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Circular hydraulic jumps: where does surface tension matter?

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Recently, an unusual scaling law has been observed in circular hydraulic jumps and has been attributed to a supposed missing term in the local energy balance of the flow (Bhagat *et al.*, *J. Fluid Mech.*, vol. 851, 2018, R5). In this paper, we show that – though the experimental observation is valuable and interesting – this interpretation is presumably not the right one. When transposed to the case of an axial sheet formed by two impinging liquid jets, the assumed principle leads in fact to a velocity distribution in contradiction with the present knowledge for this kind of flow. We show here how to correct this approach by maintaining consistency with surface tension thermodynamics: for Savart–Taylor sheets, when adequately corrected, we recover the well-known $1/r$ liquid thickness with a constant and uniform velocity dictated by Bernoulli's principle. In the case of circular hydraulic jumps, we propose here a simple approach based on Watson's description of the flow in the central region (Watson, *J. Fluid Mech.*, vol. 20, 1964, pp. 481–499), combined with appropriate boundary conditions on the circular front formed. Depending on the specific condition, we find in turn the new scaling by Bhagat *et al.* (2018) and the more conventional scaling law found long ago by Bohr *et al.* (*J. Fluid Mech.*, vol. 254, 1993, pp. 635–648). We clarify here a few situations in which one should hold rather than the other, hoping to reconcile the observations of Bhagat *et al.* with the present knowledge of circular hydraulic jump modelling. However, the question of a possible critical Froude number imposed at the jump exit and dictating logarithmic corrections to scaling remains an open and unsolved question.

Key words: capillary flows

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1. Introduction

Stationary axisymmetric liquid structures formed by jet impacts have motivated an enormous amount of literature. Three examples that will be important here are sketched in figure 1. The first is the well-known circular hydraulic jump (Rayleigh 1914; Tani 1949; Watson 1964; Craik *et al.* 1981; Bohr, Dimon & Putkaradze 1993; Bush & Aristoff 2003; Duchesne, Lebon & Limat 2014; Mohajer & Li 2015; Bhagat *et al.* 2018; Salah *et al.* 2018; Wang & Khayat 2019, 2021), shown in figure 1(*a*), with a well-developed liquid film extending all around. The second, in figure 1(*b*), is its equivalent on a 'dry' surface (atomization rim), obtained by superhydrophobic treatment (Maynes, Johnson & Webb 2011; Sen *et al.* 2019). A related, but different, configuration can be reached by simply impinging a vertical jet from below on a horizontal ceiling (Button *et al.* 2010; Jameson *et al.* 2010). The final example is the well-known radial liquid sheet (Savart 1833; Huang 1970; Clanet & Villermaux 2002; Villermaux, Pistre & Lhuissier 2013), formed either by impinging two opposite symmetrical liquid jets having the same central axis, or by impinging a liquid jet on a solid surface with a diameter similar to the jet diameter; this is depicted in figure 1(*c*).

These three geometries are of course linked by the same general equation for the energy balance. In this article, we will therefore show that apparent paradoxes raised by the modelling of the surface tension on the circular hydraulic jump by Bhagat *et al.* (2018) may be solved or at least clarified by considering the geometry depicted in figure 1(*c*).

The selection of jump radius R_J in the circular hydraulic jump case (figure 1*a*) has motivated many studies. The two best-known approaches are that of Watson and Bush (Watson 1964; Bush & Aristoff 2003), in which the height of the outer film remains a control parameter, and that of Bohr *et al.* (1993), devised instead for a liquid film extending all around at a large distance, and inspired by boundary-layer theories. As is well known, this second approach leads to a scaling law dependence of R_J upon the flow rate Q and the physical parameters (the kinematic viscosity of the fluid ν, the gravity g), which reads as follows:

$$R_J \sim Q^{5/8}\nu^{-(3/8)}g^{-(1/8)}. \tag{1.1}$$

Later, Duchesne *et al.* (2014) emphasized the importance of logarithmic corrections to scaling, due to viscous dissipation in the outer film, yet observed numerically by Bohr. They also showed that the prefactor was experimentally linked to the value of the Froude number at the jump location (precisely at the immediate exit of the jump, in the outer part of the flow), which seemed to be locked to a critical value. This phenomenon was recovered by Mohajer & Li (2015) and by Argentina *et al.* (2017) with a nonlinear modelling of film flow equations including the first finite slope terms.

Very recently, an attempt to revise this picture has been published by Bhagat *et al.* (2018), who performed new experiments and reported the observation of a different scaling, where surface tension γ was involved, but not gravity:

$$R_J \sim Q^{3/4}\rho^{1/4}\nu^{-(1/4)}\gamma^{-(1/4)}, \tag{1.2}$$

where ρ is the liquid density. To rationalize this finding, these authors claimed that most available approaches to the influence of surface tension lead to only small corrections (Bush & Aristoff 2003) and that the description of the circular hydraulic jump had thus to be completely reconsidered. They introduced an energy balance, between two radii r and

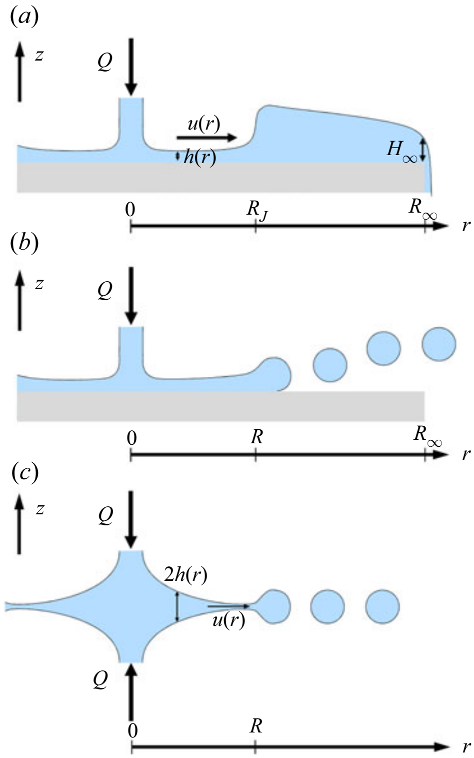

Figure 1. Three axisymmetric film flows are discussed in the present article: (*a*) the classical circular hydraulic jump formed by a jet impacting a solid disk at its centre; (*b*) atomization ring formed by a jet impacting a dry surface, possibly superhydrophobic; (*c*) liquid sheet formed by impact of two liquid jets of opposite direction.

$r + \delta r$, which reads

$$\left[\rho \frac{\bar{u}^2}{2} \bar{u} r h\right]_r^{r+\delta r} = [\gamma r \bar{u}]_r^{r+\delta r} - [p \bar{u} h]_r^{r+\delta r} - \left[\rho g \frac{h^2}{2} r \bar{u}\right]_r^{r+\delta r} - r \tau_W \bar{u} \delta r, \tag{1.3}$$

with the notation $[A]_r^{r+\delta r} = A(r + \delta r) - A(r)$, and where $\bar{u}$ designates the flux-average radial velocity, r the distance to the axis, $h(r)$ the thickness of the liquid layer, $p(r)$ the pressure at $z = 0$ and τ_W the wall shear stress. The last term on the right designates the viscous dissipation by friction on the substrate, while the first one is an additional term compared to previous approaches, which is presumed to be 'at the origin' of the new scaling (1.2). This conjecture has been contested (Duchesne, Andersen & Bohr 2019; Bohr & Scheichl 2021) (see also the answer in Bhagat & Linden (2020)), and it is also known that a scaling like (1.2) can also appear without such an assumption, as shown for instance by Button *et al.* (2010) for liquid bells formed below a ceiling.

Here it is useful to have a look at what would happen in the simplified geometry of figure 1(*c*), when applying this principle. As we shall show in § 2, this modelling leads to a velocity distribution in complete contradiction to the present knowledge of liquid sheets (and with Bernoulli's principle), which suggests that the argument of Bhagat *et al.* is flawed. In fact the flow field obtained is not new; it was previously proposed by Bouasse (1923), who attributed the calculation to Hagen (1849) (it is worth noticing that this error has recently been pointed out by Bohr & Scheichl 2021). It will be instructive here to

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recall the argument followed by Hagen and Bouasse in a Lagrangian frame, analysing an expanding circular piece of film; we describe this argument in § 3. We will then show, in the same section, how one can correct the argument to get the more classical and now admitted result deduced from Bernoulli's principle of a uniform radial velocity around the impact point, and how, if some terms in the balance are missed, one can get the flawed result of Bouasse and Hagen. Finally, coming back to an Eulerian description, we will explain how these considerations affect the principle proposed in (1.3). We will show that an extra term exactly equal and opposite to the capillary contribution should cancel this one, in a way consistent with classical thermodynamics (i.e. surface tension can only produce a non-zero amount of global work when the surface area of the associated interface changes), leading to the expression usually written from the balance of momentum.

This does not mean, however, that the scaling discovery of Bhagat *et al.* is of no interest. In §§ 4 and 5, we will try to specify to which capillary structures – different from the stationary hydraulic jump observed by Bohr – it could apply, and a possible way to justify its occurrence.

2. A look at a simple situation: the axisymmetric liquid sheet

Let us try to apply the principle suggested in (1.3) to the case suggested in figure 1(*c*), i.e. to an axisymmetric sheet formed by the coaxial impact of two jets in a situation of negligible gravity. The viscous shear on the substrate having disappeared, (1.3) reduces to a very simple balance that reads as follows:

$$\left[\rho r h \frac{u^3}{2}\right]_r^{r+\delta r} = \left[\gamma r u\right]_r^{r+\delta r}, \tag{2.1}$$

where the horizontal velocity u has no dependence upon the transverse direction, and coincides with any of its average values. This implies that the following quantity is constant all over the sheet:

$$\rho r h \frac{u^3}{2} - \gamma r u = Cte. \tag{2.2}$$

Combined with the mass balance $Q = 2\pi r h u$, this leads to the following expression for u:

$$u = 2\pi \frac{\gamma}{\rho Q} r + \sqrt{u_0^2 - 4\pi \frac{\gamma}{\rho Q} r_0 u_0 + 4\pi^2 \frac{\gamma^2}{\rho^2 Q^2} r^2}, \tag{2.3}$$

where r_0 designates the jet radius at impact and u_0 the asymptotic value for u, reached when $r = r_0$, which satisfies the equality $Q = \pi r_0^2 u_0$ in a quasi-elastic shock approximation (Villermaux *et al.* 2013). In the limit of large jet velocity, i.e. $u_0^2 \gg 2\gamma/(\rho r_0)$, this expression reduces to the following approximation, which is slowly varying upon r:

$$u \approx u_0 + 2\pi \frac{\gamma}{\rho Q}(r - r_0). \tag{2.4}$$

This is known to be false, as it has been checked experimentally that the velocity is constant all over the sheet, recovering Bernoulli's principle (see in particular figure 3 in Villermaux *et al.* 2013). It is surprising, however, that a similar expression was proposed by

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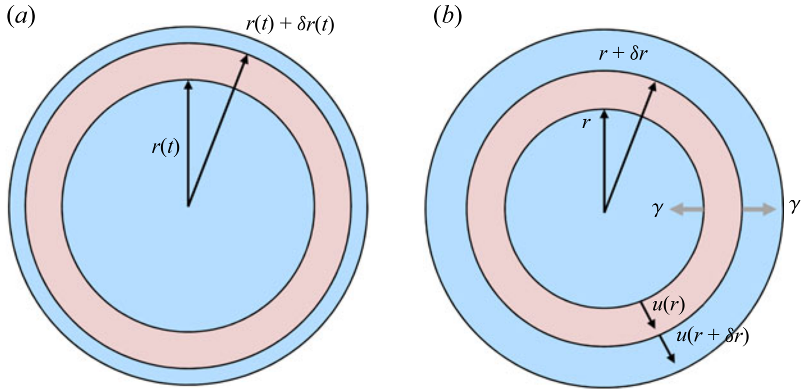

Figure 2. Lagrangian (*a*) and Eulerian (*b*) frames used in the text for discussing energy balance in an annular portion of a liquid film.

Bouasse (1923), who attributed this result to Hagen (1849), but with a slight sign change that is in fact due to a mistake of his own:

$$u \approx u_0 - 2\pi \frac{\gamma}{\rho Q}(r - r_0). \tag{2.5}$$

Though obtained erroneously, this expression is very seductive, and Bouasse used it to calculate the radius of the liquid sheet R_{LS} assuming that the sheet border should stay at the place where u vanishes, which leads to $R_{LS} = (\rho Q u_0)/(2\pi\gamma) = (\rho r_0^2 u_0^2)/(2\gamma)$. Surprisingly, this result coincides with the correct one, which is in fact obtained, now, by assuming a constant velocity, dictated by Bernoulli's principle, and the balance of momentum at the sheet perimeter, i.e. $\rho h u^2 = \gamma$ (Villermaux *et al.* 2013). But on the other hand, we would like to stress that the radial velocity is uniform in the sheet of figure 1(*c*), which means that the principle proposed in (1.3), and therefore the basis of the theory developed by Bhagat *et al.* (2018), is flawed.

3. Reconsidering Hagen's argument, and its implications for hydraulic jump

We now try to understand the fault underlying the principle of Bouasse and Hagen. Their line of thought is easier to explain if we consider a Lagrangian frame, and more precisely the balance of energy on an annular piece of fluid convected by the radial flow; this is in fact the method proposed by Bouasse himself in his treatise on fluid mechanics (Bouasse 1923).

Let us consider an annular piece of film as in figure 2(*a*), convected and distorted by the flow. Mass conservation implies that, at any time $hr\delta r = Cte$, while the balance of energy for the whole annulus reads, in the limit of δr small enough to satisfies the condition $\delta r(\partial u/\partial r) \ll u$ of a slowly varying velocity field:

$$\frac{\partial}{\partial t}\left[2\pi\left(\frac{1}{2}\rho u^2 rh\delta r + \gamma r\delta r\right)\right] \approx 2\pi\gamma\delta r u. \tag{3.1}$$

The first term in the left-hand side of this equation stands for kinetic energy, and the second for the surface energy enclosed between r and $r + \delta r$. The right-hand term comes from the work of surface forces, and does not vanish. Indeed, the same surface tension force is pulling on a different arc length, as the external boundary has a larger perimeter

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than the other (note that this is the intuitive argument underlying the analysis of Bhagat *et al.*). Still in the limit of a slowly varying velocity field at the scale δr, after noting that $\partial/\partial t = u\partial/\partial r$, (3.1) reads

$$rh\delta ru\frac{\partial}{\partial r}\left(\frac{\rho u^2}{2}\right) + \gamma u\delta r \approx \gamma u\delta r. \tag{3.2}$$

In fact, the two surface tension terms are cancelling each other, which means that the work provided to the annulus by the surface tension of the outer interfaces is completely transformed into the surface energy stored at the free surface of the annulus, in agreement with simple thermodynamic considerations. As a result, the fluid velocity is unaffected by surface tension balance and remains constant, as one would deduce from a more classical argument in terms of Bernoulli's principle; i.e. $u(r)$ is in fact independent of r:

$$u(r) = u_0 = Cte. \tag{3.3}$$

Note here that skipping from (3.1) to (3.2) is not completely trivial, as there is an extra γ term remaining, but this vanishes for the constant and uniform u_0 solution.

To reconnect with Bouasse, if instead one forgets the internal surface energy contribution in the left-hand member of (3.2), one obtains the following equation for u:

$$rh\delta r\frac{\partial}{\partial r}\left(\frac{\rho u^2}{2}\right) \approx \gamma\delta r. \tag{3.4}$$

After simplifying δr, and using the fact that $Q = 2\pi ruh$, this equation leads to

$$\frac{\partial u}{\partial r} \approx 2\pi\frac{\gamma}{\rho Q}, \tag{3.5}$$

which leads finally to (2.4). Alternatively, (2.5) is obtained when one forgets the work provided to the annulus by the outer parts of the liquid sheet, i.e. when one neglects the right-hand member of (3.2), following the intuitive but erroneous idea of Hagen (1849) that surface tension could slow down the flow. Historically, Bouasse followed the first argument but made a sign error, obtaining (2.5), which was presumably physically more natural, in view of what Hagen said long ago.

To summarize, a correct treatment of the expansion of liquid annuli in the flow leads to the classical result of a uniform velocity, while the approximations defended by Hagen and Bouasse would follow from neglecting one or the other of the capillary terms. We believe that a similar problem is involved in (1.3). If we now consider an Eulerian description of the flow, as suggested in figure 2(*b*), the balance of energy will instead read

$$\left[\rho\frac{\bar{u}^2}{2}\bar{u}rh + \gamma r\bar{u}\right]_r^{r+\delta r} = \left[\gamma r\bar{u}\right]_r^{r+\delta r} - \left[p\bar{u}h\right]_r^{r+\delta r} - \left[\rho g\frac{h^2}{2}r\bar{u}\right]_r^{r+\delta r} - r\tau_W\bar{u}\delta r, \tag{3.6}$$

where, in the left-hand side, we have added the surface energy convected by the film. It is true that one can consider a capillary force, as in Bhagat *et al.*, in the right-hand member, but in this case, one should not miss the surface flux crossing the two circles displayed in figure 2(*b*) in the left-hand side of the equation. And just as in a Lagrangian frame, the physics being the same in both frames, the capillary effects should exactly compensate for each other in this equation, which should then reduce to the more conventional form

$$\left[\rho\frac{\bar{u}^2}{2}\bar{u}rh\right]_r^{r+\delta r} = -\left[p\bar{u}h\right]_r^{r+\delta r} - \left[\rho g\frac{h^2}{2}r\bar{u}\right]_r^{r+\delta r} - r\tau_W\bar{u}\delta r. \tag{3.7}$$

This, apart from some coefficients that will depend on the detailed structure of the flow profile, is consistent with what people are used to writing starting instead from the balance of momentum (Bohr *et al.* 1993). Therefore, in the interpretation of (1.2), we do not consider that one should add a new capillary force distributed throughout space as proposed by Bhagat *et al.* (2018). In our opinion, this would merely reproduce the initial mistake of Hagen and Bouasse. However, in the same way that Bouasse's approach leads to the correct radius for the liquid sheet but with a biased approach, we may be able to show that the scaling of Bhagat *et al.* can be recovered by properly taking into account the boundary conditions. We now develop this idea further.

4. Alternative explanation of unusual scaling: the boundary condition at the 'jump' radius; comparison with atomization rings

To interpret the occurrence of the scaling of Bhagat *et al.* (2018), we propose an alternative approach. We simply treat the two ideal situations of figures 1(*a*) and 1(*b*) with the same method, and see what happens. We will then see that the situation obtained in figure 1(*b*) may be compared to the one suggested by Bhagat *et al.* (2018).

To simplify the analysis, the 'internal' flow for $r_0 < r < R_J$ is assimilated to the one discussed long ago by Watson (1964), in which fluid inertia is progressively dissipated by viscous friction; i.e. for $r < R_J$,

$$u(r, z) = \frac{27c^3}{8\pi^4}\frac{Q^2}{\nu(r^3 + l^3)}f\left(\frac{z}{h}\right), \tag{4.1}$$

where $c \approx 1.402$, $l = 0.567 r_0 R$ (with R the Reynolds number of the jet) and f is the function $f(\eta) = \sqrt{3} + 1 - 2\sqrt{3}/(1 + cn(3^{1/4}c(1 - \eta)))$. Mass conservation implies that the flux of momentum is given by

$$\rho h\langle u^2\rangle = \frac{27\sqrt{3}c^3}{16\pi^6}\frac{\rho Q^3}{r\nu(r^3 + l^3)}, \tag{4.2}$$

where $\langle u^2\rangle = \int_0^h u^2\, \mathrm{d}z$. In the case of figure 1*a*, this flow must be matched for $r > R_J$ to a film flow under the action of gravity that, according to lubrication (Duchesne *et al.* 2014), has a thickness distribution $H(r)$ given by

$$H(r)^4 = H_\infty^4 + \frac{6}{\pi}\frac{\nu Q}{g}\ln\left(\frac{R_\infty}{r}\right), \tag{4.3}$$

where R_∞ designates the outer radius of the substrate and the thickness H reaches a value called H_∞ that will depend on the specific geometrical conditions of the flow there (see figure 1*a* for the graphical definition). At $r = R_J$, one must write some matching condition that is consistent with the approximations made on each side of $r = R_J$, and stands for a shock (Bélanger 1841; Rayleigh 1914). If we assume $h \ll H$ and neglect the surface tension at the shock (i.e. for sufficiently large circular hydraulic jumps , such as those considered by Bhagat *et al.* 2018), this shock condition reads

$$\rho h(r)\langle u(R_J)^2\rangle \approx \rho g H(R_J)^2. \tag{4.4}$$

In the limit of negligible values for H_∞ and r_0, compared to the other scales, it is easy to check that these equations lead to the following scaling law for R_J:

$$R_J ln\left(\frac{R_\infty}{R_J}\right)^{1/8} = \frac{(3c)^{3/4}}{2^{9/8}\pi^{11/8}}\frac{Q^{5/8}}{\nu^{3/8}g^{1/8}}. \tag{4.5}$$

This is the scaling obtained by Bohr *et al.* (1993), modified by logarithmic corrections.

We now consider the regime described in figure 1(*b*), which may be obtained in a stationary regime with a particular superhydrophobic treatment (Maynes *et al.* 2011; Sen *et al.* 2019). In this regime the force opposed to fluid inertia at the boundaries is dictated only by surface tension and not by gravity; there is no developed shock, no liquid 'wall'. In other words, the flux of momentum is balanced only by surface tension, which means that (4.3) and (4.4) are now simply replaced by

$$\rho h(r)\langle u(R_J)^2\rangle \approx \gamma(1-\cos\theta), \tag{4.6}$$

with θ the static contact angle. This equation also applies to water bells obtained from the impact of a vertical jet below a ceiling, as detailed in Button *et al.* (2010). In that case, the local contact angle has no influence and the flux of momentum is simply balanced by γ. This means that (4.6) is still valid in this configuration if we assume $\theta = \pi/2$.

Using (4.2) in the limit $r = R_J \gg r_0$, this condition yields a new scaling that reads as follows:

$$R_J = \left(\frac{27\sqrt{3}c^3}{16\pi^6}\right)^{1/4}(1-\cos\theta)^{1/4}Q^{3/4}\nu^{-(1/4)}\rho^{1/4}\gamma^{-(1/4)}. \tag{4.7}$$

This scaling is very close to the one suggested by Bhagat *et al.* (2018) and previously by Button *et al.* (2010). The only difference is an additional factor linked to the contact angle (= 1 for water bells as previously explained). This explains why the scaling obtained by Bhagat *et al.* (2018) applies to the experimental data of Jameson *et al.* (2010) even if the theory leading to this scaling is not the right one.

We thus do not believe that there is a 'universal' scaling that should hold for any circular 'print' formed around an impacting jet. Sometimes one may find Bohr's scaling and sometimes that of Bhagat *et al.* and Button *et al.*; what matters is the analysis of the conditions surrounding the impact.

5. Another possible occurrence of the scaling of Bhagat *et al.* and Button *et al.*

We now show that the scaling of Bhagat *et al.* may also be observed in classical circular hydraulic jumps. In Bhagat *et al.* (2018), the authors consider an intermediate regime where the liquid has not yet reached the edge of the plate (see figure 3). In their experimental evidence the authors consider partial wetting conditions (they use Perspex, glass and Teflon) and aqueous solutions. Given that the front propagation speed is rather small, we can suppose that the liquid front height is approximately given by

$$h_{cap} \approx \sqrt{2}\left(\frac{\gamma}{\rho g}\right)^{1/2}(1-\cos\theta)^{1/2}, \tag{5.1}$$

as explained, for instance, in de Gennes, Brochard-Wyart & Quéré (2004).

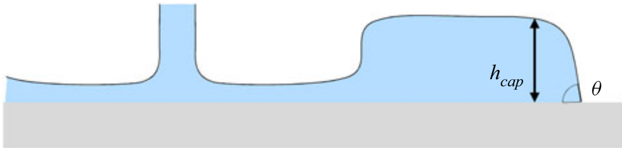

Figure 3. Sketch of the intermediate regime for a low-viscosity liquid in partial wetting.

According to Duchesne *et al.* (2014) (and as also used in Mohajer & Li (2015) and Ipatova, Smirnov & Mogilevskiy (2021)), at low viscosity and moderate flow rate, $H(r)$ is nearly constant and approximately reduces to

$$H(r) \approx H_\infty \approx h_{cap}. \tag{5.2}$$

Therefore the (simplified) shock condition (4.4) previously obtained leads to

$$\rho h(r)\langle u(R_J)^2\rangle \approx \tfrac{1}{2}\rho g h_{cap}^2. \tag{5.3}$$

Surprisingly, this argument leads again to the 'surface-tension-dominated' scaling with only a factor of $2^{1/4}$ in between:

$$R_J = \left(\frac{27\sqrt{3}c^3}{16\pi^6}\right)^{1/4} 2^{1/4}(1-\cos\theta)^{1/4}Q^{3/4}\nu^{-(1/4)}\rho^{1/4}\gamma^{-(1/4)}. \tag{5.4}$$

Now, recovering a result from Bhagat *et al.* (2018), one can also deduce that the Weber number satisfies

$$We \approx \frac{\rho h(r)\langle u(R_J)^2\rangle}{\gamma} \approx Cte; \tag{5.5}$$

i.e. a constant Weber number replaces the constant Froude number encountered in a fully established hydraulic jump with a complete, flowing outer film.

Taking $\theta = \pi/2$, we obtain

$$We \approx 1, \tag{5.6}$$

which is the result for the Weber number obtained by Bhagat *et al.* (2018).

6. Conclusion

In summary, we have reconsidered the problem of scaling law selection of the 'radius of influence' in the problem of vertical jet impact on a horizontal solid surface. In our opinion, the ideal law (1.1) proposed by Bohr and colleagues (to which one should not forget to add logarithmic corrections as in Duchesne *et al.* 2014) corresponds to the ideal situation of a stationary hydraulic jump formed inside a liquid film extending across the whole solid surface. On the other hand, the scaling (1.2) suggested in Bhagat *et al.* (2018) holds in different situations, such as the following:

(i) stationary impact of a jet on a dry surface, possibly superhydrophobic (fully or partially as in Sen *et al.* 2019), without formation of the outer film (atomization ring);
(ii) stationary impact of a jet on a dry surface in inverse gravity (impact of a jet on a ceiling);

(iii) transient regime of circular hydraulic jump formation for low-viscosity liquids in partial wetting.

It would be interesting to explore in more detail these three situations, and to identify possible other ones. In our opinion, there is no need to imagine some universal extra capillary term imposing the scaling (1.2) as imagined in Bhagat *et al.* (2018). Though this extra term really exists, it is in practice compensated for by another one (in a way consistent with classical thermodynamics) when the control volume contains the free surface of the film instead of excluding it. As usual in free surface flows, there is no increase or decrease of velocity that could be due solely to the action of surface tension, except when Marangoni effects are involved (Marmottant, Villermaux & Clanet 2000). To continue in this direction would be simply to reproduce, for thin film flows on a solid, the initial mistake of Hagen and Bouasse.

Returning to the matter of Bohr scaling, we have set aside somewhat the questions of the logarithmic corrections and the possible existence of a critical Froude number at the jump exit, as suggested in Duchesne *et al.* (2014). The possible existence of this critical Froude number leads to a different exponent for the logarithmic corrections ($3/8$ instead of $1/8$), and this question is still not resolved. As stated in the introduction, recent nonlinear analytical treatment of the film flow suggests that such a critical Froude number could exist, but this remains to be established and convincingly explained.

A specific problem of great interest where these considerations should matter is the question of jet impacts on inclined plates. It is not obvious in this kind of problem whether or not a perfect hydraulic jump can exist, and the two scalings should compete against each other in a way that merits investigation. The influence of an external field, here the tangent component of gravity, on a circular shock is a fundamental question of great interest. Specific efforts should be made in this direction (Wilson *et al.* 2012; Duchesne, Lebon & Limat 2013).

Declaration of interests. The authors report no conflict of interest.

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