tilde{S}})$ of its anticanonical class. The degree of a singular del Pezzo surface S is defined to be the degree of its minimal desingularization. For $n \in \mathbb{N}$, a $(-n)$-curve (or simply a *negative curve*) on a non-singular projective surface is a rational curve with self-intersection number $-n$. On generalized del Pezzo surfaces, only (-1)- or (-2)-curves may occur (see [**10**, p. 29]). Moreover, a generalized del Pezzo surface is ordinary if and only if it contains no (-2)-curves.

A theorem of Demazure (see [**10**, Proposition 0.4]) states that any generalized del Pezzo surface $\tilde{S}$ is isomorphic to either $\mathbb{P}^2$ (degree 9), $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$ (both of degree 8) or is obtained from $\mathbb{P}^2$ by a sequence

$$\tilde{S} = \tilde{S}_r \xrightarrow{\rho_r} \tilde{S}_{r-1} \to \cdots \to \tilde{S}_1 \xrightarrow{\rho_1} \tilde{S}_0 = \mathbb{P}^2$$

of $r \leqslant 8$ blow-ups $\rho_i \colon \tilde{S}_i \to \tilde{S}_{i-1}$ of points $p_i \in \tilde{S}_{i-1}$ not lying on a (-2)-curve on $\tilde{S}_{i-1}$ for $i = 1, \ldots, r$ (with $\tilde{S}$ of degree $9 - r$). The Picard group $\mathrm{Pic}(\tilde{S})$ of a generalized del Pezzo surface is a torsion-free abelian group of rank $10 - \deg(\tilde{S})$. For a generalized del Pezzo surface $\tilde{S}$ of degree at least 3, the anticanonical linear system defines a birational morphism $\pi \colon \tilde{S} \to S \subset \mathbb{P}^{\deg(\tilde{S})}$ to a surface S. If $\tilde{S}$ is ordinary, then π is in fact a closed immersion. Otherwise, π contracts precisely the (-2)-curves on $\tilde{S}$ to the singularities of S, and S is a singular del Pezzo surface with minimal desingularization $\tilde{S}$.

The singularity type of a singular del Pezzo surface S is defined to be the dual graph of the configuration of (-2)-curves on the minimal desingularization $\tilde{S}$. These graphs are always Dynkin diagrams and are labelled by (sums of) $\boldsymbol{A}_n$ for $n \geqslant 1$, $\boldsymbol{D}_n$ for $n \geqslant 4$, $\boldsymbol{E}_6$, $\boldsymbol{E}_7$, $\boldsymbol{E}_8$. Moreover, in each degree, there are only finitely many possibilities for the configurations of the negative curves that may occur on generalized del Pezzo surfaces; these *types* can be distinguished by the $\boldsymbol{ADE}$-types of the Dynkin diagrams for the (-2)-curves and the number of lines. The latter can be left out in most cases. Exceptions are two $\boldsymbol{A}_3$-types (with four, respectively, five, lines) in degree 4 and two $\boldsymbol{A}_1$-types (with three, respectively, four, lines) in degree 6; all other exceptions have degree 1 and 2 and are not relevant for us. For some types there are infinitely many isomorphism classes, but for all types that turn out to be equivariant compactifications of homogeneous spaces for $\mathbb{G}_a \rtimes \mathbb{G}_m$ we see that there is precisely one such surface up to isomorphism.

4.2. Actions on generalized del Pezzo surfaces

We now consider the classification of those generalized del Pezzo surfaces that admit a generically transitive action of G_d for some d, with the aim of proving Theorem 1.1. It turns out that such surfaces must satisfy a special geometric condition.

Lemma 4.1. *Let $\tilde{S}$ be a generalized del Pezzo surface that is an equivariant compactification of a homogeneous space for G_d for some d. Then,*

$$\#\{negative\ curves\ on\ \tilde{S}\} \leqslant \operatorname{rk}\operatorname{Pic}\tilde{S} + 1.$$

Proof. First note that if $\tilde{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\tilde{S} \cong \mathbb{F}_2$, then there is at most one negative curve and the inequality trivially holds. So, we may assume that $\tilde{S}$ is obtained from $\mathbb{P}^2$ by a sequence of r blow-ups. To prove the inequality in this case, it suffices to show that the boundary of the action consists of $r + 2 = \operatorname{rk}\operatorname{Pic}\tilde{S} + 1$ irreducible curves. Indeed, let E be a negative curve on $\tilde{S}$. By Lemma 2.3, the line bundle $\mathcal{O}_{\tilde{S}}(E)$ admits a G_d-linearization; in particular, the divisor class of E is invariant under the action of G_d. As E is the unique effective curve in its divisor class, we see that E itself is invariant under the action of G_d, and therefore E must lie on the boundary. The fact that the boundary consists of $r + 2$ irreducible curves then gives the required inequality.

To prove the claim we proceed by induction. Let X be a smooth projective equivariant compactification of a homogeneous space for G_d that contains a (-1)-curve E, and let $\pi\colon X \to Y$ be the map given by contracting E. Note that we may assume that Y is an equivariant compactification of a homogeneous space for G_d and that π is G_d-equivariant by Lemma 2.6. As π is an isomorphism outside E, we see that X has exactly one more boundary component than Y. Applying this inductively to $\tilde{S}$, we see that the boundary of the action on $\tilde{S}$ consists of $r + n$ irreducible curves, where n is the number of irreducible curves on the boundary of the action on $\mathbb{P}^2$. However, by the classification given in Theorem 3.3, we know that $n = 2$. This proves the claim and, hence, completes the proof of the lemma. $\qquad\square$

From the classification of generalized del Pezzo surfaces that can be found in [**13**], for example, it is straightforward to write the list of surfaces that satisfy the condition of Lemma 4.1. These are shown in Figure 1.

We note that, for each of the types of degree *at most three* given in Figure 1, there exists a *unique* surface over K with that type, up to isomorphism. Indeed, for the surfaces of degree 3 this follows from the classification given in [**8**]. This also implies uniqueness for all surfaces of degree greater than 3, except perhaps for the quartic del Pezzo surface of type $\boldsymbol{A}_3$ with four lines. However, we also have uniqueness in this case on noting that such a surface is obtained by contracting a unique (-1)-curve on a del Pezzo surface of degree 3 and type $\boldsymbol{A}_4$. There, again, exists a unique surface of this type by [**8**]. Note that this result does not hold for some of the lower degree surfaces in Figure 1; for example, there are infinitely many generalized del Pezzo surfaces of degree 2 and type $\boldsymbol{D}_6$ up to isomorphism (see [**23**, Theorem 5.7]).

Next, it follows from Lemma 2.6 that we need only consider the 'extremal' surfaces in Figure 1, namely, if a surface is an equivariant compactification of (a homogeneous space
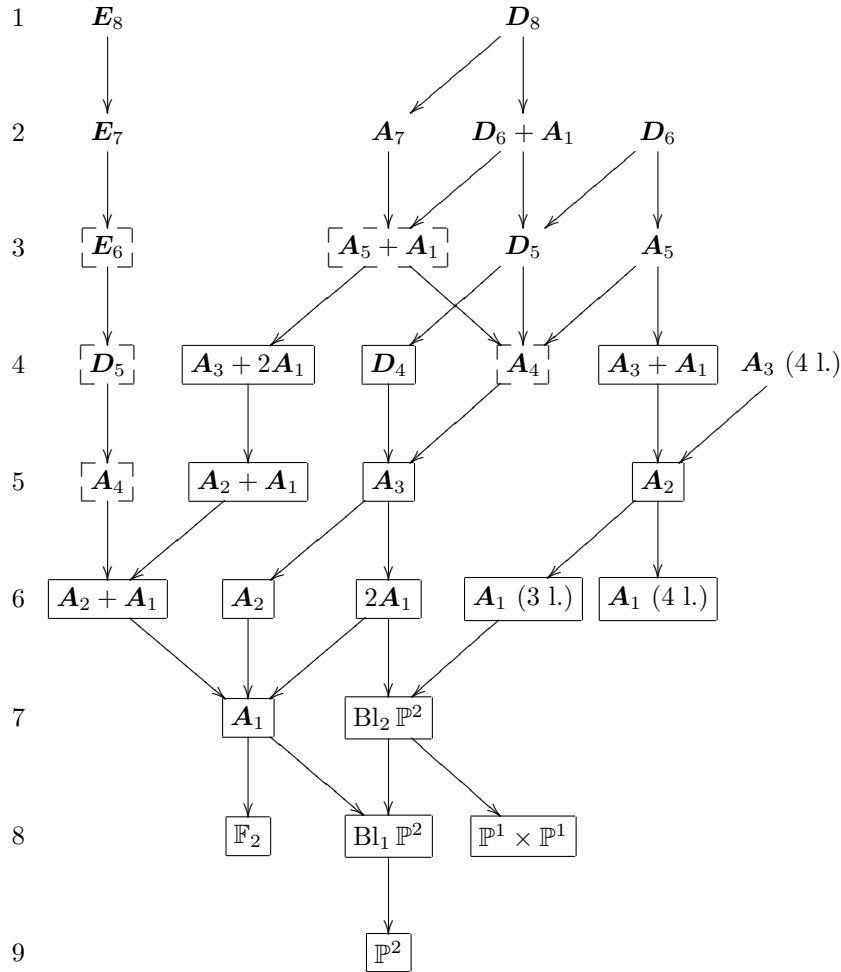
1    $\boldsymbol{E}_8$ $\boldsymbol{D}_8$

2    $\boldsymbol{E}_7$ $\boldsymbol{A}_7$ $\boldsymbol{D}_6 + \boldsymbol{A}_1$ $\boldsymbol{D}_6$

3    $\boldsymbol{E}_6$ $\boldsymbol{A}_5 + \boldsymbol{A}_1$ $\boldsymbol{D}_5$ $\boldsymbol{A}_5$

4    $\boldsymbol{D}_5$ $\boldsymbol{A}_3 + 2\boldsymbol{A}_1$ $\boldsymbol{D}_4$ $\boldsymbol{A}_4$ $\boldsymbol{A}_3 + \boldsymbol{A}_1$ $\boldsymbol{A}_3$ (4 l.)

5    $\boldsymbol{A}_4$ $\boldsymbol{A}_2 + \boldsymbol{A}_1$ $\boldsymbol{A}_3$ $\boldsymbol{A}_2$

6    $\boldsymbol{A}_2 + \boldsymbol{A}_1$ $\boldsymbol{A}_2$ $2\boldsymbol{A}_1$ $\boldsymbol{A}_1$ (3 l.) $\boldsymbol{A}_1$ (4 l.)

7           $\boldsymbol{A}_1$ $\mathrm{Bl}_2\,\mathbb{P}^2$

8           $\mathbb{F}_2$ $\mathrm{Bl}_1\,\mathbb{P}^2$ $\mathbb{P}^1 \times \mathbb{P}^1$

9           $\mathbb{P}^2$

Figure 1. Generalized del Pezzo surfaces $\tilde{S}$ in increasing degree with #{negative curves on $\tilde{S}$} $\leqslant \mathrm{rk}\,\mathrm{Pic}(\tilde{S}) + 1$. Those in solid boxes are exactly the equivariant compactifications of G_d for some d. Those in dashed boxes are exactly the equivariant compactifications of a homogeneous space for G_d for some d. Arrows denote blow-up maps (in degree at least 4, only maps used in our proofs are included). The shorthand 'l.' stands for 'lines'.

for) some G_d, then so is any surface that lies below it in Figure 1. Conversely, if a surface is not an equivariant compactification of (a homogeneous space for) G_d, then no surface in Figure 1 that lies above it is either.

We now proceed to classify the generically transitive actions of G_d on some of the surfaces in Figure 1 up to equivalence for each $d \in \mathbb{Z}$. We briefly outline the method that we use. Suppose that $\rho \colon \tilde{S} \to \mathbb{P}^2$ is the composition of $r \leqslant 6$ blow-ups of $\mathbb{P}^2$, and that

$\tilde{S}$ admits a generically transitive action of G_d for some d. Then, by Lemma 2.6, we obtain a generically transitive action of G_d on $\mathbb{P}^2$ in such a way that ρ is G_d-equivariant. Also, in every case we consider, we are able to choose ρ in such a way that the line $\{z = 0\}$ and the point $(1:0:0)$ are images of negative curves on $\tilde{S}$. As the negative curves on $\tilde{S}$ are invariant under the action (see the proof of Lemma 4.1), the line $\{z = 0\}$ and the point $(1:0:0)$ must also be invariant under the action on $\mathbb{P}^2$, and hence the action has the form given by Lemma 3.2.

Therefore, we are reduced to the following question: for which of the actions given in Lemma 3.2 is the map ρ G_d-equivariant? This is equivalent to asking whether the inverse of ρ is a G_d-equivariant birational map $\rho^{-1} \colon \mathbb{P}^2 \dashrightarrow \tilde{S}$. Also, by Proposition 2.7, this is again equivalent to asking whether or not $\pi \circ \rho^{-1}$ is G_d-equivariant, where $\pi \colon \tilde{S} \to S$ denotes the map to the associated singular del Pezzo surface. As $r \leqslant 6$, however, we see that $S \subset \mathbb{P}^{9-r}$ and, moreover, the map $\pi \circ \rho^{-1}$ is given by choosing a basis for some linear series $V \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$. We may, therefore, appeal to Lemma 2.4, and reduce to determining whether or not V is invariant under the action of G_d on $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$.

We now show this method in action by considering the extremal surfaces given in Figure 1, beginning with the one such surface of degree 5.

Lemma 4.2. *The quintic del Pezzo surface of type $\boldsymbol{A}_4$ admits a unique structure as an equivariant compactification of a homogeneous space for G_1 (but none for G_0). It is not an equivariant compactification of G_d for any d.*

Proof. The quintic type $\boldsymbol{A}_4$ is defined by the equations

$$x_2 x_4 - x_1^2 = x_3 x_4 - x_0 x_1 = x_0 x_2 - x_1 x_3 = x_1 x_2 + x_0^2 + x_4 x_5 = x_2^2 + x_0 x_3 + x_1 x_5 = 0.$$

The associated rational map from $\mathbb{P}^2$ is given by

$$(x{:}y{:}z) \mapsto (xz^2{:}yz^2{:}y^2 z{:}xyz{:}z^3{:} - (y^3 + x^2 z)).$$

This is not defined at $(1:0:0)$, and, moreover, the line $\{z = 0\}$ is mapped to the singularity $(0:0:0:0:0:1)$. Therefore, the associated action on $\mathbb{P}^2$ must leave these subvarieties invariant, hence is of the form given in Lemma 3.2. For it to be equivariant, the associated linear series of cubic forms must be invariant of the action of G_d, by Lemma 2.4. One can check that this happens if and only if $2k_1 = 3k_2$ (this condition comes from the term $-(y^3 + x^2 z)$). So, for some $k \neq 0$, we have that $(k_1, k_2) = (3k, 2k)$. If two of $\alpha_1, \alpha_2, \alpha_3$ are non-zero, this leads to $d = k = 0$, and the action is not generically transitive. If only $\alpha_1 \neq 0$, we have $d = k$, and the action is equivalent to $\tau_{d,-2d}$. If only $\alpha_2 \neq 0$, we have $d = 3k$, and the action is equivalent to $\tau_{3k,2k}$. If only $\alpha_3 \neq 0$, we have $d = 2k$, and the action is equivalent to $\tau_{2k,3k}$. In any case, the stabilizer of a general point has order at least 2, and so the action is not generically free. $\square$

We now consider the extremal surfaces of degree 4.

Lemma 4.3. *For quartic generalized del Pezzo surfaces we have the following.*

- *The surface of type $\boldsymbol{A}_3 + 2\boldsymbol{A}_1$ is an equivariant compactification of G_d for all $d \in \mathbb{Z}$.*

- *The surface of type $\boldsymbol{D}_4$ admits a unique structure as an equivariant compactification of G_2 (but none for other G_d with $d \geqslant 0$).*

- *The surface of type $\boldsymbol{A}_4$ admits a unique structure as an equivariant compactification of a homogeneous space for G_1 (but none for G_0). It is also not an equivariant compactification of G_d for any d.*

- *The surface of type $\boldsymbol{A}_3 + \boldsymbol{A}_1$ admits a unique structure as an equivariant compactification of G_1 (but none for other G_d with $d \geqslant 0$).*

- *The surface of type $\boldsymbol{A}_3$ (four lines) is not an equivariant compactification of a homogeneous space for G_d for any d.*

Proof. Type $\boldsymbol{A}_3 + 2\boldsymbol{A}_1$. The surface S is defined by

$$x_0 x_1 - x_2^2 = x_0^2 - x_3 x_4 = 0.$$

Note that this surface is toric. For each $d \in \mathbb{Z}$, the action of G_d is given by the representation

$$(b,t) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b^2 & t^{2d} & 2t^d b & 0 & 0 \\ b & 0 & t^d & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & t^{-1} \end{pmatrix},$$

which is easily checked to be generically free and generically transitive.

Type $\boldsymbol{D}_4$. The surface S can be defined by

$$x_0 x_3 - x_1 x_4 = x_0 x_1 + x_1 x_3 + x_2^2 = 0.$$

The associated rational map from $\mathbb{P}^2$ is given by

$$(x{:}y{:}z) \mapsto (xz^2{:}z^3{:}yz^2{:}-z(xz+y^2){:}-x(xz+y^2)).$$

The associated action on $\mathbb{P}^2$ must, therefore, fix $\{z = 0\}$ and $(1{:}0{:}0)$; hence, it has the form given by Lemma 3.2. By considering the term $z(xz+y^2)$, we see that we must have $k_1 = 2k_2$. Also, by considering the action on the final term, we see that for the linear series to be invariant we must have $\alpha_1 = \alpha_3 = 0$ (this is due to the appearance of the monomials y^3 and xyz if α_1 or α_3 are non-zero). Therefore, we also have $k_1 = d$. Such an action may occur only when d is even, in which case it is equivalent to $\tau_{d,-d/2}$. This

is faithful if and only if $|d| = 2$. The action when $d = 2$ may be given explicitly via the representation

$$(b, t) \mapsto \begin{pmatrix} t^2 & b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & -b & 0 & t^2 & 0 \\ -bt^2 & -b^2 & 0 & bt^2 & t^4 \end{pmatrix}.$$

Type $\boldsymbol{A}_4$. The surface S can be defined by

$$x_0 x_1 - x_2 x_3 = x_0 x_4 + x_1 x_2 + x_3^2 = 0.$$

The associated rational map from $\mathbb{P}^2$ is given by

$$(x{:}y{:}z) \mapsto (z^3{:}xyz{:}xz^2{:}yz^2{:} - y(x^2 + yz)).$$

The associated action on $\mathbb{P}^2$ must, therefore, fix $\{z = 0\}$, $(1{:}0{:}0)$ and $(0{:}1{:}0)$. This implies, in particular, that $\alpha_1 = 0$. By considering the final term, we see that we must have $2k_1 = k_2$ and $\alpha_3 = 0$ (due to a term of the form $x^2 z$ if $\alpha_3 \neq 0$). Such an action is, therefore, equivalent to $\tau_{d,2d}$ for some d. This is faithful if and only if $|d| = 1$, in which case the stabilizer of a generic point has order 2. Explicitly, the action for $d = 1$ is given by

$$(b, t) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & t^3 & 0 & bt^2 & 0 \\ b & 0 & t & 0 & 0 \\ 0 & 0 & 0 & t^2 & 0 \\ 0 & -2bt^3 & 0 & -b^2t^2 & t^4 \end{pmatrix}.$$

Type $\boldsymbol{A}_3{+}\boldsymbol{A}_1$. Note that we originally considered this surface in [**14**, § 5]. The equations are given by

$$x_1 x_3 - x_2^2 = x_0 x_3 + x_2 x_4 + x_0^2 = 0.$$

The associated rational map from $\mathbb{P}^2$ is given by

$$(x{:}y{:}z) \mapsto (xyz{:}y^3{:}y^2 z{:}yz^2{:} - xz(x + z)).$$

The action on $\mathbb{P}^2$ must fix $\{y = 0\}$, $\{z = 0\}$ and $(0{:}0{:}1)$. Hence, we must have $\alpha_2 = \alpha_3 = 0$. The linear series is invariant if and only if $k = -d$, in which case this action is equivalent to $\tau_{d,d}$. This is faithful if and only if $|d| = 1$, and the action in the case $d = -1$ is given in [**14**, § 5].

Type $\boldsymbol{A}_3$. This is given by the equations

$$x_0 x_1 - x_2^2 = (x_0 + x_1 + x_3)x_3 - x_2 x_4 = 0.$$

This surface is described in [**12**, § 6.4]. The associated rational map from $\mathbb{P}^2$ is given by

$$(x{:}y{:}z) \mapsto (z^3{:}x^2 z{:}xz^2{:}xyz - z^3{:}(x + y)(xy - z^2)).$$

The action on $\mathbb{P}^2$ must, therefore, fix $(1{:}0{:}0)$, $(0{:}1{:}0)$, $(1{:}-1{:}0)$ and the lines $\{x = 0\}$, $\{z = 0\}$. Using Lemma 3.2, we must have $\alpha_1 = \alpha_2 = 0$ and, moreover, $k_1 = k_2 = d$, as there are three fixed points. Considering the term $(x + y)(xy - z^2)$, we deduce that the linear series is invariant only if $d = 0$, which does not give a generically transitive action.   □

Finally, we consider the cubic surfaces.

Lemma 4.4. *For cubic generalized del Pezzo surfaces we have the following.*

- *The surface of type $\boldsymbol{E}_6$ admits a unique structure as an equivariant compactification of a homogeneous space for G_2 (but none for G_0 or G_1).*

- *The surface of type $\boldsymbol{A}_5 + \boldsymbol{A}_1$ admits a unique structure as an equivariant compactification of a homogeneous space for G_1 (but none for G_0).*

Moreover, given any generically transitive action of G_d on these surfaces, any fixed point that lies on a (-1)-curve must also lie on a (-2)-curve.

Proof. Type $\boldsymbol{E}_6$. This is defined by $x_3 x_0^2 - x_0 x_2^2 + x_1^3 = 0$. It is the closure of the image of $\mathbb{P}^2$ under the rational map

$$(x{:}y{:}z) \mapsto (z^3{:}yz^2{:}xz^2{:}x^2 z - y^3),$$

with $(1{:}0{:}0)$ and $\{z = 0\}$ in $\mathbb{P}^2$ fixed. The only questionable part of the linear series is $x^2 z - y^3$, which maps to an element of the linear series under the matrix in Lemma 3.2 if and only if $2k_1 = 3k_2$ and $\alpha_1 = \alpha_3 = 0$. So, there exists an integer k with $(k_1, k_2) = (3k, 2k)$. This gives an action that is equivalent to $\tau_{2k,3k}$, which is faithful if and only if $|k| = 1$. In this case, the stabilizer of a general point has order 3, so the action is not generically free. The induced action on the surface in the case $k = 1$ is given by

$$(b, t) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ b & 0 & t^3 & 0 \\ b^2 & 2bt^3 & 0 & t^6 \end{pmatrix}.$$

On the only line $\{x_0 = x_1 = 0\}$, the only fixed point is the singularity $(0{:}0{:}0{:}1)$.

Type $\boldsymbol{A}_5 + \boldsymbol{A}_1$. This surface has equations $x_1^3 + x_2 x_3^2 + x_0 x_1 x_2 = 0$. The determination of all actions of any G_d on this surface was given in [**2**, Lemma 4], but we re-prove this result for completeness. The associated rational map is

$$(x{:}y{:}z) \mapsto (-z^3 - x^2 y{:}yz^2{:}y^2 z{:}xyz).$$

An associated action of G_d on $\mathbb{P}^2$ must fix the points $(1{:}0{:}0), (0{:}1{:}0)$ and the lines $\{y = 0\}$, $\{z = 0\}$. In the form of Lemma 3.2, we must have $\alpha_1 = \alpha_3 = 0$ and $k_1 = d$. The associated linear series is invariant if and only if $2k_1 = k_2$, in which case we obtain an action on $\mathbb{P}^2$ that is equivalent to $\tau_{d,-2d}$. This action is faithful if and only if $|d| = 1$, in which case

the stabilizer of a general point has order 2. The induced action on S in the case $d = 1$ is given by

$$(b, t) \mapsto \begin{pmatrix} t^4 & -b^2t^2 & 0 & -2bt^3 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & bt^2 & 0 & t^3 \end{pmatrix}.$$

On the only lines $\{x_1 = x_2 = 0\}$ and $\{x_1 = x_3 = 0\}$, the fixed points are the singularities $(0:0:1:0)$ and $(1:0:0:0)$. $\qquad\square$

Proof of Theorem 1.1. By Lemma 4.3, the quartic generalized del Pezzo surfaces of types $\boldsymbol{A}_3 + 2\boldsymbol{A}_1$, $\boldsymbol{D}_4$ and $\boldsymbol{A}_3 + \boldsymbol{A}_1$ are equivariant compactifications of G_d for some d. Therefore, all surfaces below them in Figure 1 also are, by Lemma 2.6. This is exactly the first collection of surfaces given in the statement of Theorem 1.1.

Next, by Lemma 4.4 we see that the cubic surfaces of types $\boldsymbol{E}_6$ and $\boldsymbol{A}_5 + \boldsymbol{A}_1$ are equivariant compactifications of homogeneous spaces for G_d for some d. Again by Lemma 2.6, we deduce that all surfaces below them in Figure 1 are also equivariant compactifications of homogeneous spaces for G_d for some d. Also, by Lemmas 4.2 and 4.3, we know that the quintic generalized del Pezzo surface of type $\boldsymbol{A}_4$ and the quartic generalized del Pezzo surface of type $\boldsymbol{A}_4$ are not equivariant compactifications of G_d for any d. In particular, this implies the same result for every surface lying above them in Figure 1 by Lemma 2.6.

To complete the proof of Theorem 1.1, it suffices to show that the remaining surfaces in Figure 1 are not equivariant compactifications of homogeneous spaces for G_d for any d. For the quartic surface of type $\boldsymbol{A}_3$ with four lines, this follows from Lemma 4.3. The cubic del Pezzo surfaces of types $\boldsymbol{D}_5$ and $\boldsymbol{A}_5$ have one-dimensional automorphism groups by [**21**, Table 3], so they cannot have a generically transitive action of any G_d. Surfaces of type $\boldsymbol{E}_7$ and $\boldsymbol{A}_7$ of degree 2 are blow-ups of the cubic surfaces of type $\boldsymbol{E}_6$ and $\boldsymbol{A}_5 + \boldsymbol{A}_1$ in a point on one of the (-1)-curves outside the (-2)-curves. However, by Lemma 4.4, there are no generically transitive actions fixing such points, and hence these surfaces of degree 2 cannot have such an action. Finally, surfaces of type $\boldsymbol{D}_6 + \boldsymbol{A}_1$ and $\boldsymbol{D}_6$ in degree 2 and type $\boldsymbol{E}_8$ in degree 1 are blow-ups of surfaces that have no generically transitive action of G_d, so they also cannot have such an action. This completes the proof of Theorem 1.1. $\qquad\square$

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