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ON THE DISTRIBUTION OF THE RANK STATISTIC FOR STRONGLY CONCAVE COMPOSITIONS

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Abstract

A strongly concave composition of n is an integer partition with strictly decreasing and then increasing parts. In this paper we give a uniform asymptotic formula for the rank statistic of a strongly concave composition introduced by Andrews *et al.* ['Modularity of the concave composition generating function', *Algebra Number Theory* **7**(9) (2013), 2103–2139].

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1. Introduction

A partition of a positive integer n is a sequence of nonincreasing positive integers whose sum equals n. Let $p(n)$ be the number of integer partitions of n. To explain Ramanujan's famous partition congruences with modulus 5, 7 and 11, the rank and crank statistic for integer partitions was introduced and investigated by Dyson [9] and Andrews and Garvan [2, 11]. Let $N(m,n)$ and $M(m,n)$ be the number of partitions of n with rank m and crank m, respectively. It is well known that

$$\sum_{n\geq 0} N(m,n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(3n-1)/2+|m|n}(1-q^n)$$

and

$$\sum_{n\geq 0} M(m,n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(n-1)/2+|m|n}(1-q^n),$$

where $(a;q)_\infty = \prod_{j\geq 0}(1-aq^j)$ for any $a\in\mathbb{C}$ and $|q|<1$.

In [10], Dyson conjectured an asymptotic formula for the crank statistic for integer partitions:

$$M(m,n) \sim \frac{\pi}{4\sqrt{6n}}\operatorname{sech}^2\Big(\frac{\pi m}{2\sqrt{6n}}\Big)p(n), \qquad n\to+\infty. \tag{1.1}$$

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Bringmann and Dousse [4] proved that (1.1) holds for all $|m| \le (\sqrt{n}\log n)/(\pi\sqrt{6})$. In [8], Dousse and Mertens proved the same result for $N(m,n)$. For more results on the asymptotics of the rank and crank statistic for integer partitions, see [6, 7, 13, 14].

A concave composition λ of n is a nonnegative integer sequence $\{a_r\}_{r=1}^{s}$ of the form

$$a_1 \ge a_2 \ge \cdots \ge a_{k-1} > a_k < a_{k+1} \le \cdots \le a_{s-1} \le a_s$$

and with sum n for some $s \in \mathbb{Z}_+$. Here a_k is called the central part of λ. If all the '$\ge$' and '$\le$' are replaced by '$>$' and '$<$', respectively, we refer to a strongly concave composition. The rank of λ is defined as $\mathrm{rk}(\lambda) := s - 2k + 1$; it is the analogue of the rank statistic for integer partitions and measures the position of the central part.

Let $\mathcal{V}(n)$ and $\mathcal{V}_d(n)$ be the sets of all concave compositions and all strongly concave compositions, respectively, of the nonnegative integer n. Also, let $V(n) = \#\mathcal{V}(n)$ and $V_d(n) = \#\mathcal{V}_d(n)$ be the numbers of concave compositions and strongly concave compositions of n, respectively. Andrews [1] found the generating functions

$$v(q) := \sum_{n\ge0} V(n)q^n = \sum_{n\ge0} \frac{q^n}{(q^{n+1};q)_\infty^2}$$

and

$$v_d(q) := \sum_{n\ge0} V_d(n)q^n = \sum_{n\ge0} (-q^{n+1};q)_\infty^2 q^n.$$

Andrews *et al.* [3] proved that $v(q)$ is a mixed mock modular form. More precisely, they established the following modularity properties.

Theorem 1.1. *Let $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{C}$ and $\Im(\tau) > 0$. Define $f(\tau) = q(q;q)_\infty^3 v(q)$ and*

$$\hat{f}(\tau) = f(\tau) - \frac{i}{2}\eta(\tau)^3 \int_{-\bar{\tau}}^{i\infty} \frac{\eta(z)^3}{(-i(z+\tau))^{1/2}}\,dz + \frac{\sqrt{3}}{2\pi i}\eta(\tau)\int_{-\bar{\tau}}^{i\infty} \frac{\eta(z)}{(-i(z+\tau))^{3/2}}\,dz,$$

where the Dedekind η-function is given by $\eta(\tau) = q^{1/24}(q;q)_\infty$. Then the function $\hat{f}$ transforms as a modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$.

For $v_d(q)$, Andrews [1] proved that

$$v_d(q) = 2(-q;q)_\infty^2 \sum_{n\ge0}\left(\frac{-12}{n}\right) q^{(n^2-1)/24} - \sum_{n\ge0}(-1)^n q^{n(n+1)/2},$$

where $(\frac{\cdot}{\cdot})$ is the Kronecker symbol, that is, $v_d(q) + \sum_{n\ge0}(-1)^n q^{n(n+1)/2}$ is essentially a modular function multiplied by a false theta function. So, we may expect $V_d(n)$ to be simpler to study than $V(n)$, but not to yield such precise results. For example, Andrews *et al.* [3] obtained an asymptotic formula with a polynomial error for $V(n)$ by using the circle method of Bringmann and Mahlburg [5]. They also gave an asymptotic expansion for $V_d(n)$ which is technically easier to establish:[1]

$$V_d(N) \sim 2^{-1/4}3^{-5/4}N^{-3/4}e^{2\pi\sqrt{N/6}}\left(1 + \sum_{n\ge1} c_n N^{-n/2}\right) \tag{1.2}$$

for $N \to +\infty$, where the $c_n \in \mathbb{R}, n \in \mathbb{Z}_+$, are computable constants.

[1]Note that the leading coefficient of the asymptotic expansion (1.2) is $2^{-1/4}3^{-5/4}$ rather that $2\cdot 2^{-1/4}3^{-5/4}$ as stated in [3, Theorem 1.5].

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Let $V_d(m,n)$ be the number of strongly concave compositions of n with rank equal to m. Andrews *et al.* [3] proved that[1]

$$\sum_{n\geq 0}\sum_{m\in\mathbb{Z}} V_d(m,n)x^m q^n = -\sum_{n\geq 0}(-1)^n q^{n(n+1)/2}x^{2n+1} + (-x;q)_\infty(-x^{-1}q;q)_\infty \sum_{n\geq 0}\Big(\frac{-12}{n}\Big)x^{(n-1)/2}q^{(n^2-1)/24}. \quad (1.3)$$

In this paper we investigate the asymptotics of $V_d(m,n)$ as n tends to infinity with arbitrary m, motivated by the questions in [3, pages 2108–2109] for the more complex behaviour of the distribution of concave compositions.

The first result of this paper is the following proposition.

Proposition 1.2. *Let $p(n)$ be the number of integer partitions of a nonnegative integer n and let $p(-\ell)=0$ for $\ell\in\mathbb{Z}_+$. Then, for $N,\ell\in\mathbb{Z}$,*

$$V_d\Big(\ell, N+\frac{|\ell|(|\ell|+1)}{2}\Big) = \sum_{n\geq 0}\Big(\frac{-3}{2n+1}\Big)p\Big(N-\frac{2n(n+1)}{3}-n|\ell|\Big). \quad (1.4)$$

In particular, for $m,n\in\mathbb{Z}$ with $0\leq n<\frac{1}{2}|m|(|m|+5)+4$,

$$V_d(m,n) = p\Big(n-\frac{|m|(|m|+1)}{2}\Big). \quad (1.5)$$

From Proposition 1.2, we derive the following uniform asymptotics for $V_d(m,n)$ as $n\to+\infty$.

Theorem 1.3. *Uniformly for all $\ell\in\mathbb{Z}$ and $N\to+\infty$,*

$$V_d\Big(\ell, N+\frac{|\ell|(|\ell|+1)}{2}\Big) = p(N)F\Big(\frac{\pi|\ell|}{\sqrt{6N}}\Big)(1+O(N^{-1/10})), \quad (1.6)$$

where the implied constant is absolute and

$$F(\alpha) = \frac{1+e^{-\alpha}}{1+e^{-\alpha}+e^{-2\alpha}}.$$

In particular, if the integer m satisfies $m=o(N^{3/8})$, then

$$\frac{V_d(m,N)}{V_d(N)} \sim \frac{1}{(24N)^{1/4}}\exp\Big(-\frac{\pi m^2}{\sqrt{24N}}\Big). \quad (1.7)$$

Finally, we give a limiting distribution for the rank statistic for strongly concave compositions. Define the real function $\Psi_d(x)$ by

$$\Psi_d(x) = \lim_{N\to+\infty}\frac{1}{V_d(N)}\#\Big\{\lambda\in\mathcal{V}_d(N) : \frac{\mathrm{rk}(\lambda)}{(6N/\pi^2)^{1/4}}\leq x\Big\} \quad \text{for } x\in\mathbb{R}.$$

[1] We correct some sign errors in the statement of (1.3) in [3].

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It is clear that

$$\Psi_d(x) = \lim_{N\to+\infty} \frac{1}{V_d(N)} \sum_{\substack{m\in\mathbb{Z}\\ m\le(6N/\pi^2)^{1/4}x}} \sum_{\substack{\lambda\in\mathcal{V}_d(N)\\ \mathrm{rk}(\lambda)=m}} 1 = \lim_{N\to+\infty} \sum_{\substack{m\in\mathbb{Z}\\ m\le(6N/\pi^2)^{1/4}x}} \frac{V_d(m,N)}{V_d(N)}$$

and that $\Psi_d(-\infty) = 0$ and $\Psi_d(+\infty) = 1$. Hence, by using (1.7) and the fact that $V_d(m,N) = V_d(|m|,N)$, it is easy to deduce the following corollary by Abel's summation formula.

Corollary 1.4. *The distribution function $\Psi_d(x)$ is the standard normal distribution on $\mathbb{R}$, that is,*

$$\Psi_d(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\,dx.$$

2. Proofs of the results

2.1. The proof of Proposition 1.2. By the Jacobi triple product formula,

$$(q;q)_\infty(-xq;q)_\infty(-x^{-1};q)_\infty = \sum_{n\in\mathbb{Z}} q^{n(n+1)/2}x^n,$$

the basic properties of the Kronecker symbol and (1.3),

$$\sum_{\substack{n\ge0\\ m\in\mathbb{Z}}} V_d(m,n)x^m q^n = -\sum_{n\ge0}(-1)^n q^{n(n+1)/2}x^{2n+1} + \frac{1}{(q;q)_\infty}\sum_{\ell\in\mathbb{Z}} q^{\ell(\ell+1)/2}x^{-\ell}\sum_{n\ge0}\Big(\frac{-12}{2n+1}\Big)x^n q^{n(n+1)/6}.$$

This yields, for integer $r \ge 0$,

$$\begin{aligned}\sum_{n\ge0} V_d(-r,n)q^n &= \frac{1}{(q;q)_\infty}\sum_{\substack{\ell-n=r\\ \ell\in\mathbb{Z},n\ge0}}\Big(\frac{-3}{2n+1}\Big)\Big(\frac{2}{2n+1}\Big)^2 q^{(1/6)n(n+1)+(1/2)\ell(\ell+1)}\\ &= \sum_{k\ge0}p(k)q^k\sum_{n\ge0}\Big(\frac{-3}{2n+1}\Big)q^{(1/6)n(n+1)+(1/2)(n+r)(n+r+1)}\\ &= \sum_{N\ge0}q^N\sum_{n\ge0}\Big(\frac{-3}{2n+1}\Big)p\Big(N-\frac{2n(n+1)}{3}-rn-\frac{r(r+1)}{2}\Big),\end{aligned}$$

which means that

$$V_d\Big(-\ell, N+\frac{\ell(\ell+1)}{2}\Big) = \sum_{n\ge0}\Big(\frac{-3}{2n+1}\Big)p\Big(N-\frac{2n(n+1)}{3}-n\ell\Big)$$

for all integers $\ell \ge 0$. Since $V_d(-m,n) = V_d(m,n)$, we have proved (1.4) in Proposition 1.2. Further, if $2\ell+4 > N$, then

$$V_d\Big(\ell, N+\frac{|\ell|(|\ell|+1)}{2}\Big) = p(N),$$

which gives (1.5) in Proposition 1.2.

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2.2. Asymptotic results for $p(n)$. We need the following asymptotic result for $p(n)$ proved by Hardy and Ramanujan in [12].

LEMMA 2.1. *For $n \in \mathbb{Z}_+$,*

$$p(n) - \hat{p}(n - 1/24) = O(n^{-1} e^{B\sqrt{n}/2}),$$

where $B = 2\pi/\sqrt{6}$ and

$$\hat{p}(x) = \frac{e^{B\sqrt{x}}}{4\sqrt{3}x}\Big(1 - \frac{1}{B\sqrt{x}}\Big).$$

(These definitions for B and $\hat{p}(x)$ are used throughout this section.)

We also need the following approximation for $p(X + r)$ with $r = o(X^{3/4})$.

LEMMA 2.2. *For $r = o(X^{3/4})$ and X sufficiently large,*

$$\frac{p(X + r)}{p(X)} = e^{Br/2\sqrt{X}}\Big(1 + O\Big(\frac{1}{X} + \frac{|r|}{X} + \frac{|r|^2}{X^{3/2}}\Big)\Big).$$

PROOF. From Lemma 2.1, it is clear that

$$\begin{aligned}\frac{\hat{p}(X + r)}{\hat{p}(X)} &= e^{B(\sqrt{X+r}-\sqrt{X})}\Big(1 + O\Big(\frac{|r|}{X}\Big)\Big)\\ &= e^{Br/2\sqrt{X}+O(r^2/X^{3/2})}\Big(1 + O\Big(\frac{|r|}{X}\Big)\Big)\\ &= e^{Br/2\sqrt{X}}\Big(1 + O\Big(\frac{|r|}{X} + \frac{|r|^2}{X^{3/2}}\Big)\Big)\end{aligned}$$

by the generalised binomial theorem. Since

$$\frac{p(N)}{\hat{p}(N)} = 1 + O\Big(\frac{1}{N}\Big)$$

for all $N \geq 1$,

$$\frac{p(X + r)}{p(X)} = e^{Br/2\sqrt{X}}\Big(1 + O\Big(\frac{1}{X} + \frac{|r|}{X} + \frac{|r|^2}{X^{3/2}}\Big)\Big),$$

which completes the proof of the lemma. □

2.3. The proof of Theorem 1.3.

2.3.1. Case $|\ell| > \sqrt{N}(\log N)^2$. Define

$$F(\ell, N) := V_d\Big(\ell, N + \frac{|\ell|(|\ell| + 1)}{2}\Big). \tag{2.1}$$

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For $N/2 \geq |\ell| > \sqrt{N}(\log N)^2$, from Proposition 1.2 and Lemma 2.2,

$$\begin{aligned}
F(\ell, N) &= \sum_{\substack{n\geq 0\\ 2n(n+1)/3+n|\ell|\leq N}} \Big(\frac{-3}{2n+1}\Big) p\Big(N - \frac{2n(n+1)}{3} - n|\ell|\Big)\\
&= p(N) + O\Big(\sum_{\substack{n\geq 2\\ 2n(n+1)/3+n\ell\leq N}} p(N - n|\ell|)\Big)\\
&= p(N) + O(\sqrt{N} p(N - 2|\ell|)) = p(N) + O(\sqrt{N} p(N - \lfloor \sqrt{N}(\log N)^2 \rfloor))\\
&= p(N)\Big(1 + O\Big(\sqrt{N} \exp\Big(-\frac{B\lfloor \sqrt{N}(\log N)^2\rfloor}{2\sqrt{N}}\Big)\Big)\Big),
\end{aligned}$$

where $\lfloor \cdot \rfloor$ is the greatest integer function. Hence, for $N/2 \geq |\ell| > \sqrt{N}(\log N)^2$,

$$F(\ell, N) = p(N)(1 + O(N^{-\sqrt{\log N}})). \tag{2.2}$$

2.3.2. Case $|\ell| \leq \sqrt{N}(\log N)^2$. Since

$$\Big(\frac{-3}{2n+1}\Big) = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 3,\\ 0 & \text{if } n \equiv 1 \bmod 3,\\ -1 & \text{if } n \equiv 2 \bmod 3\end{cases}$$

for $0 \leq \ell \leq \sqrt{N}(\log N)^2$,

$$\begin{aligned}
F(\ell, N) &= \sum_{n\geq 0} \Big(\frac{-3}{2n+1}\Big) p\Big(N - \frac{2n(n+1)}{3} - n\ell\Big)\\
&= \sum_{n\geq 0} [p(N - Q_1(n, \ell)) - p(N - Q_2(n, \ell))],
\end{aligned}$$

where

$$Q_1(n, \ell) = 2n(3n+1) + 3n\ell \quad \text{and} \quad Q_2(n, \ell) = Q_1(n, \ell) + (8n + 4 + 2\ell).$$

We split the sum into two parts:

$$\begin{aligned}
\frac{F(\ell, N)}{p(N)} &= \frac{1}{p(N)} \sum_{\substack{n\geq 0\\ n^2+n\ell > \sqrt{N}(\log N)^2}} [p(N - Q_1(n, \ell)) - p(N - Q_2(n, \ell))]\\
&\quad + \frac{1}{p(N)} \sum_{\substack{n\geq 0\\ n^2+n\ell \leq \sqrt{N}(\log N)^2}} [p(N - Q_1(n, \ell)) - p(N - Q_2(n, \ell))] =: R + I.
\end{aligned}$$

Noting that $Q_2(n, \ell) \geq Q_1(n, \ell) \geq n^2 + n\ell$ for all $n \geq 0$, we can estimate R by

$$\begin{aligned}
|R| &\leq \frac{2}{p(N)} \sum_{\substack{n\geq 0\\ n^2+n\ell > \sqrt{N}(\log N)^2}} p(N - Q_1(n, \ell))\\
&\leq \frac{2}{p(N)} \sum_{\substack{n\geq 0\\ n^2+n\ell > \sqrt{N}(\log N)^2}} p(N - (n^2 + n\ell)) \leq 2\sqrt{N} \frac{p(N - \lfloor \sqrt{N}(\log N)^2 \rfloor)}{p(N)}.
\end{aligned}$$

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Thus, from Lemma 2.2,

$$R \ll \sqrt{N} e^{-B\lfloor \sqrt{N}(\log N)^2 \rfloor / 2\sqrt{N}} \ll N^{-\sqrt{\log N}}.$$

To estimate I, we note that

$$0 \le Q_1(n,\ell) \le Q_2(n,\ell) \le 16(n^2 + n\ell) + 2\ell + 4 = O(\sqrt{N}(\log N)^2)$$

for $n \ge 0$ and $n^2 + n\ell \le \sqrt{N}(\log N)^2$. By Lemma 2.2,

$$\begin{aligned} I &= \sum_{n\ge 0} (e^{-BQ_1(n,\ell)/2\sqrt{N}} - e^{-BQ_2(n,\ell)/2\sqrt{N}}) \\ &\quad - \sum_{\substack{n\ge 0 \\ n^2+n\ell > \sqrt{N}(\log N)^2}} (e^{-BQ_1(n,\ell)/2\sqrt{N}} - e^{-BQ_2(n,\ell)/2\sqrt{N}}) \\ &\quad + O\bigg(\sum_{i=1}^{2} \sum_{\substack{n\ge 0 \\ n^2+n\ell \le \sqrt{N}(\log N)^2}} e^{-BQ_i(n,\ell)/2\sqrt{N}} \Big(\frac{1}{N} + \frac{Q_i(n,\ell)}{N} + \frac{Q_i(n,\ell)^2}{N^{3/2}} \Big)\bigg) = I_M + I_R \end{aligned}$$

with

$$I_M = \sum_{n\ge 0} (e^{-BQ_1(n,\ell)/2\sqrt{N}} - e^{-BQ_2(n,\ell)/2\sqrt{N}})$$

and

$$\begin{aligned} I_R &\ll \sum_{\substack{n\ge 0 \\ n^2+n\ell > \sqrt{N}(\log N)^2}} e^{-B(n^2+\ell n)/\sqrt{N}} + \sum_{\substack{n\ge 0 \\ n^2+n\ell \le \sqrt{N}(\log N)^2}} \frac{(\log N)^4}{N^{1/2}} e^{-B(n^2+n\ell)/2\sqrt{N}} \\ &\ll N^{-\sqrt{\log N}} + N^{-1/2}(\log N)^4 \sum_{\substack{n\ge 0 \\ n^2+n\ell \le \sqrt{N}(\log N)^2}} 1 \ll N^{-1/4}(\log N)^5. \end{aligned}$$

We conclude that

$$F(\ell, N)/p(N) = I_M + O(N^{-1/5}) \tag{2.3}$$

for $0 \le \ell \le \sqrt{N}(\log N)^2$. To estimate I_M, we need the following lemma.

LEMMA 2.3. *Let $0 \le \ell = o(\alpha^{-1})$. Then, as $\alpha \to 0^+$,*

$$f(\alpha) := \alpha \sum_{n\ge 0} (4n + \ell) e^{-2\alpha n^2 - \alpha n\ell} = 1 + O(\sqrt{\alpha} + |\alpha\ell|).$$

PROOF. By Abel's summation formula, or integration by parts for a Riemann–Stieltjes integral,

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$$\begin{aligned}
f(\alpha) &= 4\alpha \sum_{n\geq 0}(n+\ell/4)e^{-2\alpha(n+\ell/4)^2+\alpha\ell^2/8}\\
&= 4\alpha e^{\alpha\ell^2/8}\int_{0-}^{\infty} e^{-2\alpha(x+\ell/4)^2}\, d\Big(\sum_{0\leq n\leq x}(n+\ell/4)\Big)\\
&= 4\alpha e^{\alpha\ell^2/8}\Big(\int_0^{\infty} e^{-2\alpha(x+\ell/4)^2}\, d\Big(\frac{x^2}{2}+\frac{x\ell}{4}\Big) + O\Big(\alpha\int_0^{\infty}(x+\ell/4)^2 e^{-2\alpha(x+\ell/4)^2}\,dx\Big)\Big)\\
&= 4\alpha e^{\alpha\ell^2/8}\Big(\int_{\ell/4}^{\infty} x e^{-2\alpha x^2}\,dx + O\Big(\alpha\int_{\ell/4}^{\infty} x^2 e^{-2\alpha x^2}\,dx\Big)\Big)\\
&= e^{\alpha\ell^2/8}\int_{\alpha\ell^2/8}^{\infty} e^{-x}\,dx + O\Big(\sqrt{\alpha}e^{\alpha\ell^2/8}\int_{\alpha\ell^2/8}^{\infty} x^{1/2}e^{-x}\,dx\Big) = 1 + O(\sqrt{\alpha}+|\alpha\ell|),
\end{aligned}$$

which completes the proof of the lemma. □

We now evaluate I_M. By the definitions of $F(\alpha)$ and I_M, for $\ell \geq N^{3/8}$,

$$\begin{aligned}
I_M &= \sum_{0\leq n\leq N^{1/5}} (e^{-BQ_1(n,\ell)/2\sqrt{N}} - e^{-BQ_2(n,\ell)/2\sqrt{N}}) + O(N^{-\sqrt{\log N}})\\
&= \sum_{0\leq n\leq N^{1/5}} e^{-B(3n+1)n/\sqrt{N}}(1-e^{-B(\ell+4n)/\sqrt{N}})\, e^{-3Bn\ell/2\sqrt{N}} + O(N^{-\sqrt{\log N}})\\
&= (1+O(N^{-1/10}))\sum_{0\leq n\leq N^{1/5}} (1-e^{-B(\ell+4n)/\sqrt{N}})\, e^{-3Bn\ell/2\sqrt{N}} + O(N^{-\sqrt{\log N}})\\
&= (1+O(N^{-1/10}))\frac{1-e^{-B\ell/\sqrt{N}}}{1-e^{-3B\ell/2\sqrt{N}}} = (1+O(N^{-1/10}))F\Big(\frac{B\ell}{2\sqrt{N}}\Big)
\end{aligned}$$

and, for $0 \leq \ell \leq N^{3/8}$,

$$\begin{aligned}
I_M &= \sum_{0\leq n\leq N^{2/5}} (e^{-Bn/\sqrt{N}} - e^{-B(5n+\ell)/\sqrt{N}})\, e^{-B(6n^2+3n\ell)/2\sqrt{N}} + O(N^{-\sqrt{\log N}})\\
&= (1+O(N^{-1/10}))\sum_{0\leq n\leq N^{2/5}} \frac{B(4n+\ell)}{\sqrt{N}} e^{-B(6n^2+3n\ell)/2\sqrt{N}} + O(N^{-\sqrt{\log N}})\\
&= (1+O(N^{-1/10}))\frac{B}{\sqrt{N}}\sum_{n\geq 0}(4n+\ell)e^{-B(6n^2+3n\ell)/2\sqrt{N}} + O(N^{-\sqrt{\log N}})\\
&= \frac{2}{3}(1+O(N^{-1/10}))(1+O(N^{-1/4}+\ell N^{-1/2})) = (1+O(N^{-1/10}))F\Big(\frac{B\ell}{2\sqrt{N}}\Big)
\end{aligned}$$

by the use of Lemma 2.3. Thus, for $0 \leq \ell \leq \sqrt{N}(\log N)^2$,

$$F(\ell, N) = p(N)F\Big(\frac{\pi\ell}{\sqrt{6N}}\Big)(1+O(N^{-1/10})) \tag{2.4}$$

from (2.3) and the definition $B = 2\pi/\sqrt{6}$.

Finally, by using (2.1), (2.2), (2.4) and the fact that $V_d(m,n) = V_d(|m|,n)$, we finish the proof of (1.6). By using (1.2), (1.6) and Lemma 2.1, we obtain the proof of (1.7), which completes the proof of Theorem 1.3.

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