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Abstract

In this paper, we provide a new approach to prove some weighted-blowup formulae for genus zero orbifold Gromov–Witten invariants. As a consequence, we show the invariance of symplectically rational connectedness with respect to weighted-blowup along positive centers. Furthermore, we use this method to give a new proof to the genus zero relative-orbifold correspondence of Gromov–Witten invariants.

1. Introduction

Let X be a smooth projective variety and let $S \subseteq X$ be a smooth subvariety. Denote the blowup of X along S by $\widehat{X}$. Consider $X \times \mathbf{P}^1$ with the standard $\mathbf{C}^*$-action on $\mathbf{P}^1$ (see (8)). There are two fixed loci $X_0 := X \times \{0\}$ and $X_\infty := X \times \{\infty\}$. By blowing up $X \times \mathbf{P}^1$ along $S_\infty := S \times \{\infty\}$, we get a new space, denoted by W. With respect to the induced $\mathbf{C}^*$-action on W, the fixed locus $W^{\mathbf{C}^*}$ has three connected components, which we denote by $F_0 \cong X$, $F_\infty \cong \widehat{X}$ and $F_* \cong S$. It is then natural to expect, with the virtual localization technique (cf. [GP99, Liu13]), one might relate Gromov–Witten invariants of fix loci, in particular, such as $F_0 \cong X$ and $F_\infty \cong \widehat{X}$. But extracting the desired invariants effectively from localization formulae may not be trivial. In order to do this, the main ingredient we use in this paper is to take the weight-r blowup, which is also known as the rth root construction, along certain divisors of W, then we apply a polynomiality property in r of certain Gromov–Witten invariants when $r \gg 1$. Such a polynomiality property was discovered by Pixton (cf. [JPPZ17, JPPZ20]) and is proved to be very powerful in this kind of computation; see, for example, [FWY20, FWY21, TY20]. Note that this polynomiality property for orbifold Gromov–Witten invariants was obtained by Chen, Du and Wang (cf. [CDW22]) and Tseng and You (cf. [TY23]) independently. In this paper, we use this strategy to get a new blowup formula of genus zero Gromov–Witten invariants (see Theorem 1.3). We also allow both X and S to be Deligne–Mumford stacks and the blowup to be weighted (see Theorems 1.8 and 1.9). In order to consider higher genus invariants, one may consider $\mathbf{P}(L \oplus \mathcal{O})$ instead of $X \times \mathbf{P}^1$ for some proper chosen line bundle L over X. This will be addressed in a further paper. On the other hand, we apply the same strategy but more complicated treatments to give a new proof to the genus zero relative-orbifold correspondence of Gromov–Witten invariants.

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We now outline the results presented in this paper. As there are some significant differences between the formulae for the smooth case and the orbifold case, for convenience, we deal with the smooth case (see § 1.1) and the orbifold case (see § 1.2) separately.

1.1 A blowup formula of genus zero smooth Gromov–Witten invariants

The searching of blowup formulae of Gromov–Witten invariants is an important issue in the Gromov–Witten theory and far from being completed. This issue was usually studied by using the degeneration formula of Gromov–Witten invariants (see Remark 1.4). In this paper, the approach we propose provides a direct comparison of Gromov–Witten invariants, hence avoids the detour to the degeneration formula and relative Gromov–Witten theory (see Remark 1.6).

We first consider the smooth case, i.e. X is a smooth projective variety and $S \subseteq X$ is a smooth subvariety with codimension $\kappa \geqslant 2$. Denote by N the normal bundle of S in X.

Definition 1.1. The normal bundle N of S in X is called *nonnegative*, if for every holomorphic map $f\colon \mathbf{P}^1 \to S$, the pullback bundle f^*N over $\mathbf{P}^1$ satisfies

$$\int_{\mathbf{P}^1} c_1(f^*N) + \kappa \geqslant 0, \tag{1}$$

and when the inequality in (1) is strict, N is called *positive*.

Remark 1.2. Recall that the bundle $N \to S$ is called *convex* if for every holomorphic map $f\colon \mathbf{P}^1 \to S$, we have $H^1(\mathbf{P}^1, f^*N) = 0$. Obviously, convexity implies nonnegativity.

Our main result on the blowup formula is the following theorem.

Theorem 1.3 (See Theorem 2.3). *Let X be a smooth projective variety and $S \subseteq X$ be a smooth subvariety. Let $\widehat{X}$ be the blowup of X along S. Suppose the normal bundle N of S in X is positive and S has codimension $\kappa \geqslant 2$. Then we have the following equality of genus zero Gromov–Witten invariants*

$$\langle \alpha_1, \ldots, \alpha_n \rangle^X_{0,n,A} = \langle p^*\alpha_1, \ldots, p^*\alpha_n \rangle^{\widehat{X}}_{0,n,p^!A}, \tag{2}$$

where $p\colon \widehat{X} \to X$ is the natural projection of the blowup.

Remark 1.4. In [Hu00, Hu01], by using the degeneration formula (cf. [LR01, Li02, IP04]) of Gromov–Witten invariants, Hu showed that the blowup formula (2) holds when S is (i) a point, (ii) a higher genus curve and (iii) a genus zero curve or a surface over which the bundle $TX|_S$ is semipositive (this implies that $TX|_S$ is positive in the sense of Definition 1.1).

On the other hand, the blowup projection $p\colon \widehat{X} \to X$ induces a natural morphism between moduli spaces of stable maps of $\widehat{X}$ and X. By using this induced morphism together with degeneration formula, Lai [Lai09] showed that the blowup formula (2) holds when the normal bundle N is convex and has a rank rk $\geqslant 2$ subbundle F generated by global sections, or when every holomorphic map from $\mathbf{P}^1$ to S is constant. For both cases the normal bundle N is positive in the sense of Definition 1.1. Lai [Lai09] and Manolache [Man11] also proved the formula (2) for the case that X is a smooth projective subvariety of a homogeneous space P and S is the transversal intersection of X with a smooth subvariety of P.

The formula (2) generalizes these blowup formulae in [Hu00, Hu01, Lai09, Man11] on the way to loosen the conditions on the normal bundles of the blowup centers.

Remark 1.5. When the normal bundle N is nonnegative and the homology class A in (2) belongs to the image of $H_2(S, \mathbf{Z}) \to H_2(X, \mathbf{Z})$ and satisfies $c_1(N)(A) + \kappa = 0$, there would be some extra contributions from S and formula (2) does not hold for such an A. See, for example, [Ke20, Theorem 1.1(2)] for blowups along $(-1, -1)$-curves. This will be addressed in a future paper.

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Remark 1.6. As reviewed in Remark 1.4, the usual argument for the blowup formulae is to use the degeneration formula and then apply the dimension arguments, or compare the moduli spaces of stable maps via the blowup projection. Comparing with their work, our approach has two advantages. The first is on the geometry model: we give a short-cut model relating X and $\widehat{X}$ directly instead of detouring to the degeneration model. Second, our approach provides a way to extract expected invariants effectively from localization formulae, which at least may be viewed as a 'dimension argument' in the localization technique.

1.2 Weighted-blowup formulae of genus zero orbifold Gromov–Witten invariants

We now consider the orbifold case. Let X be a Deligne–Mumford stack with projective coarse space, and $\mathsf{S} \subseteq \mathsf{X}$ be a smooth substack of codimension $\kappa \geqslant 1$ with normal bundle denoted by N.

In this orbifold case, we could study the weighted-blowup formulae of orbifold Gromov–Witten invariants. There has been little progress in this direction. In [HH15, Du17, Du23], some weighted-blowup formulae for weighted-blowup of symplectic manifolds along points, certain curves and surfaces were studied. We now state the results in this paper.

DEFINITION 1.7. The normal bundle N of S in X is called *nonnegative*, if for every genus zero orbifold holomorphic map $\mathsf{f}: \mathsf{C} \to \mathsf{S}$, the de-singularization, denoted by $|\mathsf{f}^*\mathsf{N}|$ (cf. [CR04, § 4.2]), of the pullback bundle $\mathsf{f}^*\mathsf{N}$ over the coarse space $\mathbf{P}^1 = |\mathsf{C}|$ of C satisfies

$$\int_{|\mathsf{C}|} c_1(|\mathsf{f}^*\mathsf{N}|) + \kappa \geqslant 0, \tag{3}$$

and when the inequality in (3) is strict, N is called *positive*.

Let $\mathfrak{a} = (a_1, \dots, a_\kappa) \in \mathbf{Z}^\kappa_{\geqslant 1}$ be the blowup weight, and let $\widehat{\mathsf{X}}_\mathfrak{a}$ be the weight-$\mathfrak{a}$ blowup of X along S (cf. [MM12, CDH19]). Unlike the standard blowup, the weighted-blowup formula is nontrivial even when $\kappa = 1$. Moreover, the formulae are different for the cases $\kappa \geqslant 2$ and $\kappa = 1$. Thus, we deal with them separately in Theorems 1.8 and 1.9. For the meaning of the notation in (4) and (5), see § 3.2.

THEOREM 1.8 (See Theorem 3.7). *Let X be a smooth Deligne–Mumford stack with projective coarse space, and let $\mathsf{S} \subseteq \mathsf{X}$ be a smooth substack. Let $\widehat{\mathsf{X}}_\mathfrak{a}$ be the weight-$\mathfrak{a}$ blowup of X along S. Suppose the normal bundle N of S in X is positive and S has codimension $\kappa \geqslant 2$. Then we have the following equality of genus zero orbifold Gromov–Witten invariants:*

$$\langle \alpha_1, \dots, \alpha_n \rangle^{\mathsf{X}}_{0,\mathfrak{g},A} = \langle p^*_{[\hat{g}_1]}\alpha_1, \dots, p^*_{[\hat{g}_j]}\alpha_n \rangle^{\widehat{\mathsf{X}}_\mathfrak{a}}_{0,\hat{\mathfrak{g}},\widehat{A}}. \tag{4}$$

For the codimension $\kappa = 1$ case, when blowup weight $\mathfrak{a} = (1)$, $\widehat{\mathsf{X}}_\mathfrak{a}$ and X have the same Gromov–Witten theory. Thus, we only consider the case that $\mathfrak{a} = (a)$ with $a \geqslant 2$. Then $\widehat{\mathsf{X}}_\mathfrak{a}$ is just the ath root construction (cf. [Cad07, § 2] and [AGV08, Appendix B]) of X along the divisor S.

THEOREM 1.9 (See Theorem 3.8). *Under the same assumption as Theorem 1.8 on X, S and N, when the codimension $\kappa = 1$ and the blowup weight $\mathfrak{a} = (a)$ satisfies $a \geqslant 2$, we have the following equality of genus zero orbifold Gromov–Witten invariants:*

$$\langle \alpha_1, \dots, \alpha_n \rangle^{\mathsf{X}}_{0,\mathfrak{g},A} = \langle p^*_{[\hat{g}_1]}\alpha_1, \dots, p^*_{[\hat{g}_j]}\alpha_n, \underbrace{1, \dots, 1}_{I_A} \rangle^{\widehat{\mathsf{X}}_\mathfrak{a}}_{0,\hat{\mathfrak{g}}^A,A}. \tag{5}$$

1.3 Applications to symplectically rational connectedness

As an application of our blowup formulae, we study the symplectically rational connectedness under blowups.

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Definition 1.10. A smooth projective variety X is called *symplectic k-point rationally connected* if there is a non-zero genus zero Gromov–Witten invariant of the form

$$\langle \underbrace{[pt], \ldots, [pt]}_{k}, \alpha_{k+1}, \ldots, \alpha_{k+n} \rangle^{X}_{0,k+n,A} \neq 0, \tag{6}$$

with $A \neq 0$ and $\alpha_j \in H^*(X)$. When $k = 1$, X is also called *symplectically uniruled*, and when $k = 2$, X is also called *symplectically rationally connected*.

Remark 1.11. There are as yet no standard definitions for symplectic k-point rationally connectedness except $k = 1$ (cf. [HLR08]). In this paper, we just take (6) as a definition.

Then as a direct consequence of Theorem 1.3 we have the following result.

Theorem 1.12. *Suppose the normal bundle N of S in X is positive and X is symplectic k-point rationally connected. Then $\widehat{X}$, the blowup of X along S, is also symplectic k-point rationally connected.*

Hu and the present authors studied a more general notion of symplectic uniruledness on orbifolds (cf. [CDH19]), which motivates the following definition.

Definition 1.13. Let $\mathfrak{g} = ([g_1], \ldots, [g_k])$ be a k-tuple of indices of twisted sectors of X. We say that X is $\mathfrak{g}$-*symplectic k-point rationally connected* if there is a nonzero genus zero orbifold Gromov–Witten invariant of the form

$$\langle [pt]_{[g_1]}, \ldots, [pt]_{[g_k]}, \alpha_{k+1}, \ldots, \alpha_{k+n} \rangle^{\mathsf{X}}_{0,k+n,A} \neq 0,$$

with $A \neq 0$ and $[pt]_{[g_j]}$ being the point class of the twisted sector $\mathsf{X}[g_j]$ of X for $1 \leqslant j \leqslant k$.

Then as a direct consequence of Theorems 1.8 and 1.9, we have the following result.

Theorem 1.14. *Suppose the normal bundle N of S in X is positive and X is $\mathfrak{g}$-symplectic k-point rationally connected for $\mathfrak{g} = ([g_1], \ldots, [g_k])$. Then $\widehat{\mathsf{X}}_{\mathfrak{a}}$ is $\widehat{\mathfrak{g}}$-symplectic k-point rationally connected with $\widehat{\mathfrak{g}} = ([\hat{g}_1], \ldots, [\hat{g}_k])$. See (23) for the meaning of $[\hat{g}]$.*

1.4 Genus zero relative-orbifold correspondence

With well-chosen root constructions, our model can also give a new and direct proof of the following genus zero relative-orbifold correspondence (cf. [ACW17, FWY20, TY20, TY23, CDW22]).

Theorem 1.15 (See Theorem 4.3). *The following genus zero relative-orbifold correspondence holds:*

$$\pi_{\mathrm{orb},*}([\overline{\mathscr{M}}_{\Gamma_r}(\mathsf{X}_r)]^{\mathrm{vir}}) = \pi_{\mathrm{rel},*}([\overline{\mathscr{M}}_{\Gamma}(\mathsf{X}|\mathsf{S})]^{\mathrm{vir}}), \quad \textit{when} \quad r \gg 1. \tag{7}$$

For the meaning of notation, see § 4.2. The statements given in [ACW17, TY20, CDW22] are in terms of invariants, and the statements given in [FWY20, TY23] are in the form of (7). The case that X and S are both smooth was dealt with in [ACW17, TY20, FWY20], and the case that X and S are both orbifolds was dealt with in [TY23, CDW22].

1.5 Organization of this paper

We prove the blowup formula in Theorem 1.3 in § 2 and the weighted-blowup formulae in Theorems 1.8 and 1.9 in § 3. Then we apply our approach to give a new proof for Theorem 1.15 in § 4.

2. A blowup formula of Gromov–Witten invariants

In this section we focus on the smooth case, i.e. X is a smooth projective variety and $\widehat{X}$ is the blowup of X along a codimension κ smooth subvariety S, and prove the blowup formula in Theorem 1.3. In the rest of this section, we fix the following notation: N is the normal bundle of S, $p\colon \widehat{X} \to X$ is the blowup projection and $Z := p^{-1}(S)$ is the exceptional divisor of $\widehat{X}$, which is the projectivization $\mathbf{P}(N)$ of N.

2.1 A geometric model: a symplectic cobordism between X and $\widehat{X}$

In this subsection, we construct a symplectic cobordism W between X and its blowup $\widehat{X}$. The terminology symplectic cobordism was introduced by Guillemin and Sternberg in [GS89], and later was adapted in [HLR08] to consider symplectic birational geometry. The symplectic cobordism W is obtained as follows. Note that such a construction appears in [FL19] where S is taken to be a divisor, i.e. $\kappa = 1$. In the rest of this section, we assume $\kappa \geqslant 2$.

Consider the product $X \times \mathbf{P}^1$ of X with the projective line $\mathbf{P}^1$. Denote the homogeneous coordinates of $\mathbf{P}^1$ by $[z_0, z_1]$. Set $0 := [0, 1]$ and $\infty := [1, 0]$. Let $\mathbf{C}^*$ act on $X \times \mathbf{P}^1$ by acting on $\mathbf{P}^1$ via

$$\lambda \cdot [z_0, z_1] = [z_0, \lambda z_1]. \tag{8}$$

Then the fixed locus $(X \times \mathbf{P}^1)^{\mathbf{C}^*}$ consists of the disjoint union of $X_0 := X \times \{0\}$ and $X_\infty := X \times \{\infty\}$.

Let W be the blowup of $X \times \mathbf{P}^1$ along $S_\infty = S \times \{\infty\} \subseteq X_\infty$. The exceptional divisor of W is

$$D := \mathbf{P}(N \oplus \mathcal{O}_S).$$

Here $\mathcal{O}_S \cong S \times T_\infty \mathbf{P}^1$. Then D contains an infinity divisor $\mathbf{P}(N \oplus 0)$, identified with $Z = \mathbf{P}(N)$, and a zero section $\mathbf{P}(0 \oplus \mathcal{O}_S)$, which is a copy of S and denoted by S_*. The $\mathbf{C}^*$-action lifts to W.

Lemma 2.1. *The fixed locus $W^{\mathbf{C}^*}$ of W with respect to the induced $\mathbf{C}^*$-action consists of three disjoint components*

$$F_0 = X_0 \cong X, \quad F_\infty = \widehat{X}_\infty \cong \widehat{X}, \quad F_* = S_* \cong S.$$

The normal line bundle L_0 of F_0 in W is trivial with action weight -1; the normal line bundle L_∞ of F_∞ is $\mathcal{O}_{\widehat{X}}(-Z)$ with action weight 1; the normal bundle N_* of F_* is $N \oplus \mathcal{O}_S$ with action weight $(-1, 1)$.

Proof. The normal line bundles L'_0 and L'_∞ of X_0 and X_∞ in $X \times \mathbf{P}^1$ are both trivial and have action weights -1 and 1, respectively. As we blow up $X \times \mathbf{P}^1$ along $S_\infty \subseteq X_\infty$, the normal line bundle L_0 of F_0 is L'_0 with action weight -1. The normal bundle of F_∞ is $p^* L'_\infty \otimes \mathcal{O}_{\widehat{X}}(-Z) \cong \mathcal{O}_{\widehat{X}}(-Z)$ with action weight $1 + 0 = 1$, where $p\colon F_\infty \cong \widehat{X} \to X_\infty \cong X$ is the natural blowup projection. Finally, the lifting S_* of S_∞ has normal $N \oplus \mathcal{O}_S$, where $\mathcal{O}_S \cong L'_\infty|_{S_\infty}$ with action weight 1 and N is the normal bundle of S_* in $\mathbf{P}(N \oplus \mathcal{O}_S) = \mathbf{P}(N \oplus L'_\infty|_{S_\infty})$ with action weight -1. □

Let $r \geqslant 1$ be an integer. Let W_r be the rth root construction of W along $F_\infty \cong \widehat{X}$ (cf. [Cad07, § 2] and [AGV08, Appendix B]), which is the same as the weight-r blowup of W along F_∞ (cf. [MM12, CDH19]). The root construction/weighted blowup is essentially derived from the construction of root of line bundles. In fact, locally near F_∞, W is isomorphic to L_∞, and W_r is obtained by replacing L_∞ with its rth root. Here we take L_∞ as an example to briefly recall the

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construction of root of line bundles (cf. [AGV08, Appendix B] and [CDW22, § 2.1.1]). Let L^*_∞ be L_∞ minus the zero section F_∞, i.e. the $\mathbf{C}^*$-principal bundle of L_∞. Then

$$L_\infty = L^*_\infty \times_{\mathbf{C}^*(-1,1)} \mathbf{C}$$

and its zero section F_∞ is $L^*_\infty/\mathbf{C}^*$. Consider the following $\mathbf{Z}_r$-extension

$$1 \longrightarrow \mathbf{Z}_r \hookrightarrow \mathbf{C}^* \xrightarrow{(-)^r} \mathbf{C}^* \longrightarrow 1$$

of $\mathbf{C}^*$. The rth root of L_∞ is

$$\sqrt[r]{L_\infty} = L^*_\infty \times_{\mathbf{C}^*(-r,1)} \mathbf{C}.$$

The zero section $[L^*_\infty/\mathbf{C}^*(r)]$ of $\sqrt[r]{L_\infty}$ is called the rth root gerbe (cf. [AJT15, § 2.2]) of L_∞, and is denoted by $\sqrt[r]{L_\infty/F_\infty}$ in the literature. In this paper, we simply denote it by $\sqrt[r]{F_\infty}$, as we only consider the line bundle L_∞ over F_∞.

We have a natural projection $W_r \to W$. The $\sqrt[r]{F_\infty}$ is the divisor of W_r lying over F_∞ and its normal line bundle in W_r is $\sqrt[r]{L_\infty}$. The divisor of W_r lying over D is the rth root construction D_r of D along its infinity divisor $Z = \mathbf{P}(N)$. In fact,

$$D_r = \mathbf{P}_{(r,\dots,r,1)}(N \oplus \mathcal{O}_S)$$

is the weight-$(r,\dots,r,1)$ projectivization (cf. [CDH19]) of $N \oplus \mathcal{O}_S$, and the normal line bundle of D_r in W_r is $\mathcal{O}_{D_r}(-r)$, the pull back of the tautological line bundle $\mathcal{O}_D(-1)$ via the natural projection $D_r \to D$.

Similarly, with respect to the projection $D_r \to D$, the divisor of D_r lying over Z is the rth root gerbe $\sqrt[r]{Z}$ of $L_\infty|_Z \to Z$, which is $\mathbf{P}_{(r,\dots,r)}(N)$, the weight-$(r,\dots,r)$ projectivization of N. Then the $\sqrt[r]{F_\infty}$ intersects with D_r transversally along the $\sqrt[r]{Z}$. Again, the $\mathbf{C}^*$-action lifts to W_r.

Lemma 2.2. *The fixed locus $W_r^{\mathbf{C}^*}$ of W_r with respect to the induced $\mathbf{C}^*$-action consists of three disjoint components $F_0 \cong X$, $\sqrt[r]{F_\infty} \cong \sqrt[r]{\widehat{X}}$ and $F_* = S_* \cong S$. The normal bundles of F_0 and F_* in W_r are the same as their normal bundles in W. The normal bundle of $\sqrt[r]{F_\infty}$ in W_r is $\sqrt[r]{L_\infty}$ with action weight $1/r$.*

As an orbifold, the inertia space (see Remark 3.3) IW_r of W_r is a disjoint union of the untwisted sector W_r together with $r-1$ twisted sectors, indexed by $e^{2\pi i(j/r)}$, $1 \leqslant j \leqslant r-1$, each of which is a copy of $\sqrt[r]{F_\infty} \cong \sqrt[r]{\widehat{X}}$. We write IW_r as

$$IW_r = W_r \sqcup \bigsqcup_{1\leqslant j\leqslant r-1} \sqrt[r]{F_\infty}[j], \tag{9}$$

where for each j a point in the twisted sector $\sqrt[r]{F_\infty}[j]$ consists of a pair $(x, e^{2\pi i(j/r)})$, $x \in F_\infty$. The group element $e^{2\pi i(j/r)}$ acts on the fiber of $\sqrt[r]{L_\infty}$ by multiplication. Since $\sqrt[r]{Z} \subseteq \sqrt[r]{F_\infty}$, we also have $\sqrt[r]{Z}[j] \subseteq \sqrt[r]{F_\infty}[j]$ for $1 \leqslant j \leqslant r-1$.

2.2 The blowup formula

2.2.1 *The statement of the formula.* Let $\overline{\mathscr{M}}_{0,n,A}(X)$ be the moduli space of genus zero, degree $A \in H_2(X;\mathbf{Z})$, n marked, stable maps into X. Let $\alpha_j \in H^*(X)$ for $1 \leqslant j \leqslant n$. A genus zero Gromov–Witten invariant of X is

$$\langle \alpha_1, \dots, \alpha_n \rangle^X_{0,n,A} := \int_{[\overline{\mathscr{M}}_{0,n,A}(X)]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(\alpha_j),$$

where ev_j is the jth evaluation map associated to the jth marking.

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Recall that $p\colon \widehat{X} \to X$ is the blowup projection. Then we have the following Gromov–Witten invariant of $\widehat{X}$

$$\langle p^*\alpha_1, \ldots, p^*\alpha_n \rangle^{\widehat{X}}_{0,n,p^!A} := \int_{[\overline{\mathscr{M}}_{0,n,p^!A}(\widehat{X})]^{\mathrm{vir}}} \prod_{j=1}^{n} ev_j^*(p^*\alpha_j).$$

Our first main result on the blowup formula of Gromov–Witten invariants is stated as follows.

THEOREM 2.3. *Let X be a smooth projective variety and $S \subseteq X$ be a smooth subvariety. Let $\widehat{X}$ be the blowup of X along S. Suppose the normal bundle N of S in X is positive (see Definition 1.1) and S has codimension $\kappa \geqslant 2$. Then we have the following equality of genus zero Gromov–Witten invariants*

$$\langle \alpha_1, \ldots, \alpha_n \rangle^{X}_{0,n,A} = \langle p^*\alpha_1, \ldots, p^*\alpha_n \rangle^{\widehat{X}}_{0,n,p^!A}. \tag{10}$$

Remark 2.4. The virtual dimension of $\overline{\mathscr{M}}_{0,n,A}(X)$ is the same as that of $\overline{\mathscr{M}}_{0,n,p^!A}(\widehat{X})$. Therefore, *a priori*, we assume that the total degree of insertions in (10) matches the dimension constraint, i.e.

$$\frac{1}{2}\sum_{j=1}^{n} \deg \alpha_j = \operatorname{vdim} \overline{\mathscr{M}}_{0,n,A}(X). \tag{11}$$

In fact, when the dimension constraint is not satisfied, both Gromov–Witten invariants in (10) are zero and the blowup formula (10) holds trivially.

Remark 2.5 (Remark on psi-class). Here the blowup formula (10) deals with primary Gromov–Witten invariants. The computation in § 2.4 shows that this formula also holds for certain descendent invariants, as we now explain. First, we have the natural projection

$$\pi\colon \overline{\mathscr{M}}_{0,n,p^!A}(\widehat{X}) \to \overline{\mathscr{M}}_{0,n,A}(X).$$

Let $\bar{\psi}_j$ be the psi-class of $\overline{\mathscr{M}}_{0,n,A}(X)$ associated to the jth marking, and $\pi^*\bar{\psi}_j$ be the pullback class. Then under the assumption of Theorem 2.3, we have the equality

$$\langle \bar{\tau}_{d_1}\alpha_1, \ldots, \bar{\tau}_{d_n}\alpha_n \rangle^{X}_{0,n,A} = \langle \bar{\tau}_{d_1}p^*\alpha_1, \ldots, \bar{\tau}_{d_n}p^*\alpha_n \rangle^{\widehat{X}}_{0,n,p^!A},$$

where $\bar{\tau}_{d_j}$ means the additional insertion $\bar{\psi}_j^{d_j}$ for X and $\pi^*\bar{\psi}_j^{d_j}$ for $\widehat{X}$.

2.2.2 *Proof of the blowup formula.* We prove the blowup formula modulo Propositions 2.8, 2.9 and 2.10, which are proved in § 2.4. The main ingredient is the vanishing of a certain Gromov–Witten invariant of W_r for all r.

Viewing A as a homology class of $X \times \mathbf{P}^1$ via the inclusion $X \to X \times \{0\} \subseteq X \times \mathbf{P}^1$, we get a homology class $p_W^! A \in H_2(W;\mathbf{Z})$ with $p_W\colon W \to X \times \mathbf{P}^1$ the blowup projection. Recall that W_r is the rth root stack of W along F_∞. Its coarse space is W. Consider the moduli space $\overline{\mathscr{M}}_{0,n,p_W^!A}(W_r)$ of genus zero, degree $p_W^!A$, orbifold stable maps into W_r with n smooth markings (cf. [CR02, AGV08]), and the following Gromov–Witten invariant of W_r

$$\langle q^*\alpha_1, \ldots, q^*\alpha_n \rangle^{W_r}_{0,n,p_W^!A} := \int_{[\overline{\mathscr{M}}_{0,n,p_W^!A}(W_r)]^{\mathrm{vir}}} \prod_{j=1}^{n} ev_j^*(q^*\alpha_j), \tag{12}$$

where q is the projection $q\colon W_r \to W \to X \times \mathbf{P}^1 \to X$.

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Remark 2.6. We will compute the integration (12) by the virtual localization technique (cf. [GP99, Liu13]). Since the projection $q\colon W_r \to X$ is equivariant with respect to the $\mathbf{C}^*$-action, where the action on the target X is trivial, $q^*\alpha_j$ is also an equivariant cohomology class. Thus, the integration (12) can be treated as equivariant integration.

Lemma 2.7. *Under the dimension constraint assumption* (11)*, the Gromov–Witten invariant* (12) *vanishes for all* $r \in \mathbf{Z}_{\geqslant 1}$.

Proof. Note that $c_1^{W_r}(p_W^! A) = c_1^X(A)$, $\operatorname{vdim} \overline{\mathscr{M}}_{0,n,p_W^! A}(W_r) = \operatorname{vdim} \overline{\mathscr{M}}_{0,n,A}(X) + 1$. Therefore, by the dimension constraint assumption (11), we have

$$\frac{1}{2}\sum_{j=1}^{n} \deg \alpha_j = \operatorname{vdim} \overline{\mathscr{M}}_{0,n,A}(X) < \operatorname{vdim} \overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$$

and, consequently, the Gromov–Witten invariant (12) vanishes. □

This lemma says that the Gromov–Witten invariant (12) vanishes. By Remark 2.6, we will compute this vanishing invariant by the virtual localization technique via the $\mathbf{C}^*$-action on W_r. We denote by t the equivariant parameter of the $\mathbf{C}^*$-action.

By the virtual localization technique, the invariant (12) is contributed from the fixed locus of the $\mathbf{C}^*$-action on $\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$. As usual, each component of the fixed locus of $\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$ is indexed by a decorated graph Φ and the component is denoted by $\overline{\mathscr{M}}_\Phi$. Then the total contributions to the invariant (12) of all components of the fixed locus of $\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$ are

$$0 = \int_{[\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)]^{\mathrm{vir}}} \prod_{j=1}^{n} ev_j^*(q^*\alpha_j) = \sum_{\Phi} \int_{\overline{\mathscr{M}}_\Phi} \prod_{j=1}^{n} ev_j^*(q^*\alpha_j) \cdot \frac{1}{e_{\mathbf{C}^*}(\mathscr{N}_\Phi)} =: \sum_{\Phi} \operatorname{Cont}(\Phi),$$

where $\mathscr{N}_\Phi$ is the virtual normal bundle of $\overline{\mathscr{M}}_\Phi$ in $\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$.

All components $\overline{\mathscr{M}}_\Phi$ are described explicitly in § 2.3. The contribution $\operatorname{Cont}(\Phi)$ of each component $\overline{\mathscr{M}}_\Phi$ to the invariant (12) is computed in § 2.4. In particular, there are two special graphs, Φ_0 and Φ_∞, for which every stable map in $\overline{\mathscr{M}}_{\Phi_0}$ (and $\overline{\mathscr{M}}_{\Phi_\infty}$, respectively) has images in F_0 (and $\sqrt[r]{F_\infty}$, respectively). We will prove the following propositions in §§ 2.4.1, 2.4.2 and 2.4.3, respectively.

Proposition 2.8. *The contribution of* $\overline{\mathscr{M}}_{\Phi_0}$ *to the invariant* (12) *is*

$$\operatorname{Cont}(\Phi_0) = -\frac{1}{t} \cdot \langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,A}^X.$$

Proposition 2.9. *The contribution of* $\overline{\mathscr{M}}_{\Phi_\infty}$ *to the invariant* (12) *is*

$$\operatorname{Cont}(\Phi_\infty) = \frac{1}{t} \cdot \langle p^*\alpha_1, \ldots, p^*\alpha_n \rangle_{0,n,p^! A}^{\widehat{X}}.$$

Proposition 2.10. *Suppose the normal bundle N of S in X is positive and S has codimension $\kappa \geqslant 2$. Then for a decorated graph Φ other than Φ_0 and Φ_∞, under the transformation $\mathfrak{s} = tr$, the contribution* $\operatorname{Cont}(\Phi)$ *of* $\overline{\mathscr{M}}_\Phi$ *to the invariant* (12) *is a polynomial in r with lowest degree at least 2 when $r \gg 1$.*

With these three propositions, we finish the proof of Theorem 2.3.

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Proof of Theorem 2.3. Under the assumption of the theorem, by Propositions 2.8, 2.9 and 2.10 we see that, after the transformation $\mathfrak{s} = tr$, when $r \gg 1$,

$$\begin{aligned} 0 &= \mathrm{Cont}(\Phi_0) + \mathrm{Cont}(\Phi_\infty) + \sum_{\Phi \neq \Phi_0, \Phi_\infty} \mathrm{Cont}(\Phi) \\ &= \frac{r}{\mathfrak{s}}\left(\left\langle p^*\alpha_1, \dots, p^*\alpha_n\right\rangle^{\widehat{X}}_{0,n,p^!A} - \left\langle \alpha_1, \dots, \alpha_n\right\rangle^{X}_{0,n,A}\right) + \sum_{\Phi \neq \Phi_0, \Phi_\infty} \mathrm{Cont}(\Phi) \end{aligned}$$

with the right-hand side being a polynomial in r. Thus, the vanishing of the coefficient of r on the right-hand side implies the blowup formula. □

Remark 2.11. To consider the blowup formulae for a higher genus Gromov–Witten invariant $\left\langle \alpha_1, \dots, \alpha_n\right\rangle^{X}_{g,n,A}$, $g \geqslant 1$, we choose a suitable line bundle $L \to X$, such that $\left\langle c_1(L), A\right\rangle = gK$ for some $K \in \mathbf{Z}_{\geqslant 1}$, to construct the geometric model W and W_r. Then we also use localization technique to compute a vanishing invariant of W_r similar to (12) for $r \gg 1$. However, it requires a more delicate work to extract $\left\langle \alpha_1, \dots, \alpha_n\right\rangle^{X}_{g,n,A}$ and $\left\langle p^*\alpha_1, \dots, p^*\alpha_n\right\rangle^{\widehat{X}}_{g,n,p^!A}$ from the localization computation. Some certain integrals over DR-cycles associated to (X, L) also appear naturally in the localization computation. This will be addressed in a further paper.

2.3 Description of $\overline{\mathscr{M}}_\Phi$ for $r \gg 1$

2.3.1 *Decorated graphs.* It is standard to associate connected components of fixed locus of $\overline{\mathscr{M}}_{0,n,p^!_W A}(W_r)$ with decorated graphs. See, for example, [Liu13]. We assume $r \gg 1$ in this subsection.

Recall that the fixed locus $W_r^{\mathbf{C}^*}$ consists of the disjoint union of $F_0 \cong X$, $\sqrt[r]{F_\infty} \cong \sqrt[r]{\widehat{X}}$ and $F_* \cong S$, whose coarse spaces are $|F_0| = F_0$, $|\sqrt[r]{F_\infty}| = F_\infty$ and $|F_*| = F_*$. Now consider a $\mathbf{C}^*$-fixed stable map $f\colon (C, x_1, \dots, x_n) \to W_r$ in $\overline{\mathscr{M}}_{0,n,p^!_W A}(W_r)$. We associate it with a decorated graph Φ which consists of the set $V(\Phi)$ of vertices with labels, the set $E(\Phi)$ of edges with labels, and the set of n tails $T = \{\iota_1, \dots, \iota_n\}$. The decorated graph Φ is described as the following.

Vertices. Assign a vertex v to a connected component C_v in $f^{-1}(W_r^{\mathbf{C}^*})$. Moreover, for each v we associate it with the following labels:

- label $c_v = 0$, ∞ or $*$ depending on whether $|f(C_v)| \subseteq F_0, F_\infty$ or F_*, respectively;
- label $A_v = f_*[|C_v|] \in H_2(F_{c_v}; \mathbf{Z})$, the homology class of $f_*([|C_v|])$;
- label $T_v \subseteq T$, the set of markings on C_v.

We define three subsets V_0, V_∞ and V_* of $V(\Phi)$ by $V_\bullet := \{v \mid c_v = \bullet\}$.

Edges. Assign an edge e to each connected component of $C \setminus \bigcup_{v \in V(\Gamma)} C_v$. Let C_e be the closure of the corresponding component. Then C_e is an orbifold $\mathbf{P}^1$ and the map $f\colon C_e \to D_r$ is a branched covering of a fiber of $D_r \to S_*$ ramified over S_* and $\sqrt[r]{Z}$. We assume $0 = f^{-1}(S_*)$, $\infty = f^{-1}(I\sqrt[r]{Z})$. We associate e the following labels:

- label d_e to be the degree of f, i.e. $f_*[|C_e|] = d_e[F] \in H_2(D; \mathbf{Z})$, where $[F]$ is the fiber class of $D \to S_*$;
- label twisted sectors $\sqrt[r]{Z}[d_e]$ of $\sqrt[r]{Z}$, which is the image of ∞; here $d_e < r$ since $r \gg 1$.

As previous subsection, for each decorated graph Φ, we denote by $\overline{\mathscr{M}}_\Phi$ the corresponding component of the fixed locus of $\overline{\mathscr{M}}_{0,n,p^!_W A}(W_r)$ and the virtual normal bundle of $\overline{\mathscr{M}}_\Phi$ by $\mathscr{N}_\Phi$. Denote by $\mathrm{Aut}(\Phi)$ the group of automorphism of Φ that fixes all labels.

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Definition 2.12. For a decorated graph Φ, a vertex $v \in V(\Phi)$ is called stable if $val(v) + n_v \geqslant 3$ or $A_v \neq 0$, where $val(v)$ is the number of edges adjacent to v and $n_v = |T_v|$ is the number of markings on C_v. Let $V^S(\Phi)$ be the set of stable vertices in $V(\Phi)$, and $V^U(\Phi)$ be the set of unstable vertices in $V(\Phi)$.

The set of unstable vertices $V^U(\Phi)$ is the union of the following three types of vertices:

$$V^1(\Phi) = \{v \in V(\Phi) \mid val(v) = 1,\ n_v = 0,\ A_v = 0\},$$
$$V^{1,1}(\Phi) = \{v \in V(\Phi) \mid val(v) = n_v = 1,\ A_v = 0\},$$
$$V^2(\Phi) = \{v \in V(\Phi) \mid val(v) = 2,\ n_v = 0,\ A_v = 0\}.$$

Lemma 2.13. *When $r \gg 1$, for a decorated graph Φ, we have $V_\infty \subseteq V^S(\Phi)$ or, equivalently, $V^U(\Phi) \subseteq V_*$.*

Proof. We show that any v with $c_v = \infty$ must be stable. Let v be a vertex with $c_v = \infty$. If it is not stable, then v belongs to $V^1(\Phi), V^{1,1}(\Phi)$ or $V^2(\Phi)$.

If $v \in V^1(\Phi) \sqcup V^{1,1}(\Phi)$, let e be the unique edge adjacent to v. This e then corresponds to an orbifold map $f : C_e \to D_r$ with C_v being the point $\infty = f^{-1}(I\sqrt[r]{Z})$ in C_e. Then we see that $C_v = \infty$ is mapped into $\sqrt[r]{Z}[d_e]$ since $r \gg 1$ and, hence, is an orbifold marking over the domain curve C. This contradicts to the assumption that stable maps in $\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$ contain only smooth markings.

Similarly, an unstable vertex $v \in V^2(\Phi)$ with $c_v = \infty$ must correspond to a balanced orbifold nodal point whose two branches are mapped into $\sqrt[r]{Z}[d_{e_1}]$ and $\sqrt[r]{Z}[d_{e_2}]$, respectively, where e_1 and e_2 are the only two edges adjacent to v. Since for a balanced orbifold nodal point, its two branches must be mapped to two twisted sectors of $\sqrt[r]{Z}$ decorated by two elements in μ_r that are inverses of each other, therefore we must have $e^{2\pi i(d_{e_1}/r)} \cdot e^{2\pi i(d_{e_2}/r)} = 1$, i.e.

$$d_{e_1} + d_{e_2} \equiv 0 \pmod{r},$$

which is impossible when $r \gg 1$. This finishes the proof. □

Definition 2.14. Let Φ_0 (and Φ_∞, respectively) denote the graph consisting of one single vertex v with $c_v = 0$ (and $c_v = \infty$, respectively). Then

$$\overline{\mathscr{M}}_{\Phi_0} = \overline{\mathscr{M}}_{0,n,A}(F_0), \quad \text{and} \quad \overline{\mathscr{M}}_{\Phi_\infty} = \overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty}).$$

Now consider a graph Φ other than Φ_0, Φ_∞. Since the homology class is $p_W^! A$, there is no vertex $v \in V(\Phi)$ with $c_v = 0$. For this graph Φ, suppose

$$V_\infty = \{u_1, \dots, u_{\mathfrak{p}}\}, \quad V_* = \{v_1, \dots, v_{\mathfrak{q}}\},$$

and the set of edges is

$$E = \{e_{jk} \mid 1 \leqslant j \leqslant \mathfrak{p},\ 1 \leqslant k \leqslant \mathfrak{q}\}.$$

Set the degree of e_{jk} to be d_{jk}. We remark that V_∞ may be empty.

2.3.2 *Explicit expression of $\overline{\mathscr{M}}_\Phi$.* Roughly speaking, $\overline{\mathscr{M}}_\Phi$ is certain (fiber) products of moduli spaces associated to vertices and edges, which we denote by $\overline{\mathscr{M}}_v$ and $\mathscr{F}_e$, respectively.

The moduli space $\overline{\mathscr{M}}_u$ for each vertex $u \in V_\infty$. The labels on u are $c_u = \infty$, A_u and $T_u = \{\iota_{u1}, \dots, \iota_{ua}\}$. Suppose the set of edges adjacent to u is $E_u = \{e_{ul_1}, \dots, e_{ul_b}\}$. Then E_u provides b nodal points on the domain curve C whose branches on C_u are viewed as (unordered)

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orbifold markings. These orbifold markings on C_u are mapped into the twisted sectors

$$\sqrt[r]{Z}[r-d_{ul_k}] \subseteq \sqrt[r]{F_\infty}[r-d_{ul_k}]$$

for $1 \leqslant k \leqslant b$. Thus, we get

$$\overline{\mathscr{M}}_u := \overline{\mathscr{M}}_{0,T_u,E_u,A_u}(\sqrt[r]{F_\infty}) \cap \bigwedge_{k=1}^{b} [\sqrt[r]{Z}[r-d_{ul_k}]],$$

where $\overline{\mathscr{M}}_{0,T_u,E_u,A_u}(\sqrt[r]{F_\infty})$ is the moduli space of genus zero stable maps in $\sqrt[r]{F_\infty}$ with original T_u-markings and E_u (unordered) orbifold markings, $[\sqrt[r]{Z}[r-d_{ul_k}]]$ is the class of $\sqrt[r]{Z}[r-d_{ul_k}]$ in $\sqrt[r]{F_\infty}[r-d_{ul_k}]$ which equals to the pullback of the class of Z in F_∞ via the natural projection $\sqrt[r]{F_\infty}[r-m_{ul_k}] \to F_\infty$.

Lemma 2.15. *For each edge $e_{ul_k} \in E_u$, there is a natural map*

$$\widetilde{ev}_{ul_k} : \overline{\mathscr{M}}_u \to F_* = S_*$$

induced from the evaluation map $ev_{ul_k} : \overline{\mathscr{M}}_{0,T_u,E_u,A_u}(\sqrt[r]{F_\infty}) \to I\sqrt[r]{F_\infty}$.

Proof. Note that after taking intersection with $[\sqrt[r]{Z}[r-d_{ul_k}]]$, the image of evaluation map ev_{ul_k} is $\sqrt[r]{Z}[r-d_{ul_k}]$. Then by composing $ev_{d_{ul_k}}$ with the projection $\sqrt[r]{Z}[r-d_{ul_k}] \to F_* = S_*$, we get $\widetilde{ev}_{ul_k}$. □

The moduli space $\overline{\mathscr{M}}_v$ for each vertex $v \in V_*$. If v is unstable, then $\overline{\mathscr{M}}_v = F_* = S_*$. Now suppose v is stable, and is labelled by $c_v = *$, A_v and $T_v = \{\iota_{v1}, \dots, \iota_{va}\}$. Similarly, suppose the set of edges adjacent to v is $E_v = \{e_{l_1 v}, \dots, e_{l_b v}\}$. Then we get a moduli space

$$\overline{\mathscr{M}}_v := \overline{\mathscr{M}}_{0,T_v,E_v,A_v}(F_*) = \overline{\mathscr{M}}_{0,T_v,E_v,A_v}(S_*).$$

Here T_v stands for the original marked points on C_v and E_v stands for the unordered markings on C_v from the nodal points determined by edges in E_v as above.

The moduli space $\mathscr{F}_e$ for each edge $e \in E$. Let $e = e_{jk}$ be the edge connecting $u_j \in V_\infty$ and $v_k \in V_*$ with degree d_{jk}. If $v_k \notin V^1(\Phi)$, $\mathscr{F}_{e_{jk}}$ is the fixed locus of

$$\overline{\mathscr{M}}_{0,1,[d_{jk}],d_{jk}[F]}(D_r) \subseteq \overline{\mathscr{M}}_{0,1,[d_{jk}],d_{jk}[F]}(W_r),$$

where 0 is the genus, 1 is the number of smooth markings, $[d_{jk}]$ indicates the twisted sector $\sqrt[r]{Z}[d_{jk}]$ of the unique orbifold marking, $d_{jk}[F]$ is the degree of maps. If $v_k \in V^1(\Phi)$, $\mathscr{F}_{e_{jk}}$ is the fix locus of

$$\overline{\mathscr{M}}_{0,[d_{jk}],d_{jk}[F]}(D_r) \subseteq \overline{\mathscr{M}}_{0,[d_{jk}],d_{jk}[F]}(W_r).$$

Furthermore, $\mathscr{F}_{e_{jk}}$ is a fibration over S_* with fiber being the $\mathbf{C}^*$-fixed locus of the moduli space stable maps into $\mathbf{P}_{(r,\dots,r,1)}(\mathbf{C}^{\kappa+1})$ with topological data $(0,1,[d_{jk}],d_{jk}[F])$ or $(0,[d_{jk}],d_{jk}[F])$. This fixed locus $\mathscr{F}_{e_{jk}}$ is determined by the marking mapped into $\sqrt[r]{Z}[d_{jk}]$. It is straightforward to see that $\mathscr{F}_{e_{jk}}$ is a $\mathbf{Z}_{d_{jk}}$-gerbe over Z.

The moduli space $\overline{\mathscr{M}}_\Phi$

Proposition 2.16. *For a graph Φ other than Φ_0 and Φ_∞, set $d_\Phi := \prod_{e\in E} d_e$. Then*

$$\overline{\mathscr{M}}_\Phi = \frac{r^{|E|}}{d_\Phi \cdot |\mathrm{Aut}(\Phi)|} \cdot \prod_{u_j \in V_\infty} \overline{\mathscr{M}}_{u_j} \times_{S_*^{|E|}} \prod_{v_k \in V_*} \overline{\mathscr{M}}_{v_k}. \tag{13}$$

Here we use the map defined in Lemma 2.15 for the fiber product.

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Proof. First, we have

$$\overline{\mathscr{M}}_{\Phi} = \frac{1}{|\mathrm{Aut}(\Phi)|} \cdot \prod_{u_j \in V_\infty} \overline{\mathscr{M}}_{u_j} \times_{(I\sqrt[r]{Z})^{|E|}} \prod_{e_{jk}\in E} \mathscr{F}_{e_{jk}} \times_{S_*^{|E|}} \prod_{v_k \in V_*} \overline{\mathscr{M}}_{v_k} \tag{14}$$

where $I\sqrt[r]{Z}$ is the inertia space of $\sqrt[r]{Z}$.

Let $\{\beta_l\}$ be a basis of $H^*(F_*) = H^*(S_*)$ and $\{\check{\beta}_l\}$ be its dual basis. Let H be the hyperplane class of $Z \to S_*$, then

$$\{\gamma_{lk} = \beta_l H^k \mid 1 \leqslant k \leqslant \kappa - 1\}$$

is a basis of $H^*(Z)$ with $\{\check{\gamma}_{lk}\}$ being its dual basis. We now simplify (14). For simplicity, we only consider the case $V_\infty = \{u\}$, $V_* = \{v\}$, $E = \{e_{uv}\}$, and omit the factor $1/|\mathrm{Aut}(\Phi)|$. For the general cases, the computation is identical. From (14), we have

$$\overline{\mathscr{M}}_{\Phi} = r \cdot \sum_{\gamma_{lk}, \beta_{l'}} ([\overline{\mathscr{M}}_u \cap ev_{uv}^* \check{\gamma}_{lk}] \cdot [\mathscr{F}_{e_{uv}} \cap (\gamma_{lk} \cup \check{\beta}_{l'})] \cdot [\overline{\mathscr{M}}_v \cap ev_{uv}^* \beta_{l'}]),$$

where r comes from the orbifold Poincaré dual of $\sqrt[r]{Z}$. Since $\mathscr{F}_{e_{uv}}$ is a $\mathbf{Z}_{d_{e_{uv}}}$-gerbe over Z, only when $\gamma_{lk} = \gamma_{l'(\kappa-1)} = \beta_{l'} H^{\kappa-1}$ the term $[\mathscr{F}_{e_{uv}} \cap (\gamma_{lk} \cup \check{\beta}_{l'})]$ is nonzero, and contributes $1/d_{e_{uv}}$. Therefore,

$$\overline{\mathscr{M}}_{\Phi} = \frac{r}{d_{e_{uv}}} \sum_{\beta_{l'}} [\overline{\mathscr{M}}_u \cap \widetilde{ev}_{uv}^* \check{\beta}_{l'}] \cdot [\overline{\mathscr{M}}_v \cap ev_{uv}^* \beta_{l'}] = \frac{r}{d_{e_{uv}}} \cdot (\overline{\mathscr{M}}_u \times_{S_*} \overline{\mathscr{M}}_v).$$

Here, we use the facts that $\check{\gamma}_{lk} = \check{\beta}_{l'}$ when $\gamma_{lk} = \beta_{l'} H^{\kappa-1}$, and $ev_{uv}^* \check{\gamma}_{lk} = ev_{uv}^* \check{\beta}_{l'} = \widetilde{ev}_{uv}^* \check{\beta}_{l'}$. □

2.4 Contributions Cont(Φ)

We prove Propositions 2.8, 2.9 and 2.10 in this subsection.

2.4.1 *Proof of Proposition 2.8.* For the graph Φ_0 we have $\overline{\mathscr{M}}_{\Phi_0} = \overline{\mathscr{M}}_{0,n,A}(F_0)$. Let $\mathcal{C} \to \overline{\mathscr{M}}_{\Phi_0}$ be the universal curve and $f\colon \mathcal{C} \to F_0$ be the universal map. The virtual normal bundle of $\overline{\mathscr{M}}_{\Phi_0}$ in $\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$ is the induced index bundle $\mathcal{R}\pi_* f^* L_0$ by the normal line bundle L_0 of $F_0 \cong X$ in W_r. We denote this index bundle simply by $(L_0)_{\Phi_0}$. It is a line bundle with action weight -1. Thus, since $(q^*\alpha_j)|_{F_0} = \alpha_j$, by the dimension constraint (11), the contribution of Φ_0 is

$$\begin{aligned}\mathrm{Cont}(\Phi_0) &= \int_{[\overline{\mathscr{M}}_{0,n,A}(F_0)]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*((q^*\alpha_j)|_{F_0})}{c_1((L_0)_{\Phi_0} \otimes \mathcal{O}(-1))} = \int_{[\overline{\mathscr{M}}_{0,n,A}(F_0)]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*((q^*\alpha_j)|_{F_0})}{-t + c_1((L_0)_{\Phi_0})} \\ &= \frac{1}{-t} \cdot \int_{[\overline{\mathscr{M}}_{0,n,A}(F_0)]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(\alpha_j) = -\frac{1}{t} \cdot \langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,A}^X.\end{aligned}$$

This finishes the proof of Proposition 2.8.

2.4.2 *Proof of Proposition 2.9.* For the graph Φ_∞, we have $\overline{\mathscr{M}}_{\Phi_\infty} = \overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty})$. The virtual normal bundle of $\overline{\mathscr{M}}_{\Phi_\infty}$ in $\overline{\mathscr{M}}_{0,n,p_W^! A}(W_r)$ is the induced index bundle $\mathcal{R}\pi_* f^* \sqrt[r]{L_\infty}$ by the normal bundle of $\sqrt[r]{F_\infty}$ in W_r, i.e. the rth root $\sqrt[r]{L_\infty}$ of the line bundle L_∞. We denote this index bundle simply by $(\sqrt[r]{L_\infty})_{\Phi_\infty}$. It is a line bundle with action weight $1/r$ since $r \gg 1$.

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Moreover, we have the following commutative diagram

$$\begin{array}{ccccc} \overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty}) & \xrightarrow{ev_j} & \sqrt[r]{F_\infty} & & \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \pi} & \searrow^{q|_{\sqrt[r]{F_\infty}}} & \\ \overline{\mathscr{M}}_{0,n,p^! A}(F_\infty) & \xrightarrow{ev_j} & F_\infty & \xrightarrow{p} & X \end{array}$$

and $q^*\alpha_j|_{\sqrt[r]{F_\infty}} = (q|_{\sqrt[r]{F_\infty}})^*\alpha_j = \pi^* p^*(\alpha_j)$. Thus, the contribution of Φ_∞ is

$$\begin{aligned} \mathrm{Cont}(\Phi_\infty) &= \int_{[\overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty})]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*((q^*\alpha_j)|_{\sqrt[r]{F_\infty}})}{c_1\big((\sqrt[r]{L_\infty})_{\Phi_\infty} \otimes \mathcal{O}(1/r)\big)} \\ &= \int_{[\overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty})]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*(\pi^* p^*(\alpha_j))}{t/r + c_1((\sqrt[s]{L_\infty})_{\Phi_\infty})} \\ &= \frac{r}{t} \cdot \int_{[\overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty})]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(\pi^* p^*(\alpha_j)) \\ &= \frac{r}{t} \cdot \frac{1}{r} \cdot \int_{[\overline{\mathscr{M}}_{0,n,p^! A}(F_\infty)]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(p^*\alpha_j) \\ &= \frac{1}{t} \cdot \langle p^*\alpha_1, \ldots, p^*\alpha_n \rangle_{0,n,p^! A}^{\widehat{X}}, \end{aligned}$$

where the third equality follows from the dimension constraint assumption (11) and

$$\mathrm{vdim}\, \overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty}) = \mathrm{vdim}\, \overline{\mathscr{M}}_{0,n,p^! A}(F_\infty) = \mathrm{vdim}\, \overline{\mathscr{M}}_{0,n,A}(X),$$

and the fourth equality follows from the computation of Gromov–Witten invariants of root gerbes in [AJT15, Theorem 4.3], i.e. $\epsilon_*([\overline{\mathscr{M}}_{0,n,p^! A}(\sqrt[r]{F_\infty})]^{\mathrm{vir}}) = (1/r)[\overline{\mathscr{M}}_{0,n,p^! A}(F_\infty)]^{\mathrm{vir}}$. This finishes the proof of Proposition 2.9.

2.4.3 *Proof of Proposition 2.10.* We now consider a general graph Φ other than Φ_0 and Φ_∞. For such a graph, the contribution is

$$\mathrm{Cont}(\Phi) = \frac{1}{|\mathrm{Aut}(\Phi)|} \cdot \prod_{j=1} ev_j^*(q^*\alpha_j) \cap \frac{[\overline{\mathscr{M}}_\Phi]^{\mathrm{vir}}}{e_{\mathbf{C}^*}(\mathscr{N}_\Phi)} = \frac{1}{|\mathrm{Aut}(\Phi)|} \cdot \prod_{j=1} ev_j^*(q^*\alpha_j) \cap \epsilon_*\bigg(\frac{[\overline{\mathscr{M}}_\Phi]^{\mathrm{vir}}}{e_{\mathbf{C}^*}(\mathscr{N}_\Phi)}\bigg),$$

with $\epsilon\colon \overline{\mathscr{M}}_{0,n,p_W^! A}(W_r) \to \overline{\mathscr{M}}_{0,n,p_W^! A}(W)$ the natural projection.

As usual, the contribution from the virtual normal bundle can be decomposed into a combination of contributions of vertices, edges and nodal points. Before we give explicit computations, we introduce some notation:

– the set V_* is decomposed into

$$V_* = V_*^S \sqcup V_*^1 \sqcup V_*^{1,1} \sqcup V_*^2,$$

the disjoint union of stable vertices and three types of unstable vertices (see Definition 2.12);

– the set E of edges is decomposed into

$$E = E^S \sqcup E^1 \sqcup E^{1,1} \sqcup E^2$$

according to the vertices in V_* that the edges are adjacent to;

– nodal points attached to $C_u, u \in V_\infty$ are denoted by n with lower indices; nodal points attached to $C_v, v \in V_*$ are denoted by $\bar{\mathsf{n}}$ with lower indices.

(1) *Contributions from stable vertices.* For a stable vertex $v \in V^S(\Phi)$, $\overline{\mathscr{M}}_v$ is a moduli space of stable maps into the fixed component F_0, $\sqrt[r]{F_\infty}$ or $F_* \subseteq W_r$ according to $c_v = 0$, ∞ or $*$. Let $\mathscr{N}_v$ be the virtual normal bundle of $\overline{\mathscr{M}}_v$ in the moduli space of stable maps into W_r with the same topological data. By the contribution of a stable vertex we mean the equivariant Euler class of $\mathscr{N}_v$. The explicit formula for the contribution of each stable vertex is given as follows.

(1a) If $u \in V_\infty$,

$$\mathrm{cont}_u := e_{\mathbf{C}^*}((\mathcal{R}\pi_* f^* \sqrt[r]{L_\infty})_u)^{-1} = c_{\mathrm{rk}(u)}\left(- (\mathcal{R}\pi_* f^* \sqrt[r]{L_\infty})_u \otimes \mathcal{O}\left(\frac{1}{r}\right)\right),$$

where $(\mathcal{R}\pi_* f^* \sqrt[r]{L_\infty})_u$ means the induced index bundle over $\overline{\mathscr{M}}_u$, and $\mathrm{rk}(u) = |E_u| - 1$ since $r \gg 1$.

(1b) If $v \in V_*^S$,

$$\mathrm{cont}_v := e_{\mathbf{C}^*}((\mathcal{R}\pi_* f^*(N \oplus \mathcal{O}_S))_v)^{-1},$$

where $(\mathcal{R}\pi_* f^*(N \oplus \mathcal{O}_S))_v$ means the induced index bundle over $\overline{\mathscr{M}}_v$ whose rank is $\mathrm{rk}(v) = \int_{\mathbf{P}^1} c_1(f_v^* N) + \kappa + 1$ for a stable map $f_v \colon \mathbf{P}^1 \to S_*$ in $\overline{\mathscr{M}}_v$, hence $\mathrm{rk}(v) = c_1(N)(A_v) + \kappa + 1$.

(2) *Contributions from edges.* Let $e \in E$ be an edge. By the contribution of the edge e, we mean the equivariant Euler class of the virtual normal bundle of $\mathscr{F}_e$ in the corresponding moduli space of W_r. Suppose the edge e connects the vertex $u \in V_\infty$ and the vertex $v \in V_*$ and has degree d_e. By the standard computations, when $v \notin V_*^1$, the contribution of e is

$$\mathrm{cont}_e := \left(\left(\prod_{0 \leqslant j \leqslant d_e - 1} \frac{(d_e - j)t}{d_e}\right)^\kappa + \cdots\right)^{-1} \cdot \left(\prod_{1 \leqslant l \leqslant d_e - 1} \frac{-lt}{d_e} + \cdots\right), \tag{15}$$

where those '$\cdots$' represent lower degree (but nonnegative) terms in t. When $v \in V_*^1$, in addition to cont_e there is an extra contribution t/d_e coming from the automorphisms of the domain curve.

Lemma 2.17. *The* cont_e *is a polynomial in* t^{-1} *with lowest degree* $\mathrm{rk}_e := d_e(\kappa - 1) + 1$.

(3) *Contributions from nodal points.* Nodal points correspond to some half edges or vertices with exactly two adjacent edges. Be precise, there are three cases.

(3a) Let $u \in V_\infty$ and $e = e_{uv}$ be an edge adjacent to u with degree d_e. Then we have a half edge (u, e) corresponding to a nodal point n_e attached to C_u. The contribution is

$$\mathrm{cont}_{\mathsf{n}_e} := \frac{1}{(t + ev_\infty^*(c_1(L_\infty)))/d_e - \bar{\psi}_u} = \frac{d_e}{t + ev_\infty^*(c_1(L_\infty)) - d_e \bar{\psi}_u}, \tag{16}$$

where ∞ in ev_∞ is the ∞ marking on C_e, $\bar{\psi}_u$ is the first Chern class of the cotangent line bundle over $\overline{\mathscr{M}}_u$ associated to the unordered orbifold marking corresponding to the nodal point n_e, whose fiber is the cotangent space of the coarse space of C_u at n_e.

(3b) Let $v \in V_*^S$ and $e = e_{uv}$ be an edge adjacent to v with degree d_e. Then we have a half edge (v, e) corresponding to a nodal point $\bar{\mathsf{n}}_e$ attached to C_v. The contribution is

$$\mathrm{cont}_{\bar{\mathsf{n}}_e} := \frac{(-1)^\kappa t^{\kappa+1}}{(-t - ev_\infty^*(c_1(L_\infty)))/d_e - \bar{\psi}_v} = \frac{d_e \cdot (-1)^\kappa t^{\kappa+1}}{-t - ev_\infty^*(c_1(L_\infty)) - d_e \bar{\psi}_v},$$

where ∞ in ev_∞ is the ∞ marking on C_e, $\bar{\psi}_v$ is the first Chern class of the cotangent line bundle over $\overline{\mathscr{M}}_v$ associated to the unordered marking corresponding to the nodal point $\bar{\mathsf{n}}_e$,

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whose fiber is the cotangent space of C_v at $\bar{\mathsf{n}}_e$ and the $t^{\kappa+1}$ corresponds to the deformation of stable maps at the nodal point n_e along the normal direction of S_* in W_r.

(3c) Let $v \in V_*^2$ and $e_1 = e_{u_1 v}$, $e_2 = e_{u_2 v}$ be the only two adjacent edges with degrees d_{e_1} and d_{e_2}, respectively. The nodal point $\bar{\mathsf{n}}_v$ is the intersection of the two components C_{e_1} and C_{e_2} corresponding to the two edges. The contribution is

$$\mathrm{cont}_{\bar{\mathsf{n}}_v} := \frac{(-1)^\kappa t^{\kappa+1}}{(-t - ev_\infty^* c_1(L_\infty))/d_{e_1} + (-t - ev_\infty^* c_1(L_\infty))/d_{e_2}},$$

where ∞ in ev_∞ is the ∞ marking on C_{e_1} and C_{e_2}, respectively, and the $t^{\kappa+1}$ corresponds to the deformation of stable maps at $\bar{\mathsf{n}}_v$ along the normal direction of S_* in W_r.

We conclude that

$$\frac{1}{e_{\mathbf{C}^*}(\mathscr{N}_\Phi)} = \prod_{u\in V_\infty}\Big(\mathrm{cont}_u \prod_{e\in E_u}\mathrm{cont}_{\mathsf{n}_e}\Big)\prod_{v\in V_*^S}\Big(\mathrm{cont}_v\prod_{e\in E_v}\mathrm{cont}_{\bar{\mathsf{n}}_e}\Big)\prod_{v\in V_*^2}\mathrm{cont}_{\bar{\mathsf{n}}_v}\prod_{e\in E}\mathrm{cont}_e\prod_{\substack{e\in E_v\\ v\in V_*^1}}\frac{t}{d_e}.$$

For a stable vertex $u \in V_\infty$, we set

$$\mathrm{Cont}_u := \frac{r^{|E_u|}}{d_u}\Big(\mathrm{cont}_u \cdot \prod_{e\in E_u}\mathrm{cont}_{\mathsf{n}_e}\Big)\cap [\overline{\mathscr{M}}_u]^{\mathrm{vir}}, \tag{17}$$

with $d_u = \prod_{e\in E_u} d_e$. For a vertex $v \in V_*$, we set

$$\mathrm{Cont}_v := [\overline{\mathscr{M}}_v]^{\mathrm{vir}} \cap \begin{cases} \mathrm{cont}_v\cdot\prod_{e\in E_v}\mathrm{cont}_{\bar{\mathsf{n}}_e}\cdot\prod_{e\in E_v}\mathrm{cont}_e & \text{if } v\in V_*^S,\\ \mathrm{cont}_{\bar{\mathsf{n}}_v}\cdot\prod_{e\in E_v}\mathrm{cont}_e & \text{if } v\in V_*^2,\\ \mathrm{cont}_e & \text{if } v\in V_*^{1,1},\\ \dfrac{t}{d_e}\cdot\mathrm{cont}_e & \text{if } v\in V_*^1,\end{cases} \tag{18}$$

where when $v \in V_*^{1,1} \sqcup V_*^1$, $E_v = \{e\}$. Then we have the following.

PROPOSITION 2.18. *When $r \gg 1$, the contribution* $\mathrm{Cont}(\Phi)$ *is*

$$\frac{1}{|\mathrm{Aut}(\Phi)|}\cdot\sum_{l_e, e\in E}\prod_{j=1}^n ev_j^*(q^*\alpha_j)\prod_{e\in E}\widetilde{ev}_{e,+}^*\beta_{l_e} ev_{e,-}^*\check{\beta}_{l_e}\prod_{u\in V_\infty}\epsilon_*\mathrm{Cont}_u\prod_{v\in V_*}\mathrm{Cont}_v,$$

where $\{\beta_l\}$ is a basis of $H^*(F_*) = H^*(S_*)$ with $\{\check{\beta}_l\}$ the dual basis as in the proof of Proposition 2.16, $ev_{e,+}$ and $ev_{e,-}$ are the evaluation maps corresponding respectively to the two special points ∞ and 0 on C_e for an edge e, and $\widetilde{ev}_{e,+}$ is obtained from $ev_{e,+}$ via Lemma 2.15.

To prove Proposition 2.10, we need the following two technical lemmas. The first was proved in [JPPZ20, § 3].

LEMMA 2.19. *When $r \gg 1$, under the transformation $\mathfrak{s} = tr$, for each vertex $u \in V_\infty$, the term $t\cdot\epsilon_*\mathrm{Cont}_u$ is a polynomial in r and rational in $\mathfrak{s}$.*

LEMMA 2.20. *When the normal bundle N of S in X is positive and S has codimension $\kappa \geqslant 2$ in X, the product $\prod_{v\in V_*}\mathrm{Cont}_v$ is a polynomial in t^{-1} with lowest degree at least $|V_*|$. Moreover, when V_*^S is not empty, the lowest degree is at least $|V_*| + 1$.*

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Proof. For a polynomial in t^{-1}, denote its lowest degree in t^{-1} by $\deg^{\mathrm{low}}_{t^{-1}}(\cdot)$. We next compute $\deg^{\mathrm{low}}_{t^{-1}}(\mathrm{Cont}_v)$ for all $v \in V_*$.

For a vertex $v \in V_*^S$, we have

$$\begin{aligned}\deg^{\mathrm{low}}_{t^{-1}}(\mathrm{Cont}_v) &= \deg^{\mathrm{low}}_{t^{-1}}(\mathrm{cont}_v) + \sum_{e\in E_v} (\deg^{\mathrm{low}}_{t^{-1}}(\mathrm{cont}_{\bar{\mathfrak{n}}_e}) + \deg^{\mathrm{low}}_{t^{-1}}(\mathrm{cont}_e)) \\ &= (c_1(N)(A_v) + \kappa + 1) + \sum_{e\in E_v} (1 - (\kappa+1) + \mathrm{rk}_e) \\ &= (c_1(N)(A_v) + \kappa + 1) + \sum_{e\in E_v} (1 - (\kappa+1) + d_e(\kappa-1) + 1) \\ &\geqslant (1+1) + \sum_{e\in E_v} (d_e - 1)(\kappa - 1) \geqslant 2.\end{aligned}$$

Here we use the key assumption $c_1(N)(A_v) + \kappa = \int_{\mathbf{P}^1} c_1(f_v^* N) + \kappa > 0$.

For a vertex $v \in V_*^2$, let e_1 and e_2 be the only two adjacent edges. Then the degrees of these two edges satisfy $d_{e_1}, d_{e_2} \geqslant 1$. Consequently, we have

$$\begin{aligned}\deg^{\mathrm{low}}_{t^{-1}}(\mathrm{Cont}_v) &= \mathrm{rk}_{e_1} + \mathrm{rk}_{e_2} + 1 - (\kappa+1) \\ &= (d_{e_1}(\kappa-1) + 1 + d_{e_2}(\kappa-1) + 1) - \kappa \\ &= (d_{e_1} + d_{e_2} - 1)(\kappa - 1) + 1 \geqslant \kappa \geqslant 2.\end{aligned}$$

For a vertex $v \in V_*^{1,1}$ with the unique adjacent edge e of degree d_e, we have $d_e \geqslant 1$ and

$$\deg^{\mathrm{low}}_{t^{-1}}(\mathrm{Cont}_v) = \mathrm{rk}_e = d_e(\kappa-1) + 1 \geqslant \kappa \geqslant 2.$$

For a vertex $v \in V_*^1$ with the unique adjacent edge e of degree d_e, we have $d_e \geqslant 1$ and

$$\deg^{\mathrm{low}}_{t^{-1}}(\mathrm{Cont}_v) = \mathrm{rk}_e - 1 = d_e(\kappa-1) + 1 - 1 \geqslant \kappa - 1 \geqslant 1.$$

The lemma follows. ☐

Now we prove Proposition 2.10.

Proof of Proposition 2.10. By Proposition 2.18, when $r \gg 1$, $\mathrm{Cont}(\Phi)$ is

$$\mathrm{Cont}(\Phi) = \frac{1}{|\mathrm{Aut}(\Phi)|} \cdot \sum_{l_e, e\in E} \prod_{j=1}^n ev_j^* q^* \alpha_j \prod_{e\in E} \widetilde{ev}^*_{e,+} \beta_{l_e} ev^*_{e,-} \check{\beta}_{l_e} \prod_{u\in V_\infty} \epsilon_* \mathrm{Cont}_u \prod_{v\in V_*} \mathrm{Cont}_v.$$

Under the transformation $\mathfrak{s} = tr$, i.e. $t^{-1} = r\mathfrak{s}^{-1}$, for V_* we have

$$\prod_{v\in V_*} \mathrm{Cont}_v = \left(\frac{1}{t}\right)^{|V_*|} G(t^{-1}) = \left(\frac{r}{\mathfrak{s}}\right)^{|V_*|} G(r, \mathfrak{s}),$$

where $G(r, \mathfrak{s})$ is a polynomial in r by Lemma 2.20. Similarly, when V_∞ is nonempty, by Lemma 2.19, for each vertex $u \in V_\infty$,

$$\epsilon_* \mathrm{Cont}_u = \frac{1}{t} F_u(r, \mathfrak{s}) = \frac{r}{\mathfrak{s}} F_u(r, \mathfrak{s}),$$

where $F_u(r, \mathfrak{s})$ is also a polynomial in r when $r \gg 1$, which implies that

$$\prod_{u\in V_\infty} \epsilon_* \mathrm{Cont}_u = \left(\frac{r}{\mathfrak{s}}\right)^{|V_\infty|} F(r, \mathfrak{s}),$$

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with $F(r,\mathfrak{s})$ being a polynomial in r when $r \gg 1$. Put them together, we conclude that when $V_\infty \neq \varnothing$ and $r \gg 1$, $\mathrm{Cont}(\Phi)$ is a polynomial in r with lowest degree at least $|V_*| + |V_\infty| \geqslant 2$. If V_∞ is empty, V_* contains at least one stable vertex, i.e. $V_*^S \neq \varnothing$. By Lemma 2.20, we have the fact that $\mathrm{Cont}(\Phi)$ is a polynomial in r with lowest degree at least $|V_*| + 1 \geqslant 2$. This finishes the proof of Proposition 2.10. □

3. Weighted-blowup formulae of orbifold Gromov–Witten invariants

In this section we focus on the orbifold case. Let X be a Deligne–Mumford stack with a projective coarse space and $\mathsf{S} \subseteq \mathsf{X}$ be a codimension κ smooth substack with normal bundle N. Let $\widehat{\mathsf{X}}_{\mathfrak{a}}$ be the weight-$\mathfrak{a}$ blowup (cf. [MM12, CDH19]) of X along S with weight

$$\mathfrak{a} = (a_1, \ldots, a_\kappa) \in \mathbf{Z}^{\kappa}_{\geqslant 1}.$$

Denote by Z the exceptional divisor of $\widehat{\mathsf{X}}_{\mathfrak{a}}$, which is the weight-$\mathfrak{a}$ projectivization $\mathbf{P}_{\mathfrak{a}}(\mathsf{N})$ of N.

We treat Deligne–Mumford stacks via proper étale Lie groupoids, which are called orbifold groupoids. There are some nice references on orbifold groupoids. See, for example, [ALR07] and [MP97]. One can see also [CDH19, § 2] for a brief introduction to orbifold groupoids, Chen–Ruan cohomology, weighted blowups, etc. The strategy to prove the weighted-blowup formulae is exactly the same as the proof of the blowup formula in § 2.

3.1 The symplectic cobordism between X and $\widehat{\mathsf{X}}_{\mathfrak{a}}$

As in § 2.1, let W be the weight-$(\mathfrak{a},1)$ blowup of $\mathsf{X} \times \mathbf{P}^1$ along $\mathsf{S}_\infty = \mathsf{S} \times \{\infty\}$. The exceptional divisor is

$$\mathsf{D} := \mathbf{P}_{(\mathfrak{a},1)}(\mathsf{N} \oplus \mathscr{O}_{\mathsf{S}}).$$

Here $\mathscr{O}_{\mathsf{S}} \cong \mathsf{S} \times T_\infty \mathbf{P}^1$. Then D contains a copy of $\mathsf{Z} = \mathbf{P}_{\mathfrak{a}}(\mathsf{N})$ as the infinity divisor $\mathbf{P}_{(\mathfrak{a},1)}(\mathsf{N} \oplus 0)$ and a copy of S as the zero section $\mathbf{P}_{(\mathfrak{a},1)}(0 \oplus \mathscr{O}_{\mathsf{S}})$, which we denote by S_*. The $\mathbf{C}^*$-action on $\mathbf{P}^1$ given by (8) induces an action on $\mathsf{X} \times \mathbf{P}^1$, which lifts to an action on W. Similar to Lemma 2.1 we have the following lemma.

LEMMA 3.1. *The fixed locus $\mathsf{W}^{\mathbf{C}^*}$ of W with respect to the induced $\mathbf{C}^*$-action consists of three disjoint components*

$$\mathsf{F}_0 \cong \mathsf{X}, \quad \mathsf{F}_\infty \cong \widehat{\mathsf{X}}_{\mathfrak{a}}, \quad \mathsf{F}_* = \mathsf{S}_* \cong \mathsf{S}.$$

The normal line bundle L_0 of F_0 in W is a trivial line bundle with action weight -1; the normal line bundle L_∞ of F_∞ is $\mathscr{O}_{\widehat{\mathsf{X}}_{\mathfrak{a}}}(-\mathsf{Z})$ with action weight 1; the normal bundle N_* of F_* is $\mathsf{N} \oplus \mathscr{O}_{\mathsf{S}}$ with action weight $(-1,1)$.

Let W_r be the rth root construction of W along $\mathsf{F}_\infty \cong \widehat{\mathsf{X}}_{\mathfrak{a}}$. With respect to the natural projection $\mathsf{W}_r \to \mathsf{W}$, the divisor of W_r lying over F_∞ is the rth root gerbe induced from the line bundle $\mathsf{L}_\infty \to \mathsf{F}_\infty$. We denote it by $\sqrt[r]{\mathsf{F}_\infty}$. Its normal line bundle in W_r is the rth root line bundle $\sqrt[r]{\mathsf{L}_\infty}$ of L_∞. The divisor of W_r lying over D is the rth root construction D_r of D along Z. Explicitly,

$$\mathsf{D}_r = \mathbf{P}_{(r\mathfrak{a},1)}(\mathsf{N} \oplus \mathscr{O}_{\mathsf{S}}),$$

the weight-$(r\mathfrak{a},1)$ projectivization of $\mathsf{N} \oplus \mathscr{O}_{\mathsf{S}}$. The normal line bundle of D_r in W_r is $\mathscr{O}_{\mathsf{D}_r}(-r)$, which is the pullback of the tautological line bundle $\mathscr{O}_{\mathsf{D}}(-1)$ via the natural projection $\mathsf{D}_r \to \mathsf{D}$. The divisor of D_r lying over Z is the rth root gerbe of $\mathsf{L}_\infty|_{\mathsf{Z}} \to \mathsf{Z}$, denoted by $\sqrt[r]{\mathsf{Z}}$, which is also the weight-$r\mathfrak{a}$ projectivization $\mathbf{P}_{r\mathfrak{a}}(\mathsf{N})$ of N. The $\sqrt[r]{\mathsf{F}_\infty}$ intersects with D_r along the $\sqrt[r]{\mathsf{Z}}$. The $\mathbf{C}^*$-action on W lifts to W_r.

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Lemma 3.2. *The fixed locus* $\mathsf{W}_r^{\mathbf{C}^*}$ *of* W_r *with respect to the induced* $\mathbf{C}^*$*-action consists of three disjoint components*

$$\mathsf{F}_0 \cong \mathsf{X}, \quad \sqrt[r]{\mathsf{F}_\infty} \cong \sqrt[r]{\widehat{\mathsf{X}}_{\mathfrak{a}}}, \quad \mathsf{F}_* = \mathsf{S}_* \cong \mathsf{S}.$$

The normal bundles of F_0 *and* F_* *in* W_r *are the same as their normal bundles in* W*. The normal bundle of* $\sqrt[r]{\mathsf{F}_\infty}$ *in* W_r *is* $\sqrt[r]{\mathsf{L}_\infty}$ *with action weight* $1/r$.

Remark 3.3. In general, given an orbifold X its inertia space IX is defined to be the union of the twisted sectors of X (cf. [CR04]). Being precise, let $\mathscr{T}^{\mathsf{X}}$ denote the index set of twisted sectors. For any $[g] \in \mathscr{T}^{\mathsf{X}}$, the twisted sector is denoted by $\mathsf{X}[g]$, then

$$\mathsf{IX} = \bigsqcup_{[g]\in\mathscr{T}^{\mathsf{X}}} \mathsf{X}[g].$$

We now describe the inertia space of W_r and related subspaces.

Twisted sectors of S. Let $[g]$ be an element in $\mathscr{T}^{\mathsf{S}}$. It is an equivalence class of some group elements. Let g be any representative of the class. Denote the order of g by $\mathfrak{o}(g)$. Suppose its action weight on the fiber direction of N is

$$\mathfrak{b}(g) = (\mathfrak{b}^1(g), \ldots, \mathfrak{b}^\kappa(g)) \in \mathbf{Z}^\kappa_{\geqslant 1}.$$

Note that here we take $1 \leqslant \mathfrak{b}^w(g) \leqslant \mathfrak{o}(g)$ for $1 \leqslant w \leqslant \kappa$. Note also that $\mathscr{T}^{\mathsf{S}} \subseteq \mathscr{T}^{\mathsf{X}}$.

Lemma 3.4. *For each* $[g] \in \mathscr{T}^{\mathsf{S}}$, $\mathsf{S}[g]$ *is a suborbifold of* $\mathsf{X}[g]$ *whose normal bundle is the restriction of the inertia normal bundle* $\mathsf{IN} \to \mathsf{IS}$ *on* $\mathsf{S}[g]$.

Proof. Set

$$I_{[g]} := \{w \mid \mathfrak{b}(g)^w = \mathfrak{o}(g),\ 1 \leqslant w \leqslant \kappa\}. \tag{19}$$

It determines an obvious subbundle $\mathsf{N}_{I_{[g]}}$ of N over which g acts trivially. If $I_{[g]}$ is empty, then $\mathsf{X}[g] = \mathsf{S}[g]$, otherwise, $\mathsf{S}[g]$ is a suborbifold of $\mathsf{X}[g]$ whose normal bundle is the 'restriction' of $\mathsf{N}_{I_{[g]}}$ on $\mathsf{S}[g]$, which is the same as the restriction of the inertia normal bundle $\mathsf{IN} \to \mathsf{IS}$ on $\mathsf{S}[g]$. ☐

The codimension of $\mathsf{S}[g]$ in $\mathsf{X}[g]$ is $|I_{[g]}|$, the cardinality of $I_{[g]}$. Motivated by Lemma 3.4, we set

$$\mathscr{T}^{\mathsf{S}}_0 := \{[g] \in \mathscr{T}^{\mathsf{S}} \mid I_{[g]} = \varnothing\}. \tag{20}$$

Twisted sectors of Z. Recall that $\mathsf{Z} = \mathbf{P}_{\mathfrak{a}}(\mathsf{N})$. The projection $\pi\colon \mathsf{Z} \to \mathsf{S}$ induces projections

$$\mathsf{I}\pi\colon \mathsf{IZ} \to \mathsf{IS} \quad \text{and} \quad \Lambda\colon \mathscr{T}^{\mathsf{Z}} \to \mathscr{T}^{\mathsf{S}}.$$

Let $[g] \in \mathscr{T}^{\mathsf{S}}$, we describe $\Lambda^{-1}([g])$. Note that $\mathscr{T}^{\mathsf{Z}} \subseteq \mathscr{T}^{\mathsf{S}} \times S^1$, we write an element in $\mathscr{T}^{\mathsf{Z}}$ in the form

$$[g, e^{2\pi i R}], \quad \text{where } [g] \in \mathscr{T}^{\mathsf{S}},\ 0 < R \leqslant 1.$$

Here we abuse the notation $[g]$ and g. Then by direct computations, we have

$$\Lambda^{-1}([g]) = \left\{[g, e^{2\pi i R}] \in \mathscr{T}^{\mathsf{S}} \times S^1 \,\middle|\, \exists\, 1 \leqslant w \leqslant \kappa,\ \text{s.t. } \frac{\mathfrak{b}^w(g)}{\mathfrak{o}(g)} + R a_w \in \mathbf{Z}\right\}.$$

We set

$$\mathcal{R}_{[g]} := \{R \in (0, +\infty) \mid [g, e^{-2\pi i R}] \in \Lambda^{-1}([g])\} \quad \text{and} \quad R_{[g]} := \min \mathcal{R}_{[g]}. \tag{21}$$

We now describe twisted sector $\mathsf{Z}[h]$ for an $[h] = [g, e^{2\pi i R}] \in \mathscr{T}^{\mathsf{Z}}$. Set

$$I_{[h]} := \left\{w \,\middle|\, 1 \leqslant w \leqslant \kappa,\ \frac{\mathfrak{b}^w(g)}{\mathfrak{o}(g)} + R a_w \in \mathbf{Z}\right\}. \tag{22}$$

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The $I_{[h]}$ determines an obvious subbundle of N, denoted by $\mathsf{N}_{[h]}$. The blowup weight $\mathfrak{a}$ induces a subweight $\mathfrak{a}_{[h]} := (a_w)_{w\in I_{[h]}}$ on $\mathsf{N}_{[h]}$.

Lemma 3.5. *We have* $\mathsf{Z}[h] = \mathbf{P}_{\mathfrak{a}_{[h]}}(\mathsf{N}_{[h]})$.

We skip the proof.

Twisted sectors of $\widehat{\mathsf{X}}_{\mathfrak{a}}$. Since $\widehat{\mathsf{X}}_{\mathfrak{a}} \setminus \mathsf{Z} \cong \mathsf{X} \setminus \mathsf{S}$, we have $\mathscr{T}^{\widehat{\mathsf{X}}_{\mathfrak{a}}} \setminus \mathscr{T}^{\mathsf{Z}} = \mathscr{T}^{\mathsf{X}} \setminus \mathscr{T}^{\mathsf{S}}$. Hence, if $[h] \notin \mathscr{T}^{\mathsf{Z}}$, $\widehat{\mathsf{X}}_{\mathfrak{a}}[h] \cong \mathsf{X}[h]$. Now consider an $[h] = [g, e^{2\pi iR}] \in \mathscr{T}^{\mathsf{Z}}$. The normal bundle of Z in $\widehat{\mathsf{X}}_{\mathfrak{a}}$ is isomorphic to $\mathscr{O}_{\mathsf{Z}}(-1)$. The action of $[h]$ on the fiber direction of $\mathscr{O}_{\mathsf{Z}}(-1)$ is by multiplying $e^{-2\pi iR}$. We conclude as follows.

Lemma 3.6. *For an* $[h] = [g, e^{2\pi iR}] \in \mathscr{T}^{\mathsf{Z}}$*, when* $e^{2\pi iR} \neq 1$, $\widehat{\mathsf{X}}_{\mathfrak{a}}[h] = \mathsf{Z}[h]$*. Otherwise,* $\mathsf{Z}[h]$ *is a divisor of* $\widehat{\mathsf{X}}_{\mathfrak{a}}[h]$ *whose normal bundle is* $\mathscr{O}_{\mathsf{Z}[h]}(-1)$.

Twisted sectors of W. Since W is a weight-$(\mathfrak{a}, 1)$ blowup of $\mathsf{X} \times \mathbf{P}^1$ along $\mathsf{S}_\infty = \mathsf{S} \times \{\infty\}$ and the exceptional divisor is $\mathsf{D} = \mathbf{P}_{(\mathfrak{a},1)}(\mathsf{N} \oplus \mathscr{O}_{\mathsf{S}})$, we may apply the same strategy to get $\mathscr{T}^{\mathsf{D}}$ and $\mathscr{T}^{\mathsf{W}}$. Then we have

$$\mathscr{T}^{\mathsf{W}} = (\mathscr{T}^{\mathsf{X}\times\mathbf{P}^1} \setminus \mathscr{T}^{\mathsf{S}_\infty}) \cup \mathscr{T}^{\mathsf{D}}.$$

Furthermore, since Z and S_* are two special suborbifolds of D, it is easy to conclude that

$$\mathscr{T}^{\mathsf{D}} = \mathscr{T}^{\mathsf{Z}} \cup \mathscr{T}^{\mathsf{S}}.$$

Twisted sectors of W_r. Since the rth root construction (i.e. weight-r blowup) is on $\mathsf{F}_\infty \cong \widehat{\mathsf{X}}_{\mathfrak{a}}$, the difference of twisted sectors between W and W_r happens at $\mathsf{F}_\infty \cong \widehat{\mathsf{X}}_{\mathfrak{a}}$. In fact, we have

$$\mathscr{T}^{\sqrt[r]{\mathsf{F}_\infty}} \cong \mathscr{T}^{\mathsf{F}_\infty} \times \mathbf{Z}_r, \quad \text{and} \quad \sqrt[r]{\mathsf{F}_\infty}[h] \cong \sqrt[r]{\mathsf{F}_\infty[\Lambda(h)]}.$$

3.2 Weighted-blowup formulae

Let $A \in H_2(|\mathsf{X}|; \mathbf{Z})$ and $\mathfrak{g} := ([g_1], \ldots, [g_n]) \in (\mathscr{T}^{\mathsf{X}})^n$. Consider the moduli space $\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X})$ of genus zero, n marked, degree A orbifold stable maps into X (cf. [CR02, AGV08]) with jth marking mapped into $\mathsf{X}[g_j]$. For $\alpha_j \in H^*(\mathsf{X}[g_j])$, $1 \leqslant j \leqslant n$, a Gromov–Witten invariant of X is

$$\langle \alpha_1, \ldots, \alpha_n \rangle^{\mathsf{X}}_{0,\mathfrak{g},A} := \int_{[\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X})]^{\mathrm{vir}}} \prod_{j=1}^{n} ev_j^*(\alpha_j).$$

In order to consider a corresponding invariant of $\widehat{\mathsf{X}}_{\mathfrak{a}}$, we need some preparations. For the projection $\mathscr{T}^{\widehat{\mathsf{X}}_{\mathfrak{a}}} \to \mathscr{T}^{\mathsf{X}}$ we fix one of its right inverse

$$\hat{\cdot}\colon\ \mathscr{T}^{\mathsf{X}} \to \mathscr{T}^{\widehat{\mathsf{X}}_{\mathfrak{a}}} \tag{23}$$

as the following:

(i) if $[g] \in \mathscr{T}^{\mathsf{S}}_0$, set $[\hat{g}] := [g, e^{-2\pi iR_{[g]}}]$, where $R_{[g]} = \min \mathcal{R}_{[g]}$ is given in (21);
(ii) if $[g] \in \mathscr{T}^{\mathsf{S}} \setminus \mathscr{T}^{\mathsf{S}}_0$, set $[\hat{g}] := [g, 1]$;
(iii) if $[g] \in \mathscr{T}^{\mathsf{X}} \setminus \mathscr{T}^{\mathsf{S}}$, set $[\hat{g}] := [g]$.

Now set

$$\widehat{\mathfrak{g}} := ([\hat{g}_1], \ldots, [\hat{g}_n]). \tag{24}$$

For each $[g_j]$, denote the projection over the corresponding twisted sectors by

$$p_{[\hat{g}_j]}\colon\ (\widehat{\mathsf{X}}_{\mathfrak{a}})[\hat{g}_j] \to \mathsf{X}[g_j].$$

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This is the restriction of the projection of inertia spaces. Then the corresponding insertions are transformed as

$$\alpha_i \Longrightarrow p^*_{[\hat{g}_i]}\alpha_i. \tag{25}$$

For simplicity of exposition, in $\mathfrak{g}$ we assume

$$[g_1],\ldots,[g_{n_0(\mathfrak{g})}] \in \mathscr{T}_0^{\mathsf{S}}, \quad [g_{n_0(\mathfrak{g})+1}],\ldots,[g_n] \in \mathscr{T}^{\mathsf{X}} \setminus \mathscr{T}_0^{\mathsf{S}}. \tag{26}$$

We next explain the transformation of A. When $\kappa = 1$, $|\widehat{\mathsf{X}}_{\mathfrak{a}}| = |\mathsf{X}|$, A remains the same. We next consider the case that $\kappa \geqslant 2$. For this case, we need extract from $p^! A$ some fiber class of $\mathsf{Z} \to \mathsf{S}$. We first explain the fiber class $[F] \in H_2(\mathsf{Z}; \mathbf{R})$ of $\mathsf{Z} \to \mathsf{S}$. It is the dual of $c_1(\mathscr{O}_{\mathsf{Z}}(1))$ (cf. [CCLT09, p. 141] for the case of weighted projective space). Similarly, we have fiber class of $\mathsf{D} \to \mathsf{S}$, which is the dual of $c_1(\mathscr{O}_{\mathsf{D}}(1))$ and restricts to the fiber class of $\mathsf{Z} \to \mathsf{S}$, so we also denote it by $[F]$. Then when $\kappa \geqslant 2$, we set

$$\widehat{A} := p^! A - \sum_{1 \leqslant j \leqslant n_0(\mathfrak{g})} R_{[g_j]}[F]. \tag{27}$$

The weighted-blowup formula of the orbifold Gromov–Witten invariants for the codimension $\kappa \geqslant 2$ case is stated as follows.

Theorem 3.7. *Let X be a smooth Deligne–Mumford stack with projective coarse space and $\mathsf{S} \subseteq \mathsf{X}$ be a codimension κ smooth substack. Let $\widehat{\mathsf{X}}_{\mathfrak{a}}$ be the weight-$\mathfrak{a}$ blowup of X along S. Suppose the normal bundle N of S in X is positive (see Definition 1.7). Then when $\kappa \geqslant 2$, we have the following equality of genus zero orbifold Gromov–Witten invariants*

$$\langle \alpha_1,\ldots,\alpha_n \rangle^{\mathsf{X}}_{0,\mathfrak{g},A} = \langle p^*_{[\hat{g}_1]}\alpha_1,\ldots,p^*_{[\hat{g}_n]}\alpha_n \rangle^{\widehat{\mathsf{X}}_{\mathfrak{a}}}_{0,\hat{\mathfrak{g}},\widehat{A}}. \tag{28}$$

Next we state the formula for the case $\kappa = 1$, i.e. S is a divisor of X. Unlike the case $\kappa \geqslant 2$, in addition to the transformations given in (24) and (25), we need add extra orbifold markings for $\widehat{\mathsf{X}}_{\mathfrak{a}}$, which comes from the intersection information of stable maps in the moduli space $\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X})$ with the divisor S. Set

$$I_A := [\mathsf{S} \cdot A]_{\mathrm{orb}} - \sum_{1 \leqslant i \leqslant n_0(\mathfrak{g})} \frac{\mathfrak{b}(g_i)}{\mathfrak{o}(g_i)},$$

where $n_0(\mathfrak{g})$ is given in (26), $\mathfrak{b}(g_i)$ is the action weight of g_i on the normal line bundle of S in X, and $[\mathsf{S} \cdot A]_{\mathrm{orb}}$ is the intersection number of an orbifold stable map $\mathsf{f}\colon \mathsf{C} \to \mathsf{X}$ in $\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X})$ with S. The I_A is a nonnegative integer. Then we add I_A orbifold markings associated with the twisted sector corresponding to

$$\bar{1}_{\mathsf{Z}} := [1, e^{-2\pi i/a}] \in \mathscr{T}^{\mathsf{Z}},$$

where $\mathfrak{a} = (a)$ is the blowup weight. Then we set

$$\hat{\mathfrak{g}}^A := ([\hat{g}_1],\ldots,[\hat{g}_n], \underbrace{\bar{1}_{\mathsf{Z}},\ldots,\bar{1}_{\mathsf{Z}}}_{I_A}).$$

The insertions of the I_A extra markings are set to be 1.

Theorem 3.8. *Under the same assumption as that in Theorem 3.7 on X, S and N, when the codimension $\kappa = 1$ and the blowup weight $\mathfrak{a} = (a)$ satisfies $a \geqslant 2$, we have the following equality of genus zero orbifold Gromov–Witten invariants*

$$\langle \alpha_1,\ldots,\alpha_n \rangle^{\mathsf{X}}_{0,\mathfrak{g},A} = \langle p^*_{[\hat{g}_1]}\alpha_1,\ldots,p^*_{[\hat{g}_n]}\alpha_n, \underbrace{1,\ldots,1}_{I_A} \rangle^{\widehat{\mathsf{X}}_{\mathfrak{a}}}_{0,\hat{\mathfrak{g}}^A,A}. \tag{29}$$

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Remark 3.9. As the smooth case, under the assumptions in Theorems 3.7 and 3.8, the virtual dimension of $\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X})$ equals to that of $\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},\widehat{A}}(\widehat{\mathsf{X}}_{\mathfrak{a}})$ and $\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}}^A,A}(\widehat{\mathsf{X}}_{\mathfrak{a}})$. Therefore, *a priori*, we assume that the total degree of insertions matches the dimension constraint in both (28) and (29), i.e.

$$\frac{1}{2}\sum_{j=1}^{n}\deg\alpha_j = \operatorname{vdim}\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X}). \tag{30}$$

Otherwise, the invariants in (28) and (29) are all zero, and the formulae hold trivially.

Remark 3.10. As in Remark 2.5, the blowup formulae (28) and (29) also hold for descendent invariants with psi-classes over $\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},\widehat{A}}(\widehat{\mathsf{X}}_{\mathfrak{a}})$ (or $\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}}^A,A}(\widehat{\mathsf{X}}_{\mathfrak{a}})$) being pullbacks of those of $\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X})$ via the natural projections

$$\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},\widehat{A}}(\widehat{\mathsf{X}}_{\mathfrak{a}}) \to \overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X}) \quad \text{and} \quad \overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}}^A,A}(\widehat{\mathsf{X}}_{\mathfrak{a}}) \to \overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X}).$$

We next give a brief proof of the weighted-blowup formulae in Theorems 3.7 and 3.8.

Proof of the weighted-blowup formulae. The strategy to prove the weighted-blowup formulae is almost the same as the proof of the blowup formula (10) given in § 2. As $\mathsf{X} \cong \mathsf{F}_0 \hookrightarrow \mathsf{W}_r$, $\mathscr{T}^{\mathsf{X}} \subseteq \mathscr{T}^{\mathsf{W}_r}$. A twisted sector of X can be treated as a part of a twisted sector of W_r, and we have a natural projection

$$q\colon\ \mathsf{W}_r[g] \to \mathsf{W}[g] \to (\mathsf{X}\times\mathbf{P}^1)[g] \to \mathsf{X}[g], \quad \forall\ [g]\in\mathscr{T}^{\mathsf{X}}.$$

Moreover, this projection is $\mathbf{C}^*$-equivariant with $\mathbf{C}^*$ acting on $\mathsf{X}[g]$ trivially. Consider the following orbifold Gromov–Witten invariant of W_r

$$\langle q^*\alpha_1,\ldots,q^*\alpha_n\rangle^{\mathsf{W}_r}_{0,\mathfrak{g},p_W^!A} := \int_{[\overline{\mathscr{M}}_{0,\mathfrak{g},p_W^!A}(\mathsf{W}_r)]^{\mathrm{vir}}} \prod_{j=1}^{n} ev_j^*(q^*\alpha_j). \tag{31}$$

As Remark 2.6, we could treat the integration (31) as equivariant integration. We also have the following result.

LEMMA 3.11. *We have* $\operatorname{vdim}\overline{\mathscr{M}}_{0,\mathfrak{g},p_W^!A}(\mathsf{W}_r) = \operatorname{vdim}\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X})+1$, *and consequently by the dimension constraint* (30) *the invariant* (31) *vanishes for all* $r\in\mathbf{Z}_{\geqslant 1}$.

As in § 2 we next compute the vanishing invariant (31) by the virtual localization technique via the $\mathbf{C}^*$-action on W_r. As in § 2.3 we represent each component of the fixed locus of $\overline{\mathscr{M}}_{0,\mathfrak{g},p_W^!A}(\mathsf{W}_r)$ by a decorated graph Φ (see § 3.3), which we denote by $\overline{\mathscr{M}}_{\Phi}$. Then we compute the contribution $\mathrm{Cont}(\Phi)$ of each component $\overline{\mathscr{M}}_{\Phi}$ to the invariant (31). As in Propositions 2.8 and 2.9, we also have two special graphs Φ_0 and Φ_∞. We will prove the following propositions in § 3.4.

PROPOSITION 3.12. *The contribution of* $\overline{\mathscr{M}}_{\Phi_0}$ *to the invariant* (31) *is*

$$\mathrm{Cont}(\Phi_0) = -\frac{1}{t}\cdot\langle\alpha_1,\ldots,\alpha_n\rangle^{\mathsf{X}}_{0,\mathfrak{g},A}.$$

PROPOSITION 3.13. *When* $r\gg 1$, *the contribution of* $\overline{\mathscr{M}}_{\Phi_\infty}$ *to the invariant* (31) *is*

$$\mathrm{Cont}(\Phi_\infty) = \frac{1}{t}\cdot\langle p^*_{[\hat g_1]}\alpha_1,\ldots,p^*_{[\hat g_n]}\alpha_n\rangle^{\widehat{\mathsf{X}}_{\mathfrak{a}}}_{0,\widehat{\mathfrak{g}},\widehat{A}} \tag{32}$$

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if $\kappa \geqslant 2$, *and*

$$\mathrm{Cont}(\Phi_\infty) = \frac{1}{t} \cdot \big\langle p^*_{[\hat{g}_1]}\alpha_1, \ldots, p^*_{[\hat{g}_n]}\alpha_n, \underbrace{1, \ldots, 1}_{I_A} \big\rangle^{\widehat{\mathsf{X}}_{\mathfrak{a}}}_{0,\hat{\mathfrak{g}}^A,A} \tag{33}$$

if $\kappa = 1$.

For the blowup weight $\mathfrak{a} = (a_1, \ldots, a_\kappa)$, we set $\|\mathfrak{a}\| := \sum_{w=1}^{\kappa} a_w$.

Proposition 3.14. *Suppose the normal bundle* N *of* S *in* X *is positive and the blowup weight satisfies* $\|\mathfrak{a}\| \geqslant 2$. *Then when* $r \gg 1$, *the contributions of all decorated graphs other than* Φ_0 *and* Φ_∞ *are polynomials in* r *with lowest degree at least* 2 *under the transformation* $\mathfrak{s} = tr$.

Finally, with these three propositions, the argument in the end of § 2.2.2 proves the weighted-blowup formulae in Theorems 3.7 and 3.8. We omit it here. □

3.3 Description of $\overline{\mathscr{M}}_\Phi$ for $r \gg 1$

3.3.1 *Decorated graphs.* As in § 2.3.1 we use decorated graphs to describe connected components of the fixed locus of $\overline{\mathscr{M}}_{0,\mathfrak{g},p^!_W A}(\mathsf{W}_r)$. The principle of the assignment of decorated graphs to connected components of fixed locus is the same as that in § 2.3.1. We omit it here.

However, we remark that the case is slightly complicated due to the labels of twisted sectors on half edges as we now explain. As in § 2.3.1, each edge e corresponds to a fixed curve represented by an orbifold C_e mapped onto a fiber of $\mathsf{D}_r \to \mathsf{S}_*$:

$$\mathsf{u}\colon\ \mathsf{C}_e \to \mathsf{D}_r.$$

Here C_e is an orbifold $\mathbf{P}^1$. Then $\mathsf{u}(0) \in \mathsf{IS}_*$ and $\mathsf{u}(\infty) \in \mathsf{I}\sqrt[r]{\mathsf{Z}}$. The labels associated to e then are:

- label d_e to be the degree of u, i.e. $|\mathsf{u}(\mathsf{C}_e)| = d_e[F]$, where $[F]$ is the fiber class of $\mathsf{D} \to \mathsf{S}$;
- label twisted sectors $[g_e]$ and $[h_e]$ associated to $\mathsf{u}(0)$ and $\mathsf{u}(\infty)$, respectively.

On the other hand, by projecting to S_* we get a stable map $\mathsf{C}_e \to \mathsf{S}_*$ with two orbifold markings which is constant on coarse space, therefore $I(\mathsf{S}_*[g_e]) = \mathsf{S}_*[\Lambda([h_e])]$ where $I\colon\ \mathsf{IS}_* \to \mathsf{IS}_*$ is the canonical involution of inertia orbifolds. Consequently, these labels are related by

$$[h_e] = [g_e^{-1}, e^{2\pi i d_e/r}].$$

This suggests that the label $[h_e]$ encodes all information of d_e, g_e and h_e. It also implies that

$$d_e \in \mathcal{R}_{[g_e]}.$$

We follow the lines in § 2.3.1 and use same notation. As Lemma 2.13 we have following result.

Lemma 3.15. *For a decorated graph* Φ, *when* $r \gg 1$, $V_\infty \subseteq V^S(\Phi)$.

Let Φ_0 be the graph consisting of only one single vertex v with $c_v = 0$. Then

$$\overline{\mathscr{M}}_{\Phi_0} = \overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{F}_0) \cong \overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X}).$$

It is the Φ_0 in Proposition 3.12.

Now let Φ be a graph other than Φ_0. It then has no vertex with $c_v = 0$. Suppose

$$V_\infty = \{u_1, \ldots, u_{\mathfrak{p}}\}, \quad V_* = \{v_1, \ldots, v_{\mathfrak{q}}\}$$

and the set of edges is

$$E = \{e_{jk} \mid 1 \leqslant j \leqslant \mathfrak{p}, 1 \leqslant k \leqslant \mathfrak{q}\}.$$

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Remark 3.16. Unlike the case in Definition 2.14 in § 2.3.1, we may not have a graph Φ_∞ consisting of a single vertex with $c_v = \infty$. In fact, note that for a $[g] \in \mathscr{T}_0^{\mathsf{S}}$ the twisted sector $\mathsf{X}_\infty[g] = \mathsf{S}_\infty[g]$ of $\mathsf{X} \times \mathbf{P}^1$ transforms into $\mathsf{S}_*[g]$. Therefore, such a graph Φ_∞ exists if and only if (1) $\kappa \geqslant 2$ and $n_0(\mathfrak{g}) = 0$ in (26) or (2) $\kappa = 1$ and $[\mathsf{S} \cdot A]_{\mathrm{orb}} = 0$.

3.3.2 *Explicit expression of* $\overline{\mathscr{M}}_\Phi$. We next describe the component $\overline{\mathscr{M}}_\Phi$ for a general $\Phi \neq \Phi_0$.

The moduli space $\overline{\mathscr{M}}_u$ *for each vertex* $u \in V_\infty$. For a vertex $u \in V_\infty$ labelled with $c_u = \infty, A_u$ and $T_u = \{\iota_{u1}, \ldots, \iota_{ua}\}$, suppose the set of adjacent edges is $E_u = \{e_{ul_1}, \ldots, e_{ul_b}\}$. Recall that for an edge e_{ul_k}, there are labels $d_{ul_k}, [g_{ul_k}]$ and $[h_{ul_k}]$ with

$$[h_{ul_k}] = [g_{ul_k}^{-1}, e^{2\pi i d_{ul_k}/r}].$$

The E_u provides b nodal points on the domain curve whose branches over C_u are viewed as (unordered) orbifold markings and are mapped into the twisted sectors

$$\sqrt[r]{\mathsf{Z}}[h_{ul_k}^{-1}] \subseteq \sqrt[r]{\mathsf{F}_\infty}[h_{ul_k}^{-1}]$$

with $[h_{ul_k}^{-1}] = [g_{ul_k}, e^{-2\pi i d_{ul_k}/r}]$ for $1 \leqslant k \leqslant b$. Thus, we get

$$\overline{\mathscr{M}}_u := \overline{\mathscr{M}}_{0,T_u,\mathfrak{h}_u,A_u}(\sqrt[r]{\mathsf{F}_\infty}) \cap \bigwedge_{k=1}^{b} [\sqrt[r]{\mathsf{Z}}[h_{ul_k}^{-1}]],$$

where $\overline{\mathscr{M}}_{0,T_u,\mathfrak{h}_u,A_u}(\sqrt[r]{\mathsf{F}_\infty})$ is the moduli space of genus zero stable maps into $\sqrt[r]{\mathsf{F}_\infty}$ with original T_u-markings and $|E_u|$ (unordered) orbifold markings decorated by

$$\mathfrak{h}_u := ([h_{ul_1}^{-1}], \ldots, [h_{ul_b}^{-1}]).$$

Here $[\sqrt[r]{\mathsf{Z}}[h_{ul_k}^{-1}]]$ is the class of $\sqrt[r]{\mathsf{Z}}[h_{ul_k}^{-1}]$ in $\sqrt[r]{\mathsf{F}_\infty}[h_{ul_k}^{-1}]$, which equals to the pullback of the class of $\mathsf{Z}[g_{ul_k}, e^{-2\pi i d_{ul_k}}]$ in $\mathsf{F}_\infty[g_{ul_k}, e^{-2\pi i d_{ul_k}}]$ via the natural projection $\sqrt[r]{\mathsf{F}_\infty}[h_{ul_k}^{-1}] \to \mathsf{F}_\infty[g_{ul_k}, e^{-2\pi i d_{ul_k}}]$. As Lemma 2.15 we have the following result.

Lemma 3.17. *For each edge* $e_{ul_k} \in E_u$, *there is a natural map*

$$\widetilde{ev}_{ul_k} \colon \overline{\mathscr{M}}_u \to \mathsf{IS}_*$$

induced from the evaluation map $ev_{ul_k} \colon \overline{\mathscr{M}}_{0,T_u,\mathfrak{h}_u,A_u}(\sqrt[r]{\mathsf{F}_\infty}) \to \mathsf{I}\sqrt[r]{\mathsf{F}_\infty}$.

The moduli space $\overline{\mathscr{M}}_v$ *for each vertex* $v \in V_*$. If v is unstable, then we set $\overline{\mathscr{M}}_v := \mathsf{S}_*$. Now suppose v is stable, labelled by $c_v = *$, A_v, $T_v = \{\iota_{v1}, \ldots, \iota_{va}\}$ and attached by $E_v = \{e_{l_1 v}, \ldots, e_{l_b v}\}$. We get a moduli space

$$\overline{\mathscr{M}}_v := \overline{\mathscr{M}}_{0,T_v,\mathfrak{g}_v,A_v}(\mathsf{S}_*),$$

where $\mathfrak{g}_v := ([g_{l_1 v}^{-1}], \ldots, [g_{l_b v}^{-1}])$ and $[g_{l_k v}]$ is the label of the edge $e_{l_k v} \in E_v$ for $1 \leqslant k \leqslant b$.

The moduli space $\mathscr{F}_e$ *for each edge* $e \in E$. Suppose $e = e_{jk}$ connects $u_j \in V_\infty$ and $v_k \in V_*$ and has labels $d_{jk}, [g_{jk}]$ and $[h_{jk}]$. Then, if $v_k \notin V^1(\Phi)$, $\mathscr{F}_{e_{jk}}$ is the fixed locus of

$$\overline{\mathscr{M}}_{0,[g_{jk}],[h_{jk}],d_{jk}[F]}(\mathsf{D}_r) \subseteq \overline{\mathscr{M}}_{0,[g_{jk}],[h_{jk}],d_{jk}[F]}(\mathsf{W}_r),$$

and if $v_k \in V^1(\Phi)$, then $[g_{jk}] = [1]$ and $\mathscr{F}_{e_{jk}}$ is the fixed locus of

$$\overline{\mathscr{M}}_{0,[h_{jk}],d_{jk}[F]}(\mathsf{D}_r) \subseteq \overline{\mathscr{M}}_{0,[h_{jk}],d_{jk}[F]}(\mathsf{W}_r).$$

The moduli space $\overline{\mathscr{M}}_\Phi$. As (13) in Proposition 2.16 we have the following result.

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Proposition 3.18. *For a decorated graph $\Phi \neq \Phi_0$, i.e. with $V_0 = \varnothing$, set $d_\Phi := \prod_{e\in E} d_e$. Then we have*

$$\overline{\mathscr{M}}_\Phi = \frac{r^{|E|}}{d_\Phi \cdot |\mathrm{Aut}(\Phi)|} \cdot \prod_{u_j \in V_\infty} \overline{\mathscr{M}}_{u_j} \times_{(\mathsf{IS}_*)^{|E|}} \prod_{v_k \in V_*} \overline{\mathscr{M}}_{v_k}.$$

Here we use the map in Lemma 3.17 for the fiber product.

3.4 Contributions Cont(Φ)

In this subsection, we compute $\mathrm{Cont}(\Phi)$, i.e. prove Propositions 3.12, 3.13 and 3.14.

3.4.1 $\mathrm{Cont}(\Phi_0)$. For the graph Φ_0 we have $\overline{\mathscr{M}}_{\Phi_0} = \overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{F}_0)$. As the smooth case its virtual normal bundle in $\overline{\mathscr{M}}_{0,\mathfrak{g},p_W^! A}(\mathsf{W}_r)$ is the induced index bundle $(\mathsf{L}_0)_{\Phi_0}$ by the trivial normal line bundle $\mathsf{L}_0 = \mathsf{F}_0 \times T_0\mathbf{P}^1$ of $\mathsf{F}_0 \cong \mathsf{X}$ in W_r. The $(\mathsf{L}_0)_{\Phi_0}$ is a line bundle with action weight -1. Note that $(q^*\alpha_j)|_{\mathsf{IF}_0} = \alpha_j$. Thus, the contribution of Φ_0 is

$$\begin{aligned}\mathrm{Cont}(\Phi_0) &= \int_{[\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{F}_0)]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*((q^*\alpha_j)|_{\mathsf{IF}_0})}{c_1((\mathsf{L}_0)_{\Phi_0}\otimes \mathcal{O}(-1))} = \int_{[\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{F}_0)]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*(\alpha_j)}{-t + c_1((\mathsf{L}_0)_{\Phi_0})} \\ &= \frac{1}{-t}\cdot \int_{[\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{F}_0)]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(\alpha_j) = -\frac{1}{t}\cdot \langle \alpha_1,\dots,\alpha_n\rangle^{\mathsf{X}}_{0,\mathfrak{g},A}.\end{aligned}$$

This finishes the proof of Proposition 3.12.

3.4.2 $\mathrm{Cont}(\Phi_\infty)$*: case I.* Note that by Remark 3.16 the graph Φ_∞ consisting of a single v vertex with $c_v = \infty$ exists if and only if (1) $\kappa \geqslant 2$ and $n_0(\mathfrak{g}) = 0$ in (26) or (2) $\kappa = 1$ and $[\mathsf{S}\cdot A]_{\mathrm{orb}} = 0$, which both implies $n_0(\mathfrak{g}) = 0$. Suppose this is the case, which we call case I.

Then each $[\hat{g}_j] \in \hat{\mathfrak{g}}$ determined by $[g_j] \in \mathfrak{g}$ in (24) determines a twisted sector $\mathsf{F}_\infty[\hat{g}_j]$ of F_∞ and also a twisted sector $\sqrt[r]{\mathsf{F}_\infty}[\hat{g}_j]$ of $\sqrt[r]{\mathsf{F}_\infty}$. We have

$$\mathsf{W}_r[g_j] \cap \mathsf{I}\sqrt[r]{\mathsf{F}_\infty} = \sqrt[r]{\mathsf{F}_\infty}[\hat{g}_j].$$

Then $\overline{\mathscr{M}}_{\Phi_\infty} = \overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},p^! A}(\sqrt[r]{\mathsf{F}_\infty})$. Its virtual normal bundle in $\overline{\mathscr{M}}_{0,\mathfrak{g},p_W^! A}(\mathsf{W}_r)$ is the index bundle $(\sqrt[r]{\mathsf{L}_\infty})_{\Phi_\infty}$ induced from the normal bundle $\sqrt[r]{\mathsf{L}_\infty}$ of $\sqrt[r]{\mathsf{F}_\infty}$ in W_r. The $(\sqrt[r]{\mathsf{L}_\infty})_{\Phi_\infty}$ is a line bundle with action weight $1/r$. Note that we have the following commutative diagram

$$\begin{array}{ccccc} \overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},p^! A}(\sqrt[r]{\mathsf{F}_\infty}) & \xrightarrow{ev_j} & \sqrt[r]{\mathsf{F}_\infty}[\hat{g}_j] & & \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \pi} & \searrow^{q|_{\sqrt[r]{\mathsf{F}_\infty}[\hat{g}_j]}} & \\ \overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},p^! A}(\mathsf{F}_\infty) & \xrightarrow{ev_j} & \mathsf{F}_\infty[\hat{g}_j] & \xrightarrow{p_{[\hat{g}_j]}} & \mathsf{X}[g_j] \end{array} \tag{34}$$

and $q^*\alpha_j|_{\sqrt[r]{\mathsf{F}_\infty}[\hat{g}_j]} = (q|_{\sqrt[r]{\mathsf{F}_\infty}[\hat{g}_j]})^*\alpha_j = \pi^* p^*_{[\hat{g}_j]}(\alpha_j)$. Thus, the contribution of Φ_∞ is

$$\begin{aligned}\mathrm{Cont}(\Phi_\infty) &= \int_{[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},p^! A}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*(q^*\alpha_j|_{\sqrt[r]{\mathsf{F}_\infty}[\hat{g}_j]})}{c_1((\sqrt[r]{\mathsf{L}_\infty})_{\Phi_\infty}\otimes\mathcal{O}(1/r))} = \int_{[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},p^! A}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}} \frac{\prod_{j=1}^n ev_j^*(\pi^* p^*_{[\hat{g}_j]}\alpha_j)}{t/r + c_1((\sqrt[r]{\mathsf{L}_\infty})_{\Phi_\infty})} \\ &= \frac{r}{t}\cdot \int_{[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},p^! A}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}} \prod_{j=1}^n \pi^* ev_j^*(p^*_{[\hat{g}_j]}\alpha_j) = \frac{r}{t}\cdot\frac{1}{r}\cdot \int_{[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},p^! A}(\mathsf{F}_\infty)]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(p^*_{[\hat{g}_j]}\alpha_j) \\ &= \frac{1}{t}\cdot \langle p^*_{[\hat{g}_1]}\alpha_1,\dots,p^*_{[\hat{g}_1]}\alpha_1\rangle^{\hat{\mathsf{X}}_{\mathfrak{a}}}_{0,\hat{\mathfrak{g}},p^! A},\end{aligned}$$

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where the third equality follows from the dimension constraint (30) and

$$\operatorname{vdim}\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},p^!A}(\sqrt[r]{\mathsf{F}_\infty}) = \operatorname{vdim}\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},p^!A}(\mathsf{F}_\infty) = \operatorname{vdim}\overline{\mathscr{M}}_{0,\mathfrak{g},A}(\mathsf{X}),$$

and the fourth equality follows from the computation of orbifold Gromov–Witten invariants of banded gerbes in [TT21, Theorem 3.3], i.e. $\epsilon_*([\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},p^!A}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}) = (1/r)[\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},p^!A}(\mathsf{F}_\infty)]^{\mathrm{vir}}$.

This proves (32) and (33) of Proposition 3.13 for the current case.

3.4.3 *Proof of Proposition 3.14, and the completion of the proof of Proposition 3.13.* The main task is to compute $\mathrm{Cont}(\Phi)$ for graph Φ with nonempty V_*. We will prove that when $r \gg 1$, $\mathrm{Cont}(\Phi)$ is a polynomial in r with lowest degree 1 under the transformation $\mathfrak{s} = tr$. Moreover, the lowest degree 1 is achieved only when (1) $\kappa \geqslant 2$ and $n_0(\mathfrak{g}) > 0$ or (2) $\kappa = 1$ and $[\mathsf{S}\cdot A]_{\mathrm{orb}} > 0$, and it is achieved by exactly only one graph, denoted by Φ_∞. Then we verify that for such Φ_∞, $\mathrm{Cont}(\Phi_\infty)$ is that given in Proposition 3.13. This completes the proof of propositions.

Similar to the smooth case, the contribution of Φ is

$$\mathrm{Cont}(\Phi) = \frac{1}{|\mathrm{Aut}(\Phi)|}\cdot\prod_{j=1}^{n} ev_j^*(q^*\alpha_j) \cap \epsilon_*\bigg(\frac{[\overline{\mathscr{M}}_\Phi]^{\mathrm{vir}}}{e_{\mathbf{C}^*}(\mathscr{N}_\Phi)}\bigg)$$

with $\epsilon\colon \overline{\mathscr{M}}_{0,\mathfrak{g},p_W^!A}(\mathsf{W}_r) \to \overline{\mathscr{M}}_{0,\mathfrak{g},p_W^!A}(\mathsf{W})$ the natural projection, and

$$\mathrm{Cont}(\Phi) = \frac{1}{|\mathrm{Aut}(\Phi)|}\cdot\sum_{l_e, e\in E}\prod_{j=1}^{n} ev_j^*(q^*\alpha_j)\prod_{e\in E}\widetilde{ev}_{e,+}^*\beta_{l_e} ev_{e,-}^*\check{\beta}_{l_e}\prod_{u\in V_\infty}\epsilon_*(\mathrm{Cont}_u)\prod_{v\in V_*}\mathrm{Cont}_v,$$

where the following hold.

(i) The $\{\beta_l\}$ is a basis of $H^*_{\mathrm{CR}}(\mathsf{S})$ with $\{\check{\beta}_l\}$ the dual basis, and $ev_{e,+}$ and $ev_{e,-}$ correspond to the two special points ∞ and 0 of the curve C_e of an edge e, respectively.

(ii) For a vertex $u \in V_\infty$

$$\mathrm{Cont}_u := \frac{r^{|E_u|}}{d_u}\bigg(\mathrm{cont}_u\cdot\prod_{e\in E_u}\mathrm{cont}_{\mathsf{n}_e}\bigg)\cap[\overline{\mathscr{M}}_u]^{\mathrm{vir}}$$

with $d_u = \prod_{e\in E_u} d_e$;

(iii) For a vertex $v \in V_*$,

$$\mathrm{Cont}_v := [\overline{\mathscr{M}}_v]^{\mathrm{vir}} \cap \begin{cases} \mathrm{cont}_v\cdot\prod_{e\in E_v}\mathrm{cont}_{\bar{\mathsf{n}}_e}\cdot\prod_{e\in E_v}\mathrm{cont}_e & \text{if } v\in V_*^S,\\ \mathrm{cont}_{\bar{\mathsf{n}}_v}\cdot\prod_{e\in E_v}\mathrm{cont}_e & \text{if } v\in V_*^2,\\ \mathrm{cont}_e & \text{if } v\in V_*^{1,1},\\ \dfrac{t}{d_e}\cdot\mathrm{cont}_e & \text{if } v\in V_*^1,\end{cases}$$

and

- for each $u \in V_\infty$,

$$\mathrm{cont}_u = c_{\mathrm{rk}(u)}\bigg(-(\mathcal{R}\pi_*\mathsf{f}^*\sqrt[r]{\mathsf{L}_\infty})_u\otimes\mathcal{O}\Big(\frac{1}{r}\Big)\bigg),$$

  of which $(\mathcal{R}\pi_*\mathsf{f}^*\sqrt[r]{\mathsf{L}_\infty})_u$ means the induced index bundle over $\overline{\mathscr{M}}_u$ and $\mathrm{rk}(u) = |E_u| - 1$ as $r \gg 1$,

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– for each stable vertex $v \in V_*$,

$$\mathrm{cont}_v = e_{\mathbf{C}^*}((\mathcal{R}\pi_* \mathsf{f}^*(\mathsf{N} \oplus \mathcal{O}_{\mathsf{S}}))_v)^{-1},$$

of which $(\mathcal{R}\pi_* \mathsf{f}^*(\mathsf{N} \oplus \mathcal{O}_{\mathsf{S}}))_v$ means the induced index bundle over $\overline{\mathscr{M}}_v$ with rank $\mathrm{rk}(v) = \int_{|\mathsf{C}_v|} c_1(|\mathsf{f}_v^*\mathsf{N}|) + \kappa + 1$ and $|\mathsf{f}_v^*\mathsf{N}|$ is the de-singularization of the pullback bundle over C_v via a stable map $\mathsf{f}_v \colon \mathsf{C}_v \to \mathsf{S}_*$ in $\overline{\mathscr{M}}_v$,

– for each edge $e \in E$,

$$\mathrm{cont}_e := \Bigg(\prod_{\substack{1 \leqslant w \leqslant \kappa \\ j \in \left[0, d_e a_w - \overline{\mathfrak{b}(g_e)}^w / \mathfrak{o}(g_e)\right] \cap \mathbf{Z} \\ j \neq d_e a_w - \overline{\mathfrak{b}(g_e)}^w / \mathfrak{o}(g_e)}} \frac{(d_e a_w - \overline{\mathfrak{b}(g_e)}^w / \mathfrak{o}(g_e) - j)t}{d_e} + \cdots \Bigg)^{-1} \cdot \Bigg(\prod_{1 \leqslant l \leqslant d_e - 1 + \{-d_e\}} \frac{-lt}{d_e} + \cdots \Bigg) \tag{35}$$

of which

$$\overline{\mathfrak{b}(g_e)}^w = \begin{cases} \mathfrak{b}(g)^w & \text{if } \mathfrak{b}(g_e)^w < \mathfrak{o}(g_e), \\ 0 & \text{if } \mathfrak{b}(g_e)^w = \mathfrak{o}(g_e), \end{cases}$$

and those '$\cdots$' represent lower degree (but nonnegative) terms in t,

– for each $e \in E_u$ with $u \in V_\infty$, the associated nodal point n_e contributes

$$\mathrm{cont}_{\mathsf{n}_e} = \frac{d_e}{t + ev_\infty^* c_1(\mathsf{L}_\infty) - d_e \bar{\psi}_e},$$

where $\bar{\psi}_e$ is the psi-class associated to the cotangent line bundle over $\overline{\mathscr{M}}_u$, whose fiber is the cotangent space of the coarse space $|\mathsf{C}_u|$ at the special point associated to n_e,

– for each $e \in E_v$ with $v \in V_*^S$, the associated nodal point $\bar{\mathsf{n}}_e$ contributes

$$\mathrm{cont}_{\bar{\mathsf{n}}_e} = \frac{d_e \cdot t^{|I_{[g_e]}|+1}}{-t - ev_\infty^* c_1(\mathsf{L}_\infty) - d_e \bar{\psi}_e} \cdot \prod_{1 \leqslant w \leqslant \kappa,\, \mathfrak{b}(g_e)^w = \mathfrak{o}(g_e)} (-a_w),$$

where a_w is the blowup weight, $\mathfrak{b}(g_e)$ and $I_{[g_e]}$ are defined in § 3.1, $\bar{\psi}_e$ is the psi-class over $\overline{\mathscr{M}}_v$ and $t^{|I_{[g_e]}|+1}$ corresponding to the deformation of stable maps at the nodal point $\bar{\mathsf{n}}_e$ along the normal direction of S_* in W_r, and

– for each $v \in V_*^2$ with exactly two adjacent edges e_1, e_2, the associated nodal point $\bar{\mathsf{n}}_v$ contributes

$$\mathrm{cont}_{\bar{\mathsf{n}}_v} = \frac{t^{|I_{[g_{e_1}]}|+1}}{(-t - ev_\infty^* c_1(\mathsf{L}_\infty))/d_{e_1} + (-t - ev_\infty^* c_1(\mathsf{L}_\infty))/d_{e_2}} \cdot \prod_{1 \leqslant w \leqslant \kappa,\, \mathfrak{b}(g_v)^w = \mathfrak{o}(g_v)} (-a_w),$$

where $t^{|I_{[g_v]}|+1}$ corresponds to the deformation of stable maps at the nodal point $\bar{\mathsf{n}}_v$ along the normal direction of S_* in W_r.

The following result was proved in [CDW22, § 3.2.5] and [TY23, § 2.2.2].

Lemma 3.19. *When $r \gg 1$, under the transformation $\mathfrak{s} = tr$, for each vertex $u \in V_\infty$, the term $t \cdot \epsilon_* \mathrm{Cont}_u$ is a polynomial in r and rational in $\mathfrak{s}$.*

We next consider the product $\prod_{v \in V_*} \mathrm{Cont}_v$.

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LEMMA 3.20. *Suppose $\|\mathfrak{a}\| \geqslant 2$ and the normal bundle N of S in X is positive. If Φ has nonempty V_*, as a polynomial in t^{-1} the lowest degree in t^{-1} of the product $\prod_{v\in V_*} \mathrm{Cont}_v$ satisfies*

$$\deg^{\mathrm{low}}_{t^{-1}}\left(\prod_{v\in V_*}\mathrm{Cont}_v\right) \geqslant \sum_{v\in V_*^2\sqcup V_*^{1,1}} |I_{[g_v]}| + 2|V_*^S| + (\kappa-1)|V_*^1|$$

and equality holds implies $d_e = R_{[g_e]}$ for every $e \in E$.

Proof. For each edge $e \in E$, as a polynomial in t^{-1}, the lowest degree in t^{-1} of the contribution cont_e in (35) is

$$\mathrm{rk}_e := \left(\sum_{1\leqslant w\leqslant\kappa}\left[d_e a_w - \frac{\mathfrak{b}(g_e)^w}{\mathfrak{o}(g_e)}\right] + |I_{[g_e]}| + \kappa - \sum_{1\leqslant w\leqslant\kappa, d_e a_w - \mathfrak{b}(g_e)^w/\mathfrak{o}(g_e)\in\mathbf{Z}} 1\right) - (d_e - 1 + \{-d_e\}),$$

where $[\cdot]$ (respectively, $\{\cdot\}$) means the integer part (respectively, fractional part) of a real number, and $d_e \in \mathcal{R}_{[g_e]}$.

We claim that

$$\mathrm{rk}_e \geqslant |I_{[g_e]}| \tag{36}$$

and equality holds implies $d_e = R_{[g_e]}$ for every edge e.

Proof of the claim. Recall that for a $[g] \in \mathscr{T}^{\mathsf{S}}$

$$\mathcal{R}_{[g]} = \{R\in(0,\infty) \mid [g, e^{-2\pi i R}] \in \Lambda^{-1}([g])\} = \left\{\frac{k\mathfrak{o}(g)+\mathfrak{b}(g)^w}{a_w\mathfrak{o}(g)} \,\middle|\, 1\leqslant w\leqslant\kappa,\ k\in\mathbf{Z}_{\geqslant 0}\right\}.$$

Consider the function

$$\mathrm{rk}(R) = \left(\sum_{1\leqslant w\leqslant\kappa}\left[Ra_w - \frac{\mathfrak{b}(g)^w}{\mathfrak{o}(g)}\right] + |I_{[g]}| + \kappa - \sum_{1\leqslant w\leqslant\kappa, Ra_w - \mathfrak{b}(g)^w/\mathfrak{o}(g)\in\mathbf{Z}} 1\right) - (R-1+\{-R\})$$

defined on $\mathcal{R}_{[g]}$ for a fixed $[g] \in \mathscr{T}^{\mathsf{S}}$. Note that if $R \in \mathcal{R}_{[g]}$, then $R+1 \in \mathcal{R}_{[g]}$, and

$$\mathrm{rk}(R+1) = \mathrm{rk}(R) + \|\mathfrak{a}\| - 1.$$

Therefore, by the assumption that $|\mathfrak{a}| \geqslant 2$, $\mathrm{rk}(R)$ takes its minimal value at some certain points in $(0,1]\cap\mathcal{R}_{[g]}$. For an $R \in (0,1]\cap\mathcal{R}_{[g]}$, suppose there are exactly $(w_1,k_1),\ldots,(w_l,k_l)$ such that

$$R = \frac{k_j\mathfrak{o}(g)+\mathfrak{b}(g)^{w_j}}{a_{w_j}\mathfrak{o}(g)}, \quad \text{for } 1\leqslant j\leqslant l.$$

Then

$$\begin{aligned}\mathrm{rk}(R) &= \sum_{1\leqslant j\leqslant l} k_j + \sum_{1\leqslant w\leqslant\kappa,\ w\notin\{w_1,\ldots,w_l\}}\left[Ra_w - \frac{\mathfrak{b}(g)^w}{\mathfrak{o}(g)}\right] + |I_{[g]}| + \kappa - l\\ &\geqslant -1\cdot(\kappa-l) + |I_{[g]}| + \kappa - l = |I_{[g]}|,\end{aligned}$$

where equality holds if and only if $R = \mathfrak{b}(g)^{w_j}/a_{w_j}\mathfrak{o}(g)$, i.e. $k_j = 0$ for $1\leqslant j\leqslant l$, and R satisfies

$$-1 < Ra_w - \frac{\mathfrak{b}(g)^w}{\mathfrak{o}(g)} < 0$$

for all $w \notin \{w_1,\ldots,w_l\}$, i.e. $R = R_{[g]} = \min\mathcal{R}_{[g]}$. This finishes the proof of the claim. □

By applying the same argument as in the proof of Lemma 2.20 coupled with the claim (36) we get this lemma, thus we skip the details. □

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After we combine the contributions together, we get the following result.

Lemma 3.21. *Suppose* $\|\mathfrak{a}\| \geqslant 2$ *and the normal line bundle of* S *in* X *is positive. Then when* $r \gg 1$, *with respect to the transformation* $\mathfrak{s} = tr$, *for a graph* Φ *with nonempty* V_*, *the* $\mathrm{Cont}(\Phi)$ *is a polynomial in* r *with lowest degree*

$$\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) \geqslant |V_\infty| + \sum_{v \in V_*^2 \sqcup V_*^{1,1}} |I_{[g_v]}| + 2|V_*^S| + (\kappa - 1)|V_*^1|,$$

where the equality holds only if every edge e *in* Φ *satisfies* $d_e = R_{[g_e]}$.

We now come to the key lemma.

Lemma 3.22. *When* $\kappa \geqslant 2$, *under the assumption of Lemma 3.21, for a graph* Φ *with nonempty* V_*, *we have*

$$\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) \geqslant 1.$$

Moreover, $\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) = 1$ *if and only if* $|V_\infty| = 1$ *and* $V_* = V_*^{1,1} = \{v_1, \ldots, v_{n_0(\mathfrak{g})}\}$ *where each vertex* v_j *is labelled by* $[g_j]$ *and has a unique adjacent edge* e_j *with labels* $d_{e_j} = R_{[g_j]}$, $[g_{e_j}] = [g_j]$ *and* $[h_{e_j}] = [h_j] := [g_j^{-1}, e^{2\pi i R_{[g_j]}/r}]$.

Proof. When Φ contains no edges, we must have $|V_\infty| = 0, |V_*| = |V_*^S| = 1$, and consequently $\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) \geqslant 2$.

Now suppose Φ has edges. Then as $\kappa \geqslant 2$, we get $\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) \geqslant |V_\infty| \geqslant 1$, with equality when

$$\sum_{v \in V_*^2 \sqcup V_*^{1,1}} |I_{[g_v]}| + (\kappa + 1)|V_*^S| + (\kappa - 1)|V_*^1| = 0, \quad \text{and} \quad |V_\infty| = 1,$$

which implies $|V_*^S| = |V_*^1| = |V_*^2| = 0$ and $\sum_{v \in V_*^2 \sqcup V_*^{1,1}} |I_{[g_v]}| = \sum_{v \in V_*^{1,1}} |I_{[g_v]}| = 0$. Hence, $|I_{[g_v]}| = 0$ for all vertices $v \in V_*^{1,1}$. Then, by the definitions of $I_{[g_v]}$ in (19) and of $\mathscr{T}_0^{\mathsf{S}}$ in (20), $V_*^{1,1}$ contains exactly those markings decorated by $[g_j]$ for $1 \leqslant j \leqslant n_0(\mathfrak{g})$. Then by Lemma 3.20 the result follows. □

Let Φ_∞ be the graph in the previous lemma such that $\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) = 1$. When $\kappa \geqslant 2$, the graph exists only if $n_0(\mathfrak{g}) > 0$ and, it is unique if exists. We next compute $\mathrm{Cont}(\Phi_\infty)$. Denote the unique vertex in V_∞ by u_∞. Then we have $A_{u_\infty} = \widehat{A}$ in (27). The corresponding moduli space is

$$\overline{\mathscr{M}}_{u_\infty} \cong \overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},\widehat{A}}(\sqrt[r]{\mathsf{F}_\infty}).$$

For this case we also have a commutative diagram similar to (34). We have $|\mathrm{Aut}(\Phi_\infty)| = 1$ and

$$\begin{aligned}\mathrm{Cont}(\Phi_\infty) &= r^{n_0(\mathfrak{g})} \int_{[\overline{\mathscr{M}}_{u_\infty}]^{\mathrm{vir}}} \frac{c_{\mathrm{rk}}(-(\mathcal{R}\pi_*\mathsf{f}^*\sqrt[r]{\mathsf{L}_\infty})_{u_\infty} \otimes \mathcal{O}(1/r)) \cdot \prod_{j=1}^n ev_j^*(q^*\alpha_j|_{\mathsf{I}\sqrt[r]{\mathsf{F}_\infty}})}{\prod_{1 \leqslant j \leqslant n_0(\mathfrak{g})}(t + ev_j^* c_1(\mathsf{L}_\infty) - R_{[g_j]}\bar{\psi}_j)} \\ &= r^{n_0(\mathfrak{g})} \cdot \int_{[\overline{\mathscr{M}}_{0,\widehat{\mathfrak{g}},\widehat{A}}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}} \frac{c_{\mathrm{rk}}(-(\mathcal{R}\pi_*\mathsf{f}^*\sqrt[r]{\mathsf{L}_\infty})_{u_\infty} \otimes \mathcal{O}(1/r))}{\prod_{1 \leqslant j \leqslant n_0(\mathfrak{g})}(t + ev_j^* c_1(\mathsf{L}_\infty) - R_{[g_j]}\bar{\psi}_j)} \\ &\quad \cdot \prod_{j=1}^n ev_j^*(\pi^* p_{[\hat{g}_j]}^* \alpha_j) \wedge \prod_{j=1}^{n_0(\mathfrak{g})} ev_j^*([\sqrt[r]{\mathsf{Z}}[h_j^{-1}]])\end{aligned}$$

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$$= \frac{r^{n_0(\mathfrak{g})}}{t^{n_0(\mathfrak{g})}} \cdot \int_{[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},\hat{A}}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}} \frac{c_{\mathrm{rk}}(-(\mathcal{R}\pi_*\mathsf{f}^*\sqrt[r]{\mathsf{L}_\infty})_{u_\infty} \otimes \mathcal{O}(1/r)) \cdot \prod_{j=1}^n ev_j^*(\pi^* p_{[\hat{g}_j]}^* \alpha_j)}{\prod_{1\leqslant j\leqslant n_0(\mathfrak{g})}(1+(ev_j^* c_1(\mathsf{L}_\infty) - R_{[g_j]}\bar{\psi}_j)/t)}$$

$$= \frac{r^{n_0(\mathfrak{g})}}{t^{n_0(\mathfrak{g})}} \cdot \frac{t^{n_0(\mathfrak{g})-1}}{r^{n_0(\mathfrak{g})-1}} \cdot \int_{[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},\hat{A}}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(\pi^* p_{[\hat{g}_j]}^* \alpha_j)$$

$$= \frac{r}{t} \cdot \frac{1}{r} \cdot \int_{[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},\hat{A}}(\mathsf{F}_\infty)]^{\mathrm{vir}}} \prod_{j=1}^n ev_j^*(p_{[\hat{g}_j]}^* \alpha_j) = \frac{1}{t}\langle p_{[\hat{g}_1]}^*\alpha_1, \ldots, p_{[\hat{g}_n]}^*\alpha_n\rangle_{0,\hat{\mathfrak{g}},\hat{A}}^{\widehat{\mathsf{X}}_{\mathfrak{a}}}, \tag{37}$$

where:

- the third equality follows form

$$[\sqrt[r]{\mathsf{Z}}[h_j^{-1}]] = 1, \quad \forall\, 1 \leqslant j \leqslant n_0(\mathfrak{g})$$

since $\sqrt[r]{\mathsf{Z}}[h_j^{-1}] = \sqrt[r]{\mathsf{F}_\infty}[h_j^{-1}]$, which follows from the fact that $[g_j] \in \mathscr{T}_0^{\mathsf{S}}$ and $[h_j^{-1}] = [g_j, e^{-2\pi i R_{[g_j]}/r}]$ acts on the normal line bundle $\mathcal{O}_{\sqrt[r]{\mathsf{Z}}}(-r)$ of $\sqrt[r]{\mathsf{Z}}$ in $\sqrt[r]{\mathsf{F}_\infty}$ by multiplying $e^{2\pi i R_{[g_j]}} \neq 1$;
- the fourth equality follows from the dimension constraint and the fact that

$$\mathrm{rk}(-(\mathcal{R}\pi_*\mathsf{f}^*\sqrt[r]{\mathsf{L}_\infty})_{u_\infty}) = -\bigg(-\sum_{1\leqslant j\leqslant n_0(\mathfrak{g})} \frac{R_{[g_j]}}{r} + 1 - \sum_{1\leqslant j\leqslant n_0(\mathfrak{g})} \bigg\{-\frac{R_{[g_j]}}{r}\bigg\}\bigg) = n_0(\mathfrak{g}) - 1,$$

when $r \gg 1$;
- the fifth equality follows from the computation of orbifold Gromov–Witten invariants of banded gerbes in [TT21, Theorem 3.3], i.e. $\epsilon_*([\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},\hat{A}}(\sqrt[r]{\mathsf{F}_\infty})]^{\mathrm{vir}}) = (1/r)[\overline{\mathscr{M}}_{0,\hat{\mathfrak{g}},\hat{A}}(\mathsf{F}_\infty)]^{\mathrm{vir}}$.

We next consider the case that $\kappa = 1$. Thus, the blowup weight $\mathfrak{a} = (a)$ and $\|\mathfrak{a}\| = a$. As Lemma 3.22 we have the following result.

LEMMA 3.23. *When $\kappa = 1$, under the assumption of Lemma 3.21, for a graph Φ with nonempty V_*, we have*

$$\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) \geqslant 1.$$

Moreover, $\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) = 1$ if and only if:

- *$V_*^{1,1} = \{v_1, \ldots, v_{n_0(\mathfrak{g})}\}$ have exactly $n_0(\mathfrak{g})$ vertices where each vertex v_j is labelled by $[g_j]$ and has the unique adjacent edge e_j with labels $d_{e_j} = R_{[g_j]} = \mathfrak{b}(g_j)/a\mathfrak{o}(g_j)$, $[g_{e_j}] = [g_j]$ and $[h_{e_j}] = [g_j^{-1}, e^{2\pi i R_{[g_j]}/r}]$; and*
- *V_*^1 has exactly I_A unmarked vertices with each vertex having a unique adjacent edge e with labels $d_e = R_{[1]} = 1/a$, $[g_e] = [1]$ and $[h_e] = [1, e^{2\pi i/ra}]$.*

For this case, there is also exactly one graph with $\deg_r^{\mathrm{low}}(\mathrm{Cont}(\Phi)) = 1$, which we also denote by Φ_∞. By similar calculation as (37), the contribution of this Φ_∞ is

$$\mathrm{Cont}(\Phi_\infty) = \frac{1}{t} \cdot \langle p_{[\hat{g}_1]}^*\alpha_1, \ldots, p_{[\hat{g}_n]}^*\alpha_n, \underbrace{1, \ldots, 1}_{I_A}\rangle_{0,\hat{\mathfrak{g}}^A,A}^{\widehat{\mathsf{X}}_{\mathfrak{a}}}.$$

We skip the details of the proof.

This finishes the proof of Propositions 3.12, 3.13 and 3.14, and hence the weighted-blowup formulae in Theorems 3.7 and 3.8.

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4. The genus zero relative-orbifold correspondence

As one may note, our approach consists of two main ingredients: one is that the model W provides a relation between X and $\widehat{\mathsf{X}}_{\mathfrak{a}}$; second, we use a '$r$th root construction + polynomiality' argument to certain divisors to extract expected invariants. Along this line, in this section we apply our approach to give a new, direct proof of the genus zero relative-orbifold correspondence, i.e. Theorem 1.15. This time we apply root constructions twice to W to get a new space $\mathsf{W}_{r,s}$ and apply the virtual localization technique to a certain relative invariant of $\mathsf{W}_{r,s}$. To avoid tedious computations caused by orbifold structures, we only give a proof by assuming that X is smooth. The orbifold case can be dealt with similarly by following the computations in § 3.

We fix the following notation. Let S be a divisor of X, N be the normal line bundle of S in X and X_r be the rth root construction of X along S, i.e. the weight-r blowup of X along S. Thus, the exceptional divisor of X_r is the rth root gerbe $\sqrt[r]{N/S}$ of $N \to S$, denoted simply by $\sqrt[r]{S}$.

4.1 The symplectic cobordism between X and X_r

Let $W, D, Z, S_*, F_0, F_\infty, F_*$ be as in § 2.1. Let $B \subseteq W$ be the strict transform of $S \times \mathbf{P}^1 \subseteq X \times \mathbf{P}^1$ with respect to the blowup. Since S is a divisor, we have some simple facts:

- $F_\infty = \widehat{X} = X$, and $Z = S$ has normal line bundle N in $\widehat{X} = X$;
- B is a divisor of W; denote its normal line bundle by L_B;
- $B \cong S \times \mathbf{P}^1$, and $B \cap D = S_*$, $B \cap F_0 \cong S_0$.

Let W_r be the rth root construction of W along D (not F_∞ in § 2.1). Under the natural projection $W_r \to W$, the divisors lying over F_0, F_∞, D and B are

$$F_0, \quad F_{\infty,r} \cong X_r, \quad \sqrt[r]{D} := \sqrt[r]{\mathcal{O}_D(-1)/D} = \mathbf{P}_{(r,r)}(N \oplus \mathcal{O}_S), \quad \text{and} \quad B_r,$$

where B_r is the rth root construction of B along S_*. The $Z \subseteq D$ also lifts to $\sqrt[r]{Z} = \mathbf{P}_{(r,r)}(N \oplus 0) \cong \sqrt[r]{S}$. The $\mathbf{C}^*$-action lifts to W_r.

Lemma 4.1. *The fixed locus $W_r^{\mathbf{C}^*}$ of W_r with respect to the induced $\mathbf{C}^*$-action consists of three disjoint components*

$$F_0, \quad F_{\infty,r} \cong X_r, \quad \text{and} \quad \sqrt[r]{F_*} = \sqrt[r]{\mathcal{O}_D(-1)|_{F_*}/F_*} \cong \sqrt[r]{\mathcal{O}_S/S}.$$

The normal line bundle of F_0 in W_r is the same as its normal line bundle in W, the normal line bundle of $F_{\infty,r} \cong X_r$ in W_r is the pullback of L_∞ (still denoted by L_∞), and the normal bundle of $\sqrt[r]{F_*}$ in W_r is $N \oplus \sqrt[r]{\mathcal{O}_S}$ with action weight $(-1, 1/r)$.

We next take the sth root construction of W_r along $F_{\infty,r} \cong X_r$. Denote the resulting stack by $W_{r,s}$. Therefore, under the natural projection $W_{r,s} \to W_r \to W$, the divisors lying over F_0, F_∞, D and B are

$$F_0, \quad \sqrt[s]{F_{\infty,r}} := \sqrt[s]{L_\infty/F_{\infty,r}} \cong \sqrt[s]{X_r}, \quad \sqrt[r]{D_s} = \mathbf{P}_{(sr,r)}(N \oplus \mathcal{O}_S), \quad \text{and} \quad B_r,$$

where the normal line bundle of F_0 in $W_{r,s}$ is L_0; the normal line bundle of $\sqrt[s]{F_{\infty,r}} \cong \sqrt[s]{X_r}$ in $W_{r,s}$ is the sth root $\sqrt[s]{L_\infty}$ of $L_\infty \to F_{\infty,r}$; the $\sqrt[r]{D_s}$ is the sth root construction of $\sqrt[r]{D}$ along $\sqrt[r]{Z} = \mathbf{P}_{(r,r)}(N \oplus 0)$ and its normal line bundle in $W_{r,s}$ is the rth root line bundle $\sqrt[r]{\mathcal{O}_{D_s}(-s)}$ with $D_s = \mathbf{P}_{(s,1)}(N \oplus \mathcal{O}_S)$; and the normal line bundle of B_r in W_r is the pullback of L_B, which we still denote by L_B.

Lemma 4.2. *The $\mathbf{C}^*$-action lifts to $W_{r,s}$, whose fixed locus consists of three disjoint components F_0, $\sqrt[s]{F_{\infty,r}} \cong \sqrt[s]{X_r}$, and $\sqrt[r]{F_*}$.*

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In $W_{r,s}$, the intersection of $\sqrt[s]{F_{\infty,r}} \cong \sqrt[s]{X_r}$ and $\sqrt[r]{D_s} = \mathbf{P}_{(sr,r)}(N \oplus \mathcal{O}_S)$ is a two-step root gerbe over Z induced by the two line bundles $L_\infty|_Z$ and N. Note that $L_\infty|_Z$ and N are dual line bundles of each other. We denote the resulting root gerbe by $\sqrt[r,s]{Z}$. Its normal bundle in $W_{r,s}$ is $\sqrt[r]{N} \oplus \sqrt[s]{L_\infty|_Z}$. The twisted sectors of $\sqrt[r,s]{Z}$, $\sqrt[r]{F_*} \cong \sqrt[r]{S}$, etc. have similar descriptions as twisted sectors of W_r in (9). For example,

$$\sqrt[r,s]{Z}[j,k]$$

means the twisted sector of $\sqrt[r,s]{Z}$, whose action on $\sqrt[r]{N}$ and $\sqrt[s]{L_\infty|_Z}$ are multiplying by $e^{2\pi i(j/r)}$ and $e^{2\pi i(k/s)}$, respectively.

4.2 The genus zero relative-orbifold correspondence

Let $\Gamma = (0, A, m, \mu)$ be a topological data for genus zero relative stable maps into (X, S) where $\mu = (\mu_1, \dots, \mu_\rho) \in \mathbf{Z}^\rho_{\geqslant 1}$ is a partition of $d := S \cdot A$. Suppose $r \gg 1$. We next view Γ as a topological data Γ_r of absolute stable maps into X_r, by viewing the jth $(j = 1, \dots, \rho)$ relative marking decorated by μ_j as an orbifold absolute marking mapped into

$$\sqrt[r]{S}[\mu_j] \subseteq X_r[\mu_j],$$

and the m absolute markings as smooth markings. We write $\Gamma_r = (0, A, m, [\mu])$.

Thus, associated to Γ there is a relative moduli space $\overline{\mathscr{M}}_\Gamma(X \mid S)$, an absolute moduli space $\overline{\mathscr{M}}_{\Gamma_r}(X_r)$, and an absolute moduli space $\overline{\mathscr{M}}_{0,A,m+\rho}(X)$, together with two natural projections

$$\pi_{\mathrm{rel}} \colon \overline{\mathscr{M}}_\Gamma(X \mid S) \to \overline{\mathscr{M}}_{0,A,m+\rho}(X) \times_{X^\rho} S^\rho$$

by forgetting the relative information μ, and

$$\pi_{\mathrm{orb}} \colon \overline{\mathscr{M}}_{\Gamma_r}(X_r) \to \overline{\mathscr{M}}_{0,A,m+\rho}(X) \times_{X^\rho} S^\rho$$

by forgetting the rth root structures.

Theorem 4.3. *The following genus zero relative-orbifold correspondence holds:*

$$\pi_{\mathrm{orb},*}([\overline{\mathscr{M}}_{\Gamma_r}(X_r)]^{\mathrm{vir}}) = \pi_{\mathrm{rel},*}([\overline{\mathscr{M}}_\Gamma(X|S)]^{\mathrm{vir}}) \quad \text{when } r \gg 1. \tag{38}$$

We outline the proof of this theorem here and the details of the proof occupies the rest of this section.

By the two inclusions $X \to F_0, F_\infty \subseteq W$, from $A \in H_2(X; \mathbf{Z})$ we get $A_0, A_\infty \in H_2(W; \mathbf{Z})$. Let $[F]$ the fiber class of $D \to S_*$. Then we have $A_0 = A_\infty + d[F]$. Note that $B \cdot A_0 = S \cdot A = d$. Take the relative topological data

$$\Gamma_W := (0, A_0, m, \mu)$$

of $(W_{r,s}, B_r)$ with all absolute and relative markings being smooth markings. Let $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s}|B_r)$ be the corresponding relative moduli space. Then we have a natural projection

$$\pi_{\text{orb-rel}} \colon \overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \mid B_r) \to \overline{\mathscr{M}}_{0,A_0,m+\rho}(W_{r,s}) \times_{W^\rho_{r,s}} B^\rho_r \to \overline{\mathscr{M}}_{0,A,m+\rho}(X) \times_{X^\rho} S^\rho.$$

This map is $\mathbf{C}^*$-equivariant with $\mathbf{C}^*$ acting on $\overline{\mathscr{M}}_{0,A,m+\rho}(X) \times_{X^\rho} S^\rho$ trivially. Therefore, the push-forward of the virtual cycle of $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \mid B_r)$ will be a polynomial in the equivariant parameter t. In particular, the coefficient of t^{-1} will vanish, which gives us the genus zero relative-orbifold correspondence in (38) when $r, s \gg 1$.

4.3 Description of components of fixed locus of $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \mid B_r)$ for $r, s \gg 1$

The relative Gromov–Witten theory are defined by expanded degeneration (degenerated targets) (cf. [LR01, Li01, IP03, CLSZ11, AF16]). For $(W_{r,s}, B_r)$, an expanded degeneration is obtained

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by gluing $W_{r,s}$ along B_r with a chain of orbifold $\mathbf{P}^1$-bundle $\mathbf{P}(L_B \oplus \mathcal{O}_{B_r})$ (of length $l, l < \infty$) over B_r. These chains are called rubbers associated to $L_B \to B_r$. A rubber has two ends. We call the end glued with B_r in $W_{r,s}$ the infinity section of the rubber, and the other end the zero section of the rubber. There is a natural $(\mathbf{C}^*)^l$-action on the length l chain of $\mathbf{P}(L_B \oplus \mathcal{O}_{B_r})$ with jth $\mathbf{C}^*$ acting on the jth component by scaling on L_B. There is also an induced $\mathbf{C}^*$-action on L_B from the $\mathbf{C}^*$-action on $W_{r,s}$. Thus, we have a $\mathbf{C}^*$-action on chains of $\mathbf{P}(L_B \oplus \mathcal{O}_{B_r})$. In particular, for a length l chain of $\mathbf{P}(L_B \oplus \mathcal{O}_{B_r})$, the $(\mathbf{C}^*)^l$-action commutes with the $\mathbf{C}^*$-action. Hence, the $\mathbf{C}^*$-action on $W_{r,s}$ extends to a $\mathbf{C}^*$-action on every degenerate target. Note that the B_r is not fixed by the $\mathbf{C}^*$-action. The fixed locus of the rubber component associated to $L_B \to B_r$ consists of the rubber components associated to $L_B|_{S_0} \cong N \to S_0$ and $L_B|_{\sqrt[r]{S_*}} \cong N \to \sqrt[r]{S_*}$. We next describe components of fixed locus of $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \,|\, B_r)$ for $r, s \gg 1$.

Since the homology class $A_0 = A_\infty + d[F]$, those $\mathbf{C}^*$-fixed relative stable maps in $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \,|\, B_r)$ would have images in $F_0 \cong X$ with rubbers associated to $N \to S_0 \cong S$, or in $\sqrt[s]{F_{\infty,r}} \cup_{\sqrt[r,s]{Z}} \sqrt[r]{D_s}$ with rubbers associated to $N \to \sqrt[r]{S_*}$. We call them type (I) and type (II) $\mathbf{C}^*$-fixed relative stable maps respectively. The following lemma is obvious.

Lemma 4.4. *Those type (I) $\mathbf{C}^*$-fixed relative stable maps form a connected component of the $\mathbf{C}^*$-fixed locus of $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \,|\, B_r)$. This component is identified with $\overline{\mathscr{M}}_{\Gamma}(X \,|\, S)$.*

We next consider a type (II) $\mathbf{C}^*$-fixed relative stable map $f\colon C \to (W_{r,s}, B_r)$. We can decompose it into a combination of the following three parts:

- a (possible disconnected) degree $A'_\infty \in H_2(F_\infty; \mathbf{Z})$ stable map $f_\infty\colon C_\infty \to \sqrt[s]{F_{\infty,r}}$;
- a disjoint union of degree $d_j[F]$, $1 \le j \le \rho'$, simple[1] $\mathbf{C}^*$-fixed relative stable maps
$$f_j\colon (C_j, x_j, y_j) \to (\sqrt[r]{D_s}, \sqrt[r]{S_*}) \tag{39}$$
with an absolute marking x_j and a relative marking y_j;
- a degree $A_* \in H_2(S_*; \mathbf{Z})$ relative stable map $f^\sim\colon C^\sim \to \mathbf{P}^\sim$ into the rubber component $\mathbf{P}^\sim$ associated to $N \to \sqrt[r]{S_*}$.

Set $\mathbf{d} =: (d_1, \ldots, d_{\rho'})$ and $\|\mathbf{d}\| := \sum_{j=1}^{\rho'} d_j$. We have
$$A_0 = A'_\infty + \|\mathbf{d}\| \cdot [F] + A_*; \quad Z \cdot A'_\infty = \|\mathbf{d}\|; \quad S_* \cdot A_* = d - \|\mathbf{d}\|. \tag{40}$$

On the curve C, there are ρ' nodal points $(\mathsf{n}_1, \ldots, \mathsf{n}_{\rho'})$ connecting C_∞ with $\bigsqcup_{j=1}^{\rho'} C_j$. They provide corresponding markings on C_∞ and $\bigsqcup_{j=1}^{\rho'} C_j$, which are denoted by $(\check{x}_1, \ldots, \check{x}_{\rho'})$ and $(x_1, \ldots, x_{\rho'})$, respectively. Here $x_j \in C_j, 1 \le j \le \rho'$. Similarly, there are ρ' relative markings $(\check{y}_1, \ldots, \check{y}_{\rho'})$ on $C^\sim$ and relative markings $(y_1, \ldots, y_{\rho'})$ where $y_j \in C_j, 1 \le j \le \rho'$ that form ρ' nodal points $(\bar{\mathsf{n}}_1, \ldots, \bar{\mathsf{n}}_{\rho'})$ connecting $C^\sim$ with $\bigsqcup_{j=1}^{\rho'} C_j$. These $\check{y}_j, 1 \leqslant j \leqslant \rho'$, are mapped into the infinity section of the rubber. The $C^\sim$ also has $\ell(\mu)$ smooth relative markings decorated by μ and mapped into the zero section of the rubber.

The original m smooth absolute markings are distributed only on C_∞ and $C^\sim$. Suppose there are m_∞ (respectively, $m^\sim$) smooth absolute markings on C_∞ (respectively, $C^\sim$). Then $m = m_\infty + m^\sim$.

Now for each $f_j\colon (C_j, x_j, y_j) \to (\sqrt[r]{D_s}, \sqrt[r]{S_*})$ in (39), the contact order at the relative marking y_j is d_j. Suppose that the relative marking y_j is mapped into $\sqrt[r]{S_*}[\lambda_j]$ with $0 \leqslant \lambda_j \leqslant r-1$, and the absolute marking x_j is mapped into
$$\sqrt[r,s]{Z}[\nu_j, d_j] = \sqrt[r,s]{Z}[\nu_j, d_j]$$

[1] Here a simple S^1-fixed stable map means it is fixed by the S^1 and the target is not expanded. See [GV05].

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for a certain $0 \leqslant \nu_j \leqslant r-1$. We label the absolute marking x_j by $[\nu_j, d_j]$, and the relative marking y_j by $([\lambda_j], d_j)$. By applying the orbifold Riemann–Roch theorem (cf. [CR04, Proposition 4.2.2]) to the pullback bundle $f_j^* \sqrt[r]{\mathcal{O}_{D_s}(-s)} \to C_j$, we see that these λ_j, ν_j, d_j satisfy the relation

$$r \mid \lambda_j + \nu_j + d_j. \tag{41}$$

Therefore, λ_j is determined by ν_j and d_j.

Since the absolute marking $\check{x}_j$ over C_∞ and the absolute marking x_j over C_j form the nodal point n_j, the absolute marking $\check{x}_j$ is mapped into

$$\sqrt[r,s]{Z}[r-\nu_j, s-d_j].$$

We label the absolute marking $\check{x}_j$ by $[r-\nu_j, s-d_j]$, and set

$$\mathfrak{h} := ([r-\nu_1, s-d_1], \ldots, [r-\nu_{\rho'}, s-d_{\rho'}]).$$

Similarly, as $\check{y}_j$ and y_j forms the nodal point $\bar{\mathsf{n}}_j$, the relative marking $\check{y}_j$ over $C^\sim$ is mapped into $\sqrt[r]{S_*}[r-\lambda_j]$ and also has contact order d_j. We label the relative marking $\check{y}_j$ by $([r-\lambda_j], d_j)$, and set

$$\mathfrak{r} := (([r-\lambda_1], d_1), \ldots, ([r-\lambda_{\rho'}], d_{\rho'})).$$

To summarize, the three parts of the decomposition of a type (II) fixed relative stable map $f\colon C \to (W_{r,s}, B_r)$ have topological data

$$\Gamma_\infty := (0, A'_\infty, m_\infty, \mathfrak{h}), \quad \Gamma_E := \bigsqcup_{j=1}^{\rho'} (0, d_j[F], [\nu_j, d_j], ([\lambda_j], d_j)), \quad \Gamma^\sim := (0, A_*, m^\sim, \mathfrak{r}, \mu). \tag{42}$$

The decomposed stable maps may have disconnected domain curves. We use the superscript '$\bullet$' to denote the moduli space of stable maps with possible disconnected domain curves.

Lemma 4.5. *Those type (II) $\mathbf{C}^*$-fixed relative stable maps corresponding to the same decomposition of topological data as given by (42) form a connected component of the fixed locus of $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \,|\, B_r)$, which is*

$$\overline{\mathscr{M}}_{\Gamma_\infty, \Gamma_E, \Gamma^\sim} := \frac{\overline{\mathscr{M}}_{\Gamma_\infty}(\sqrt[s]{X_r})^\bullet \times_{(I\sqrt[r,s]{Z})^{\rho'}} \overline{\mathscr{M}}^\bullet_{\Gamma_E}(\sqrt[r]{D_s} \,|\, \sqrt[r]{S_*})^{\mathbf{C}^*} \times_{(I\sqrt[r]{S_*})^{\rho'}} \overline{\mathscr{M}}^\sim_{\Gamma^\sim}}{|\mathrm{Aut}(\Gamma_\infty, \Gamma_E, \Gamma^\sim)|}, \tag{43}$$

where $\mathrm{Aut}(\Gamma_\infty, \Gamma_E, \Gamma^\sim)$ is the automorphism group of the decomposition $(\Gamma_\infty, \Gamma_E, \Gamma^\sim)$ of Γ_W, and $\overline{\mathscr{M}}^\sim_{\Gamma^\sim}$ is the moduli space of stable maps into the rubber component associated to $N \to \sqrt[r]{S_*}$.

There is a special component for which the target is not expanded. For this component the type (II) $\mathbf{C}^*$-fixed relative stable maps decompose into two parts with topological data

$$\Gamma^*_\infty := (0, A_\infty, m, \mathfrak{h}^*), \quad \Gamma^*_E := \bigsqcup_{j=1}^{\rho} (0, \mu_j[F], [r-\mu_j, \mu_j], ([0], \mu_j)), \tag{44}$$

where $\mathfrak{h}^* = ([\mu_1, s-\mu_1], \ldots, [\mu_\rho, s-\mu_\rho])$. The corresponding component of the fixed locus is

$$\overline{\mathscr{M}}_{\Gamma^*_\infty, \Gamma^*_E} = \overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r}) \times_{(I\sqrt[r,s]{Z})^\rho} \overline{\mathscr{M}}^\bullet_{\Gamma^*_E}(\sqrt[r]{D_s} \,|\, \sqrt[r]{S_*})^{\mathbf{C}^*}. \tag{45}$$

4.4 Contributions of components of fixed locus

We now compute the equivariant Euler class of the virtual normal bundle of each component of the fixed locus of $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \,|\, B_r)$, hence its contribution $\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s} \,|\, B_r)$, when $r, s \gg 1$.

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4.4.1 *The component* $\overline{\mathscr{M}}_\Gamma(X\,|\,S)$. For this component of fixed locus, its normal bundle is induced from the normal bundle $L_0 = F_0 \times T_0\mathbf{P}^1$ of F_0, a trivial line bundle with action weight -1. For this case, we also have the universal curve and the universal map

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} & \longrightarrow & X \\ {\scriptstyle\pi}\downarrow & & \downarrow & & \\ \overline{\mathscr{M}}_\Gamma(X|S) & \longrightarrow & \mathcal{T} & & \end{array}$$

where $\mathcal{C}$ is the universal curve, $\mathcal{X}$ is the universal family over $\mathcal{T}$, and $\mathcal{T}$ is the stack of expanded relative pairs of (X, S) (cf. [GV05, § 2.5] and [Li01, § 4.1]), from which we have the induced index bundle $(L_0)_\Gamma := \mathcal{R}^*\pi_*f^*L_0$ over $\overline{\mathscr{M}}_\Gamma(X\,|\,S)$. The bundle $(L_0)_\Gamma$ is a line bundle with action weight -1. Thus, the contribution of the component $\overline{\mathscr{M}}_\Gamma(X\,|\,S)$ to $[\overline{\mathscr{M}}_\Gamma(W_{r,s}\,|\,B_r)]^{\mathrm{vir}}$ is

$$\mathrm{Cont}_0 := \frac{[\overline{\mathscr{M}}_\Gamma(X\,|\,S)]^{\mathrm{vir}}}{c_1((L_0)_\Gamma \otimes \mathcal{O}(-1))} = \frac{[\overline{\mathscr{M}}_\Gamma(X\,|\,S)]^{\mathrm{vir}}}{-t + c_1((L_0)_\Gamma)}. \tag{46}$$

4.4.2 *The component* $\overline{\mathscr{M}}_{\Gamma^*_\infty,\Gamma^*_E}$. In (45), $\overline{\mathscr{M}}^{\bullet}_{\Gamma^*_E}(\sqrt[r]{D_s}\,|\,\sqrt[r]{S_*})^{\mathbf{C}^*}$ is a disjoint union of the simple fixed locus of ρ moduli spaces of relative stable maps of $(\sqrt[r]{D_s}, \sqrt[r]{S_*})$, with jth one having the topological data

$$(0, \mu_j[F], [r - \mu_j, \mu_j], ([0], \mu_j))$$

in (44). Thus, the relative marking y_j is smooth with contact order μ_j, and the absolute marking x_j is an orbifold marking mapped into $\sqrt[r,s]{Z}[r - \mu_j, \mu_j]$. We denote the simple fixed locus of the jth one by $\mathscr{F}_{\mu_j}$. There is a natural corresponding moduli space of relative stable maps into (D_s, S_*) whose topological data are

$$(0, \mu_j[F], [\mu_j], \mu_j).$$

We denote the corresponding simple fixed locus by $\mathscr{F}'_{\mu_j}$. Since $\sqrt[r]{D_s}$ is a root gerbe (hence, banded) over D_s, by the analysis of moduli space of banded gerbes in [TT21, § 5], we see that $\mathscr{F}_{\mu_j}$ is a $\mathbf{Z}_r$-gerbe over $\mathscr{F}'_{\mu_j}$, and

$$[\mathscr{F}_{\mu_j}]^{\mathrm{vir}} = \frac{1}{r} \cdot [\mathscr{F}'_{\mu_j}]^{\mathrm{vir}}.$$

Moreover, as $\mathscr{F}'_{\mu_j}$ is a $\mathbf{Z}_{\mu_j}$-gerbe over $S_* = Z$, we have

$$[\mathscr{F}_{\mu_j}]^{\mathrm{vir}} = \frac{1}{r} \cdot [\mathscr{F}'_{\mu_j}]^{\mathrm{vir}} = \frac{1}{r\mu_j} S_*. \tag{47}$$

Since $\sqrt[r,s]{Z}$ is a $\mathbf{Z}_r \times \mathbf{Z}_s$-gerbe over $Z = S_*$, combining with (47), we get

$$\overline{\mathscr{M}}_{\Gamma^*_\infty,\Gamma^*_E} = \overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r}) \times_{(I\,\sqrt[r,s]{Z})^\rho} \prod_{1\leqslant j\leqslant\rho} \mathscr{F}_{\mu_j} = \frac{s^\rho}{\prod_j \mu_j} \cdot \overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r}). \tag{48}$$

We now consider the virtual normal bundle, which consists of two parts.

– An index bundle $(\sqrt[s]{L_\infty})_{\Gamma^*_\infty} := \mathcal{R}\pi_*f^*\sqrt[s]{L_\infty}$ over $\overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r})$ induced from $\sqrt[s]{L_\infty}$ with action weight $1/s$. As $s \gg 1$, the rank of the bundle is $1 - \rho$. Thus, the equivariant Euler class is

$$c_{\mathrm{rk}}\bigg(-(\sqrt[s]{L_\infty})_{\Gamma^*_\infty} \otimes \mathcal{O}\Big(\frac{1}{s}\Big)\bigg) = \sum_{0\leqslant k\leqslant\rho-1} \Big(\frac{t}{s}\Big)^{\rho-1-k} c_k(-(\sqrt[s]{L_\infty})_{\Gamma^*_\infty}).$$

– For $1 \leqslant j \leqslant \rho$, the nodal point n_j contributes a subline bundle whose equivariant Euler class is

$$\frac{1}{(t + ev_{\mathsf{n}_j}^*(c_1(L_\infty\,|\,_Z)))/\mu_j - \bar{\psi}_j} = \frac{\mu_j}{t + ev_{\mathsf{n}_j}^*(c_1(L_\infty\,|\,_Z)) - \mu_j\bar{\psi}_j}.$$

Thus, the contribution of the component $\overline{\mathscr{M}}_{\Gamma_\infty^*, \Gamma_E^*}$ is

$$\mathrm{Cont}_* := [\overline{\mathscr{M}}_{\Gamma_\infty^*}(\sqrt[s]{X_r})]^{\mathrm{vir}} \cap \frac{\sum_{0\leqslant k\leqslant \rho-1} t^{\rho-1-k} s^{k+1} c_k(-(\sqrt[s]{L_\infty})_{\Gamma_\infty^*})}{\prod_{1\leqslant j\leqslant \rho}(t + ev_{\mathsf{n}_j}^* c_1(L_\infty|_Z) - \mu_j\bar{\psi}_j)}. \tag{49}$$

4.4.3 *The component* $\overline{\mathscr{M}}_{\Gamma_\infty,\Gamma_E,\Gamma^\sim}$. Now consider a general decomposition (42). For $1 \leqslant j \leqslant \rho'$, each

$$(0, d_j[F], [\nu_j, d_j], ([\lambda_j], d_j))$$

in (42) determines a simple fixed locus of the corresponding moduli space of relative stable maps into $(\sqrt[r]{D_s}, \sqrt[r]{S_*})$, which we denote by $\mathscr{F}_j$. As (47) we have

$$[\mathscr{F}_j]^{\mathrm{vir}} = \frac{S_*}{rd_j}.$$

Thus, the component $\overline{\mathscr{M}}_{\Gamma_\infty,\Gamma_E,\Gamma^\sim}$ is

$$\begin{aligned}\overline{\mathscr{M}}_{\Gamma_\infty,\Gamma_E,\Gamma^\sim} &= \frac{\overline{\mathscr{M}}^\bullet_{\Gamma_\infty}(\sqrt[s]{X_r}) \times_{(I\,\sqrt[r]{\sqrt[s]{Z}})^{\rho'}} \left(\prod_{1\leqslant j\leqslant\rho'} \mathscr{F}_j\right) \times_{(I\,\sqrt[r]{S_*})^{\rho'}} \overline{\mathscr{M}}^\sim_{\Gamma^\sim}}{|\mathrm{Aut}(\Gamma_\infty, \Gamma_E, \Gamma^\sim)|} \\ &= \frac{(rs)^{\rho'}}{\prod_{j=1}^{\rho'} d_j} \cdot \frac{\overline{\mathscr{M}}^\bullet_{\Gamma_\infty}(\sqrt[s]{X_r}) \times_{S_*^{\rho'}} \overline{\mathscr{M}}^\sim_{\Gamma^\sim}}{|\mathrm{Aut}(\Gamma_\infty, \Gamma_E, \Gamma^\sim)|}.\end{aligned}$$

The equivariant Euler class of the virtual normal bundle consists of the following.

– Suppose the number of connected components of Γ_∞ is κ, and each component is indexed by $\Gamma_{\infty,j}$, $1 \leqslant j \leqslant \kappa$. Suppose the ρ' nodal points on C_∞ are distributed on each component by the partition

$$(\rho'_1, \ldots, \rho'_\kappa),$$

i.e. $\sum_j \rho'_j = \rho'$. Then the equivariant Euler class of the induced index bundle $(\sqrt[s]{L_\infty})_{\Gamma_{\infty,j}} := \mathcal{R}\pi_* f^* \sqrt[s]{L_\infty}$ over the component of $\Gamma_{\infty,j}$ is

$$c_{\rho'_j-1}\bigg(-(\sqrt[s]{L_\infty})_{\Gamma_{\infty,j}} \otimes \mathcal{O}\Big(\frac{1}{s}\Big)\bigg).$$

Therefore, the total equivariant Euler class is

$$\prod_{j=1}^{\kappa} c_{\rho'_j-1}\bigg(-(\sqrt[s]{L_\infty})_{\Gamma_{\infty,j}} \otimes \mathcal{O}\Big(\frac{1}{s}\Big)\bigg) = \prod_{j=1}^{\kappa}\sum_{k=0}^{\rho'_j-1}\Big(\frac{t}{s}\Big)^{\rho'_j-1-k} c_k(-(\sqrt[s]{L_\infty})_{\Gamma_{\infty,j}}).$$

– For $1 \leqslant j \leqslant \rho'$, the nodal point n_j corresponding a subline bundle and its equivariant Euler class is

$$\frac{1}{(t + ev_{\mathsf{n}_j}^*(c_1(L_\infty|_Z)))/d_j - \bar{\psi}_j} = \frac{d_j}{t + ev_{\mathsf{n}_j}^*(c_1(L_\infty|_Z)) - d_j\bar{\psi}_j}.$$

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– The rubber component contributes a subline bundle and its equivariant Euler class is

$$-\prod_{j=1}^{\rho'} d_j \cdot \frac{1}{t+\Psi_\infty}$$

where Ψ_∞ is the target Psi class; see, for example, [JPPZ20, § 3.4].

Thus, the total contribution of this component of the fixed locus is

$$\mathrm{Cont}_{\Gamma_\infty,\Gamma_E,\Gamma^\sim} := \frac{[\overline{\mathscr{M}}^\bullet_{\Gamma_\infty}(\sqrt[s]{X_r}) \times_{S_*^{\rho'}} \overline{\mathscr{M}}^\sim_{\Gamma^\sim}]^{\mathrm{vir}}}{|\mathrm{Aut}(\Gamma_\infty,\Gamma_E,\Gamma^\sim)|} \cap \\ -\frac{\prod_{j=1}^{\rho'} d_j}{t+\Psi_\infty} \cdot \frac{\prod_{j=1}^{\kappa}\sum_{k=0}^{\rho'_j-1} t^{\rho'_j-1-k} s^{k+1} c_k(-(\sqrt[s]{L_\infty})_{\Gamma_{\infty,j}})}{\prod_{1\leqslant j\leqslant \rho'}(t+ev^*_{\mathsf{n}_j}c_1(L_\infty|_Z)-d_j\bar\psi_j)}. \tag{50}$$

4.5 Proof of the genus zero relative-orbifold correspondence

By the localization formula, we have

$$[\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s}\,|\,B_r)]^{\mathrm{vir}} = \mathrm{Cont}_0 + \mathrm{Cont}_* + \sum_{(\Gamma_\infty,\Gamma_E,\Gamma^\sim)} \mathrm{Cont}_{\Gamma_\infty,\Gamma_E,\Gamma^\sim}. \tag{51}$$

Now we push it forward to $\overline{\mathscr{M}}_{0,n+\rho,A}(X)\times_{X^\rho} S^\rho$ via the natural projection

$$\pi_{\text{orb-rel}}\colon \overline{\mathscr{M}}_{\Gamma_W}(W_{r,s}\,|\,B_r) \to \overline{\mathscr{M}}_{0,n+\rho,A}(X)\times_{X^\rho} S^\rho.$$

Thus, since the $\mathbf{C}^*$-action on $\overline{\mathscr{M}}_{0,n+\rho,A}(X)\times_{X^\rho} S^\rho$ is trivial, the push-forward

$$\pi_{\text{orb-rel},*}(t\cdot[\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s}\,|\,B_r)]^{\mathrm{vir}})$$

is a polynomial in t with vanishing constant term. We next extract the coefficient of t^0 in $\pi_{\text{orb-rel},*}(t\cdot[\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s}\,|\,B_r)]^{\mathrm{vir}})$. By (51), we have

$$\pi_{\text{orb-rel},*}(t\cdot[\overline{\mathscr{M}}_{\Gamma_W}(W_{r,s}\,|\,B_r)]^{\mathrm{vir}}) = \pi_{\text{orb-rel},*}(t\cdot\mathrm{Cont}_0)+\pi_{\text{orb-rel},*}(t\cdot\mathrm{Cont}_*) \\ + \sum_{(\Gamma_\infty,\Gamma_E,\Gamma^\sim)} \pi_{\text{orb-rel},*}(t\cdot\mathrm{Cont}_{\Gamma_\infty,\Gamma_E,\Gamma^\sim}). \tag{52}$$

For the coefficient of t^0 in each term on the right-hand side of (52) we have the following simplifications.

(i) The constant term of the first term is

$$\big[\pi_{\text{orb-rel},*}(t\cdot\mathrm{Cont}_0)\big]_{t^0} = \left[\pi_{\text{orb-rel},*}\left(\frac{t[\overline{\mathscr{M}}_\Gamma(X\,|\,S)]^{\mathrm{vir}}}{-t+c_1((L_0)_\Gamma)}\right)\right]_{t^0} = -\pi_{\mathrm{rel},*}[\overline{\mathscr{M}}_\Gamma(X\,|\,S)]^{\mathrm{vir}},$$

where we have used the fact that the restriction of $\pi_{\text{orb-rel}}$ on $\overline{\mathscr{M}}_\Gamma(X\,|\,S)$ is π_{rel}.

(ii) For the second term, we first have the simplification

$$\pi_{\text{orb-rel},*}(t\cdot\mathrm{Cont}_*) = \pi_{\mathrm{orb},*}\left([\overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r})]^{\mathrm{vir}} \cap \frac{\sum_{0\leqslant k\leqslant \rho-1} t^{-k}s^{k+1}c_k(-(\sqrt[s]{L_\infty})_{\Gamma^*_\infty})}{\prod_{1\leqslant j\leqslant\rho}(1+(ev^*_{\mathsf{n}_j}c_1(L_\infty|_Z)-\mu_j\bar\psi_j)/t)}\right),$$

where we have used the fact that the restriction of $\pi_{\text{orb-rel}}$ on $\overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r})$ is exactly the $\pi_{\mathrm{orb}}\colon \overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r}) \to \overline{\mathscr{M}}_\Gamma(X)\times_{X^\rho} S^\rho$. So the constant term in t is

$$[\pi_{\text{orb-rel},*}(t\cdot\mathrm{Cont}_*)]_{t^0} = \pi_{\mathrm{orb},*}\big(s\cdot[\overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r})]^{\mathrm{vir}}\big).$$

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Note that the projection $\pi_{\mathrm{orb}}\colon \overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r}) \to \overline{\mathscr{M}}_{0,A,m+\rho}(X) \times_{X^\rho} S^\rho$ splits into

$$\overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r}) \xrightarrow{\pi'_{\mathrm{orb}}} \overline{\mathscr{M}}_{\Gamma_r}(X_r) \xrightarrow{\pi_{\mathrm{orb}}} \overline{\mathscr{M}}_{0,A,m+\rho}(X) \times_{X^\rho} S^\rho,$$

since the Γ^*_∞ (see (44)) is a lifting of $\Gamma_r = (0, A, m, [\mu])$ to $\sqrt[s]{X_r}$. Therefore we have

$$\begin{aligned}\pi_{\mathrm{orb},*}\big(s \cdot [\overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r})]^{\mathrm{vir}}\big) &= \pi_{\mathrm{orb},*} \circ \pi'_{\mathrm{orb},*}\big(s \cdot [\overline{\mathscr{M}}_{\Gamma^*_\infty}(\sqrt[s]{X_r})]^{\mathrm{vir}}\big) \\ &= \pi_{\mathrm{orb},*}\left(\frac{1}{s} \cdot s \cdot [\overline{\mathscr{M}}_{\Gamma_r}(X_r)]^{\mathrm{vir}}\right) = \pi_{\mathrm{orb},*}([\overline{\mathscr{M}}_{\Gamma_r}(X_r)]^{\mathrm{vir}}),\end{aligned}$$

where for the second equality we have used the computation of push-forward of virtual fundamental classes of moduli spaces of stable maps of banded gerbes (cf. [TT21, Theorem 3.3]). We conclude that

$$[\pi_{\text{orb-rel},*}(t \cdot \mathrm{Cont}_*)]_{t^0} = \pi_{\mathrm{orb},*}([\overline{\mathscr{M}}_{\Gamma_r}(X_r)]^{\mathrm{vir}}). \tag{53}$$

(iii) The third term on the right-hand side of (52) is

$$\pi_{\text{orb-rel},*}(t \cdot \mathrm{Cont}_{\Gamma_\infty,\Gamma_E,\Gamma^\sim}) = \pi_{\text{orb-rel},*}\Bigg(\frac{[\overline{\mathscr{M}}^{\bullet}_{\Gamma_\infty}(\sqrt[s]{X_r}) \times_{S^{\rho'}_*} \overline{\mathscr{M}}^{\bullet,\sim}_{\Gamma^\sim}]^{\mathrm{vir}}}{|\mathrm{Aut}(\Gamma_\infty,\Gamma_E,\Gamma^\sim)|} \cap \\ -\frac{t \cdot \prod_{j=1}^{\rho'} d_j}{t+\Psi_\infty} \cdot \frac{\prod_{j=1}^{\kappa}\sum_{h=0}^{\rho'_j-1} t^{\rho'_j-1-h} s^{h+1} c_h(-(\sqrt[s]{L_\infty})_{\Gamma_{\infty,j}})}{\prod_{1\leqslant j\leqslant \rho'}(t + ev^*_{\mathsf{n}_j} c_1(L_\infty|_Z) - m_j\bar\psi_j)}\Bigg).$$

It is straightforward to find that as a polynomial in t^{-1} its lowest degree is $\kappa \geqslant 1$. Thus, the constant term of the third term of the right-hand side of (52) vanishes.

Therefore, by the vanishing of the constant term of $\pi_{\text{orb-rel},*}(t \cdot [\overline{\mathscr{M}}_\Gamma(W_r \mid B_r)]^{\mathrm{vir}})$ we get

$$\pi_{\mathrm{orb},*}([\overline{\mathscr{M}}_{\Gamma_r}(X_r)]^{\mathrm{vir}}) = \pi_{\mathrm{rel},*}[\overline{\mathscr{M}}_\Gamma(X \mid S)]^{\mathrm{vir}}.$$

This finishes the proof of Theorem 4.3. The proof of the orbifold case is similar and follows from similar computations as those in § 3.

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CONFLICTS OF INTEREST

None.

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