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THE REAL SPECTRUM OF HIGHER LEVEL OF A COMMUTATIVE RING

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ABSTRACT. The following paper defines a new type of ordering of higher level on a commutative ring. This definition allows the set of all orderings of level n to be given a topology which we show is consistent with the topology of the real spectrum.

Introduction. In the 1930's Artin, Schreier and Baer [1,2,3] each studied orders on a field. Recently Becker broadened this to orderings of higher level on a field [5] (with further work in [7] by Becker, Harman, and Rosenberg). Here orders are called orderings of level 1. Orders were also generalized by Coste and Coste-Roy to orderings of level 1 on a commutative ring with unit [11] (see also Becker [4], and Lam [15]). For a commutative ring A, these can be viewed as pairs $(\wp, \chi)$ where $\wp$ is a prime ideal of A and χ is a signature from the quotient field of $A/\wp$ into $\{\pm 1\}$. The resulting structure is called the real spectrum. The work in this area has evolved into a foundation for Real Semi-Algebraic Geometry, where the objects of study are sets which can be defined by a finite number of polynomial equalities and inequalities.

This paper extends the order concept to that of orderings of level n on a commutative ring, and is modeled on Becker's presentation of the level 1 case for commutative rings [4]. Orderings of level n on a ring A are defined so as to be in one to one correspondence with the sets $(\wp, \chi)$ where $\wp$ is a prime ideal of A and χ is a signature from the quotient field of $A/\wp$ into the $2n^{\text{th}}$ roots of unity. Note, we identify an ordering with a signature. A second type of ordering of level n on a ring, which may be identified with the kernel of a signature, has also been defined [6, 16].

We define the real spectrum of level n of A, denoted R_n-spec A, to be the set of all orderings of level n on A. We examine two topologies on R_n-spec A. And show that R_n-spec A is a contravariant functor from the category of commutative rings with unit into the category of topological spaces. In future papers we hope to further investigate real algebraic geometry of higher level. For other work in this area see also Berr [9].

1. **Defining R_n-spec A.** Given a ring A, let $\dot{A} = A \setminus \{0\}$. Further, given any prime ideal of $\wp$ of A, let $k(\wp)$ denote the quotient field of $A/\wp$, and $\dot{k}(\wp)$ denote $k(\wp) \setminus \{0\}$. Finally by $\bar{a}$ we shall denote $a + \wp$ in $A/\wp \subseteq k(\wp)$.

Let $\mu(m) = \{z \in \mathbb{C} \mid z^m = 1\}$ be the group of m^{th} roots of unity, and let $\mu = \varinjlim \mu(m)$ be the group of all roots of unity. A signature χ is a homomorphism of the

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multiplicative group of a field, with values in μ, whose kernel is additively closed. This implies that if a signature has image $\mu(t)$ then t is even [7]. We say a signature has level n if its image is a subset of $\mu(2n)$, and exact level n if its image equals $\mu(2n)$.

An order on a field F is a subset P of F, such that $P+P \subseteq P$, $P \cdot P \subseteq P$, $P \cap -P = 0$, and $P \cup -P = F$. It can also be defined as the kernel of a signature $\chi\colon \dot{F} \longrightarrow \mu(2)$, where every such signature yields an order $P = \ker \chi \cup \{0\}$.

An order on a ring A is a subset α of A, such that $\alpha+\alpha \subseteq \alpha$, $\alpha \cdot \alpha \subseteq \alpha$, $\alpha \cap -\alpha =$ a prime ideal, and $\alpha \cup -\alpha = A$. Orders on rings are equivalent to pairs $(\wp, \bar{P})$ where $\wp$ is a prime ideal and $\bar{P}$ is an order on $k(\wp)$. Indeed, if $\pi\colon A \longrightarrow A/\wp$ is the canonical projection, then $\alpha = \pi^{-1}(\bar{P} \cap A/\wp)$, [4]. Since $\bar{P} = \ker \chi \cup \{0\}$ where $\chi\colon \dot{k}(\wp) \longrightarrow \mu(2)$, it is clear that $\bar{P}$ completely determines χ (or $\ker \chi$) and that χ (or $\ker \chi$) completely determines $\bar{P}$. Therefore the pairs $(\wp, \bar{P})$, $(\wp, \chi)$ and $(\wp, \ker \chi)$ are equivalent, furthermore it has been shown all are equivalent to α where $\alpha = \{a \in A \mid (a+\wp) \in \bar{P} = \ker \chi \cup \{0\}\}$ [4].

We now consider the pairs $(\wp, \chi)$, where χ is a signature of level n on $\dot{k}(\wp)$. Again there are homomorphisms $A \longrightarrow k(\wp)$ and $\dot{k}(\wp) \longrightarrow \mu(2n)$ and we have

THEOREM 1.1. *Let A be a ring, $\wp \in$ spec A, χ a signature of level n on $\dot{k}(\wp)$ and ζ a primitive $2n^{\text{th}}$ root of unity. If $\alpha_i = \{a \in A \mid a+\wp = \bar{a} \in \chi^{-1}(\zeta^i) \cup \{0\}\}$, then the family of subsets $\alpha_1, \ldots, \alpha_{2n}$ obeys the following rules:*

(i) $A = \alpha_1 \cup \cdots \cup \alpha_{2n}$

(ii) $\alpha_i \cap \alpha_j = \wp$ *if* $i \neq j$, *and denoting* $\alpha_i \setminus \wp$ *by* α_i^*

(iii) $\alpha_i^* + \alpha_i^* \subseteq \alpha_i^*$

(iv) $\alpha_i^* \cdot \alpha_j^* \subseteq \alpha_k^*$ *where* $k = i+j$ *if* $i+j \leq 2n$, *and* $k = i+j-2n$ *if* $i+j > 2n$.

PROOF. The first two statements are clear, and the fourth is easily verified. To prove the third statement we let a_i and b_i be in α_i^*, so $\chi(\bar{a_i}) = \zeta^i$ and $\chi(\bar{b_i}) = \zeta^i$. To compute $\chi(\overline{a_i+b_i})$, we first note that $\bar{a_i} = \bar{b_i} \cdot \bar{c}/\bar{d}$ where $\bar{c}/\bar{d} \in k(\wp)$. Furthermore, $\bar{c}/\bar{d}$ is in the kernel of χ since $\zeta^i = \chi(\bar{a_i}) = \chi(\bar{b_i} \cdot \bar{c}/\bar{d}) = \chi(\bar{b_i}) \cdot \chi(\bar{c}/\bar{d}) = \zeta^i \cdot \chi(\bar{c}/\bar{d})$. Therefore $\chi(\overline{a_i+b_i}) = \chi(\bar{a_i}+\bar{b_i}) = \chi\big(\bar{b_i}(\bar{c}/\bar{d}+1)\big) = \chi(\bar{b_i}) \cdot \chi(\bar{c}/\bar{d}+1) = \chi(\bar{b_i}) \cdot 1 = \zeta^i$ and $(a_i+b_i) \in \alpha_i^*$.

LEMMA 1.2. *Given a ring A and a family of subsets $\alpha_1, \ldots, \alpha_{2n}$ satisfying 1.1(i) through (iv), then $1 \in \alpha_{2n}^*$ and $-1 \in \alpha_n^*$.*

PROOF. Assume $1 \in \alpha_i^*$. If $i \leq n$ then $1 = 1 \cdot 1 \in \alpha_{2i}^*$, and this yields the contradiction $1 \in \alpha_i^* \cap \alpha_{2i}^* = \emptyset$. So $i > n$ and $1 = 1 \cdot 1 \in \alpha_{2i-2n}$. Therefore $1 \in \alpha_i^* \cap \alpha_{2i-2n}^*$, which implies $i = 2i-2n$, that is $i = 2n$.

Now assume $-1 \in \alpha_j^*$, then $1 = -1 \cdot -1$ lies in α_{2j}^* if $j \leq n$ and in α_{2j-2n}^* if $j > n$. Since $1 \in \alpha_{2n}^*$ either $j = n$ or $j = 2n$. If $j = 2n$ then -1 and 1 both lie in α_{2n}^*, which contradicts $\alpha_{2n}^* + \alpha_{2n}^* \subseteq \alpha_{2n}^*$. Therefore $j = n$, and $-1 \in \alpha_n^*$.

Given a collection of subsets $\alpha_1, \ldots, \alpha_{2n}$ of A so that (i) through (iv) of 1.1 hold, the next theorem yields a signature of level n on $k(\wp)$, where $\wp = \alpha_i \cap \alpha_j$. Subsequently, given $\alpha_1, \ldots, \alpha_{2n}$ found from a signature via Theorem 1.1 we will find a method for obtaining the original signature from the subsets α_i.

THEOREM 1.3. *Let A be a ring with a family of subsets $\alpha_1, \ldots, \alpha_{2n}$ such that*
(i) $A = \alpha_1 \cup \cdots \cup \alpha_{2n}$
(ii) $\alpha_i \cap \alpha_j = \wp$ a prime ideal for all $i \neq j$, and defining $\alpha_i^* = \alpha_i^* \setminus \wp$
(iii) $\alpha_i^* + \alpha_i^* \subseteq \alpha_i^*$
(iv) $\alpha_i^* \cdot \alpha_j^* \subseteq \alpha_k^*$ where $k = i+j$ if $i+j \leq 2n$, and $k = i+j-2n$ if $i+j > 2n$.
Then for any primitive $2n^{\text{th}}$ root of unity, ξ, the map $\chi \colon \dot{k}(\wp) \longrightarrow \mu(2n)$ given by $\chi\left(\frac{\bar{a}_i}{\bar{b}_i}\right) = \xi^{i-j}$ is a signature.

PROOF. Using Lemma 1.2 we see there is a unitary homomorphism of semigroups $\varphi \colon A^{\cdot}/\wp \longrightarrow \mu(2n)$ given by $\varphi(\bar{a}_i) = \xi^i$. This extends to a group homomorphism $\chi \colon \dot{k}(\wp) \longrightarrow \mu(2n)$ given by $\chi\left(\frac{\bar{a}_i}{\bar{b}_j}\right) = \xi^{i-j}$ for $a_i \in \alpha_i$ and $b_j \in \alpha_j$. We need to show the kernel of χ is additively closed.

Let $a_i \in \alpha_i^*$ and $b_j \in \alpha_j^*$, then $\chi\left(\frac{\bar{a}_i}{\bar{b}_j}\right) = \xi^{i-j} = 1$ if and only if $i = j$. Thus $\ker \chi = \left\{ \frac{\bar{a}_h}{\bar{b}_h} \mid a_h \text{ and } b_h \text{ are in } \alpha_h^* \right\}$. Let $\frac{\bar{a}_i}{\bar{b}_i}$ and $\frac{\bar{c}_j}{\bar{d}_j}$ be in the kernel of χ where a_i and b_i are in α_i^*, and c_j and d_j are in α_j^*. Then if k is as in hypothesis (iv), $\frac{\bar{a}_i}{\bar{b}_i} + \frac{\bar{c}_j}{\bar{d}_j} = \frac{\overline{a_i d_j} + \overline{b_i c_j}}{\overline{b_i d_j}} = \frac{\overline{a_i d_j + b_i c_j}}{\overline{b_i d_j}} = \frac{\bar{r}_k}{\bar{s}_k}$ for some r_k and s_k in α_k^*. Therefore the kernel of χ is additively closed and χ is a signature.

DEFINITION AND REMARK 1.4. Let the primitive root $\zeta \in \mu(2n)$ from Theorem 1.1, and the primitive root $\xi \in \mu(2n)$ in Theorem 1.3 be the same. Then we have a one-to-one correspondence between the ordered pairs $(\wp, \chi)$ where $\wp$ is in spec A and χ is a signature of $\dot{k}(\wp)$, and collections of subsets $\alpha_1, \ldots, \alpha_{2n}$ obeying 1.1(i) through (iv).

Let $i = \sqrt{-1}$, from this point on we will always use $e^{\pi i/n}$ for the primitive $2n^{\text{th}}$ root of unity used in Theorems 1.1 and 1.3, and will consistently denote it ζ. It is not necessary to use this particular root, but we need to fix ζ in order to get a one-to-one correspondence. Furthermore, we shall henceforth denote by χ_α the signature obtained from $\alpha_1, \ldots, \alpha_{2n}$ using Theorem 1.3, and $\zeta = e^{\pi i/n}$.

Note, if two collections are the same except for indexing they yield different signatures and so we will consider them distinct.

DEFINITION 1.5. An *ordering* α of level n on a ring A is an ordered collection of subsets of A; $\alpha_1, \ldots, \alpha_{2n}$, such that
(i) $A = \alpha_1 \cup \cdots \cup \alpha_{2n}$
(ii) $\alpha_i \cap \alpha_j = \wp_\alpha$ a prime ideal, and denoting $\alpha_i^* = \alpha_i \setminus \wp_\alpha$
(iii) $\alpha_i^* + \alpha_i^* \subseteq \alpha_i^*$
(iv) $\alpha_i^* \cdot \alpha_j^* \subseteq \alpha_k^*$ where $k = i+j$ if $i+j \leq 2n$, and $k = i+j-2n$ if $i+j > 2n$.

We call $\wp_\alpha$ the *support* of α, and write it $\operatorname{supp}(\alpha)$. As in Remark 1.4, we consider the indexing part of the ordering. Therefore, there is a bijection between orderings of level n and pairs $(\wp, \chi)$, where $\wp$ is in spec A and χ is a signature of level n on $\dot{k}(\wp)$, given by $\alpha \longrightarrow (\operatorname{supp}(\alpha), \chi_\alpha)$.

DEFINITION 1.6. Let $R_n\text{-spec}\, A = \{ \alpha \mid \alpha \text{ is an ordering of level } n \}$.

The following lemma yields an equivalent definition of an ordering.

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LEMMA 1.7. *The four conditions of Definition 1.5 are equivalent to*

(i) $A = \alpha_1 \cup \cdots \cup \alpha_{2n}$.

(ii) $\alpha_i \cap \alpha_j = \wp_\alpha$ *a prime ideal.*

(iii) $\alpha_i + \alpha_i \subseteq \alpha_i$.

(iv) $\alpha_i \cdot \alpha_j \subseteq \alpha_k$ *where* $k = i + j$ *if* $i + j \leq 2n$, *and* $k = i + j - 2n$ *if* $i + j > 2n$.

(v) $-1 \notin \alpha_{2n}$ *(equivalently* $-1 \in \alpha_n$*).*

DEFINITION AND REMARKS 1.8. If α is an ordering of level n whose associated signature χ_α maps $\dot{k}(\wp_\alpha)$ onto $\mu(2n)$, then we define both α and χ_α to have *exact* level n. Note, if α is an ordering of exact level n on a ring A, then $\chi_\alpha^{-1}(\zeta^j) \neq \emptyset$ so $\alpha_j \neq \wp_\alpha$. Therefore if α_i^* is empty for some i, then $\mathrm{Im}(\chi_\alpha) = \mu(2m) \underset{\neq}{\subset} \mu(2n)$. Thus there exists an ordering β of level $m < n$ associated with χ_α.

If α is an ordering of level n on A associated with a signature χ mapping $\dot{k}(\wp_\alpha)$ into $\mu(2n)$, then $\alpha_j = \{a \in A \mid a + \wp_\alpha = \bar{a} \in \chi^{-1}\big((e^{\pi i/n})^j\big) \cup \{0\}\}$. If χ maps into $\mu(2m) \underset{\neq}{\subset} \mu(2n)$, so $n/m \in Z$, then there is an ordering β of level m associated to χ. Here $\beta_j = \{a \in A \mid \bar{a} \in \chi^{-1}\big((e^{\pi i/m})^j\big) \cup \{0\}\} = \{a \in A \mid \bar{a} \in \chi^{-1}\big((e^{\pi in/nm})^j\big) \cup \{0\}\} = \alpha_{j \cdot n/m}$. Similarly if n/m divides j then $\alpha_j = \beta_{jm/n}$, and if n/m does not divide j then $\alpha_j = \wp_\beta$. Therefore every ordering α of level n associated with a signature χ which maps onto $\mu(2m) \subset \mu(2n)$ is associated to an ordering β of level m. Henceforth *we shall not distinguish between orderings obtained from the same signature,* and will indicate the level in which we are writing by a left subscript where needed. Thus, for β as above, $\beta = {}_m\alpha$. That is, when α is written as an ordering of level m it equals β.

EXAMPLE 1.9. If ${}_6\alpha = \alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ is an ordering of level 6, then we identify α with ${}_{12}\alpha$ where ${}_{12}\alpha = \{{}_{12}\alpha_1 = \wp_\alpha, {}_{12}\alpha_2 = \alpha_1, {}_{12}\alpha_3 = \wp_\alpha, {}_{12}\alpha_4 = \alpha_2, {}_{12}\alpha_5 = \wp_\alpha, {}_{12}\alpha_6 = \alpha_3, {}_{12}\alpha_7 = \wp_\alpha, {}_{12}\alpha_8 = \alpha_4, {}_{12}\alpha_9 = \wp_\alpha, {}_{12}\alpha_{10} = \alpha_5, {}_{12}\alpha_{11} = \wp_\alpha, {}_{12}\alpha_{12} = \alpha_6\}$.

2. The topologies of R_n-spec A.

In Section 1 we defined the real spectrum of higher level of a commutative ring, R_n-spec A. In this section we will define two topologies on R_n-spec A and examine the resulting structures.

DEFINITION 2.1. In order to topologize R_n-spec A we will view $f \in A$ as a function from R_n-spec A into $\Pi k(\wp_\alpha)$ by setting $f(\alpha) = f + \wp_\alpha \in k(\wp_\alpha)$. Given χ_α and ζ as in Definition 1.4, we define $D(f,t) = \{\alpha \in R_n\text{-spec}\, A \mid \chi_\alpha\big(f(\alpha)\big) = \zeta^t\} = \{\alpha \in R_n\text{-spec}\, A \mid \chi_\alpha(f + \wp_\alpha) = \zeta^t\} = \{\alpha \mid f \in \alpha_t^*\}$ where $1 \leq t \leq 2n$.

DEFINITION 2.2. If $\alpha \in R_n$-spec A, then $\alpha_t \neq \wp_\alpha$ for some t, so there exists $f \in A$ such that $f \in \alpha_t^*$. Thus every α is contained in some $D(f,t)$, so the sets $D(f,t)$ make up a subbasis for a topology on R_n-spec A. By definition its basis is given by the sets $\bigcap_{i=1}^r D(f_i, t_i)$. We will call this the Coste-Roy (C-R) topology on R_n-spec A.

REMARK 2.3. If A is a field then the only prime ideal is the zero ideal, so by Definition 1.5 the orderings of A are bijective with the signatures of $\dot{A}$. Let X_n denote the set of all signatures of level n on $\dot{A}$, where $\Phi: R_n\text{-spec}\,A \rightarrowtail\!\!\twoheadrightarrow X_n$ via $\alpha \longrightarrow \chi_\alpha$ is the above bijection. We topologize X_n, as usual, by giving the character group of $\dot{A}$ the standard compact open topology, using the discrete topology on $\dot{A}$ and $\mu(2n)$, and then giving X_n the subspace topology. A subbasis of this topology is given by the sets $\{\chi \in X_n \mid \chi(c) = \zeta^i\}$ for c in $\dot{A}$ and $\zeta = e^{\pi i/n}$, [see 8, Definition 1.3(ii)]. Now $\Phi^{-1}(\{\chi \in X_n \mid \chi(c) = \zeta^i\}) = \{\alpha \in R_n\text{-spec}\,A \mid c \in \alpha_i^*\} = D(c,i)$, and $\Phi\big(D(c,i)\big) = \{\chi \in X_n \mid \chi(c) = \zeta^i\}$ so that Φ is a homeomorphism of R_n-spec A with X_n, [8]. Therefore, when A is a field, the C-R topology on R_n-spec A is equivalent to the compact open topology on X_n.

We will now look at slightly more general sets than the $D(f,t)$. They will form a subbasis of a finer topology of R_n-spec A, and will allow us to arrive at conclusions about the C-R topology. To simplify our notation we first note the following fact.

LEMMA 2.4. *Let A be a ring, then* $\bigcap_{i=1}^m \{\alpha \mid f_i \in \wp_\alpha\} = \{\alpha \mid \sum_{i=1}^m f_i^2 \in \wp_\alpha\}$ *for* $f_1, \ldots, f_m$ *in* A.

This may be proved by considering the images of the f_i under the map $A \longrightarrow A/\wp_\alpha \longrightarrow k(\wp_\alpha)$ and using the fact that $k(\wp_\alpha)$ is formally real [5].

DEFINITION 2.5. Let $\{\alpha \mid f \in \wp_\alpha, g_{i,1}, \ldots, g_{i,t_i} \in \alpha_i^* \text{ for } 1 \le i \le 2n \text{ and } t_i \ge 1\} = \{\alpha \mid f \in \wp_\alpha\} \cap \bigcap_{i=1}^{2n} \bigcap_{j=1}^{t_i} \{\alpha \mid g_{i,j} \in \alpha_i^*\} = \{\alpha \mid f \in \wp_\alpha\} \cap \bigcap_{i=1}^{2n} \bigcap_{j=1}^{t_i} D(g_{i,j}, i)$. Note that by Lemma 2.4 it does not increase the generality of the set to include $\bigcap_{i=1}^m \{\alpha \mid f_i \in \wp_\alpha\}$.

DEFINITION 2.6. A subset of R_n-spec A is *constructible* if it can be obtained from the sets $D(f,t)$ by a finite sequence of taking unions, intersections, and complements.

PROPOSITION 2.7. *A set is constructible if and only if it can be written as a finite union of sets of the form*

(2.8) $$\{\alpha \mid f \in \wp_\alpha, \text{ and } g_{i,1}, \ldots, g_{i,t_i} \in \alpha_i^*, \text{ for } 1 \le i \le 2n \text{ and } t_i \ge 1\}.$$

PROOF. Let $\mathcal{T}$ be the collection of all subsets of R_n-spec A that are finite unions of the sets of the form (2.8). If $T \in \mathcal{T}$, then T is constructible if and only if a set of the form 2.8 is constructible. But this follows from Definition 2.5 and the fact that $\{\alpha \mid f \in \wp_\alpha\}$ equals $[D(f,1)^c \cap D(f,2)^c \cap \cdots \cap D(f,2n)^c]$.

Conversely, since the $D(f,t)$ satisfy 2.8, to show all constructible sets lie in $\mathcal{T}$, it is enough to show $\mathcal{T}$ is closed under finite intersection, complement and union. Since by definition $\mathcal{T}$ is closed under finite unions, we proceed to intersections.

Let $A = \bigcup_{k=1}^n T_k$ and $B = \bigcup_{\ell=1}^m W_\ell$ be in $\mathcal{T}$ where T_k and W_ℓ are of the form 2.8. Then $A \cap B = \big[\bigcup_{k=1}^n T_k\big] \cap \big[\bigcup_{\ell=1}^m W_\ell\big] = \bigcup_{k=1}^n \bigcup_{\ell=1}^m (T_k \cap W_\ell)$. This lies in $\mathcal{T}$ since by 2.4 and 2.5 it is clear that $T_k \cap W_\ell$ is of the form 2.8. Now induction yields $\mathcal{T}$ is closed under finite intersections.

Again assume $A = \bigcup_{k=1}^n T_k$ lies in $\mathcal{T}$ where the T_k are of the form 2.8. To show $\mathcal{T}$ is closed under finite complement it is enough to show $A^c = \bigcap_{k=1}^n (T_k)^c$ lies in $\mathcal{T}$. But

$\mathcal{T}$ is closed under finite intersection, therefore, we need only show $(T_k)^c \in \mathcal{T}$. Since T_k is of the form 2.8 this follows from the fact that $\{\alpha \mid h \in \alpha_i^*\}^c = \{\alpha \mid h \in \wp_\alpha\} \cup \bigcup_{\substack{j=1 \\ j\neq i}}^{2n} \{\alpha \mid h \in \alpha_j^*\} \in \mathcal{T}$ and $\{\alpha \mid f \in \wp_\alpha\}^c = \bigcup_{i=1}^{2n} \{\alpha \mid f \in \alpha_i^*\} \in \mathcal{T}$.

REMARK 2.9. We now put another topology on R_n-spec$A$ which we will show has the constructible sets as a basis. This shall involve mapping R_n-spec$A$ into $\prod_{f\in A}\{0,1,\ldots,2n\}$, where $Z = \prod_{f\in A}\{0,1,\ldots,2n\}$ is given the Tychonoff topology. That is, it is given the product topology using the discrete topology on $\{0,1,\ldots,2n\}$. By Tychonoff's theorem [14], Z is compact and Hausdorff in this topology.

DEFINITION AND PROPOSITION 2.10. *Define* $\varphi: R_n$-spec$A \longrightarrow \prod_{f\in A}\{0,1,\ldots,2n\}$ *by* $\varphi(\alpha) = \prod_{f\in A}\{e_\alpha(f)\}$ *where* $e_\alpha(f)$ *is* 0 *iff* $f \in \wp_\alpha$, *and equals* i *iff* $f \in \alpha_i^*$. *Then* φ *is injective, and the image of* φ *is closed in the Tychonoff topology.*

PROOF. Suppose $\varphi(\alpha) = \varphi(\beta)$, then $e_\alpha(f) = e_\beta(f)$ for all f in A. Thus $\{f \mid e_\alpha(f) = i\} = \{f \mid e_\beta(f) = i\}$ for $0 \le i \le 2n$. This implies $\wp_\alpha = \wp_\beta$ and $\alpha_i^* = \beta_i^*$ for $1 \le i \le 2n$. Therefore $\alpha_i = \beta_i$ for all i, so $\alpha = \beta$, and φ is injective.

Now let $x = (x_f)$ lie in $\overline{\mathrm{Im}\varphi}$, the closure of $\mathrm{Im}\varphi$. For $i = 1,\ldots,2n$ let $\alpha_i = \{f \mid x_f = 0 \text{ or } i\}$. To show $x \in \mathrm{Im}\varphi$ we must show that the α_i obey the four statements of Definition 1.5.

The first statement is clear. To show the second statement we must first show $\wp$ is closed under subtraction. Assume $f \in \wp$ and $g \in \wp$ but $f - g \notin \wp$. Then $x_f = 0$, $x_g = 0$ and $x_{f-g} = h \neq 0$, for some natural number h, where $1 \le h \le 2n$. Let $U = \{y \in Z \mid y_f = y_g = 0, y_{f-g} = h\}$, then U is an open neighbourhood of $x = (x_f) \in \overline{\mathrm{Im}\varphi}$ since the topology is discrete. Therefore $U \cap \mathrm{Im}\varphi \neq \emptyset$. Let $\beta \in R_n$-spec$A$ such that $\varphi(\beta) \in U$, then f and g lie in supp(β), and $f - g \notin$ supp(β). This is a contradiction since supp(β) is an ideal. Therefore, $f \in \wp$ and $g \in \wp$ implies $f - g \in \wp$.

The rest of the proof is similar. That is, we define an appropriate open set containing x, as above it intersects $\mathrm{Im}\varphi$, and this gives a contradiction.

DEFINITION 2.11. The Tychonoff topology on R_n-spec$A$ is the topology inherited via the pullback of φ, when $Z = \prod_{f\in A}\{0,1,\ldots,2n\}$ is topologized as in Remark 2.9. That is, a subset T of R_n-spec$A$ is open in the Tychonoff topology of R_n-spec$A$ if and only if $\varphi(T)$ equals the image of φ intersected with an open subset of the Tychonoff topology of Z.

PROPOSITION 2.12.

(i) *The constructible subsets of* R_n-spec$A$ *form a basis of the Tychonoff topology.*

(ii) *In the Tychonoff topology* R_n-spec$A$ *is compact and Hausdorff.*

(iii) *A set is constructible if and only if it is clopen in the Tychonoff topology.*

PROOF. (i) Let $I_{2n} = \{0,1,\ldots,2n\}$. A subbasis of the product topology on $\prod_{f\in A}\{0,1,\ldots,2n\}$ is given by $I_{2n} \times I_{2n} \times \cdots \times I_{2n} \times \{i\} \times I_{2n} \times \ldots$ where $0 \le i \le 2n$. Therefore a subbasis of the Tychonoff topology on R_n-spec$A$ is given by sets of the form

$\{\alpha \mid f \in \alpha_i^*\} = D(f,i)$ or $\{\alpha \mid f \in \wp_\alpha\}$. A basis thus consists of all possible finite intersections of sets of these forms. Hence by Definition 2.5 and Proposition 2.7 the constructible subsets of R_n-spec A form a basis of the Tychonoff topology.

(ii) Let $Z = \prod_{f\in A}\{0,1,\ldots,2n\}$, and $\varphi: R_n\text{-spec}\,A \longrightarrow Z$ as in Definition 2.10. Since the image of φ is closed in Z, and Z is a compact Hausdorff space, then the image of φ is compact in Z. Therefore R_n-spec A is compact in the topology inherited from φ. But any bijective continuous map from a compact space into a Hausdorff space is a homeomorphism [14], therefore R_n-spec A is also Hausdorff in the Tychonoff topology.

(iii) By Definition 2.6 the complement of a constructible set is constructible. Therefore since in the Tychonoff topology constructible sets are open, they are also closed. Conversely, assume S is clopen in the Tychonoff topology. As an open set it is a union of constructible sets. But S is also closed, and hence compact. Therefore it is a finite union of constructible sets, so by definition S is constructible.

REMARK 2.13. Any basis element, $\bigcap_{i=1}^r D(f_i,t_i)$, of the C-R topology on R_n-spec A is constructible by definition. Therefore, by Proposition 2.12 (i), it is open in the Tychonoff topology. That is, the Tychonoff topology is finer than the C-R topology.

THEOREM 2.14. *In the* C-R *topology,* R_n-spec A *is quasi-compact, and every constructible set is quasi-compact.*

The proof of compactness follows from 2.12 and 2.13. In Example 2.17 we shall show that R_n-spec A is not necessarily Hausdorff in the C-R topology. From now on, unless otherwise stated, we will only consider the C-R topology on R_n-spec A.

LEMMA 2.15. *Let α and β be members of R_n-spec$A$, then $\alpha_i \subseteq \beta_i$ for all $i = 1,\ldots,2n$ if and only if $\beta_i^* \subseteq \alpha_i^*$ for all $i = 1,\ldots,2n$.*

PROOF. Suppose $\alpha_i \subseteq \beta_i$ for $i = 1,\ldots,2n$. If there exists an $f \in \beta_j^*$ with $f \notin \alpha_j^*$ for some j between 1 and $2n$, then $f \in \alpha_t$ for some $t \neq j$. But since $\alpha_t \subseteq \beta_t$ we have $f \in \beta_t$. This contradicts the fact $f \in \beta_j^*$, so $\beta_i^* \subseteq \alpha_i^*$ for $i = 1,\ldots,2n$. The converse is proved similarly.

LEMMA 2.16. *Let α and β be members of R_n-spec A, and let $\overline{\{\alpha\}}$ denote the closure of α in R_n-spec A. Then $\beta \in \overline{\{\alpha\}}$ if and only if $\alpha_i \subseteq \beta_i$ for all $i = 1,\ldots,2n$.*

PROOF. Suppose $\alpha_i \not\subseteq \beta_i$ for some i between 1 and $2n$. Then there exists $f \in \alpha_i$ such that $f \in \beta_j^*$ for some $j \neq i$. Hence $D(f,j) = \{\gamma \mid f \in \gamma_j^*\}$ is an open set containing β but not α. Thus $\beta \notin \overline{\{\alpha\}}$.

Conversely, suppose $\alpha_i \subseteq \beta_i$ if $1 \leq i \leq 2n$, so that by Lemma 2.15 we have $\beta_i^* \subseteq \alpha_i^*$. It is enough to show that every basis element containing β contains α. Let $U = \bigcap_{i=1}^m D(f_i,t_i) = \bigcap_{i=1}^m \{\gamma \mid f_i \in \gamma_{t_i}^*\}$ be an arbitrary basis element containing β. If $\alpha \notin U$, then $f_i \notin \alpha_{t_i}^*$ for some i, so that $\beta_{t_i}^* \subseteq \alpha_{t_i}^*$ implies $f_i \notin \beta_{t_i}^*$. But this contradicts the fact $\beta \in U$, therefore, $\beta \in \overline{\{\alpha\}}$.

Note that if R_n-spec A is Hausdorff then its points are closed. By Lemma 2.16, $\beta \in \overline{\{\alpha\}}$ if and only if $\alpha_i \subseteq \beta_i$ for all $i = 1,\ldots,2n$. Therefore, to ascertain whether R_n-spec A

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is Hausdorff it is not enough to examine α_{2n} and β_{2n} we need the added structure found in Definition 1.5.

Let A be a field and assume there are two orderings of level n, α and β. In the field case the α_i^* and the β_i^* both form partitions of $\dot{A}$, so $\alpha_i \subseteq \beta_i$ for all i implies $\alpha = \beta$. Thus, by Lemma 2.16, when A is a field the points of R_n-spec A are closed. This agrees with the fact that in the field case the topology on R_n-spec A is homeomorphic with the compact open topology on the space of signatures (Remark 2.3), and this topology is Hausdorff [8, Prop. 1.4].

To give an example of a ring A such that R_n-spec A is not Hausdorff, it is enough to find a ring with two distinct orderings α and β in R_n-spec A such that $\alpha_i \subseteq \beta_i$ if $1 \leq i \leq 2n$. Becker [4, p. 34] shows that if $A = \mathbb{R}[t]$ then points are not closed in R_1-spec A. The next example is of a ring A such that the points of R_n-spec A are not closed for any n.

EXAMPLE 2.17. Let $A = \mathbb{R}(x)[[t]]$, the ring of formal power series in one variable over the field $\mathbb{R}(x)$. We shall show R_n-spec A is not Hausdorff by finding two distinct orderings of level n, α and β, such that $\alpha_i \subseteq \beta_i$ for $i = 1, \ldots, 2n$. We first consider the orderings of level n on A with support zero. These are merely the restrictions of orderings of level n on $\mathbb{R}(x)((t))$ to $\mathbb{R}(x)[[t]]$, where $\mathbb{R}(x)((t))$ is the field of formal power series in t over $\mathbb{R}(x)$. To obtain a signature of exact level n on $\mathbb{R}(x)((t))/\{0\}$ we start with a signature χ of exact level n on $\mathbb{R}(x)/\{0\}$ [7, Prop. 2.9(ii)], and let χ' map $\mathbb{R}(x)((t))/\{0\}$ into $\mu(2n)$ via $\chi'[f_0(x)t^k(1+f_1(x)t+f_2(x)t^2+\ldots)] = \chi\big(f_0(x)\big)\cdot\zeta^k$. We shall show χ' is a signature of exact level n on $\mathbb{R}(x)((t))/\{0\}$.

Clearly χ' is a character, it remains to show the kernel of χ' is additively closed. Let $\chi'[f_0(x)t^k(1+f_1(x)t+f_2(x)t^2+\cdots)] = 1 = \chi'[g_0(x)t^j(1+g_1(x)t+g_2(x)t^2+\cdots)]$, so that $\chi\big(f_0(x)\big)\cdot\zeta^k = \chi\big(g_0(x)\big)\cdot\zeta^j = 1$. Either $j \neq k$ or $j = k$. In the first case, we may assume $j > k$ so that $\chi'[f_0(x)t^k(1+f_1(x)t+\cdots)+g_0(x)t^j(1+g_1(x)t+\cdots)] = \chi'[f_0(x)t^k(1+r_1(x)t+r_2(x)t^2+\cdots)]$ for appropriate $r_i(x)$. But this is $\chi\big(f_0(x)\big)\cdot\zeta^k = 1$. In the second case, we have $j = k$ so that $\chi\big(f_0(x)\big) = \chi\big(g_0(x)\big)$ and $\chi\big(g_0(x)/f_0(x)\big) = 1$. Now $\chi'[f_0(x)t^k(1+f_1(x)t+\cdots)+g_0(x)t^k(1+g_1(x)t+\cdots)] = \chi'[(f_0(x)+g_0(x))t^k(1+s_1(x)t+\cdots)]$ for appropriate $s_i(x)$. But this equals $\chi\big(f_0(x)+g_0(x)\big)\cdot\zeta^k = \chi\big(f_0(x)\big)\cdot\chi[1+(g_0(x)/f_0(x))]\cdot\zeta^k = \chi\big(f_0(x)\big)\cdot 1\cdot\zeta^k = 1$. Therefore the kernel of χ' is additively closed, so χ' is a signature. It has exact level n since $\mu(2n) = \chi\big(\mathbb{R}(x)/\{0\}\big) = \chi'\big(\mathbb{R}(x)/\{0\}\big) \subseteq \chi'\big(\mathbb{R}(x)((t))/\{0\}\big) \subseteq \mu(2n)$. Let α be the ordering on $\mathbb{R}(x)[[t]]$ determined by $\alpha_i = (\chi')^{-1}(\zeta^i)$ restricted to $\mathbb{R}(x)[[t]]$. Thus $\alpha_i = \{f(x,t) = f_0(x)t^k(1+f_1(x)t+\cdots) \text{ in } \mathbb{R}(x)[[t]] \mid \chi'\big(f(x,t)\big) = \zeta^i\}$.

Our second ordering of exact level n on $\mathbb{R}(x)[[t]]$, shall have suppport $t\mathbb{R}(x)[[t]]$, a maximal ideal of $\mathbb{R}(x)[[t]]$. To define it we first determine an ordering of exact level n on $K = \mathbb{R}(x)[[t]]/t\mathbb{R}(x)[[t]] \cong \mathbb{R}(x)$. Let χ be the signature of exact level n on $\mathbb{R}(x)/\{0\}$ used above. Let $\theta: \dot{K} \longrightarrow \mathbb{R}(x)/\{0\}$ be the above isomorphism, and let $\pi: R(x)[[t]] \longrightarrow K$ be the canonical projection. Then $\chi\theta$ is a signature on $\dot{K}$, and by Theorem 1.1, we have $\beta_i = \{f(x,t) \in \mathbb{R}(x)[[t]] \mid \pi\big(f(x,t)\big) \text{ lies in } (\chi\theta)^{-1}(\zeta^i) \cup t\mathbb{R}(x)[[t]]\}$ yields an ordering of level n on $\mathbb{R}(x)[[t]]$. Simplifying we see $\beta_i = \{f(x,t) = f_0(x)t^k\big(1+f_1(x)t+\cdots\big) \in \mathbb{R}(x)[[t]] \mid \pi\big(f(x,t)\big) \in \theta^{-1}\chi^{-1}(\zeta^i) \text{ or } k > 0\} = \{f(x,t) = f_0(x)t^k\big(1+f_1(x)t+\cdots\big) \in$

$\mathbb{R}(x)[[t]] \mid k > 0$, or $\theta\pi\big(f(x,t)\big) \in \chi^{-1}(\zeta^i)\big\} = \big\{f_0(x)t^k\big(1 + f_1(x)t + f_2(x)t^2 + \cdots\big) \in \mathbb{R}(x)[[t]] \mid k > 0$, or $k = 0$ and $f_0(x) \in \chi^{-1}(\zeta^i)\big\}$.

We now have two orderings α and β of exact level n on $\mathbb{R}(x)[[t]]$ such that $\big\{f_0(x)t^k\big(1 + f_1(x)t + f_2(x)t^2 + \cdots\big) \in \mathbb{R}(x)[[t]] \mid \chi\big(f_0(x)\big)\zeta^k = \zeta^i\big\} = \alpha_i \underset{\neq}{\subset} \beta_i = \big\{f_0(x)t^k\big(1 + f_1(x)t + f_2(x)t^2 + \cdots\big) \in \mathbb{R}(x)[[t]] \mid k > 0$, or $k = 0$ and $\chi\big(f_0(x)\big) = \zeta^i\big\}$. Therefore points are not closed and R_n-spec $\big(\mathbb{R}(x)[[t]]\big)$ is not Hausdorff.

DEFINITION 2.18. Let α and β be in R_n-spec A. If α and β lie in disjoint open subsets of R_n-spec A, then we say α and β can be *separated.*

PROPOSITION 2.19. *Two orderings α and β in R_n-*spec A *can be separated if and only if there exist i such that $\alpha_i \not\subseteq \beta_i$ and there exist j such that $\beta_j \not\subseteq \alpha_j$.*

PROOF. If α and β can be separated then they lie in mutually disjoint open sets, so $\beta \notin \overline{\{\alpha\}}$ and $\alpha \notin \overline{\{\beta\}}$. Therefore, by Lemma 2.16, there eixst i such that $\alpha_i \not\subseteq \beta_i$ and there exist j such that $\beta_j \not\subseteq \alpha_j$.

Conversely, suppose $\alpha_i \not\subseteq \beta_i$ and $\beta_j \not\subseteq \alpha_j$ for some i and j. Since $\alpha_i \not\subseteq \beta_i$ there exists $f \in \alpha_i$ such that $f \in \beta_m^*$ for some $m \neq i$. If $f \notin \wp_\alpha$, then $\beta \in D(f,m)$ and $\alpha \in D(f,i)$ so α and β lie in two disjoint open sets, and thus can be separated. Therefore, we need only consider the case $f \in \wp_\alpha$, with $f \in \beta_m^*$. Similarly we may assume there exists $g \in \wp_\beta$ with $g \in \alpha_\ell^*$.

If $\ell \neq m$ then $(f{+}g) \in (\wp_\alpha{+}\alpha_\ell^*) \subseteq \alpha_\ell^*$ and $(f{+}g) \in (\beta_m^*{+}\wp_\beta) \subseteq \beta_m^*$, so $\alpha \in D(f{+}g,\ell)$ and $\beta \in D(f+g,m)$. These are disjoint open sets so α and β can be separated. If $\ell = m$ then $(f-g) \in (\beta_m^* - \wp_\beta) \subseteq \beta_m^*$, and since $-1 \in \alpha_n$ we have $(f-g) \in (\wp_\alpha + \alpha_t^*) \subseteq \alpha_t^*$ where $t = \ell + n$ if $\ell + n \leq 2n$ and $t = \ell + n - 2n$ if $\ell + n > 2n$. Therefore $\alpha \in D(f-g,t)$ and $\beta \in D(f-g,m)$ which are disjoint open sets since $m \neq t$. Therefore α and β can be separated.

DEFINITION 2.20. We will write $\alpha \subseteq \beta$ for $\alpha_i \subseteq \beta_i$ for all i, and we will say β *specializes* α, or is a *specialization* of α if $\alpha \subseteq \beta$. In this situation, α is called a *generalization* of β, or is said to *generalize* β.

Let α and β be orderings of levels m and n respectively, where $s = \operatorname{lcm}(m,n)$. Then following the logic of Remark 1.8 we write $\alpha \subseteq \beta$ if ${}_s\alpha \subseteq {}_s\beta$. Note $\alpha \subseteq \beta$ if and only if ${}_t\alpha \subseteq {}_t\beta$ for all t divisible by s.

THEOREM 2.21. *Given a ring A, let α be an ordering of level n and β and ordering of level m. Assume m divides n, then $\alpha \supseteq {}_n\beta$ implies $\alpha = {}_n\gamma$ for some γ an ordering of level m. That is, specializations do not increase level, α specializes an ordering of level m if and only if* $\operatorname{Im}\chi_\alpha \subseteq \mu(2m)$.

PROOF. Assume $\alpha \supseteq {}_n\beta$ for β an ordering of level m, but $\alpha \neq {}_n\gamma$ for γ an ordering of level m. By Remark 1.8 there exists an i not divisible by n/m such that $\alpha_i \underset{\neq}{\supset} \wp_\alpha$. Let $a \in \alpha_i \backslash \wp_\alpha = \alpha_i^*$. Since $\alpha \supseteq {}_n\beta$ we have $\alpha_i^* \subseteq {}_n\beta_i^*$, therefore $a \in {}_n\beta_i^*$. But this is impossible, since ${}_n\beta_i^*$ is the empty set whenever n/m does not divide i. Therefore, $\alpha = {}_n\gamma$ for some γ an ordering of level m.

In order to put R_n-spec A in context, and allow the use of previously proved theorems, we make the following remark.

REMARK 2.22. R_n-spec A is an example of a spectral space [12] (or coherent space [13]). *I.e.* it is T_0 and quasi-compact, the quasi-compact open subsets are closed under finite intersection and form an open basis, and every nonempty irreducible closed subset has a generic point. The patch topology of the spectral space is the Tychonoff topology. *I.e.* the Tychonoff topology has the quasi-compact open subsets and their complements as an open basis. These assertions easily follow from Propositions 2.12 and 2.19, Theorem 2.14, and [12, Prop. 4]. In fact we shall see R_n-spec A is a normal spectral space [10]. That is, we shall see that each point has a unique maximal specialization.

PROPOSITION 2.23.

(i) The specializations of an ordering form a chain under inclusion of ordering.

(ii) $\{\alpha\}$ is closed if and only if α is a maximal ordering

(iii) An ordering is contained in a unique maximal specialization.

PROOF. (i) Let $\alpha \subseteq \beta$ and $\alpha \subseteq \gamma$. If $\beta \not\subseteq \gamma$ and $\gamma \not\subseteq \beta$ then by Proposition 2.19 we see that β and γ are contained in two mutually disjoint open sets. But this is impossible since by Lemma 2.16 these sets must both contain α. Therefore $\beta \supseteq \gamma$ or $\gamma \supseteq \beta$, so the specializations of α form a chain under inclusion.

(ii) If $\{\alpha\}$ is closed, then by Lemma 2.16 there does not exist a β properly containing α. Hence α is a maximal ordering. Conversely, if there does not exist β such that $\beta \supsetneq \alpha$ then $\overline{\{\alpha\}} = \{\alpha\}$ so α is closed.

(iii) This is equivalent [10, Prop. 2] to showing any two distinct closed points of R_n-spec A may be separated. But closed points are maximal orderings, and by Proposition 2.19 these can be separated.

Note that Proposition 2.23 states that the closure of a singleton $\{\alpha\}$, which is the set of all specializations of α, is a totally ordered set, containing a minimum element, α, and a maximum element.

DEFINITION 2.24. Let R_n-specm A be the closed points of R_n-spec A. Then R_n-specm A consists of the maximal orderings of R_n-spec A, and is called the *maximal real spectrum* of level n on A. We give R_n-specm A the subspace topology inherited from R_n-spec A.

PROPOSITION 2.25. *R_n-specm A is a compact and Hausdorff space.*

The proof uses Theorem 2.14 to show R_n-specm A is compact, and Proposition 2.19 to show it is Hausdorff.

3. **The topology of** $Y \subset R_n$**-spec** A**.** We now consider the topology of subspaces Y of R_n-spec A, where Y is closed in the Tychonoff topology of R_n-spec A. Note that by Proposition 2.12 (iii) these results will hold for constructible sets. Unless otherwise specified we give R_n-spec A the C-R topology and give Y the subspace topology.

In addition, we also have a topology inherited from the Tychonoff topology of R_n-spec A, we call this topology the Tychonoff topology of Y. A subset U of Y is Tychonoff open in Y if $U = V \cap Y$ where V is Tychonoff open in R_n-spec A. Similarly a subset U of Y is Tychonoff closed in Y if $U = V \cap Y$ where V is Tychonoff closed in R_n-spec A.

A set is called constructible in Y if it is the intersection of Y with a constructible set of R_n-spec A. By definition these sets form a basis of the Tychonoff topology of Y. By Proposition 2.12 (iii) the constructible sets of R_n-spec A are clopen in the Tychonoff topology of R_n-spec A, therefore a set which is constructible in Y is clopen in the Tychonoff topology of Y. Furthermore, if V is constructible in R_n-spec A then $Y \setminus (Y \cap V) = Y \cap V^c$. That is, if U is constructible in Y then $Y \setminus U$ is constructible in Y.

In the coarser C-R topology, we note only that constructible sets in Y are compact subsets of Y. This follows from the Tychonoff topology, as was seen for the corresponding statement in R_n-spec A (2.14), since constructible sets of Y are Tychonoff closed in R_n-spec A, which is compact Hausdorff in the Tychonoff topology.

REMARK 3.1 [12]. Using the language of Remark 2.22; if Y is closed in the Tychonoff topology of R_n-spec A, then Y, when given the subspace topology, is a spectral space. Furthermore the patch topology on Y agrees with the topology inherited from the Tychonoff (or patch) topology on R_n-spec A.

LEMMA 3.2. *If Y is closed in the Tychonoff topology of R_n-spec A, then it is the intersection of constructible subsets of R_n-spec A.*

PROOF. Since Y^c is open in the Tychonoff topology, we have $Y^c = \bigcup_{i \in I} A_i$, for some index set I, where the A_i are constructible sets. By 2.6 the complement of a constructible set is constructible, so $Y = [\bigcup_{i \in I} A_i]^c = \bigcap_{i \in I} A_i^c$ is the intersection of constructible sets.

PROPOSITION 3.3. *Let Y be a subspace of R_n-spec A, with Y closed in the Tychonoff topology, and let α be an element of Y.*

(i) α is a maximal ordering in Y if and only if $\{\alpha\}$ is closed in Y.

(ii) α admits a unique maximal specialization in Y.

(iii) $Y^{\max} = \{\alpha \in Y \mid \alpha$ is maximal in $Y\}$ is a compact Hausdorff space.

The proof is like that of Propositions 2.23 and 2.25.

REMARK 3.4. It is clear from Proposition 3.3 and Remark 2.22 that if Y is closed in the Tychonoff topology of R_n-spec A, then Y is a normal spectral space.

PROPOSITION 3.5 [10, PROP. 3]. *The map $\lambda : R_n$-spec $A \longrightarrow R_n$-specm A sending α to the maximal specialization of α, and the map λ_Y from Y to $Y^{\max}$ which sends α to its maximal specialization in Y, are continuous and closed.*

PROPOSITION 3.6 [13, CHAPTER II, PROP. 4.6]. *A subset X of Y is closed (respectively open) in Y if and only if it is Tychonoff closed (respectively Tychonoff open) in Y, and it is closed with respect to specializations (respectively generalizations) in Y.*

DEFINITION 3.7. We shall call a set *open constructible* in Y if it is the finite union of sets of the form $Y \cap \bigcap_{i=1}^{r} D(f_i, t_i)$. We shall call a set *closed constructible* in Y if it is the finite union of sets of the form $Y \cap \bigcap_{i=1}^{r} D(f_i, t_i)^c$.

Note that the open constructible sets are by definition the basis elements of the topology of Y, that is the topology inherited from the C-R topology of R_n-spec A. Furthermore, X is open constructible if and only if $Y \backslash X$ is closed constructible.

PROPOSITION 3.8. *Let X be constructible in Y, then X is open (respectively closed) in Y if and only if X is open constructible (respectively closed constructible) in Y.*

PROOF. It is enough to prove the theorem for open constructible sets. Let X be constructible in Y with X open in Y. Since a basis of Y is given by sets of the form $Y \cap \bigcap_{i=1}^{r} D(f_i, t_i)$, we have $X = \bigcup_J \Big[Y \cap \bigcap_{i_j=1}^{r_j} D(f_{i_j}, t_{i_j}) \Big]$ for some index set J. But X is compact in Y so this is a finite union and X is open constructible in Y. The converse is clear from the definition of open constructible.

4. **The contravariant functor** R_n-spec.

In this section we see that R_n-spec is a contravariant functor on the category of commutative rings with unit into the category of topological spaces.

LEMMA 4.1. *Let A and B be commutative rings and $\varphi: A \longrightarrow B$ be a unitary ring homomorphism. Then if $\beta \in R_n$-spec B, and α is the indexed collection of subsets of A where $\alpha_i = \varphi^{-1}(\beta_i)$, then α lies in R_n-spec A.*

The proof is straightforward.

DEFINTIION 4.2. For A and B commutative rings and $\varphi: A \longrightarrow B$ a ring homomorphism, let $\varphi_*: R_n\text{-spec}\, B \longrightarrow R_n\text{-spec}\, A$ be defined by $\varphi_*(\beta) = \alpha$ where $\alpha_i = \varphi^{-1}(\beta_i)$.

We now note several facts which allow us to show that φ_* is continuous and that R_n-spec acts as a contravariant functor.

Let $\varphi: A \longrightarrow B$, as above, with $\beta \in R_n$-spec B and $\alpha = \varphi_*(\beta)$. Let $\bar{\alpha}$ be the ordering on $A/\wp_\alpha$ given by $\bar{\alpha}_i = \{ a + \wp_\alpha \mid a \in \alpha_i \}$, and let $\bar{\beta}$ be defined similarly. Then we have the monomorphism $\bar{\varphi}: (A/\wp_\alpha, \bar{\alpha}) \longrightarrow (\beta/\wp_\beta, \bar{\beta})$, given by $(a + \wp_\alpha) \longrightarrow \big(\varphi(a) + \wp_\beta\big)$. Now $\varphi^{-1}(\beta_i) = \alpha_i$ implies $\bar{\varphi}(a_i + \wp_\alpha) = \big(\varphi(a_i) + \wp_\beta\big) = (b_i + \wp_\beta)$ for some $b_i \in \beta_i$. Therefore $\bar{\varphi}(\bar{\alpha}_i) \subseteq \bar{\beta}_i$.

If we use the function notation $f(\alpha) = f + \wp_\alpha \in k(\wp_\alpha)$, as in Definition 2.1, we see that $\bar{\varphi}\big[f\big(\varphi_*(\beta)\big)\big] = \bar{\varphi}[f(\alpha)] = \bar{\varphi}(f + \wp_\alpha) = \varphi(f) + \wp_\beta = \varphi(f)(\beta)$.

LEMMA 4.3. *Recalling the notation $D(f, t) = \{ \gamma \mid f \in \gamma_t^* \}$ from Definition 2.1, we have: $\varphi_*^{-1}\big(D(f, t)\big) = D\big(\varphi(f), t\big)$.*

PROOF. We have $\varphi_*^{-1}\big(D(f,t)\big) = \varphi_*^{-1}(\{\gamma \mid f \in \gamma_t^*\})$. Let $\gamma = \varphi_*(\delta)$, so that $\gamma_i = \varphi^{-1}(\delta_i)$, then $\varphi_*^{-1}\big(D(f,t)\big) = \varphi_*^{-1}\big(\{\varphi_*(\delta) \mid f \in \varphi^{-1}(\delta_t)^*\} = \{\delta \mid f \in \varphi^{-1}(\delta_t^*)\} = \{\delta \mid \varphi(f) \in \delta_t^*\} = D\big(\varphi(f), t\big)$.

PROPOSITION 4.4. *Given $\varphi: A \longrightarrow B$ and $\varphi_*: R_n\text{-spec}\, B \longrightarrow R_n\text{-spec}\, A$, φ_* is continuous in the* C-R *topology of the n^{th} level real spectra, and in the Tychonoff topologies. In particular, inverse images of constructible sets are constructible.*

PROOF. To show φ_* is continuous in the C-R topologies we need to show $\varphi_*^{-1}(V)$ is open for an arbitrary basis element $V = \bigcap_{i=1}^r D(f_i, t_i)$ of R_n-spec A. But $\varphi_*^{-1}(V) = \bigcap_{i=1}^r \varphi_*^{-1}\big(D(f_i,t_i)\big) = \bigcap_{i=1}^r D\big(\varphi(f_i), t_i\big)$ by Lemma 4.3, and this is open by definition.

Similarly, to show φ_* is continuous in the Tychonoff topologies, it is enough to show $\varphi_*^{-1}(W)$ is open in the Tychonoff topology of R_n-spec B for an arbitrary basis element $W = \bigcup_{j=1}^m \Big[\bigcap_{i_j=1}^{n_j} D(f_{i_j}, t_{i_j}) \cap \{\gamma \mid f_j \in \wp_\gamma\}\Big]$. Again using Lemma 4.3 and simplifying we have $\varphi_*^{-1}(W) = \bigcup_{j=1}^m \Big[\bigcap_{i_j=1}^{n_j} D\big(\varphi(f_{i_j}, t_{i_j})\big) \cap \{\delta \mid \varphi(f_j) \in \wp_\delta\}\Big]$ which is open in the Tychonoff topology of R_n-spec B.

THEOREM 4.5. *R_n-spec is a contravariant functor from the category of commutative rings with unit into the category of topological spaces.*

PROOF. We have shown R_n-spec is a functor. Now let $\varphi: A \longrightarrow B$ and $\Psi: B \longrightarrow C$, with $\gamma \in R_n$-spec C, then $(\Psi\varphi)^{-1}(\gamma_i) = \varphi^{-1}\Psi^{-1}(\gamma_i)$. Therefore $(\Psi\varphi)_* = \varphi_*\Psi_*$ and R_n-spec is a contravariant functor.

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