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Star Versus Two Stripes Ramsey Numbers and a Conjecture of Schelp

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R. H. Schelp conjectured that if G is a graph with $|V(G)| = R(P_n, P_n)$ such that $\delta(G) > \frac{3|V(G)|}{4}$, then in every 2-colouring of the edges of G there is a monochromatic P_n. In other words, the Ramsey number of a path does not change if the graph to be coloured is not complete but has large minimum degree.

Here we prove Ramsey-type results that imply the conjecture in a weakened form, first replacing the path by a matching, showing that the star-matching–matching Ramsey number satisfying $R(S_n, nK_2, nK_2) = 3n - 1$. This extends $R(nK_2, nK_2) = 3n - 1$, an old result of Cockayne and Lorimer. Then we extend this further from matchings to connected matchings, and outline how this implies Schelp's conjecture in an asymptotic sense through a standard application of the Regularity Lemma.

It is sad that we are unable to hear Dick Schelp's reaction to our work generated by his conjecture.

1. Introduction

The path–path Ramsey number was determined in [7] and its diagonal case (stated for convenience for even paths) is that $R(P_{2n}, P_{2n}) = 3n - 1$, *i.e.*, in every 2-colouring of the edges of K_{3n-1}, the complete graph on $3n - 1$ vertices, there is a monochromatic P_{2n}, a path on $2n$ vertices. An easy example shows that K_{3n-2} can be 2-coloured with no monochromatic P_{2n}. It is a natural question to ask whether a similar conclusion is true if K_{3n-1} is replaced by some subgraph of it. One such result was obtained in [10], where it was proved that in every 2-colouring of the edges of the complete 3-partite graph $K_{n,n,n}$ there is a monochromatic $P_{(1-o(1))2n}$. The following conjecture of Schelp [15] states that K_{3n-1} can be replaced by a graph G of large minimum degree $\delta(G)$.

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Conjecture 1.1. *Suppose that n is large enough and G is a graph on $3n-1$ vertices with minimum degree larger than $\frac{3|V(G)|}{4}$. Then, in any 2-colouring of the edges of G there is a monochromatic P_{2n}.*

Schelp's conjecture is stated in its original form as in [15], but it is probably true for *every* $n \geqslant 1$. In fact, apart from Theorem 1.6, all results we prove here are valid for every n.

Schelp also noticed that the condition on the minimum degree in Conjecture 1.1 is close to best possible. Indeed, suppose that $3n-1 = 4m$ for some m and consider a graph whose vertex set is partitioned into four parts A_1, A_2, A_3, A_4 with $|A_i| = m$. Assume there are no edges from A_1 to A_2 and from A_3 to A_4; edges in $[A_1, A_3], [A_2, A_4]$ are red, edges in $[A_1, A_4], [A_2, A_3]$ are blue and edges within A_i are coloured arbitrarily. In this colouring the longest monochromatic path has $2m = \frac{3n-1}{2}$ vertices, much smaller than $2n$, while the minimum degree is $3m-1 = \frac{3(3n-1)}{4} - 1$. Thus, and this makes the conjecture surprising, even a minuscule increase in the minimum degree results in a dramatic increase in the length of the longest monochromatic path. Schelp notes in [15] (citing [16]) that he proved that there exists a $c < 1$ for which Conjecture 1.1 holds if the minimum degree is raised to $c|V(G)|$.

We will prove Ramsey-type results leading to an asymptotic version of Conjecture 1.1. As a first step, we have Theorem 1.2 and its diagonal case, Corollary 1.3, a weaker form of Conjecture 1.1, where paths are replaced by matchings. This is a 'traditional' 3-colour Ramsey-type result which strengthens significantly (the 2-colour case of) a well-known result of Cockayne and Lorimer [3].

Let nK_2 denote a matching of size n, *i.e.*, n pairwise disjoint edges, and let S_t be a star with t edges. The Ramsey number for two matchings (in fact for any number of matchings) was determined in [3] as $R(n_1K_2, n_2K_2) = 2n_1 + n_2 - 1$ for $n_1 \geqslant n_2$. The next result extends this, as it implies that the Ramsey number for two matchings does not change if a graph of maximum degree $n_1 - 1$ is deleted from $K_{2n_1+n_2-1}$. It is worth noting that the Ramsey number for many stars and *one matching* was determined in [4].

Theorem 1.2. *Suppose that $n_1 \geqslant n_2 \geqslant 1$ and $t \geqslant 1$. Then*

$$R(S_t, n_1K_2, n_2K_2) = \begin{cases} 2n_1 + n_2 - 1 & \text{if } t \leqslant n_1, \\ n_1 + n_2 - 1 + t & \text{if } t \geqslant n_1. \end{cases}$$

Corollary 1.3. $R(S_n, nK_2, nK_2) = 3n - 1$.

Next we have Theorem 1.4, which is still weaker than Conjecture 1.1 but gives a monochromatic *connected* matching (a monochromatic matching all of whose edges are in the same component of the relevant colour) of the right size. This is the main result of the paper.

Theorem 1.4. *Suppose that a graph G has $3n-1$ vertices and $\delta(G) > \frac{3|V(G)|}{4}$. Then, in every 2-colouring of the edges of G there is a monochromatic connected matching of size n.*

It is worth mentioning the following lemma, which is used in the proof of Theorem 1.4. A well-known remark of Erdős and Rado says that in a 2-coloured complete graph there is a monochromatic spanning tree. For a survey of results arising from this remark, see [8]. Lemma 1.5 extends the remark from complete graphs (where $\delta(G) = |V(G)| - 1$) to graphs of large minimum degree.

Lemma 1.5. *Suppose that the edges of a graph G with $\delta(G) \geqslant \frac{3|V(G)|}{4}$ are 2-coloured. Then there is a monochromatic component with order larger than $\delta(G)$. This estimate is sharp.*

In Section 4 we outline how Theorem 1.4 and the Regularity Lemma imply Theorem 1.6, the asymptotic form of Conjecture 1.1. This technique is established by Łuczak in [13] and used successfully in many recent results: see, *e.g.*, [2], [6], [9], [10] and [11].

Theorem 1.6. *For every $\eta > 0$ there is an $n_0 = n_0(\eta)$ such that the following is true. Suppose that G is a graph on $n \geqslant n_0$ vertices with $\delta(G) > (\frac{3}{4} + \eta)n$. Then, in every 2-colouring of the edges of G there is a monochromatic path with at least $(\frac{2}{3} - \eta)n$ vertices.*

We note that Benevides, Łuczak, Scott, Skokan and White [1] recently proved Conjecture 1.1.

2. Proof of Theorem 1.2

To see that the Ramsey number cannot be less than claimed in Theorem 1.2, consider a partition of $n_1 + n_2 + \max\{t, n_1\} - 2$ vertices into three sets, A, B, C of size $n_1 - 1, n_2 - 1, \max\{t, n_1\}$, respectively. Colour all edges incident to some vertex of B blue. From the remaining uncoloured edges, colour red those that are incident to A. If $t > n_1$ then all edges within C remain uncoloured (or might be viewed as the 'star-colour'). If $t \leqslant n_1$ then $|C| = n_1$, and in this case colour all edges red within C. (In fact this is the 2-colouring of $K_{2n_1+n_2-1}$ that does not have a monochromatic matching of size n_i in colour i.) Clearly, there is no S_t in the star-colour, there is no red n_1K_2 and no blue n_2K_2.

To prove the other direction, consider a graph G with $f(n_1, n_2, t)$ vertices, where

$$f(n_1, n_2, t) = \begin{cases} 2n_1 + n_2 - 1 & \text{if } t \leqslant n_1, \\ n_1 + n_2 - 1 + t & \text{if } t \geqslant n_1, \end{cases}$$

and consider an arbitrary red–blue colouring of the edges of G. We show that there is either a vertex non-adjacent to at least t vertices or a red matching of size n_1 or a blue matching of size n_2. Note that the case $t < n_1$ obviously follows from the case $t = n_1$, so we may assume that $|V(G)| = n_1 + n_2 - 1 + t$ and $t \geqslant n_1 \geqslant n_2$. We use induction on n_1, for $n_1 = 1$ (thus $n_2 = 1$); the statement is obvious for every t.

In the inductive step we reduce the triple (t, n_1, n_2) to $(t, n_1 - 1, n_2)$ if $n_1 > n_2$ and to $(t, n_1 - 1, n_1 - 1)$ if $n_1 = n_2$. In both cases we assume that every vertex of G is non-adjacent to at most $t - 1$ vertices. Depending on which case we have, either there is a red matching of size $n_1 - 1$ or a blue matching of size n_2 or a blue matching of size $n_1 - 1$. If there is a blue matching of size n_2 there is nothing to prove. Otherwise, by switching colours if

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necessary, we may assume that there is a red matching of size $n_1 - 1$, and our goal is to find a blue matching of size n_2.

Using the Gallai–Edmonds structure theorem (in fact the Tutte–Berge formula suffices) for the subgraph $G_R \subset G$ with the red edges, we can find $X \subset V = V(G) = V(G_R)$ such that $V \setminus X$ has $d + |X|$ odd connected components in G_R, where d is the deficiency of G_R. Using that $d = |V(G_R)| - 2\nu(G_R) = n_1 + n_2 - 1 + t - 2(n_1 - 1) = n_2 - n_1 + t + 1$, the number of odd components of $V \setminus X$ in G_R is $t - n_1 + n_2 + 1 + |X|$. We consider the union of all even connected components of $V \setminus X$ as one special component, and label the components as $C_0, C_1, \dots C_m$ so that $|C_0|$ is the largest component, and either $m = t - n_1 + n_2 + |X|$ (if all components are odd), or $m = t - n_1 + n_2 + 1 + |X|$ (if there are non-empty even components). Note that $m \geqslant 1$.

Let H be the graph with vertex set $V(G) \setminus X$ and with edge set those edges of G that connect different C_i. Obviously all edges of H are blue. We are going to prove that H has a (blue) matching of size n_2. Note that X together with one vertex from each odd component must be in $V(G)$, thus $|X| + t - n_1 + n_2 + 1 + |X| \leqslant n_1 + n_2 - 1 + t$, implying that $|X| \leqslant n_1 - 1$. Therefore $|V(H)| = |V(G)| - |X| \geqslant n_1 + n_2 - 1 + t - |X| \geqslant n_1 + n_2 - 1 + t - (n_1 - 1) \geqslant 2n_2$. If H has minimum degree at least n_2, then (using that $|V(H)| \geqslant 2n_2$) a well-known lemma in [5] implies that H has a matching of size n_2 and the proof is finished. Thus we may assume that there is a component C_i and $y \in C_i$ such that $d_H(y) < n_2$. Then,

$$n_2 > d_H(y) \geqslant (n_1 + n_2 - 1 + t) - |X| - |C_i| - (t - 1) = n_1 + n_2 - |X| - |C_i|,$$

and we get that $|C_i| > n_1 - |X|$, and since $|X| \leqslant n_1 - 1$, we can write $|C_i| = n_1 - |X| + k$ with some integer $k \geqslant 1$. In fact, $C_i = C_0$ because we cannot have any other component C_j as large as C_i, otherwise

$$\begin{aligned} |V| &\geqslant |X| + |C_i| + |C_j| + t - n_1 + n_2 + |X| - 1 \\ &> |X| + n_1 - |X| + n_1 - |X| + t - n_1 + n_2 + |X| - 1 \\ &= n_1 + n_2 + t - 1 = |V|, \end{aligned}$$

a contradiction.

Set $D = V(H) \setminus C_0$ and notice that D is non-empty because $m \geqslant 1$. One can easily estimate the degree $d_H(y)$ for $y \in C_0$ in the bipartite subgraph $[C_0, D] \subset H$ as follows:

$$d_H(y) \geqslant (n_1 + n_2 - 1 + t) - |X| - |C_0| - (t - 1) = n_1 + n_2 - |X| - (n_1 - |X| + k) = n_2 - k. \tag{2.1}$$

On the other hand, for any $y \in C_i$ with $i > 0$,

$$\begin{aligned} d_H(y) &\geqslant |C_0| + t - n_1 + n_2 + |X| - 1 - (t - 1) = |C_0| - n_1 + n_2 + |X| \\ &= n_1 - |X| + k - n_1 + n_2 + |X| = n_2 + k \end{aligned} \tag{2.2}$$

because, apart from at most $t - 1$ non-adjacency cases, y is adjacent to vertices of C_0 and to at least one vertex of at least $m - 1 \leqslant t - n_1 + n_2 + |X| - 1$ components.

We show, with the folkloric argument of the lemma in [5] cited above (in fact it is credited there to Dirac), that conditions (2.1) and (2.2) ensure a matching of size n_2 in H.

Let M be a maximum matching in the bipartite subgraph $[C_0, D] \subset H$, and assume M has $s \leqslant n_2 - 1$ edges. Let M^* be a matching of H such that it covers all vertices of $C_0 \cap M$ and among those it is largest possible. Set $Y = V(M) \cup D$.

Suppose first that M covers all vertices of C_0. If M^* has fewer than n_2 edges, then (since $Y = V(H)$ in this case and $|V(H)| \geqslant 2n_2$) at least two vertices, v, w, of H are uncovered by M^*. Now the choice of M^* implies that all edges of H from v, w must go to vertices of M^* but condition (2.2) implies that there exists $e \in M^*$ such that u, v are adjacent to two ends of e. Replacing e by these two edges, we get a matching of size one larger than the size of M^*, a contradiction.

If M does not cover C_0, select $z \in C_0 \setminus M$. By condition (2.1), z is adjacent (in H) to a set B of $n_2 - k$ vertices in $D \cap M$. Let A be the set of vertices mapped by M from B to C_0. From the choice of M, no edge of H goes from $D \setminus M$ to A or to $C_0 \setminus M$.

Suppose that $D \setminus M = \emptyset$. Then $|D| = |M| = s \leqslant n_2 - 1$ implies that $V \setminus X$ has at most n_2 odd components (vertices of D and C_0) in G_R. However, as we have seen above, $V \setminus X$ has $t - n_1 + n_2 + 1 + |X| > n_2$ odd components in G_R, a contradiction.

Using (2.2) for every $v \in D \setminus M$, the degree of v in D is at least $n_2 + k - (s - (n_2 - k)) = 2n_2 - s$. This implies that $|Y| > s + 2n_2 - s = 2n_2$, which allows us to use the same argument as in the previous paragraph, to show that M^* has size at least n_2. We conclude that G has a blue matching of size n_2. □

3. Large connected matchings, proof of Theorem 1.4

Proof of Lemma 1.5. To see that the estimate of the lemma is sharp, consider K_n from which the edges of a balanced complete bipartite graph $[A, B]$ are removed, where $|A| = |B| = m$ $\left(0 \leqslant m \leqslant \frac{n}{2}\right)$. Set $C = V(K_n) \setminus (A \cup B)$, colour all edges incident to A red, all edges incident to B blue and all edges within C arbitrarily. Now $\delta(G) = n - m - 1$ and the largest monochromatic component in both colours has $n - m$ vertices. The theorem is also sharp in the sense that $\delta(G)$ cannot be lowered. Indeed, suppose that n is divisible by four, and consider four disjoint sets S_i with $|S_i| = n/4$. Let the pairs within S_i and in $[S_1, S_2], [S_3, S_4]$ be red edges and the pairs in $[S_1, S_4], [S_2, S_3]$ be blue edges. This defines a 2-coloured graph G with n vertices, $\delta(G) = \frac{3n}{4} - 1$, and all monochromatic components have only $n/2$ vertices.

To prove that there is a monochromatic component of the claimed size, assume that $|V(G)| = n$, $\delta(G) \geqslant \frac{3n}{4}$ and let $v \in V(G)$. Let R, B denote the vertex sets of the red and blue monochromatic components containing v. Observe that there are no edges in the bipartite graphs $[B \setminus R, R \setminus B], [R \cap B, V(G) \setminus (R \cup B)]$.

Clearly, from the minimum degree condition, $|V(G) \setminus (R \cup B)| < \frac{n}{4}$. If $B \setminus R$ or $R \setminus B$ is empty, then R or B is larger than $\frac{3n}{4}$. Otherwise, both $B \setminus R$ and $R \setminus B$ are smaller than $\frac{n}{4}$. We conclude that for the largest monochromatic, say red, component C of G, we have

$$|C| > \frac{n}{2}. \tag{3.1}$$

We show that in fact $|C| > \delta(G)$. Set $D = V(G) \setminus C$. Since C is a red component, all edges of $[C, D]$ are blue. Moreover, because of (3.1) and the minimum degree condition, the set of blue neighbours of any two vertices $v, w \in D$ must intersect in C. This implies

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that $F = D \cup A$ is connected in blue, where $A = \{x \in C : \exists v \in D, xv \text{ blue}\}$. By the choice of C, $|A \cup D| \leqslant |C|$, and therefore

$$|D| \leqslant |C \setminus A| < n - \delta(G),$$

because any vertex of D is non-adjacent to all vertices of $C \setminus A$. Thus $|D| < n - \delta(G)$, implying $|C| > \delta(G)$, as desired. □

Now we are ready to prove Theorem 1.4, the extension of Corollary 1.3.

Proof of Theorem 1.4. Set $V = V(G)$ and let C_1 be a largest monochromatic, say red, component. From Lemma 1.5, $|C_1| > \frac{3|V(G)|}{4}$. If $U = V \setminus V(C_1) \neq \emptyset$ then U is covered by a blue component C_2, because from the minimum degree condition the set of blue neighbours of any two vertices in U intersects in C_1. If $U = \emptyset$, then define C_2 as a largest blue component in G. Set $p = |V(C_1) \setminus V(C_2)|, q = |V(C_2) \setminus V(C_1)|$, from the choice of C_1 $p \geqslant q$. Set $A = V(C_1) \cap V(C_2)$. Observe that there are no edges of G in the bipartite graph $[V(C_1) \setminus V(C_2), V(C_2) \setminus V(C_1)]$. Thus, if $V(C_2) \setminus V(C_1) \neq \emptyset$ then $p < \frac{3n-1}{4} < n$.

We apply Theorem 1.2 to the subgraph spanned by A in G with parameters $t = \lceil \frac{3n-1}{4} \rceil, n_1 = n - q, n_2 = n - p$. To do this, we need to check that $n_2 = n - p \geqslant 1$. This is obvious if $q > 0$, since then $p < n$ as noted in the previous paragraph. On the other hand, if $q = 0$, *i.e.*, $V(C_1) = V$, we need another argument, in fact similar to the one used in the proof of Theorem 1.2. Observe that the largest red matching in C_1 is automatically connected, and thus we may assume it has $m < n$ edges. Applying the Tutte–Berge formula for the red graph, we can find a set $X \subset V$ whose removal leaves at least $c = 3n - 1 - 2m + |X|$ odd components. Let H be the blue subgraph of G whose vertex set is $V \setminus X$ and whose edge set is the set of blue edges of G that go between the red components of $V \setminus X$. Note that $|X| \leqslant n - 1$, otherwise G has at least $c + |X| = 3n - 1 - 2m + 2|X| > 3n - 1$ vertices, a contradiction. Thus $|V(H)| \geqslant 2n$. We show that H is a connected graph. Indeed, otherwise $V(H)$ can be partitioned into two non-empty sets P, Q so that there are no edges in the bipartite subgraph $[P, Q]$ of H, without loss of generality, let $|P| \geqslant n$. If P intersects each of the c red components in $V \setminus X$, then any $v \in V(H) \setminus P$ is non-adjacent in G to at least $c - 1 \geqslant n$ vertices, one vertex in all components not containing v. On the other hand, if P does not intersect a red component, then a vertex v from that component is non-adjacent in G to all vertices of P. In both cases we find a vertex v non-adjacent to at least n vertices and that contradicts the assumption $\delta(G) > \frac{3(3n-1)}{4}$. We conclude that H is connected (in blue), *i.e.*, we may assume $|V(C_2)| \geqslant 2n$, implying $p = |V| - |V(C_2)| \leqslant n - 1$, and therefore $n_2 = n - p \geqslant 1$ as required.

We claim that with our above choices of the parameters t, n_1, n_2 we have $|A| = 3n - 1 - p - q \geqslant R(S_t, n_1K_2, n_2K_2)$. Indeed, for $t \leqslant n_1$ we have to check that $3n - 1 - p - q \geqslant 2(n - q) + n - p - 1$, which reduces to $q \geqslant 0$. For $t > n_1$ we have to check $3n - 1 - p - q \geqslant n - p + n - q - 1 + t$, which reduces to $n \geqslant t$, obviously true for our choice of t. Thus, by Theorem 1.2 we have either a vertex with t edges missing from it or a red matching of size $n - p$ or a blue matching of size $n - q$. The first possibility contradicts the minimum

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degree assumption on G. Thus we have one of the other two possibilities when we claim that the matchings are extendible to the required size.

Indeed, assume that M is a red matching of size $n-p$ in $G[A]$. Every edge from $V(C_1)\setminus A$ to A is red. The red degree of any $v\in V(C_1)\setminus A$ towards A is at least $P=\frac{3(3n-1)}{4}-p$, and we claim that $P\geqslant 2(n-p)+p$. Indeed, the inequality reduces to $\frac{3(3n-1)}{4}\geqslant 2n$, which is obvious. Thus all the p vertices in $V(C_1)\setminus A$ are adjacent to at least p vertices of $A\setminus V(M)$, and that clearly allows us to extend M by p red edges to a red matching of size n.

Similarly, a blue matching M of size $n-q$ can be extended (for $q=0$ no extension is needed of course) by checking the inequality $Q=\frac{3(3n-1)}{4}-q\geqslant 2(n-q)+q$, which in fact reduces to the same inequality as in the previous case and finishes the proof. □

4. Building paths from connected matchings

Here we sketch how to get Theorem 1.6 from Theorem 1.4 and the Regularity Lemma [17]. The material of this section is fairly standard by now, so we omit some of the details. Combining the degree form and the 2-colour version of the Regularity Lemma, we get the following version. (For these and other variants of the Regularity Lemma see [12].)

Lemma 4.1 (Regularity Lemma: 2-coloured degree form). *For every $\varepsilon>0$ and every integer m_0, there is an $M_0=M_0(\varepsilon,m_0)$ such that for $n\geqslant M_0$ the following holds. For all graphs $G=G_1\cup G_2$ with $V(G_1)=V(G_2)=V$, $|V|=n$, and real number $\rho\in[0,1]$, there is a partition of the vertex set V into $l+1$ sets (so-called clusters) $V_0,V_1,\ldots,V_l$, and there are subgraphs $G'=G_1'\cup G_2'$, $G_1'\subset G_1$, $G_2'\subset G_2$ with the following properties:*

- $m_0\leqslant l\leqslant M_0$,
- $|V_0|\leqslant\varepsilon|V|$,
- *all clusters V_i, $i\geqslant 1$, are of the same size L,*
- $\deg_{G'}(v)>\deg_G(v)-(\rho+\varepsilon)|V|$ *for all $v\in V$,*
- $G'|_{V_i}=\emptyset$ *(V_i are independent in G'),*
- *all pairs $G'|_{V_i\times V_j}$, $1\leqslant i<j\leqslant l$, are ε-regular, each with a density 0 or exceeding ρ.*
- *all pairs $G_s'|_{V_i\times V_j}$, $1\leqslant i<j\leqslant l, 1\leqslant s\leqslant 2$, are ε-regular.*

Let G be a graph on $n\geqslant n_0$ vertices with $\delta(G)>(\frac{3}{4}+\eta)n$ and consider a 2-colouring $G=G_1\cup G_2$ of G. We apply Lemma 4.1 for G, with $\varepsilon\ll\rho\ll\eta\ll 1$. We get a partition of $V=\cup_{0\leqslant i\leqslant l}V_i$. We define the following *reduced graph* G^R. The vertices of G^R are $p_1,\ldots,p_l$, and we have an edge between vertices p_i and p_j if the pair (V_i,V_j) is ε-regular in G' with density exceeding ρ. Since in G', $\delta(G')>(\frac{3}{4}+\eta-(\rho+\varepsilon))|V|$, an easy calculation shows that in G^R we have $\delta(G^R)\geqslant\left(\frac{3}{4}+\eta-2\rho\right)l>\frac{3}{4}l$ (see, e.g., [14] for a similar computation). Define an edge-colouring $G^R=G_1^R\cup G_2^R$ in the following way. The edge p_ip_j is coloured with the colour that contains more edges from $G'|_{V_i\times V_j}$, thus clearly the density of this colour is still at least $\rho/2$ in $G'|_{V_i\times V_j}$.

We remove at most two vertices from G^R to make sure that the number of vertices has the form $3k-1$. Then, applying Theorem 1.4 to the 2-coloured G^R, we get a connected monochromatic matching saturating at least $\frac{2l}{3}$ vertices of G^R. Lifting this monochromatic

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connected matching to a monochromatic path in the original graph can be done by applying the following standard lemma (a special case of Lemma 4.2 in [9]), with $c = 2/3$ and with our choices of ε, ρ and reduced graph G^R.

Lemma 4.2. *Assume that for some positive constant c there is a monochromatic connected matching M (say in G_1^R) saturating at least $c|V(G^R)|$ vertices of G^R. Then in the original G we find a monochromatic path in G_1 covering at least $c(1-3\varepsilon)n$ vertices.*

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