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CONTACT DISTRIBUTION IN A THINNED BOOLEAN MODEL WITH POWER-LAW RADII

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Abstract

We consider a weighted stationary spherical Boolean model in $\mathbb{R}^d$ to which a Matérn-type thinning is applied. Assuming that the radii of the balls in the Boolean model have regularly varying tails, we establish the asymptotic behavior of the tail of the contact distribution of the thinned germ–grain model under four different thinning procedures of the original model.

Keywords: Boolean model; hard-core thinning; power tail; contact distribution; empty space

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1. Introduction

We consider hard-core global thinning of a stationary spherical Boolean model in $\mathbb{R}^d$, constructed as follows. Let Φ be a Poisson point process on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with mean measure

$$m(\mathrm{d}\boldsymbol{x}, \mathrm{d}r, \mathrm{d}w) = \lambda \, \mathrm{d}\boldsymbol{x} \, G(\mathrm{d}r, \mathrm{d}w). \tag{1.1}$$

Here $\lambda > 0$ is the spatial intensity and G is a probability law on $\mathbb{R}_+ \times \mathbb{R}_+$. Such a Poisson point process is often simply called a Poisson process, or a Poisson random measure. In fact, the 'measure aspect' is particularly important for us in this paper, and we occasionally emphasize it in the sequel. Let $(\boldsymbol{X}_n, R_n, W_n)$, $n = 1, 2 \ldots$ be a measurable enumeration of the points of Φ. We view $\boldsymbol{X}_n \in \mathbb{R}^d$ as the center of the nth ball, and R_n its radius. In the sequel we use the notation $B_r(\boldsymbol{x})$ for a closed ball of radius $r > 0$ centered at $\boldsymbol{x} \in \mathbb{R}^d$, so the nth point of Φ corresponds to the closed ball $B_{R_n}(\boldsymbol{X}_n)$. The last component, W_n, is the weight of the nth ball, and it is used below in resolving collisions between balls. Let $F(\cdot) = G(\cdot \times \mathbb{R}_+)$ be the law of the radius marking a spatial Poissonian point. We assume that

$$\int_0^\infty r^d F(\mathrm{d}r) < \infty. \tag{1.2}$$

It is well known that, under this assumption, with probability 1, a realization of the random field Φ has the property that only finitely many balls of the type $B_{R_n}(\boldsymbol{X}_n)$ intersect any compact

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set in $\mathbb{R}^d$. This implies that the union

$$D = \bigcup_{n=1}^{\infty} B_{R_n}(\boldsymbol{X}_n) \tag{1.3}$$

is a random closed subset of $\mathbb{R}^d$. We refer the reader to [3] for this fact and as a general reference on Boolean and related models.

It is common to refer to a random configuration of the type we have constructed as a *germ–grain model*; such a model does not need to involve a spatial Poisson point process or spherical shapes. In the Boolean model above each X_n is a germ, and the corresponding closed ball $B_{R_n}(\mathbf{0})$ is its grain. The set D in (1.3) is the *grain cover* of the space.

Some of the balls $B_{R_n}(\boldsymbol{X}_n)$ in the Boolean model as above overlap. In order to obtain a *hard-core* germ–grain model, i.e. a configuration in which no two grains overlap, it is possible to thin the Boolean model, by removing (at least) one ball in each pair of balls involved in an overlap. We follow the global thinning procedure introduced in [6]; it is in this procedure that the weight component W_n of the nth ball is used. Informally, for every pair of different balls, $B_{R_n}(\boldsymbol{X}_n)$ and $B_{R_m}(\boldsymbol{X}_m)$ with a nonempty intersection, the ball $B_{R_n}(\boldsymbol{X}_n)$ is deleted if $W_n \leq W_m$; this procedure deletes both balls if $W_n = W_m$. To be a bit more formal, we use the notation borrowed from [5]: let

$$N_{\boldsymbol{x},r,w} = \{(\boldsymbol{x}', r', w') \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \setminus (\boldsymbol{x}, r, w) \colon B_{r'}(\boldsymbol{x}') \cap B_r(\boldsymbol{x}) \neq \varnothing\} \tag{1.4}$$

(the notation is somewhat informal: a ball of the type $B_r(\boldsymbol{x})$ is not a subset of $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$; it is really a subset of that product space with w fixed, so a proper notation would be $B_r^w(\boldsymbol{x})$, but as long as no confusion is likely to arise, we use the informal notation because it is simpler). We view the set $N_{\boldsymbol{x},r,w}$ as the collection of centers, radii, and weights of balls that could, potentially, intersect a reference ball $B_r(\boldsymbol{x})$ with weight w. Then the thinned Boolean model we are considering is given by

$$\Phi^{\text{th}} = \{(\boldsymbol{x}, r, w) \in \Phi \colon w > w' \text{ for all } (\boldsymbol{x}', r', w') \in \Phi \cap N_{\boldsymbol{x},r,w}\}. \tag{1.5}$$

By construction, all the remaining grains (balls) in the thinned random field Φ^{th} are disjoint. The corresponding grain cover can be written in the form

$$D_{\text{th}} = \bigcup_{(\boldsymbol{x},r,w)\in\Phi^{\text{th}}} B_r(\boldsymbol{x}). \tag{1.6}$$

The thinning procedure we are using is sometimes referred to as the Matérn type II construction. A discussion of different Matérn type constructions and their extensions has been given in [8]. This model was studied in [1] and [6]. Chiu *et al.* [3] provided a survey of Matérn thinnings. Teichmann *et al.* [10] used probabilistic thinning rules to generalize Matérn type constructions and presented a detailed second order analysis for these general models. Our inspiration for the present work came from Kuronen and Leskelä [5], and we refer the reader to this paper for an illuminating discussion of the importance and applications of hard-core germ–grain models. Specifically, the latter paper considered the case of power-law grain sizes; in our notation we can describe this setup as follows. Recalling that we denote by F the marginal distribution of the probability measure G in (1.1) corresponding to the random radius of a Poisson ball, the power-law distribution of the grain sizes of [5] is the assumption of regular variation of the tail

$$\bar{F}(r) := 1 - F(r) = r^{-\alpha} L(r), \tag{1.7}$$

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where $\alpha > d$ and L is a slowly varying function; recall that a measurable eventually positive function L is slowly varying at ∞ if, for any $c > 0$, $L(cx)/L(x) \to 1$ as $x \to \infty$. The restriction $\alpha > d$ ensures that the integrability condition (1.2) holds. We refer the reader to [9] for information on regular varying tails. In the sequel we extensively use two facts about regularly varying functions that quantify their similarity with power functions. The first property is referred to as Potter's bounds: if (1.7) holds then, for any fixed $\delta > 0$, there exists $x_0 > 0$ such that, for all $x, y \geq x_0$,

$$\frac{\bar{F}(y)}{\bar{F}(x)} \leq (1+\delta) \max\left\{\left(\frac{y}{x}\right)^{-\alpha+\delta}, \left(\frac{y}{x}\right)^{-\alpha-\delta}\right\}; \tag{1.8}$$

see [2, Theorem 1.5.6]. We also use one of the properties referred to as Karamata's theorem, specifically the version that says that if $\bar{F}$ satisfies (1.7), then it integrates similarly to a power function, namely, if $\alpha > 1$, then

$$\frac{\int_x^\infty \bar{F}(t)\,\mathrm{d}t}{x\bar{F}(x)} \to \frac{1}{\alpha-1} \quad \text{as } x \to \infty;$$

see [2, Proposition 1.5.10].

Under the assumption (1.7) of regular variation, Kuronen and Leskelä [5] discovered the appearance of power-like decay of the covariance function of the thinned grain cover (1.6) defined by

$$k_{th}(z) = \mathbb{P}(\mathbf{0} \in D_{th},\ z \in D_{th}) - \mathbb{P}(\mathbf{0} \in D_{th})\mathbb{P}(z \in D_{th}) \quad \text{as } \|z\| \to \infty.$$

This was the case under three out of four choices of the joint law G in (1.1) they considered; we return to these choices in a moment.

In this paper we are interested in the contact distribution for the thinned Boolean model described above. It is a probability law H on $\mathbb{R}_+$ whose complementary cumulative distribution function is defined by

$$\bar{H}(r) = \mathbb{P}(B_r(\mathbf{0}) \cap D_{\mathrm{th}} = \varnothing \mid \mathbf{0} \notin D_{\mathrm{th}}), \qquad r > 0. \tag{1.9}$$

Of course, the contact distribution can be defined for any germ–grain model. It differs only by a possible atom at zero from the *empty space function*, a probability law on $[0, \infty)$ defined by

$$\bar{H}_{\mathrm{e}}(r) = \mathbb{P}(B_r(\mathbf{0}) \cap D_{\mathrm{th}} = \varnothing), \qquad r \geq 0. \tag{1.10}$$

Contact distributions are important characteristics of germ–grain models; a survey on the topic can be found in [4]. Explicit formulae for the contact distributions are mostly available only for Poisson-based models such as Poisson cluster models. For example, for the nonthinned Boolean model $D_{\mathrm{th}} = D$, we have

$$\bar{H}_{\mathrm{e}}(r) = \exp\left\{-\lambda v_d\left[r^d + d\int_r^\infty x^{d-1}\bar{F}(x-r)\,\mathrm{d}x\right]\right\}, \qquad r \geq 0.$$

Here v_d is the volume of the d-dimensional unit Euclidean ball. Our goal in this paper is to understand the tail behavior of the contact distribution H for the thinned Boolean model with a power-law distribution of the grain sizes. Specifically, we are interested in answering the question whether a power-law distribution of the grain sizes results in a power-law behavior of the contact distribution for the thinned Boolean model. Note that for the original Boolean model

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with the grain cover (1.3) the tail of the contact distribution decays, obviously, exponentially regardless of the distribution of the radius of a ball. It turns out that certain choices of the joint law G of the radius of a ball and its weight lead to appearance of a power-law-like decay of the contact distribution, while other choices do not.

One possible choice of the law G in (1.1) is given by setting $W_n = R_n$ almost surely (a.s.) for all n, so that G is concentrated on the diagonal $r = w$ of $\mathbb{R}_+ \times \mathbb{R}_+$. With this choice of G, balls with a larger radius have a larger weight. We refer to this situation as the case of heavy large balls. It is useful to mention that the results concerning this case remain true if W_n is any strictly increasing function of R_n (and even more general possibilities fall under the same framework). Another possible choice of G is given by setting $W_n = 1/R_n$ a.s. for all n. With this choice of G, balls with a smaller radius have a larger weight. As above, the results concerning this case remain true if W_n is any strictly decreasing function of R_n. A third possible choice of the law G is to make it a product law, and to make the marginal law of the weights continuous (e.g. standard uniform). That is, the weights are independent of the radii of the balls. Finally, one could make the weights of the balls constant (e.g. $W_n = 1$ a.s. for all n). In this case, only isolated balls in the original Boolean model (i.e. the balls that do not overlap with any other ball) stay in the thinned germ–grain model Φ^{th}. This last thinning mechanism is known as the Matérn type I construction.

It is interesting that, as shown in [5], when the radii of the balls are regularly varying as in (1.7), the covariance function of the thinned grain cover (1.6) has a power-like decay under all of the above thinning mechanisms apart from the case of heavy small balls.

In a certain sense the above situation is preserved when one is interested in the tail of the contact distribution. In Section 2 we show that this tail has a power-like decay in all cases apart from the case of heavy small balls. Interestingly, the power of the rate of decay is different for the three thinning mechanisms considered here. Finally, in Section 3, on the other hand, we show that in the situation when a ball with a large radius has a smaller weight than a ball with a small radius, the tail of the contact distribution decays exponentially.

2. Power law of the contact distribution

We start this section by showing that, in the case when a ball with a larger radius in a Boolean model has a larger weight than a ball with a smaller radius, the contact distribution of the thinned model has a tail with power-like decay.

Theorem 2.1. *Assume that the distribution of the radii of the balls in the Boolean model satisfies (1.7) with $\alpha > d$. If $W_n = R_n$ a.s. (i.e. if larger balls have larger weights) then the contact distribution of the thinned germ–grain model satisfies*

$$0 < \liminf_{r\to\infty} \frac{\bar{H}(r)}{(r^d \bar{F}(r))^2} \le \limsup_{r\to\infty} \frac{\bar{H}(r)}{(r^d \bar{F}(r))^2} < \infty. \tag{2.1}$$

Proof. Throughout the proof, we can and do work with the tail of the empty space function (1.10) instead of the tail of the contact distribution. Further, we use c to denote a finite positive constant whose value is not important and that may change from one appearance to the next. We also introduce a notational simplification. The Poisson point process Φ is a measure in the $(d + 2)$-dimensional space $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$, but in the present context the ‘weight’ coordinate is a function of the ‘radius’ coordinate, so it is simpler to view Φ as a measure in the $(d + 1)$-dimensional space $\mathbb{R}^d \times \mathbb{R}_+$, described by the location of the center of a ball and its radius. We use the appropriate notation throughout the proof.

We start by proving the lower bound in (2.1). We construct a scenario under which the ball $B_r(\mathbf{0})$ does not intersect Φ^{th}. The idea of the construction is that a single ball with a large radius in Φ 'eliminates' all the other balls in Φ that intersect $B_r(\mathbf{0})$, and then another ball in Φ of an even larger radius 'eliminates' the first ball of a large radius, but does not itself intersect $B_r(\mathbf{0})$. That leaves $B_r(\mathbf{0})$ disjoint from Φ^{th}. The two large balls have centers in sets of sizes proportional to r, and also radii of the size proportional to r, which explains the order of magnitude of the tail in (2.1).

For $r > 0$ we consider three disjoint subsets of $\mathbb{R}^d \times \mathbb{R}_+$:

$$A_r^{(1)} = \{(\boldsymbol{x}, t) \colon t \geq \|\boldsymbol{x}\| + r\}, \tag{2.2}$$

$$A_r^{(2)} = \{(\boldsymbol{x}, t) \colon \max\{r, \|\boldsymbol{x}\| - r\} \leq t < \|\boldsymbol{x}\| + r\}, \tag{2.3}$$

$$A_r^{(3)} = \{(\boldsymbol{x}, t) \colon \|\boldsymbol{x}\| - r \leq t < r\}. \tag{2.4}$$

Note that only those balls $B_{R_n}(\boldsymbol{X}_n)$ in Φ for which $(\boldsymbol{X}_n, R_n) \in A_r^{(1)} \cup A_r^{(2)} \cup A_r^{(3)}$ intersect $B_r(\mathbf{0})$. Further, any ball $B_{R_n}(\boldsymbol{X}_n)$ in Φ for which $(\boldsymbol{X}_n, R_n) \in A_r^{(1)}$ contains the entire ball $B_r(\mathbf{0})$ as a subset. The set $A_r^{(1)}$ is most important for us in proving the lower bound in (2.1). Consider the event

$$B_r = \{\Phi(A_r^{(1)}) = 1, \Phi(A_r^{(2)}) = 0\}.$$

On the event B_r we can define a random vector $(\boldsymbol{X}^{(r)}, R^{(r)}) \in A_r^{(1)}$ corresponding to the location of the center and the radius of the single ball in Φ for which that pair is in the set $A_r^{(1)}$. We extend the definition of $(\boldsymbol{X}^{(r)}, R^{(r)})$ to the outside of the event B_r in an arbitrary measurable way (e.g. define it on B_r^c to be the pair $(\mathbf{0}, 1)$). Clearly, this vector has the law

$$\mathbb{P}((\boldsymbol{X}^{(r)}, R^{(r)}) \in \cdot \mid B_r) = \frac{(\text{Leb} \times F)(\cdot)}{(\text{Leb} \times F)(A_r^{(1)})} \quad \text{over } A_r^{(1)}.$$

Here Leb denotes d-dimensional Lebesgue measure. Note that

$$\bar{H}_{\text{e}}(r) = \mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing) \geq \mathbb{P}(B_r \cap \widehat{B}_r), \tag{2.5}$$

where

$$\begin{aligned}\widehat{B}_r = \{&\text{there is a } \Phi\text{-ball } B_{R_n}(\boldsymbol{X}_n) \text{ with } (\boldsymbol{X}_n, R_n) \in (A_r^{(1)} \cup A_r^{(2)} \cup A_r^{(3)})^c \\ &\text{and } R_n > R^{(r)} \text{ that intersects } B_{R^{(r)}}(\boldsymbol{X}^{(r)})\}.\end{aligned}$$

Recall that v_d is the volume of the unit ball in $\mathbb{R}^d$. By switching to spherical coordinates we see that, for the large r,

$$\begin{aligned}\mathbb{E}[\Phi(A_r^{(1)})] &= \lambda d v_d \int_0^\infty x^{d-1} \bar{F}(x + r)\, \mathrm{d}x \\ &= \lambda d v_d r^d \int_0^\infty t^{d-1} \bar{F}(r(t+1))\, \mathrm{d}t \\ &\sim c r^d \bar{F}(r) \\ &\to 0.\end{aligned} \tag{2.6}$$

In the last step we have used the Potter bounds for regularly varying functions; see [9] or (1.8). Therefore, for large r,

$$\mathbb{P}(B_r) \sim \mathbb{E}[\Phi(A_r^{(1)})] \sim c r^d \bar{F}(r). \tag{2.7}$$

Similarly,

$$\mathbb{E}[\Phi(A_r^{(2)})] \sim cr^d \bar{F}(r) \to 0 \quad \text{as } r \to \infty.$$

Next, for $(\boldsymbol{y}, w) \in A_r^{(1)}$, define $A_{r,(\boldsymbol{y},w)}$ to be the event

$$\{(\boldsymbol{x}, t) \in (A_r^{(1)} \cup A_r^{(2)} \cup A_r^{(3)})^c \colon t > w, \text{ the ball } B_t(\boldsymbol{x}) \text{ intersects the ball } B_w(\boldsymbol{y})\}.$$

Then since a Poisson point process, viewed as a random measure, assigns independent values to disjoint sets,

$$\begin{aligned}\mathbb{P}(\widehat{B}_r \mid B_r) &= \frac{1}{(\mathrm{Leb} \times F)(A_r^{(1)})} \int\int_{(\boldsymbol{y},w)\in A_r^{(1)}} \mathbb{P}(\Phi(A_{r,(\boldsymbol{y},w)}) > 0)\, \mathrm{d}\boldsymbol{y} F(\mathrm{d}w) \\ &\geq \frac{1}{(\mathrm{Leb} \times F)(A_r^{(1)})} \int_{B_{3r}(\mathbf{0})\setminus B_{2r}(\mathbf{0})} \int_{5r}^{5.5r} (1 - \exp\{-m(A_{r,(\boldsymbol{y},w)})\})\, \mathrm{d}\boldsymbol{y} F(\mathrm{d}w)\end{aligned}$$

because, clearly,

$$(B_{3r}(\mathbf{0}) \setminus B_{2r}(\mathbf{0})) \times (5r, 5.5r) \subset A_r^{(1)}.$$

It follows from (2.6) that

$$\frac{(\mathrm{Leb} \times F)[(B_{3r}(\mathbf{0}) \setminus B_{2r}(\mathbf{0})) \times (5r, 5.5r)]}{(\mathrm{Leb} \times F)(A_r^{(1)})} \geq c \quad \text{for all large } r.$$

Therefore, the lower bound in (2.1) will follow from (2.5) and (2.7) once we have shown that there is a constant c such that for all r large enough

$$m(A_{r,(\boldsymbol{y},w)}) \geq cr^d \bar{F}(r) \quad \text{for all } (\boldsymbol{y}, w) \in (B_{3r}(\mathbf{0}) \setminus B_{2r}(\mathbf{0})) \times (5r, 5.5r). \tag{2.8}$$

To this end, for such a pair $(\boldsymbol{y}, w)$, consider the point

$$\tilde{\boldsymbol{y}} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}(\|\boldsymbol{y}\| + w) \in \mathbb{R}^d,$$

and the ball $B_r(\tilde{\boldsymbol{y}})$. Let $\|\boldsymbol{z}\| \leq r$. Note that the distance from the point $\tilde{\boldsymbol{y}} + \boldsymbol{z} \in B_r(\tilde{\boldsymbol{y}})$ to the ball $B_w(\boldsymbol{y})$ does not exceed $\|\boldsymbol{z}\| \leq r$, while the distance from that same point to ball $B_r(\mathbf{0})$ is greater than $\|\boldsymbol{y}\| + w - r > w + r$. Taking into account the bounds on w we have chosen, we see that any ball centered at a point $\tilde{\boldsymbol{y}} + \boldsymbol{z} \in B_r(\tilde{\boldsymbol{y}})$ with a radius $t \in (5.5r, 6r)$ will intersect the ball $B_w(\boldsymbol{y})$ but not the ball $B_r(\mathbf{0})$. We conclude that for a pair $(\boldsymbol{y}, w)$ as above,

$$A_{r,(\boldsymbol{y},w)} \supset \{(\boldsymbol{x}, t) \colon \boldsymbol{x} \in B_r(\tilde{\boldsymbol{y}}),\ t \in (5.5r, 6r)\},$$

implying that

$$m(A_{r,(\boldsymbol{y},w)}) \geq cr^d(\bar{F}(5.5r) - \bar{F}(6r)) \sim cr^d \bar{F}(r) \quad \text{as } r \to \infty,$$

by the regular variation property of $\bar{F}(\cdot)$. This proves (2.8).

We now prove the upper bound in (2.1). Let $K > 0$ be a fixed number to be specified momentarily. Denote

$$A_r^{(4)}(K) = \left\{(\boldsymbol{x}, t) \colon \max\left\{\frac{r}{K}, \|\boldsymbol{x}\| - r\right\} \leq t < \|\boldsymbol{x}\| + r\right\}.$$

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The same argument using regular variation and the Potter bounds as in (2.6) shows that, for large r,

$$\mathbb{E}[\Phi(A_r^{(4)}(K))] \le cr^d \bar{F}(r) \tag{2.9}$$

for some K-dependent constant c. This bound, together with (2.6), tells us that, for large r,

$$\mathbb{P}(\Phi(A_r^{(1)}) \ge 2) \le c(r^d \bar{F}(r))^2, \qquad \mathbb{P}(\Phi(A_r^{(4)}(K)) \ge 2) \le c(r^d \bar{F}(r))^2,$$
$$\mathbb{P}(\Phi(A_r^{(1)}) \ge 1, \Phi(A_r^{(4)}(K)) \ge 1) \le c(r^d \bar{F}(r))^2.$$

Thus, the upper bound in (2.1) follows once we prove the three inequalities below for large r:

$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\mathrm{th}} = \varnothing,\ \Phi(A_r^{(1)}) = 1) \le c(r^d \bar{F}(r))^2, \tag{2.10}$$
$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\mathrm{th}} = \varnothing,\ \Phi(A_r^{(4)}(K)) = 1) \le c(r^d \bar{F}(r))^2, \tag{2.11}$$
$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\mathrm{th}} = \varnothing,\ \Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)\big) = 0) \le c(r^d \bar{F}(r))^2. \tag{2.12}$$

We show that (2.10) and (2.11) hold for any $K > 0$, specifying K when we prove (2.12).

First we prove (2.10). For the event in that probability to occur, the only Φ-ball in $A_r^{(1)}$ must overlap with another Φ-ball that has a larger radius and lies outside $A_r^{(1)}$. Since restrictions of a Poisson point process to disjoint sets are independent, and since the only Φ-ball in $A_r^{(1)}$ has radius at least r, the probability in (2.10) is bounded above by

$$\mathbb{P}(\Phi(A_r^{(1)}) = 1) \sup_{s \ge r} \mathbb{P}(\Phi(A_s^{(5)}) > 0),$$

where

$$A_s^{(5)} = \{(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}_+ \colon t > s,\ \text{the ball } B_t(\boldsymbol{x}) \text{ intersects the ball } B_s(\mathbf{0})\}, \tag{2.13}$$

the center of the ball of radius s being irrelevant due to stationarity. It is elementary that, for large s, by the regular variation of $\bar{F}$ and Karamata's theorem on integration of regularly varying functions (see, e.g. [9]),

$$\begin{aligned} m(A_s^{(5)}) &= c \int_0^\infty x^{d-1} \bar{F}(s \vee (x-s))\, \mathrm{d}x \\ &= c(2s)^d \bar{F}(s) + c \int_{2s}^\infty x^{d-1} \bar{F}(x-s)\, \mathrm{d}x \\ &\le cs^d \bar{F}(s). \end{aligned} \tag{2.14}$$

Therefore, for large s,

$$\mathbb{P}(\Phi(A_s^{(5)}) > 0) \le cs^d \bar{F}(s),$$

and (2.10) follows from (2.6). Clearly, all the ingredients involved in the proof of (2.10) are also available for the proof of (2.11), so we need only prove (2.12).

Now we explain how to choose K: we choose it together with several other constants. Choose sequentially positive real numbers $0 < \theta < d/\alpha$ and $0 < \tau < \theta(\alpha - d)$, a positive integer $I > 2(\alpha - d)/\tau - 1$, and, finally, $K > 2^{I+1}$. We start by considering the concentric balls $B_{r2^{-i}}(\mathbf{0})$, $i = 0, 1, \dots, I$. For $i = 0, 1, \dots, I$, let

$$M_i = \sup\{R_n \colon B_{R_n}(\boldsymbol{X}_n) \text{ is a } \Phi\text{-ball}, \|\boldsymbol{X}_n\| + R_n < r2^{-i}\}.$$

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Then
$$\mathbb{P}(M_i \le (r2^{-i})^\theta) = \exp\{-m(\{(\boldsymbol{x}, t)\colon \|\boldsymbol{x}\| + t < r2^{-i},\ t > (r2^{-i})^\theta\})\},$$
and since $\theta < 1$,
$$\begin{aligned} m(\{(\boldsymbol{x}, t)\colon \|\boldsymbol{x}\| + t < r,\ t > r^\theta\}) &\ge m\big(\big\{(\boldsymbol{x}, t)\colon \|\boldsymbol{x}\| < \tfrac{1}{2}r,\ r^\theta < t \le \tfrac{1}{2}r\big\}\big) \\ &= cr^d\big(\bar{F}(r^\theta) - \bar{F}\big(\tfrac{1}{2}r\big)\big) \\ &\sim cr^d\bar{F}(r^\theta) \quad \text{as } r \to \infty. \end{aligned}$$
From the choice of θ, we see that
$$\mathbb{P}(M_i \le (r2^{-i})^\theta) = o([r^d\bar{F}(r)]^2), \qquad i = 0, 1, \dots, I,$$
so (2.12) follows once we prove the finiteness of
$$\limsup_{r\to\infty} \frac{\mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing,\ \Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\ M_i > (r2^{-i})^\theta,\ i = 0, 1, \dots, I)}{[r^d\bar{F}(r)]^2}. \tag{2.15}$$

For $i = 0, 1, \dots, I$, define the events
$$\begin{aligned} H_i = \{&\text{the } \Phi\text{-ball fully inside } B_{r2^{-i}}(\mathbf{0}) \text{ and of largest radius,} \\ &\text{is eliminated by a } \Phi\text{-ball not fully inside } B_{r2^{-i}}(\mathbf{0})\}. \end{aligned}$$
Note that, on the event H_i^c, the largest Φ-ball fully inside $B_{r2^{-i}}(\mathbf{0})$ stays in the thinned process, hence, $B_r(\mathbf{0}) \cap D_{\text{th}} \ne \varnothing$. Therefore, in order to prove (2.15), it is enough to prove the finiteness of the limsup as $r \to \infty$ of
$$\frac{\mathbb{P}(\{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\ M_i > (r2^{-i})^\theta,\ i = 0, 1, \dots, I\} \cap H_0 \cap \cdots \cap H_I)}{[r^d\bar{F}(r)]^2}. \tag{2.16}$$
Consider first the probability
$$\begin{aligned} &\mathbb{P}(\{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\ M_i > (r2^{-i})^\theta,\ i = 0, 1, \dots, I\} \cap H_I) \\ &\quad \le \mathbb{P}(\{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\ M_I > (r2^{-I})^\theta\} \cap H_I). \end{aligned}$$
On the latter event H_I, we can define a random vector $(\tilde{\boldsymbol{X}}_I, \tilde{R}_I)$ as the center and the radius of the largest Φ-ball fully within $B_{r2^{-I}}(\mathbf{0})$. Note that $\tilde{R}_I > (r2^{-I})^\theta$. The random vector $(\tilde{\boldsymbol{X}}_I, \tilde{R}_I)$ is determined by the Poisson process Φ on the set
$$\{(\boldsymbol{x}, t)\colon \|\boldsymbol{x}\| + t < r2^{-I}\},$$
and the corresponding Φ-ball can only be eliminated by a Φ-ball in the complement of that set. Since restrictions of a Poisson point process to disjoint sets are independent, we conclude, in the notation of (2.13), that, for large r,
$$\begin{aligned} \mathbb{P}(\{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\ M_I > (r2^{-I})^\theta\} \cap H_I) &\le \sup_{s \ge (r2^{-I})^\theta} \mathbb{P}(\Phi(A_s^{(5)}) > 0) \\ &\le c(r2^{-I})^{\theta d}\bar{F}((r2^{-I})^\theta) \\ &\le cr^{-\tau}, \end{aligned}$$
where in the last two steps we used (2.14), the choice of τ, and the regular variation of $\bar{F}$.

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Next consider the probability

$$\begin{aligned}&\mathbb{P}(\ \{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\, M_i > (r2^{-i})^\theta,\ i = 0, 1, \dots, I\} \cap H_{I-1} \cap H_I)\\ &\quad \le \mathbb{P}(\{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\, M_i > (r2^{-i})^\theta,\ i = I-1, I\} \cap H_{I-1} \cap H_I).\end{aligned}$$

Note that the condition $\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0$ in the above event means that the largest Φ-ball completely within $B_{r2^{-I}}(\mathbf{0})$ could only be eliminated by a Φ-ball centered at a point whose norm is in the range $r2^{-I} \pm r/K$, while the largest Φ-ball completely within $B_{r2^{-(I-1)}}(\mathbf{0})$ could only be eliminated by a Φ-ball centered at a point whose norm is in the range $r2^{-(I-1)} \pm r/K$. These two ranges are disjoint by the choice of K. Use again the fact that restrictions of a Poisson point process to disjoint sets are independent, and an argument as above yields

$$\mathbb{P}(\{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\, M_i > (r2^{-i})^\theta,\ i = I-1, I\} \cap H_{I-1} \cap H_I) \le c(r^{-\tau})^2.$$

Continuing in the same manner, we finally obtain

$$\begin{aligned}&\mathbb{P}(\{\Phi(A_r^{(1)}) = \Phi(A_r^{(4)}(K)) = 0,\, M_i > (r2^{-i})^\theta,\ i = 0, 1, \dots, I\} \cap H_0 \cap \dots \cap H_I)\\ &\quad \le c(r^{-\tau})^{I+1} \quad \text{for large } r.\end{aligned}$$

From the choice of I, we see that the finiteness property of (2.16) follows. □

We turn next to consider the case of isolated balls remaining. Once again, the contact distribution has a power-like decaying tail, but the corresponding power is different from the power obtained in Theorem 2.1. This is, perhaps, not surprising, since keeping only isolated balls results in fewest balls remaining in the thinned model, hence larger 'open space'.

Theorem 2.2. *Assume that the distribution of the radii of the balls in the Boolean model satisfies (1.7) with $\alpha > d$. If. in the thinned model. only isolated balls remain then the contact distribution of this thinned germ–grain model satisfies*

$$0 < \liminf_{r\to\infty} \frac{\bar{H}(r)}{r^d \bar{F}(r)} \le \limsup_{r\to\infty} \frac{\bar{H}(r)}{r^d \bar{F}(r)} < \infty. \tag{2.17}$$

Proof. We use the same conventions as in the proof of Theorem 2.1. In particular, we work with the tail of the empty space function, and view the Poisson point process Φ as a measure in the $(d+1)$-dimensional space $\mathbb{R}^d \times \mathbb{R}_+$.

As earlier, we start with a lower bound. One scenario under which the ball $B_r(\mathbf{0})$ is disjoint from the grain cover in the thinned model is the existence of both a Φ-ball that covers the entire ball $B_r(\mathbf{0})$ and a Φ-ball that is entirely within the ball $B_r(\mathbf{0})$. Since

$$m(\{(\boldsymbol{x}, t) \colon B_t(\boldsymbol{x}) \subset B_r(\mathbf{0})\}) \to \infty \quad \text{as } r \to \infty,$$

we conclude from (2.6) that

$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing) \ge (1 - o(1))\mathbb{P}(\Phi(A_r^{(1)}) \ge 1) \sim \mathbb{E}(\Phi(A_r^{(1)})) \sim cr^d \bar{F}(r).$$

This proves the lower bound in (2.17).

The argument for the upper bound in (2.17) is based on several facts. First, since the thinned random field Φ^{th} is a.s. nonempty, for any $\varepsilon > 0$ and large enough $a > 0$,

$$\mathbb{P}(\text{there is } B_v(\boldsymbol{x}) \in \Phi^{\text{th}} \text{ with } \|\boldsymbol{x}\| \le \varepsilon \text{ and } v \le a) > 0. \tag{2.18}$$

Second, there is a constant $c > 0$ such that for any $0 < a \le r$ there exist at least $[cr^d/a^d]$ closed balls of radius a completely within $B_r(\mathbf{0})$, such that the Euclidean distance between any two different balls is at least a (this fact is readily verified by considering a regular grid of size a inside $B_r(\mathbf{0})$).

Let $M(r)$ be the largest radius of a Φ-ball intersecting $B_r(\mathbf{0})$ (defined to be 0 if no Φ-ball intersects $B_r(\mathbf{0})$). Clearly, for any $t > 0$,

$$\mathbb{P}(M(r) > t) = 1 - \exp\{-m(A^{(6)}_{r,t})\},$$

where

$$A^{(6)}_{r,t} = \{(\boldsymbol{x}, s) \in \mathbb{R}^d \times \mathbb{R}_+ : s > t, \text{ the ball } B_s(\boldsymbol{x}) \text{ intersects the ball } B_r(\mathbf{0})\}.$$

An argument similar to the one for (2.14) shows that

$$m(A^{(6)}_{r,t}) \le cr^d \bar{F}(t), \tag{2.19}$$

with a similar lower bound but with a different constant c. Write

$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing) \le \mathbb{P}(M(r) > r) + \int_0^r \mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing \mid M(r) = t) F_{M(r)}(\mathrm{d}t),$$

where $F_{M(r)}$ is the law of $M(r)$. It follows from (2.19) that we have to prove that

$$\limsup_{r\to\infty} \frac{\int_0^r \mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing \mid M(r) = t) F_{M(r)}(\mathrm{d}t)}{r^d \bar{F}(r)} < \infty. \tag{2.20}$$

It follows from Mecke's characterization of the Poisson process [7] that there is a version of the regular conditional law of the Poisson process Φ given $M(r)$ such that, on the event $\{M(r) > 0\}$, given that $M(r) = t$ for $t > 0$, the point process Φ restricted to the set $A^{(6)}_t = \{(\boldsymbol{x}, s) : s < t\}$ is still a Poisson point process on that set with the same mean measure m, restricted to that set. Take $a > 0$ such that (2.18) holds, and choose $\varepsilon = a$. Let $0 < p < 1$ be the corresponding value of the probability in (2.18). Consider $a < t < r$. There are $[cr^d/t^d]$ closed balls of radius t completely within $B_r(\mathbf{0})$, such that the Euclidean distance between any two different balls is at least t. For each of these $[cr^d/t^d]$ balls, with probability at least p, there is an isolated Φ-ball with a center in it, and radius not exceeding t. The events that such Φ-balls exist are independent, and the presence of such a Φ-ball guarantees that $B_r(\mathbf{0}) \cap D_{\text{th}} \ne \varnothing$. Therefore, for any $t > a$,

$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing \mid M(r) = t) \le (1-p)^{[cr^d/t^d]} \le (1-p)^{cr^d/t^d - 1}.$$

It is clear that

$$\mathbb{P}(M(r) \le a) \le \mathrm{e}^{-cr^d} = o(r^d \bar{F}(r)).$$

Further, $\int_a^r \mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing \mid M(r) = t) F_{M(r)}(\mathrm{d}t)$ is bounded above by

$$\begin{aligned} c\int_a^r \mathrm{e}^{-r^d/ct^d} F_{M(r)}(\mathrm{d}t) &\le cr^d \int_0^r \mathrm{e}^{-r^d/ct^d}\, t^{-(d+1)} \bar{F}_{M(r)}(t)\,\mathrm{d}t \\ &\le cr^{2d} \int_0^r \mathrm{e}^{-r^d/ct^d}\, t^{-(d+1)} \bar{F}(t)\,\mathrm{d}t \\ &= cr^d \int_0^1 \mathrm{e}^{-1/cs^d} s^{-(d+1)} \bar{F}(rs)\,\mathrm{d}s \\ &\sim cr^d \bar{F}(r) \int_0^1 \mathrm{e}^{-1/cs^d} s^{-(\alpha+d+1)}\,\mathrm{d}s \quad \text{as } r \to \infty, \end{aligned}$$

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by the regular variation of $\bar{F}$ and the Potter bounds. This completes the proof of (2.20) and, hence, of the upper bound in the theorem. □

Finally, we consider the case when the weights are independent of the radii of the balls.

Theorem 2.3. *Assume that the distribution of the radii of the balls in the Boolean model satisfies (1.7) with $\alpha > d$. If the weight of a ball in the model is independent of its radius and has a continuous distribution, then the contact distribution of the thinned germ–grain model satisfies*

$$0 < \liminf_{r\to\infty} \frac{\bar{H}(r)}{\bar{F}(r)} \le \limsup_{r\to\infty} \frac{\bar{H}(r)}{\bar{F}(r)} < \infty. \tag{2.21}$$

Proof. The structure of the argument is similar to that in Theorem 2.1. We also follow the conventions in the proof of Theorem 2.1 by working with the tail of the empty space function. However, since in this case the weight of a Φ-ball is not a function of its radius, we have to view the Poisson point process Φ as a measure in the full $(d+2)$-dimensional space $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$.

As before, we start by proving the lower bound in (2.21). We use a scenario similar to what we used to prove the lower bound in Theorem 2.1. Specifically, this scenario uses two particular balls: one ball has a large radius and large weight that 'eliminates' all the other Φ-balls that intersect $B_r(\mathbf{0})$, while the other ball in Φ, of even larger weight, 'eliminates' the first ball but does not itself intersect $B_r(\mathbf{0})$. Now recall the definition of the sets $A_r^{(1)}$, $A_r^{(2)}$, and $A_r^{(3)}$ in (2.2)–(2.4). As before, on the event

$$B_r = \{\Phi(A_r^{(1)} \times \mathbb{R}_+) = 1\},$$

we can define a random vector $(\boldsymbol{X}^{(r)}, R^{(r)}, W^{(r)})$ corresponding to the location of the center, radius, and weight of the former ball in Φ for which the pair $(\boldsymbol{X}^{(r)}, R^{(r)})$ is in the set $A_r^{(1)}$. Therefore,

$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing) \ge \mathbb{P}(B_r \cap \widehat{B}_r),$$

where now

$$\begin{aligned}\widehat{B}_r = \{W^{(r)} > \max\{w\colon (\boldsymbol{x}, t, w) \in \Phi((A_r^{(2)} \cup A_r^{(3)}) \times \mathbb{R}_+)\\ &\text{and there is } (\boldsymbol{x}, t, w) \in \Phi((A_r^{(1)} \cup A_r^{(2)} \cup A_r^{(3)})^c)\\ &\text{and } w > W^{(r)} \text{ such that } B_t(\boldsymbol{x}) \text{ intersects } B_{R^{(r)}}(\boldsymbol{X}^{(r)})\}.\end{aligned}$$

A standard computation shows that if (for example) $2r \le R^{(r)} \le 3r$, then the expected number of the Boolean balls that intersect $B_{R^{(r)}}(\boldsymbol{X}^{(r)})$ but not $B_r(\mathbf{0})$ is at least cr^d. Similarly,

$$\mathbb{P}(\Phi(A_r^{(1)} \times (0,\infty)) = 1, 2r \le R^{(r)} \le 3r) \ge cr^d \bar{F}(r).$$

Since

$$m((A_r^{(2)} \cup A_r^{(3)}) \times \mathbb{R}_+) \le cr^d,$$

in order to establish the lower bound in (2.21) it is enough to prove the following statement. Let c_1, c_2 be positive numbers, and let N_1, N_2 be independent Poisson random variables with respective means $c_1 r^d$ and $c_2 r^d$. Let $W_0, W_n^{(1)}$, $n = 1, 2, \ldots$, and $W_n^{(2)}$, $n = 1, 2, \ldots$, be independent and identically distributed standard uniform random variables independent of the Poisson random variables. Then for some positive c,

$$\mathbb{P}\Big(\sup_{n\le N_2} W_n^{(2)} > W_0 > \sup_{n\le N_1} W_n^{(1)}\Big) \ge cr^{-d}. \tag{2.22}$$

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To this end, observe that by symmetry, for any fixed $n_1 \geq 1$, $n_2 \geq 1$,

$$\mathbb{P}\Big(\sup_{n \leq n_2} W_n^{(2)} > W_0 > \sup_{n \leq n_1} W_n^{(1)}\Big) = \frac{n_2}{n_1 + n_2 + 1} \frac{1}{n_1 + 1}. \tag{2.23}$$

Now (2.22) follows from the facts that

$$\mathbb{P}\big(N_2 \geq \tfrac{1}{2} c_2 r^d\big) \to 1 \quad \text{and} \quad \mathbb{P}(N_1 \leq 2c_1 r^d) \to 1 \quad \text{as } r \to \infty.$$

This completes the proof of the lower bound.

Now we prove the upper bound in (2.21). Let K_1 be a large positive number to be specified below. Define the event $A_r^{(7)}(K_1)$ by

$$\bigg\{(\boldsymbol{x}, s, w) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ : s \geq \frac{r}{K_1}, \text{ the ball } B_s(\boldsymbol{x}) \text{ intersects the ball } B_r(\mathbf{0})\bigg\}.$$

As in (2.9), we have

$$\mathbb{E}[\Phi(A_r^{(7)}(K_1))] \leq c r^d \bar{F}(r) \tag{2.24}$$

(with a K_1-dependent constant c). Therefore, if $l_1 \geq \alpha/(\alpha - d)$ then

$$\mathbb{P}(\Phi(A_r^{(7)}(K_1)) > l_1) = o(\bar{F}(r)) \quad \text{as } r \to \infty,$$

and, hence, we need to prove that

$$\limsup_{r \to \infty} \frac{\mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing, \Phi(A_r^{(7)}(K_1)) \leq l_1)}{\bar{F}(r)} < \infty. \tag{2.25}$$

Fix $\theta \in (d/\alpha, 1)$, and let

$$A_r^{(8)}(K_1, \theta) = \bigg\{(\boldsymbol{x}, s, w) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ : r^\theta < s < \frac{r}{K_1},$$
$$\text{the ball } B_s(\boldsymbol{x}) \text{ intersects the ball } B_r(\mathbf{0})\bigg\}.$$

As above, for large r,

$$\mathbb{E}[\Phi(A_r^{(8)}(K_1, \theta))] \leq c r^d \bar{F}(r^\theta).$$

From the choice of θ, we see that if $l_2 \geq \alpha/(\theta\alpha - d)$ then

$$\mathbb{P}(\Phi(A_r^{(8)}(K_1, \theta)) > l_2) = o(\bar{F}(r)) \quad \text{as } r \to \infty.$$

Therefore, in order to establish (2.25), it is enough to prove that, for every $j = 0, 1, \ldots, l_1$,

$$\limsup_{r \to \infty} \frac{\mathbb{P}(B_r(\mathbf{0}) \cap D_{\text{th}} = \varnothing, \Phi(A_r^{(7)}(K_1)) = j, \Phi(A_r^{(8)}(K_1, \theta)) \leq l_2)}{\bar{F}(r)} < \infty. \tag{2.26}$$

Now specify K_1 by setting $K_1 > l_2$. Note that for every choice of l_2 and K_1 as above, the complement in $B_r(\mathbf{0})$ of the union of at most l_2 balls of radii not exceeding r/K_1 contains a ball of radius γr for some $\gamma = \gamma(l_2, K_1) > 0$ (we can choose $\gamma = \frac{1}{2}(1/l_2 - 1/K_1)$).

We first consider the $j = 0$ case of (2.26). Let $\Phi_{r,\theta}$ be the restriction of the Poisson point process Φ to the set $\mathbb{R}^d \times (0, r^\theta] \times \mathbb{R}_+$. Again using the property of a Poisson point process

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that its restrictions to disjoint sets are independent, we see that (2.26) with $j=0$ follows so soon as we have shown that

$$\limsup_{r\to\infty} \frac{\mathbb{P}(B_{\gamma r}(\mathbf{0}) \cap D_{r,\theta,\mathrm{th}} = \varnothing)}{\bar{F}(r)} < \infty, \tag{2.27}$$

where $D_{r,\theta,\mathrm{th}}$ is the grain cover corresponding to the thinning of $\Phi_{r,\theta}$ (with the weights still being independent of the radii). The fact that we are allowed to use the ball centered at the origin in (2.27), instead of a randomly centered ball described in the previous paragraph, is a consequence of the translation invariance of $\Phi_{r,\theta}$.

In order to prove (2.27), we need one more simple estimate. Let t be a large number, $t \le \frac{1}{3}\gamma r$. Consider concentric balls $B_{t/3}(\mathbf{0})$, $B_t(\mathbf{0})$, and $B_{3t}(\mathbf{0})$. Then there is $0 < q < 1$ such that

$$\begin{aligned}
&\mathbb{P}(\sup\{w\colon (\boldsymbol{x}, s, w) \in \Phi_{r,\theta}((B_t(\mathbf{0}) \setminus B_{t/3}(\mathbf{0})) \times \mathbb{R}_+)\} \\
&\quad > \max\{\sup\{w\colon (\boldsymbol{x}, s, w) \in \Phi_{r,\theta}((B_{t/3}(\mathbf{0}) \setminus B_{t/9}(\mathbf{0})) \times \mathbb{R}_+)\}, \\
&\qquad \sup\{w\colon (\boldsymbol{x}, s, w) \in \Phi_{r,\theta}((B_{3t}(\mathbf{0}) \setminus B_t(\mathbf{0})) \times \mathbb{R}_+)\}) \\
&\quad \ge q \quad \text{for all } t \text{ large enough.}
\end{aligned} \tag{2.28}$$

Indeed, the Poisson point process $\Phi_{r,\theta}$ assigns mean measures of the order ct^d to each of the three annuli in question (with c K_1-dependent), so (2.28) follows by using conditioning and a computation analogous to (2.23).

Now it is clear that the probability in the numerator in (2.27) can be bounded from above by $(1-q)^{cr^{1-\theta}}$ for some $c>0$ because we can fit into $B_{\gamma r}(\mathbf{0})$ triple annuli as above with the radial separation between neighboring triples exceeding r^θ, which makes, by the definition of $\Phi_{r,\theta}$, the events whose probabilities are computed in (2.28), independent. Therefore, (2.27) holds, and so we have proved (2.26) with $j=0$.

Next we consider (2.26) with $j=1$. It follows from (2.24) that

$$\mathbb{P}(\Phi(A_r^{(7)}(K_1)) = 1) = O(r^d \bar{F}(r)), \qquad r \to \infty.$$

Therefore, we need to prove the following version of (2.27): consider the grain cover $D_{r,\theta,\mathrm{th}}$ and a random variable W independent of it, whose law is the distribution of the weight in the Boolean model (W is the weight of the single ball in $\Phi(A_r^{(7)}(K_1))$. We eliminate all the balls in $D_{r,\theta,\mathrm{th}}$ whose weight is smaller or equal to W, and we call the resulting grain cover $\hat{D}_{r,\theta,\mathrm{th}}$. Then (2.26) with $j=1$ follows once we have proved that

$$\limsup_{r\to\infty} r^d \mathbb{P}(B_{\gamma r}(\mathbf{0}) \cap \hat{D}_{r,\theta,\mathrm{th}} = \varnothing) < \infty. \tag{2.29}$$

In order to see that this is true, we use an argument similar to the one used to prove (2.27). Consider the three annuli in (2.28). Since we already know that the probability that fewer than $cr^{1-\theta}$ events in (2.28) occur is $o(r^{-d})$, we need only consider what happens if at least $cr^{1-\theta}$ of the events occur. In the latter case, the only possibility for $B_{\gamma r}(\mathbf{0}) \cap \hat{D}_{r,\theta,\mathrm{th}} = \varnothing$ is that the weight of the heaviest Φ-ball in the union of $cr^{1-\theta}$ of annuli of radii of order cr and width of order cr^θ does not exceed W. Since the mean measure of the Poisson point process $\Phi_{r,\theta}$ assigns the weight of the order cr^d to that union, the latter probability does not exceed cr^{-d}, once again, by conditioning and a computation analogous to (2.23).

Therefore, (2.29) is true, and so we have proved (2.26) with $j=1$. The cases $j=2,\dots,l_1$, are similar and easier, since the probabilities

$$\mathbb{P}(\Phi(A_r^{(7)}(K_1)) = j)$$

become asymptotically smaller as j increases, proving the upper bound in (2.21). □

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3. Heavy small balls and exponential decay of the contact distribution

In this section we prove that if the weight of a ball is a strictly decreasing function of its radius then the tail of the contact distribution decays exponentially. This turns out to be unrelated to the fact that the tail of the radii of the balls is regularly varying.

Theorem 3.1. *Assume that the distribution of the radii of the balls in the Boolean model satisfies (1.2). If $W_n = 1/R_n$ a.s. (i.e. if larger balls have smaller weights), then for some $c > 0$ the contact distribution of the thinned germ–grain model satisfies*

$$\bar{H}(r) \le \mathrm{e}^{-cr^d} \quad \textit{for all large enough } r. \tag{3.1}$$

Proof. Once again, we work with the tail of the empty space function. Since the weight of a ball is a function of its radius, we again switch to viewing the Poisson point process Φ as a measure in the $(d+1)$-dimensional space $\mathbb{R}^d \times \mathbb{R}_+$. Choose a finite number $\gamma > 0$ such that

$$F((0, \gamma)) > 0.$$

Let Φ_γ be the restriction of the Poisson point process Φ to the set $\{(\boldsymbol{x}, s) \colon s \le \gamma\}$. As in the proof of Theorem 2.2, for some $c > 0$ that depends on γ, we can find at least cr^d disjoint balls of radius γ within $B_r(\mathbf{0})$, such that the distance between any two different balls exceeds 2γ. Let us call these balls B_i, $i = 1, \dots, n$, with $n \ge cr^d$. For such i, let H_i denote the event

$$\{\Phi_\gamma(\{(\boldsymbol{x}, s) \colon B_s(\boldsymbol{x}) \cap B_i \neq \varnothing\}) = 1, \Phi_\gamma(\{(\boldsymbol{x}, s) \colon B_s(\boldsymbol{x}) \cap [(B_i + B_\gamma(\mathbf{0})) \setminus B_i] \neq \varnothing\}) = 0\}.$$

Here $B_i + B_\gamma(\mathbf{0})$ is simply the ball concentric with B_i of radius 2γ. Note that, on the event H_i, the single Φ_γ-ball in the description on the event cannot be 'eliminated' by any other Φ-ball. Indeed, if another Φ_γ-ball intersected it, the latter ball would be in the set $[(B_i + B_\gamma(\mathbf{0})) \setminus B_i]$, which is impossible on the event H_i. Further, any Φ-ball which is not a Φ_γ-ball has simply too large a radius. Therefore,

$$\mathbb{P}(B_r(\mathbf{0}) \cap D_{\mathrm{th}} = \varnothing) \le \mathbb{P}\biggl(\bigcap_{i=1}^n H_i^c\biggr) = (1 - \mathbb{P}(H_1))^n \le (1 - \mathbb{P}(H_1))^{cr^d}.$$

The equality in this calculation follows from the fact that the balls B_i are sufficiently far away from each other so that different events H_i are determined by restrictions of the Poisson point process Φ_γ to disjoint sets and, hence, are independent.

In order to prove the theorem we need only check that $\mathbb{P}(H_1) > 0$. Since Φ_γ is translation invariant, we replace, in the calculation below, B_1 by $B_\gamma(\mathbf{0})$ and $B_1 + B_\gamma(\mathbf{0})$ by $B_{2\gamma}(\mathbf{0})$. Note that the event H_1 is defined as the intersection of two independent events, so we need only check that each of these events has positive probability.

It is clear that

$$\Phi_\gamma(B_\gamma(\mathbf{0}) \times (0, \gamma]) \subset \Phi_\gamma(\{(\boldsymbol{x}, s) \colon B_s(\boldsymbol{x}) \cap B_\gamma(\mathbf{0}) \neq \varnothing\}) \subset \Phi_\gamma(B_{2\gamma}(\mathbf{0}) \times (0, \gamma]),$$

so that

$$0 < \mathbb{E}[\Phi_\gamma(\{(\boldsymbol{x}, s) \colon B_s(\boldsymbol{x}) \cap B_\gamma(\mathbf{0}) \neq \varnothing\})] < \infty$$

and, hence,

$$\mathbb{P}(\Phi_\gamma(\{(\boldsymbol{x}, s) \colon B_s(\boldsymbol{x}) \cap B_\gamma(\mathbf{0}) \neq \varnothing\}) = 1) > 0.$$

Further,

$$\Phi_\gamma(\{(\boldsymbol{x}, s) \colon B_s(\boldsymbol{x}) \cap (B_{2\gamma}(\mathbf{0})) \setminus B_\gamma(\mathbf{0})) \neq \varnothing\}) \subset \Phi_\gamma(B_{3\gamma}(\mathbf{0}) \times (0, \gamma]),$$

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so that

$$\mathbb{E}[\Phi_\gamma(\{(\boldsymbol{x}, s)\colon B_s(\boldsymbol{x}) \cap (B_{2\gamma}(\mathbf{0})) \setminus B_\gamma(\mathbf{0})) \neq \varnothing\}] < \infty$$

and, hence,

$$\mathbb{P}(\Phi_\gamma(\{(\boldsymbol{x}, s)\colon B_s(\boldsymbol{x}) \cap (B_{2\gamma}(\mathbf{0})) \setminus B_\gamma(\mathbf{0})) \neq \varnothing\} = 0) > 0.$$

This implies that $\mathbb{P}(H_1) > 0$, and the proof of the theorem is complete. □

Note that a lower bound of the type

$$\bar{H}(r) \geq \mathrm{e}^{-cr^d} \quad \text{for large } r$$

(with, possibly, a different exponent c than in (3.1)) is trivially true since it holds for the original spherical Boolean model even before thinning.

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