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ARTICLE

On Erdős–Ko–Rado for random hypergraphs I[†]

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Abstract

A family of sets is *intersecting* if no two of its members are disjoint, and has the *Erdős–Ko–Rado property* (or *is EKR*) if each of its largest intersecting subfamilies has non-empty intersection.

Denote by $\mathcal{H}_k(n, p)$ the random family in which each k-subset of $\{1, \ldots, n\}$ is present with probability p, independent of other choices. A question first studied by Balogh, Bohman and Mubayi asks:

$$\text{For what } p = p(n, k) \text{ is } \mathcal{H}_k(n, p) \text{ likely to be EKR?}$$

Here, for fixed $c < 1/4$, and $k < \sqrt{cn \log n}$ we give a precise answer to this question, characterizing those sequences $p = p(n, k)$ for which

$$\mathbb{P}(\mathcal{H}_k(n, p) \text{ is EKR}) \to 1 \ \text{ as } n \to \infty.$$

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1. Introduction

One of the most interesting combinatorial trends of the last couple of decades has been the investigation of 'sparse random' versions of some of the classical theorems of the subject – that is, of the extent to which these theorems hold in a random setting. This issue has been the subject of some spectacular successes, particularly those related to the theorems of Ramsey [19], Turán [24] and Szemerédi [23]; see [2, 11, 17, 20] for origins and, for example, [5, 7, 22] (or the survey [21]) for a few of the most recent developments.

Here we are interested in what can be said in this vein for the Erdős–Ko–Rado theorem [9], another cornerstone of extremal combinatorics. This natural question has already been considered by Balogh, Bohman and Mubayi [3], and we first quickly recall a few notions from that paper.

In what follows, k and n are always positive integers with $n > 2k$. As usual we write $[n]$ for $\{1, \ldots, n\}$ and $\binom{V}{k}$ for the collection of k-subsets of the set V. A *k-graph* (or *k-uniform hypergraph*) on V is a subset (or multisubset), $\mathcal{H}$, of $\binom{V}{k}$. Members of V and $\mathcal{H}$ are called *vertices* and *edges* respectively. Here we will always take $V = [n]$ and write $\mathcal{K}$ for $\binom{V}{k}$. For a k-graph $\mathcal{H}$ on V and $x \in V$ we use $d_{\mathcal{H}}(x)$ for the *degree* of x in $\mathcal{H}$ (the number of edges of $\mathcal{H}$ containing x) and $\Delta_{\mathcal{H}}$ for

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the maximum of these degrees. We also write $\mathcal{H}_x$ for the set of edges containing x, called the *star* of x in $\mathcal{H}$.

A collection of sets is *intersecting*, or a *clique*, if no two of its members are disjoint. The Erdős–Ko–Rado theorem says that for any n and k as above, the maximum size of an intersecting k-graph on V is $\binom{n-1}{k-1}$ and, moreover, this bound is achieved only by the stars.

Following [3], we say that $\mathcal{H} \subseteq \mathcal{K}$ satisfies (*strong*) *EKR* if every largest clique of $\mathcal{H}$ is a star; thus the Erdős–Ko–Rado theorem says $\mathcal{K}$ itself satisfies EKR. (In [3] $\mathcal{H}$ is also said to satisfy *weak EKR* if *some* largest clique is a star, but this slightly weaker condition will not concern us here.)

In what follows we use $\mathcal{H} = \mathcal{H}_k(n, p)$ for the random k-graph on V in which members of $\mathcal{K}$ are present independently, each with probability p. As suggested above, we are interested in understanding when EKR holds for $\mathcal{H}$, which we state a little more formally as follows.

Question 1.1. For what $p_0 = p_0(n, k)$ is it true that $\mathcal{H}$ satisfies EKR a.s. provided $p > p_0$?

As usual, an event – really a sequence of events parametrized by n (say) – holds *almost surely* (a.s.) if its probability tends to 1 as $n \to \infty$. Note that here we are thinking of k as a function of n (see the paragraph following Theorem 1.2).

Notice that EKR is not an increasing property (*i.e.* it is not preserved by addition of edges) and that, for given n and k,

$$f_{n,k}(p) := \mathbb{P}(\mathcal{H}_k(n, p) \text{ satisfies EKR}) \tag{1.1}$$

is not increasing in p. For instance, for sufficiently tiny p (depending on n and k) it will usually be the case that *every* clique is contained in a star. In view of this non-monotonicity, it is natural to define a *threshold* for the property EKR to be the least $p_0 = p_0(n, k)$ satisfying

$$f_{n,k}(p) \geqslant \frac{1}{2} \quad \text{for all } p \geqslant p_0. \tag{1.2}$$

This follows the usage in [16] (for example), which takes the 'threshold' for an *increasing* property Q to be the unique p for which the 'p-measure' of Q is 1/2.

For the most part we will not review the contents of [3]. The focus there is mainly on small k; roughly speaking, the authors give fairly complete results for $k = o(n^{1/3})$ and more limited information for k up to $n^{1/2-\varepsilon}$ with $\varepsilon > 0$ fixed.

The nature of the problem changes around $k = n^{1/2}$, since for k smaller than this, two random k-sets are typically disjoint, while the opposite is true for larger k. Heuristically we may say that the problem becomes more interesting/challenging as k grows and the potential violations of EKR proliferate (though increasing k does narrow the range of p for which we *expect* EKR to hold). At any rate, [3] had (as noted there) little to say about k larger than $\sqrt{n}$ (or, indeed, $k > n^{1/2-\varepsilon}$). Here (in Theorem 1.3) we precisely settle the problem for k up to and even a little beyond $\sqrt{n}$. (In line with the above heuristic, it is in the region beyond $\sqrt{n}$ that our task is most delicate.)

As in [3], we will usually find it convenient to work, not directly with p, but with

$$\varphi := p\binom{n-1}{k-1},$$

the expected degree of a vertex (called ρ in [3]); this seems more natural as we are most interested in situations where p is tiny while the value of φ is more reasonable. Throughout the paper we take

$$\mathsf{m} = \mathbb{E}|\mathcal{H}| = \frac{\varphi n}{k},$$

$\Delta = \Delta_{\mathcal{H}}$ (the maximum degree in $\mathcal{H}$) and

$$\mathsf{q} = \mathbb{P}(A \cap B \neq \emptyset), \tag{1.3}$$

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where A and B are chosen uniformly and independently from $\mathcal{K}$. The next assertion is our main result, though a little further work will be needed for the aforementioned characterization given in Theorem 1.3.

Theorem 1.2. *For any fixed $c < 1/4$, if*

$$k < \sqrt{cn \log n} \tag{1.4}$$

and φ is such that

$$\binom{\mathsf{m}}{\Delta}\mathsf{q}^{\binom{\Delta}{2}} = o(1) \ \ a.s., \tag{1.5}$$

then $\mathcal{H}$ satisfies EKR a.s.

(We use $\log = \ln$ and, as usual, $\binom{a}{b} = (a)_b/b! := a(a-1)\cdots(a-b+1)/b!$ for $a \in \mathbb{R}$ and $b \in \mathbb{N}$.) Note that (here and usually in what follows) n is a 'hidden parameter'; thus in Theorem 1.2, k and φ are functions of n and, for example, both '$o(1)$' and 'a.s.' in (1.5) refer to $n \to \infty$. (Note also that what is random in (1.5) is Δ.)

Some clarification of and motivation for (1.5) are perhaps in order. Given $\varphi = \varphi(n)$, set, for $t \in \mathbb{N}$,

$$\Lambda(t) = \Lambda_\varphi(t) = \binom{\mathsf{m}}{t}\mathsf{q}^{\binom{t}{2}}. \tag{1.6}$$

Then (1.5) says there exists

$$\varepsilon = \varepsilon(n) = o(1) \text{ such that } \mathbb{P}(\Lambda(\Delta) < \varepsilon) \to 1 \text{ as } n \to \infty.$$

Its meaning – the reason it is a natural assumption in Theorem 1.2 – is as follows. We think of $\mathsf{q}^{\binom{t}{2}}$ as the ideal value of the probability that random (independent) k-sets $A_1, \ldots, A_t$ form a clique (it would be the actual value if the events $\{A_i \cap A_j \neq \emptyset\}$ were independent). Thus, since $|\mathcal{H}|$ is usually close to m, the left side of (1.5) may be thought of as the expected number of 'generic' Δ-cliques in $\mathcal{H}$, and we should perhaps not expect EKR to hold if this number is not small.

At least for k as in (1.4), this intuition turns out to be correct; that is, (1.5) is essentially *necessary* for the conclusion of Theorem 1.2. Here we should be a little careful: since all cliques of size at most 2 are trivial (*i.e.* contained in stars), failure of (1.5) with $\Delta \leqslant 2$ does *not* suggest failure of EKR. We accordingly define (again, given φ)

$$\Lambda'(t) = \Lambda'_\varphi(t) = \begin{cases} 0 & \text{if } t \leqslant 2, \\ \Lambda(t) & \text{otherwise.} \end{cases}$$

Theorem 1.3. *For c and k as in Theorem 1.2 and any φ $(= \varphi(n))$,*

$$\mathcal{H} \textit{ satisfies EKR a.s. if and only if } \Lambda'(\Delta) < o(1) \textit{ a.s.} \tag{1.7}$$

That Theorem 1.2 implies sufficiency of the condition in (1.7) is easy but not quite tautological and will be discussed in Section 10.

It is not hard to read off threshold information from Theorem 1.3 (with 'threshold' as in (1.2), here translated to the corresponding φ_0); for example, for $k = \sqrt{\zeta n} \gg \sqrt{n}$ (satisfying (1.4)), we have $\varphi_0 \sim e^{\zeta} \log n$. Other special cases include the main positive results on EKR given in [3], those in parts (i), (ii) and (iv) of their Theorem 1.1. (We do use some of these in Section 9, but this could easily be avoided.)

Recent work in [4] provides results for k up to $n/4$ but with nothing like the present accuracy. (For k as in (1.4) their upper bound on φ_0 is of the form $e^{O(k)} \log n$.)

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We believe Theorem 1.2 is true with '$c < 1/4$' replaced by '$c < 1/2$'. It is *not* true above this, roughly because: for $k = \sqrt{cn \log n}$, with $c > 1/2$, (1.5) first occurs at $\varphi \approx \log n / \log(1/\mathsf{q}) \sim n^c \log n$, where (typically) all degrees are close to φ and for each vertex x the number of edges of $\mathcal{H} \setminus \mathcal{H}_x$ meeting all edges of $\mathcal{H}_x$ is about $\varphi(n/k)\mathsf{q}^\varphi \approx n^{c+1/2-1} = n^{\Omega(1)}$, meaning stars are unlikely even to be maximal cliques.

This is, of course, reminiscent of the Hilton–Milner theorem [14], which says that the largest *non-trivial* cliques in $\mathcal{K}$ are those of the form $\{A\} \cup \{B \in \mathcal{K}_x : B \cap A \neq \emptyset\}$ (with $A \in \mathcal{K}$ and $x \in V \setminus A$). It seems not impossible that 'generic' and 'HM' cliques are the main obstructions to EKR in general; a precise, if optimistic, statement to this effect is as follows.

Conjecture 1.4. *If k and φ are such that* (1.5) *holds and*

$$\mathcal{H} \text{ a.s. does not contain a Hilton–Milner family of size } \Delta, \tag{1.8}$$

then $\mathcal{H}$ a.s. satisfies EKR.

Getting from this to the asymptotics of p_0 is routine. Essentially – not quite literally – excluding HM families of size Δ is promising that stars are maximal cliques, and a slight weakening of Conjecture 1.4, resulting in an unnoticeable change in the corresponding p_0, would replace (1.8) with the assumption that this is true a.s.

In a companion paper [12], using methods completely different from those employed here, we jump to the other end of the spectrum, taking k to be as large as possible.

Theorem 1.5. *There is a fixed $\varepsilon > 0$ such that if $n = 2k+1$ and $p > 1-\varepsilon$, then $\mathcal{H}$ satisfies EKR a.s.*

This was prompted by Question 1.4 of [3].

Question 1.6. Is it true that for $k \in (n/2 - \sqrt{n}, n/2)$ and $p = 0.99$, EKR (or weak EKR) holds a.s. for $\mathcal{H}$?

Conjecture 1.4 would say that Theorem 1.5 remains true for p at least about 3/4. (Theorem 1.5 could presumably be extended to the full range of k covered by Question 1.6, but this appears to be far short of the truth if $n \geqslant 2k+2$, so seems of less interest.)

The rest of this paper is organized as follows. The problem is most interesting when

$$k > n^{1/2-o(1)}. \tag{1.9}$$

The bulk of our discussion of Theorem 1.2 (Sections 2 and 5–8) will deal exclusively with this range, while Section 9 handles smaller k. (Section 3 reviews a few standard tools and Section 4 gives some generalities that will apply to both regimes.)

In proving Theorem 1.2 for k as in (1.9) we will find it better to deal first with φ not too far above the 'threshold' – this will account for most of our work – and then treat larger φ mostly by a reduction to what we have established for smaller φ. We thus begin in Section 2 with an outline of the argument for small φ, in particular deriving Theorem 1.2 in this range from three main assertions, Lemmas 2.1–2.3. These are proved in Sections 5–7 following the preparations of Sections 3 and 4. Section 8 then gives the extension to large φ and, as noted above, Section 9 deals with small k. Section 10 deals mainly with necessity of the condition in (1.7). This turns out to be interesting and considerably trickier than one might expect; still, the paper being already too long, we will give the argument somewhat sketchily and only for k as in (1.9). (The problem gets easier as k shrinks.) A glossary of parameters, *etc.*, can be found after Section 10.

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Usage. As already mentioned, we take $V=[n]$, $\mathcal{K}=\binom{[n]}{k}$ and $\mathcal{H}=\mathcal{H}_k(n,p)$. In addition, we set $M=\binom{n-1}{k-1}$ (so $\varphi=Mp$) and $m=|\mathcal{H}|$ (a random variable with mean m). We use v,w,x,y,z for members of V. For a hypergraph $\mathcal{G}$, we let $\mathcal{G}_{\overline{x}}=\mathcal{G}\setminus\mathcal{G}_x$ (recall $\mathcal{G}_x=\{A\in\mathcal{G}:x\in A\}$).

We use $d_{\mathcal{G}}(x)$ for the degree of x in $\mathcal{G}$, and similarly for $d_{\mathcal{G}}(x,y)$, and, where not otherwise specified, take $d(\cdot)$ to mean $d_{\mathcal{H}}(\cdot)$. (As already stated, we use Δ for $\Delta_{\mathcal{H}}$.)

We write $B(\ell,\alpha)$ for a random variable with the binomial distribution $\mathrm{Bin}(\ell,\alpha)$, log for ln and $\binom{a}{\leqslant b}$ for $\sum_{i\leqslant b}\binom{a}{i}$. We use standard asymptotic notation ('big Oh', *etc.*), but will also sometimes use $a\asymp b$ for $a=\Theta(b)$ and $a\ll b$ for $a=o(b)$. We assume throughout that n is large enough to support our arguments and, following a standard abuse, usually pretend large numbers are integers.

2. Main points

From now until the end of Section 8 we fix $c=1/4-\varepsilon$ in Theorem 1.2. Also, as noted above, the present section assumes that k satisfies (1.9) (as well as (1.4); thus $n^{1/2-o(1)}<k<\sqrt{cn\log n}$).

As stated earlier, most of our work will deal with φ fairly near the 'threshold'. Although the problem should become easier as φ grows, some parts of the main argument below break down for larger φ; this could perhaps be remedied, but we have found it easier to first deal directly with smaller φ and then use what we have learned to handle larger values. (A disadvantage of this approach is that it necessarily gives much weaker bounds on the probability that EKR fails than one might *hope* to establish using a more direct argument.)

We thus begin in this section with an outline of where we are headed in the 'small φ' regime. As we will see, the 'threshold' φ_0 $(:=Mp_0)$ is around $\log n/\log(1/\mathsf{q})$, and as a cut-off for 'small' we set (not a delicate choice)

$$\varphi^*=\frac{\log^3 n}{\log(1/\mathsf{q})}. \tag{2.1}$$

We assume in this section, and again in parts of Section 4 and all of Sections 5–7, that $\varphi\leqslant\varphi^*$ – a restriction which could be relaxed considerably without invalidating the present argument. Thus we want to show that

$$\text{for } \varphi\leqslant\varphi^* \text{ satisfying (1.5), } \mathcal{H} \text{ satisfies EKR a.s.} \tag{2.2}$$

(It *is* true that in this regime the problem is most delicate when φ is more or less at the 'threshold'. In particular, it is only here (see the proof of Lemma 2.3) that we must make precise use of (1.5).)

Call a clique *trivial* if it is contained in a star. We will show below that there are integers $\alpha=\alpha(n,\varphi)\leqslant\beta=\beta(n,\varphi)$ satisfying, *inter alia*,

$$\Delta\in[\alpha,\beta]\ \text{ a.s.} \tag{2.3}$$

and

$$\Lambda(\alpha)=o(1). \tag{2.4}$$

Thus (2.2) would follow if we could show that $\mathcal{H}$ a.s. does not contain a non-trivial clique of size α, but this is not quite true; for example, if $d(x)=\Delta$ is significantly larger than α – say closer to β than α – then an $A\in\mathcal{H}\setminus\mathcal{H}_x$ *typically* misses fewer than $\Delta-\alpha$ edges of $\mathcal{H}_x$, in which case $\{A\}\cup\{B\in\mathcal{H}_x:B\cap A\neq\emptyset\}$ is a non-trivial clique of size greater than α.

A natural way to address this is to compare each clique possessing a sufficiently high degree vertex, say x, directly with the star $\mathcal{H}_x$. This idea is implemented in the first of the following three lemmas; these assertions will easily yield (2.2) and will also do most of the work when we come to larger φ. (To be clear, the lemmas will depend on further properties of α and β to be established below.)

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Set

$$\gamma = \min\left\{\alpha, \frac{\varphi^*}{3}\right\}, \tag{2.5}$$

$$\tau = (1-\varepsilon)\gamma, \quad \text{and} \tag{2.6}$$

$$\lambda = \max\left\{\frac{\sqrt{\log n}}{\log(1/q)}, 2\sqrt{\frac{\log n}{\log(1/q)}}\right\}. \tag{2.7}$$

The actual values are not needed in this section.

In reading Lemmas 2.1–2.3 (about whose meaning we will say a little more in a moment), one should keep the following in mind: (i) Lemmas 2.1 and 2.3 are the main points, with Lemma 2.2 just making our lives a little easier when we come to Lemma 2.3; (ii) for the present purposes – *i.e.* derivation of (2.2) – the statements could be slightly simplified, dropping the alternative '$|\mathcal{C}_{\overline{x}}| \geqslant 2/\varepsilon$' in Lemma 2.1 and replacing γ with α in Lemma 2.3 (the stated versions will be used in dealing with larger φ in Section 8).

Lemma 2.1. *A.s. there do not exist (in $\mathcal{H}$) a non-trivial clique $\mathcal{C}$ and vertex x such that $|\mathcal{C}| \geqslant d(x)$, $d_{\mathcal{C}}(x) \geqslant \tau$, and either $|\mathcal{C}| \geqslant \alpha$ or $|\mathcal{C}_{\overline{x}}| \geqslant 2/\varepsilon$.*

Lemma 2.2. *A.s. $\mathcal{H}$ does not contain a clique with two vertices of degree at least λ.*

Lemma 2.3. *A.s. $\mathcal{H}$ does not contain a clique of size γ with at most one vertex of degree greater than λ and maximum degree less than τ.*

One may think of Lemmas 2.1 and 2.3 as addressing the dichotomy that informs Conjecture 1.4, *viz.* that the things we mainly need to worry about are (non-trivial) cliques that are either close to stars or somehow 'generic'.

Lemma 2.1 deals with the first possibility. As observed following (2.4), we cannot depend on there being some *a priori* value which is a.s. both a lower bound on Δ and an upper bound on the size of a largest non-trivial clique. But we *do* expect that any clique $\mathcal{C}$ causing trouble here is close to some star $\mathcal{H}_x$; that is, $\mathcal{C}$ consists mostly of $\mathcal{C}_x$, but also contains enough edges *off* x to make up for the edges of $\mathcal{H}_x$ that it misses. Thus Lemma 2.1 deals in such direct comparisons $\mathcal{C}$ versus $\mathcal{H}_x$. (Note that (2.3) allows us to restrict to $|\mathcal{C}| \geqslant \alpha$. The interested reader might also check [12] for an unrelated, far trickier treatment of such comparisons.)

Lemma 2.3 (with the auxiliary Lemma 2.2) then handles the 'generic' case of cliques without very high degrees. Of course the lemma's mild degree restrictions are far from what might be considered 'generic' but, somewhat supporting our dichotomous intuition, it is really when degrees become more typical that the problem becomes most challenging. As mentioned earlier – and as one might expect, given the condition's rationale – it is only here that we need to fully exploit (1.5). (That this *must* be the case follows from the discussion of Section 10, where failure of (1.5) is shown to imply an $\Omega(1)$ probability of violating EKR with some clique as in Lemma 2.3.)

Proof of (2.2). Since $\mathbb{P}(\Delta < \alpha) = o(1)$ (see (2.3)), it is enough to show that $\mathcal{H}$ a.s. does not contain a non-trivial clique $\mathcal{C}$ with $|\mathcal{C}| \geqslant \Delta \geqslant \alpha$. But if $\Delta \geqslant \alpha$ and $\mathcal{H}$ does contain such a $\mathcal{C}$, then at least one of the following occurs.

(a) There is an x with $d_{\mathcal{C}}(x) \geqslant \tau$ (and $|\mathcal{C}| \geqslant \Delta \geqslant \max\{\alpha, d(x)\}$), so $x, \mathcal{C}$ are as in Lemma 2.1.

(b) There are two vertices with degree at least λ in $\mathcal{C}$.

(c) There is at most one vertex x with $d_{\mathcal{C}}(x) \geqslant \lambda$ and none with $d_{\mathcal{C}}(x) \geqslant \tau$, so (since $\alpha \geqslant \gamma$) $\mathcal{C}$ is as in Lemma 2.3.

But according to Lemmas 2.1–2.3, each of (a)–(c) occurs with probability $o(1)$, so we have (2.2). □

We now pause to fill in some preliminaries.

3. Negative association and large deviations

Some parts of the analysis below seem most conveniently handled using the notion of negative association, regarding which we just recall what little we need, in particular confining ourselves to $\{0, 1\}$-valued random variables; see *e.g.* [8, 18] for further background.

Recall that events $\mathcal{A}, \mathcal{B}$ in a probability space are *negatively correlated* (denoted $\mathcal{A} \downarrow \mathcal{B}$) if $\mathbb{P}(\mathcal{A}\mathcal{B}) \leqslant \mathbb{P}(\mathcal{A})\ \mathbb{P}(\mathcal{B})$. Given a set S, set $\Omega = \Omega_S = \{0, 1\}^S$ and recall that $\mathcal{A} \subseteq \Omega$ is *increasing* if $x \geqslant y \in \mathcal{A} \Rightarrow x \in \mathcal{A}$ (where '$\geqslant$' is product order on Ω). Say $i \in S$ *affects* $\mathcal{A} \subseteq \Omega$ if there are $\eta \in \mathcal{A}$ and $\nu \in \Omega \setminus \mathcal{A}$ with $\eta_j = \nu_j$ for all $j \neq i$, and write $\mathcal{A} \perp \mathcal{B}$ if no $i \in S$ affects both $\mathcal{A}$ and $\mathcal{B}$.

Now suppose $(X_i : i \in S)$ is drawn from some probability distribution on Ω. The X_i are said to be *negatively associated* (NA) if $\mathcal{A} \downarrow \mathcal{B}$ whenever $\mathcal{A}, \mathcal{B}$ are increasing and $\mathcal{A} \perp \mathcal{B}$. If Q_i are events whose indicators are NA then we also say that the Q_i themselves are NA.

The following observation is surely not news, but as we do not know of a reference we give the easy proof.

Proposition 3.1. *Suppose that for some $V_1, \ldots, V_s \subseteq V$ and $\ell_1, \ldots, \ell_s, A_1, \ldots, A_s$ are chosen independently with A_j uniform from $\binom{V_j}{\ell_j}$. Then the random variables $X_{\nu j} = \mathbf{1}_{\{\nu \in A_j\}}$ $(\nu \in V, j \in [s])$ are negatively associated.*

Proof. (See [8, Proposition 12].) For each j the vector $(X_{\nu j} : \nu \in V)$ is chosen uniformly from the strings of weight ℓ_j in $\{0, 1\}^{V_j}$, implying that the random variables $X_{\nu j}$ $(\nu \in V)$ are NA. (This is standard and easy, though we could not find it in writing. A stronger and far more interesting statement is the main result of [25].) We may thus apply [8, Proposition 8], which says that if the collections $\{X_{\nu j} : \nu \in V\}$ $(j \in [s])$ are mutually independent and each is NA, then the entire collection $\{X_{\nu j}\}$ is also NA. □

We will use Proposition 3.1 in conjunction with the following trivial observations.

Proposition 3.2. *If the random variables $X_1, \ldots, X_m$ are NA, $I_1, \ldots, I_r$ are disjoint subsets of $[m]$, and Q_j is an increasing event determined by $\{X_i : i \in I_j\}$, then $Q_1, \ldots, Q_r$ are NA.*

Proposition 3.3. *If the events Q_i are NA, then $\mathbb{P}(\cap Q_i) \leqslant \prod \mathbb{P}(Q_i)$.*

One virtue of negative association lies in the fact that 'Chernoff-type' large deviation bounds for random variables $X = \sum X_i$, where $X_1, \ldots$ are independent Bernoulli random variables, remain valid under the (weaker) assumption that the X_i are negatively associated. As far as we know, this was first observed by Dubhashi and Ranjan [8, Proposition 7]; it is obtained via the usual argument (Markov's inequality applied to $\exp[tX]$; see *e.g.* [15, pp. 26–28]), with the identity $\mathbb{E}e^{tX} = \prod \mathbb{E}e^{tX_i}$ replaced with the inequality $\mathbb{E}e^{tX} \leqslant \prod \mathbb{E}e^{tX_i}$. In particular this gives the following bounds (see *e.g.* [15, Theorem 2.1 and Corollary 2.4]).

Theorem 3.4. *Suppose $X_1, \ldots, X_m$ are negatively associated* $\mathrm{Ber}(p)$ *random variables, $X = \sum X_i$ and $\mu = \mathbb{E}X$. Then, for any $\lambda \geqslant 0$,*

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$$\mathbb{P}(X > \mu + \lambda) < \exp\left[-\frac{\lambda^2}{2(\mu + \lambda/3)}\right],$$
$$\mathbb{P}(X < \mu - \lambda) < \exp\left[-\frac{\lambda^2}{2\mu}\right], \tag{3.1}$$

and for any $K > 1$,

$$\mathbb{P}(X > K\mu) < [e^{K-1}K^{-K}]^{\mu}. \tag{3.2}$$

Corollary 3.5. *The inequality* (3.2) *still holds if instead of* $\mathbb{E}X = \mu$ *(in Theorem* 3.4*) we assume only* $\rho := \mathbb{E}X \leqslant \mu$.

Proof. We have (using (3.2) for the inequality)

$$\mathbb{P}(X > K\mu) = \mathbb{P}\left(X > \left(\frac{K\mu}{\rho}\right)\rho\right) < \left[e^{K\mu/\rho - 1}\left(\frac{K\mu}{\rho}\right)^{-K\mu/\rho}\right]^{\rho} = e^{K\mu-\rho}K^{-K\mu}\left(\frac{\mu}{\rho}\right)^{-K\mu}.$$

The last expression is equal to the bound in (3.2) when $\mu = \rho$ and is easily seen to be decreasing in $\mu \geqslant \rho$ (provided $K \geqslant 1$). ☐

4. Generics

This section establishes basic properties of some of the parameters we will be dealing with, in particular showing that $\mathcal{H}$ a.s. satisfies a few general properties whose failure can then be more or less ignored in what follows.

To begin, we should say something about the intersection probability q (defined in (1.3)). We have $\mathsf{q} = 1 - \vartheta$ with

$$\vartheta = \frac{(n-k)_k}{(n)_k} \sim e^{-k^2/n}, \tag{4.1}$$

where, as usual, $(b)_a = b(b-1)\cdots(b-a+1)$. (The '$\sim$' is valid provided $k = o(n^{2/3})$.) This gives the asymptotics of q for $k = \Omega(\sqrt{n})$. In particular, for $k \gg \sqrt{n}$ we have

$$\log\left(\frac{1}{\mathsf{q}}\right) \sim e^{-k^2/n}. \tag{4.2}$$

For $k \ll \sqrt{n}$ we instead have

$$\mathsf{q} \sim \frac{k^2}{n}, \tag{4.3}$$

since, with $X_v = \mathbf{1}_{\{v \in A\cap B\}}$,

$$\frac{k^2}{n} = \sum \mathbb{E}X_v \geqslant \mathsf{q} \geqslant \sum \mathbb{E}X_v - \sum \mathbb{E}X_vX_w > \frac{k^2}{n} - \binom{n}{2}\left(\frac{k}{n}\right)^4.$$

Note that in any case we have

$$\varphi^* < n^{1/4-\varepsilon+o(1)}. \tag{4.4}$$

We will usually be dealing with situations in which q is slightly perturbed by information on how relevant k-sets meet some small subset of V. This negligible effect is handled by the next observation.

Proposition 4.1. *Fix $W \subseteq V$ of size at most $\mathsf{w} \ll n/\log n$ and $B \in \binom{V}{k}$, and let A be uniform from $\binom{V}{k}$. Then, conditioned on any value of $A \cap W$,*

$$\mathbb{P}(A \cap (B \setminus W) \neq \emptyset) < \left(1 + \frac{2k^2\mathsf{w}}{\mathsf{q}n^2}\right)\mathsf{q}.$$

Proof. The probability is largest when $|W| = \mathsf{w}$ and $B \cap W = A \cap W = \emptyset$, in which case its value is $q = 1 - \varsigma$, with

$$\varsigma = \frac{(n - \mathsf{w} - k)_k}{(n - \mathsf{w})_k}.$$

We have

$$\frac{\vartheta}{\varsigma} = \frac{(n-k)_k(n-\mathsf{w})_k}{(n)_k(n-\mathsf{w}-k)_k} = \prod_{i=0}^{k-1}\left(1 + \frac{k\mathsf{w}}{(n-i)(n-\mathsf{w}-k-i)}\right) = 1 + (1+o(1))\frac{k^2\mathsf{w}}{n^2},$$

that is, $\vartheta/\varsigma - 1 \sim k^2\mathsf{w}/n^2$ ($= o(1)$ because of the bound on w). Thus

$$\frac{q}{\mathsf{q}} - 1 = \frac{\vartheta - \varsigma}{1 - \vartheta} = \frac{1}{1-\vartheta}\left(\frac{\vartheta}{\varsigma} - 1\right)\varsigma \sim \frac{k^2\mathsf{w}\varsigma}{(1-\vartheta)n^2} \sim \frac{k^2\mathsf{w}}{(1-\vartheta)n^2}e^{-k^2/n}.$$

The lemma follows. □

In all that follows we assume φ satisfies (1.5). At some (indicated) points in this section, and again throughout Sections 5–7, we will also stipulate that $\varphi \leqslant \varphi^*$ (defined in (2.1)).

We also assume from now on that $\mathsf{m} = \omega(1)$, since cases with $\mathsf{m} = O(1)$ are trivial: if $\mathsf{m} = o(1)$ then (1.5) fails (the left-hand side is a.s. 1; actually in this case $\mathcal{H}$ is a.s. empty and does satisfy EKR), while if $\mathsf{m} = \Theta(1)$ then with probability $\Omega(1)$ we have $\Delta = |\mathcal{H}| = 1$ and the left-hand side of (1.5) is m (so (1.5) does not hold). Recall that $m = |\mathcal{H}|$. Let $\psi = \psi(n)$ be some slowly growing function of n (say $\psi = \log n$). Theorem 3.4 (for independent Bernoulli random variables) says that a.s.

$$m \in (\mathsf{m} - \psi\sqrt{\mathsf{m}}, \mathsf{m} + \psi\sqrt{\mathsf{m}}). \tag{4.5}$$

We henceforth write m_0 for $\mathsf{m} + \psi\sqrt{\mathsf{m}}$.

It will often be convenient to replace $\mathcal{H}$ with $A_1, \ldots, A_m$ chosen *independently* from $\mathcal{K}$ – a change that makes little difference when m is small.

Proposition 4.2. *If $m_0 \ll \sqrt{|\mathcal{K}|}$ then for any property $\mathcal{A}$ and any c, if*

$$\max_{m \models (4.5)} |\mathbb{P}(A_1, \ldots, A_m \models \mathcal{A}) - c| = o(1)$$

(where the A_i are chosen uniformly and independently from $\mathcal{K}$), then

$$\mathbb{P}(\mathcal{H} \models \mathcal{A}) = c + o(1).$$

We will eventually need something more careful in a similar vein; see the paragraph preceding Lemma 7.1.

Proof. For any l the law of $\{A_1, \ldots, A_l\}$, given that the A_i are distinct, is the same as that of $\mathcal{H}$ given $m = l$. We may thus couple $\mathcal{H}$ and $\{A_1, \ldots, A_m\}$ so that they coincide whenever the A_i are distinct. But the probability that they are *not* distinct is at most

$$\mathbb{P}(m \not\models (4.5)) + \mathbb{P}(A_1, \ldots, A_{m_0} \text{ are not distinct}) < o(1) + m_0^2/|K| = o(1)$$

and the proposition follows. □

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From now until the 'coda' at the end of this section we assume that

$$\varphi > n^{-o(1)}. \tag{4.6}$$

As we will see in the coda, this is implied by (1.5) if we assume (1.9). (Note that here we do *not* assume (1.9), since we will also need parts of what follows in Section 9, where (1.9) does not hold.)

We next need to say something about the behaviour of φ and Δ. Recall that our default for degrees is $\mathcal{H}$; thus, in addition to $\Delta = \Delta_{\mathcal{H}}$, we take $d_x = d(x) = d_{\mathcal{H}}(x)$ and $d(x,y) = d_{\mathcal{H}}(x,y)$. The properties we need will be given in Proposition 4.3 once we have introduced the parameters α and β mentioned earlier (see (2.3), (2.4)).

Let α_1 and β, respectively, be the largest integer with $\mathbb{P}(d_v \geqslant \alpha_1) \geqslant \psi/n$ and the smallest integer with $\mathbb{P}(d_v > \beta) < 1/(n\psi)$ (for any v).

Notice that $\Lambda(0) = 1$ and (since $\Lambda(t)/\Lambda(t-1) = ((\mathsf{m}-t+1)/t)\mathsf{q}^{t-1}$ is decreasing in t) there is some t_0 such that $\Lambda(t)$ is increasing up to t_0 and decreasing thereafter. Thus (1.5) says that there are $\varsigma = \varsigma(n)$ and $\upsilon = \upsilon(n)$, both $o(1)$, such that $\mathbb{P}(\Lambda(\Delta) > \varsigma) < \upsilon$. Set $\alpha_2 := \min\{t : \Lambda(t) \leqslant \varsigma\}$ and $\alpha = \max\{\alpha_1, \alpha_2\}$.

The promised Proposition 4.3 now collects properties of these parameters that we will use repeatedly in what follows.

Proposition 4.3. *For α, β as above:*

$$\alpha \leqslant \beta, \tag{4.7}$$

$$\Lambda(\alpha) = o(1), \tag{4.8}$$

$$\Delta \leqslant \beta \text{ a.s., if } \varphi \leqslant \varphi^* \text{ then } \Delta \geqslant \alpha \text{ a.s.}, \tag{4.9}$$

$$\beta/\varphi < n^{o(1)}, \tag{4.10}$$

$$\alpha > (1-o(1))\frac{\log n}{\log(1/\mathsf{q})}, \tag{4.11}$$

$$\text{if } \varphi \leqslant \varphi^* \text{ then } \beta < (1+o(1))\varphi^* \ (< n^{1/4-\varepsilon+o(1)}). \tag{4.12}$$

It is not hard to see that in fact $\alpha \sim \beta$ in all cases and $\beta \sim \varphi$ if and only if $\varphi \gg \log n$. What we actually use for the second part of (4.9) is $\alpha_1 k/n \ll 1$.

For the rest of the paper we set

$$\mathcal{P} = \{m \text{ satisfies (4.5)}\} \wedge \{\Delta \leqslant \beta\},$$

noting that (4.9) and our earlier observation that (4.5) holds a.s. give

$$\mathbb{P}(\mathcal{P}) = 1 - o(1). \tag{4.13}$$

Proof of Proposition 4.3. The first assertion in (4.9) is immediate from the definition of β. From the definition of α_2 we have $\Lambda(\alpha_2) = o(1)$ (namely $\Lambda(\alpha_2) \leqslant \varsigma$) and $\Delta \geqslant \alpha_2$ a.s. (since $\mathbb{P}(\Delta < \alpha_2) = \mathbb{P}(\Lambda(\Delta) > \varsigma) < \upsilon$), implying $\alpha_2 \leqslant \beta$. This gives (4.7) (since $\alpha_1 \leqslant \beta$ is trivial) and (4.8).

Let $\beta^* = \lceil \varphi + \eta \rceil$, with η the positive root of $x = \sqrt{2(\varphi + x/3)(\log n + \log \psi)}$. Then Theorem 3.4 gives (for any v)

$$\mathbb{P}(d_v > \beta^*) < \exp\left[-\frac{\eta^2}{2(\varphi + \eta/3)}\right] = (n\psi)^{-1}, \tag{4.14}$$

whence $\beta \leqslant \beta^*$. (The bound is very crude for smaller values of φ, but we have lots of room in such cases.) In particular, since $\eta = O(\max\{\sqrt{\varphi \log n}, \log n\})$, (4.6) now implies both (4.10) and (4.12) (and $\beta \sim \varphi$ if $\varphi \gg \log n$, but we do not need this).

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For (4.11) we have

$$\Lambda(\alpha_2) > \exp\left[\alpha_2\left(\log\left(\frac{\mathsf{m}}{\alpha_2}\right) - \frac{\alpha_2 - 1}{2}\log\left(\frac{1}{\mathsf{q}}\right)\right)\right]$$
$$> \exp\left[\frac{\alpha_2}{2}\left((1 - o(1))\log n - \alpha_2 \log\left(\frac{1}{\mathsf{q}}\right)\right)\right]$$

(since $\log(\mathsf{m}/\alpha_2) > (1/2 - o(1))\log n$, as follows from $\mathsf{m} = \varphi n/k$, $\alpha_2 \leqslant \beta$ and (4.10)), and combining this with (4.8) gives $\alpha_2 > (1 - o(1))\log n/\log(1/\mathsf{q})$.

Finally, the second assertion in (4.9) is given by the following more general statement, which we will need again in Section 9. □

Proposition 4.4. *For any n, k, φ $(= Mp)$ and $\theta \in \mathbb{N}$ satisfying $p = o(1)$ and $\theta = o(M)$, if $\mathbb{P}(d_v \geqslant \theta) = \omega(1/n)$ and $\theta k/n = o(1)$ then $\Delta \geqslant \theta$ a.s.*

The assumption $\theta = o(M)$ is a little silly: for $k \geqslant 3$ it follows from $\theta k/n = o(1)$. For (4.9) – note we already know $\Delta \geqslant \alpha_2$ a.s. – the hypothesis $\alpha_1 k/n = o(1)$ follows from $\varphi k/n < n^{-1/4}$ and $\alpha_1/\varphi \leqslant \beta/\varphi < n^{o(1)}$; see (4.4) and (4.10). For $k < n^{1/2-\Omega(1)}$ and a *fixed* θ, Proposition 4.4 is [3, Lemma 3.6].

Proof of Proposition 4.4. Let $X_v = \mathbf{1}_{\{d_v \geqslant \theta\}}$ and $X = \sum X_v$. We are assuming $\mathbb{E}X = \omega(1)$, so to finish via the second moment method we just need

$$\mathbb{E}X_v X_w \sim \mathbb{E}^2 X_v \tag{4.15}$$

(for $v \neq w$). Letting $Z = d(v, w)$, we have

$$\mathbb{E}X_v X_w < \sum_{l \geqslant 0} \mathbb{P}(Z = l)\, \mathbb{P}^2(d_v \geqslant \theta - l). \tag{4.16}$$

For equality we would replace d_v with $d(v, \overline{w}) := |\mathcal{H}_v \setminus \mathcal{H}_w|$.

Now, Z is binomial with $\mathbb{E}Z < \varphi k/n$, so

$$\mathbb{P}(Z = l) \quad (\leqslant \mathbb{P}(Z \geqslant l)) < \left(\frac{\varphi k}{n}\right)^l. \tag{4.17}$$

On the other hand, since $d_v \sim \mathrm{Bin}(M, p)$, we have, for each $t \leqslant \theta$,

$$\frac{\mathbb{P}(d_v = t - 1)}{\mathbb{P}(d_v = t)} = \frac{t(1-p)}{(M - t + 1)p} \sim \frac{t}{\varphi}, \tag{4.18}$$

implying $\mathbb{P}(d_v \geqslant t - 1) < (1 + \theta/\varphi)\ \mathbb{P}(d_v \geqslant t)$. Thus (since $\theta k/n = o(1)$) the sum in (4.16) is asymptotic to its zeroth term, and we have (4.15).

We pickily add – to make sure that $\varphi k/n = o(1)$ – that we may assume $\theta \geqslant \varphi$: there is nothing to prove if $\theta = 0$, and $\Delta \geqslant \varphi$ is easy if $\varphi \geqslant 1$ (and $k = o(n)$, which follows from $\theta > 0$ and $\theta k/n = o(1)$). □

We will also eventually (in Section 7) need the easy: if $\varphi \leqslant \varphi^*$ then

$$\binom{m_0}{\alpha} \sim \binom{\mathsf{m}}{\alpha}. \tag{4.19}$$

The ratio of the left- and right-hand sides is

$$\frac{(m_0)_\alpha}{(\mathsf{m})_\alpha} < \left(\frac{m_0 - \alpha + 1}{\mathsf{m} - \alpha + 1}\right)^\alpha < \exp\left[O\left(\frac{\psi\alpha}{\sqrt{\mathsf{m}}}\right)\right]$$

and $\psi\alpha/\sqrt{\mathsf{m}} \leqslant \psi\beta/\sqrt{\mathsf{m}} < n^{-\varepsilon + o(1)}$ (using $\mathsf{m} = \varphi n/k$, (4.6) and (4.12)).

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For $x \in V$, let $W_x = \{y : d(x,y) \geqslant 2\}$ (a random set determined by $\mathcal{H}_x$). Let $\mathcal{R}$ be the intersection of $\mathcal{P}$ and the events $\{\Delta \geqslant \alpha\}$,

$$\{d(x,y) \leqslant 8 \text{ for all } x, y\} \tag{4.20}$$

and

$$\left\{|W_x| < \max\left\{\frac{\varphi^2 k^2}{n},\ 6\log n\right\} \text{ for all } x\right\}. \tag{4.21}$$

Although defined here in general, $\mathcal{R}$ is only of interest when φ is small.

Proposition 4.5. *If $\varphi \leqslant \varphi^*$, then $\mathbb{P}(\mathcal{R}) = 1 - o(1)$.*

Proof. We have already seen (in (4.13) and (4.9)) that $\mathcal{P}$ and $\{\Delta \geqslant \alpha\}$ hold a.s. That (4.20) does as well follows (via the union bound) from the fact, already noted in (4.17), that $\mathbb{P}(d(x,y) \geqslant l) \ll n^{-l/4}$. To deal with (4.21), it is, according to Proposition 4.2, enough to show the following.

Claim. If m satisfies (4.5) and $A_1, \ldots, A_m$ are chosen independently (and uniformly) from $\mathcal{K}$, then (4.21) holds a.s.

Here of course d in the definition of W_x now refers to $\{A_1, \ldots, A_m\}$ rather than $\mathcal{H}$.

Proof of Claim. For a given x we have, for each $y \neq x$,

$$\mathbb{P}(y \in W_x) < \binom{m}{2}\left(\frac{k}{n}\right)^4 < \left(\frac{1}{2} + o(1)\right)\left(\frac{\varphi k}{n}\right)^2$$

(using $m \sim \mathsf{m} = \varphi n/k$), implying $\mathbb{E}|W_x| < (1+o(1))\varphi^2 k^2/(2n)$. On the other hand, the events $\{y \in W_x\}$ are NA (by Propositions 3.1 and 3.2), and a little calculation, with Corollary 3.5, bounds the probability that a particular x violates (4.21) by $o(1/n)$. In more detail: if $\mu := \varphi^2 k^2/(2n) \geqslant 3\log n$, then (3.1) bounds the probability by $\exp[-(9/8)\log n]$; otherwise $K := 6\log n/\mu > 2$, and (3.2) bounds the probability by $(e^{K-1}K^{-K})^\mu = (e^{1-1/K}K^{-1})^{K\mu} \leqslant (\sqrt{e}/2)^{6\log n} = o(1/n)$. □

Coda. Finally, we say why the combination of (1.5) and (1.9) implies (4.6). Suppose instead that the first two conditions hold but $\varphi < n^{-\Omega(1)}$. Then $\Delta < O(1)$ a.s. But if $\Delta = O(1)$, then $\mathsf{q} > n^{-o(1)}$ (see (4.3)) implies $\Lambda(\Delta) = \Omega(\mathsf{m}^\Delta)n^{-o(1)}$, so that (1.5) implies $\mathsf{m} < n^{o(1)}$ (note $\Delta \geqslant 1$ a.s. since we assume $\mathsf{m} = \omega(1)$). But then (since $\mathsf{m} = \varphi n/k$ and we assume (1.9)) $\varphi < n^{-1/2+o(1)}$, implying that in fact $\Delta \leqslant 2$ a.s.

Now suppose $\Lambda(2) = o(1)$. Then $k \ll \sqrt{n}$ (otherwise $\mathsf{q} = \Omega(1)$ and $\mathsf{m} = o(1)$, contrary to assumption), and $\Lambda(2) \asymp (\varphi n/k)^2(k^2/n) = \varphi^2 n$, implying $\varphi \ll n^{-1/2}$ and $\Delta = 1$ a.s. But $\Lambda(1) = \mathsf{m}$, so we contradict (1.5) . □

5. Proof of Lemma 2.1

We recall the lemma.

Lemma 2.1. *A.s. there do not exist (in $\mathcal{H}$) a non-trivial clique $\mathcal{C}$ and vertex x such that $|\mathcal{C}| \geqslant d(x)$, $d_{\mathcal{C}}(x) \geqslant \tau$, and either $|\mathcal{C}| \geqslant \alpha$ or $|\mathcal{C}_{\overline{x}}| \geqslant 2/\varepsilon$.*

Although the proof of this requires some care, the basic idea is simple enough: the event in question requires that for some very large $\mathcal{A} \subseteq \mathcal{H}_x$ (below this will be $\{A_i : i \in I\}$), every $B \in \mathcal{C} \setminus \mathcal{H}_x$ meets every $A \in \mathcal{A}$, and we just try to show that such 'cross-intersection' is unlikely (sacrificing the clique requirement within $\mathcal{C} \setminus \mathcal{H}_x$). In outline this goes as follows. We think of choosing:

(i) $\mathcal{H}_x$, which by Proposition 4.5 we may assume respects the genericity conditions $\mathcal{R}$ (or those parts involving $\mathcal{H}_x$, called $\mathcal{R}_x$ below);

(ii) $\mathcal{H}_{\bar{x}}$, some $r := |\mathcal{C}_{\bar{x}}| \geqslant 1$ edges of which must meet every edge of some specified $\mathcal{A}$ as above.

Under $\mathcal{R}_x$ the probability that a uniform B from $\mathcal{K}_{\bar{x}}$ meets all members of a given $\mathcal{A}$ is close to its natural value (see Corollary 5.2), which, unless r is quite small, already gives a good bound on the probability of seeing $x, \mathcal{C}$ as in the lemma; see (5.10).

For smaller r the bounds in (5.10) are not quite adequate; here we are saved by the requirement that $|\mathcal{C}|$ be at least α (the technical $|\mathcal{C}_{\bar{x}}| \geqslant 2/\varepsilon$ plays no role at this stage), which limits possibilities for x since $d(x) \geqslant \alpha - r$ is unlikely when r is small.

We now turn to the actual argument. Here and in Section 6 we take

$$\mathsf{w} = \max\left\{\frac{\varphi^2 k^2}{n},\ 6\log n\right\} \tag{5.1}$$

and

$$q = \left(1 + \frac{2k^2\mathsf{w}}{\mathsf{q}n^2}\right)\mathsf{q}; \tag{5.2}$$

thus w is the bound on the $|W_x|$ in (4.21) and q is the probability bound in Proposition 4.1. We will need to say that q is close to q; here and in Section 6 we could get by with, for example, $\log(1/q) \sim \log(1/\mathsf{q})$, but for the more delicate situation in Section 7 we will need

$$q^{\binom{\alpha}{2}} \sim \mathsf{q}^{\binom{\alpha}{2}}. \tag{5.3}$$

That is, $k^2\mathsf{w}\alpha^2/(\mathsf{q}n^2) = o(1)$; in fact, $k^2\mathsf{w}\alpha^2/(\mathsf{q}n^2) < n^{-4\varepsilon+o(1)}$ since $\alpha < n^{1/4-\varepsilon+o(1)}$ (see (4.12)), $\mathsf{w} < n^{1/2-2\varepsilon+o(1)}$ (see (4.4)) and $k^2/(\mathsf{q}n) < \log n$.

We will use part (a) of the following observation in the present section and the variant (b) in Section 6.

Proposition 5.1.

(a) *Suppose* $\mathcal{A} = \{A_1, \ldots, A_d\} \subseteq \mathcal{K}_x$ *satisfies*

$$d_{\mathcal{A}}(z) \leqslant 8 \quad \text{for all } z \in V \setminus \{x\} \text{ and } |\{z \in V \setminus \{x\} : d_{\mathcal{A}}(z) \geqslant 2\}| < \mathsf{w}. \tag{5.4}$$

Then, for B *uniform from* $\mathcal{K}_{\bar{x}}$,

$$\mathbb{P}(B \cap A_i \neq \emptyset \text{ for all } i \in [d]) < (1 + o(1))q^d.$$

(b) *The same conclusion holds if* $\mathcal{A} \subseteq \{A \in \mathcal{K}_x : y \notin A\}$ *satisfies* (5.4) *and* B *is uniform from* $\{A \in \mathcal{K}_y : x \notin A\}$.

Of course, the '8' in (5.4) is just the value we happen to have below.

Proof. The proofs of (a) and (b) are essentially identical and we just give the former. Set $W = \{z \in V \setminus \{x\} : d_{\mathcal{A}}(z) \geqslant 2\}$. Since the events $\{z \in B\}$ $(z \in V \setminus \{x\})$ are negatively associated (see Proposition 3.1), Proposition 3.3 and the second condition in (5.4) give

$$\mathbb{P}(|B \cap W| = s) \leqslant \binom{\mathsf{w}}{s}\left(\frac{k}{n}\right)^s < \left(\frac{\mathsf{w}k}{n}\right)^s < n^{-(2\varepsilon - o(1))s}. \tag{5.5}$$

On the other hand we assert that, with $\mathcal{Q} = \{B \cap A_i \neq \emptyset \text{ for all } i \in [d]\}$, we have

$$\mathbb{P}(\mathcal{Q} \mid |B \cap W| = s) < q^{d-8s}. \tag{5.6}$$

To see this, condition on the value, Z, of $B \cap W$ (with $|Z| = s$), and let

$$I = \{i \in [d] : B \cap A_i \cap W = \emptyset\}.$$

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Then $|I| \geqslant d-8s$ (by the first condition in (5.4)) and B must meet the members of $\{A_i : i \in I\}$ in $V \setminus W$, where they are pairwise disjoint. By Proposition 4.1,

$$\mathbb{P}(B \cap (A_i \setminus W) \neq \emptyset \mid B \cap W = Z) < q \quad \text{for each } i.$$

But, given $\mathcal{R}_Z := \{B \cap W = Z\}$, $B \setminus Z$ is a uniformly chosen $(k-s)$-subset of $V \setminus W$, so by Propositions 3.1 and 3.2 the events $Q_i = \{B \cap (A_i \setminus W) \neq \emptyset\}$ are conditionally NA given $\mathcal{R}_Z$ (with $\mathcal{Q} = \cap_{i\in I} \mathcal{Q}_i$); thus Proposition 3.3 gives

$$\mathbb{P}(\mathcal{Q} \mid \mathcal{R}_Z) < q^{|I|} \leqslant q^{d-8s},$$

which implies (5.6).

Finally, combining (5.5) and (5.6), we have

$$\begin{aligned}\mathbb{P}(\mathcal{Q}) &= \sum_{s\geqslant 0} \mathbb{P}(|B \cap W| = s)\, \mathbb{P}(\mathcal{Q} \mid |B \cap W| = s) \\ &< \sum_{s\geqslant 0} n^{-(2\varepsilon - o(1))s} q^{d-8s} \\ &= q^d \sum_{s\geqslant 0} (n^{-(2\varepsilon-o(1))} q^{-8})^s \sim q^d \end{aligned}$$ □

Corollary 5.2. *Suppose either $\mathcal{A}$ is as in Proposition 5.1(a) and $\mathcal{B}$ is chosen uniformly from the b-subsets of $\mathcal{K}_{\overline{x}}$, or $\mathcal{A}$ is as in Proposition 5.1(b) and $\mathcal{B}$ is chosen uniformly from the b-subsets of $\{A \in \mathcal{K} : y \in A, x \notin A\}$. Then*

$$\mathbb{P}(B \cap A_i \neq \emptyset \quad \textit{for all } B \in \mathcal{B}, i \in [d]) < (1+o(1))^b q^{db}.$$

Proof. Again we just discuss the first case. We may take $\mathcal{B} = \{B_1, \ldots, B_b\}$ with B_i uniform from $\mathcal{K}_{\overline{x}} \setminus \{B_1, \ldots, B_{i-1}\}$. Then, with $\mathcal{Q}_i = \{B_i \cap A_j \neq \emptyset \text{ for all } j \in [d]\}$, we have

$$\mathbb{P}(\cap \mathcal{Q}_i) \leqslant \prod \mathbb{P}(\mathcal{Q}_i) < (1+o(1))^b q^{db},$$

with the second inequality given by Proposition 5.1. (The first is obvious: since the B_i are drawn without replacement, the probability that all are drawn from those members of $\mathcal{K}_{\overline{x}}$ that meet all A_j is less than it would be if they were drawn independently.) □

Terminology. Recall that $\mathcal{A}$, $\mathcal{B}$ (two families of sets) are *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}, B \in \mathcal{B}$.

Proof of Lemma 2.1. Let $\mathcal{Q}(x, r)$ be the event that there is some $\mathcal{C}$ as in the lemma, with $|\mathcal{C}_{\overline{x}}|$ $(= |\mathcal{C}| - d_{\mathcal{C}}(x)) = r$, and let $\mathcal{Q}(x) = \cup_{r\geqslant 1} \mathcal{Q}(x, r)$. By Proposition 4.5 it is enough to show that (for any x)

$$\mathbb{P}(\mathcal{Q}(x) \wedge \mathcal{R}) = o\left(\frac{1}{n}\right). \tag{5.7}$$

(Recall that $\mathcal{R}$ was defined in the paragraph containing (4.20) and (4.21).) Let

$$\mathcal{R}_x = \{m \leqslant m_0;\, d(x) \leqslant \beta;\, d(x,z) \leqslant 8 \text{ for all } z \in V \setminus \{x\};\, |W_x| \leqslant \mathrm{w}\}.$$

Then $\mathcal{R}_x \supseteq \mathcal{R}$, so for (5.7) it will be enough to bound

$$\mathbb{P}(\mathcal{Q}(x) \wedge \mathcal{R}_x) \leqslant \sum_{r\geqslant 1} \mathbb{P}(\mathcal{Q}(x, r) \wedge \mathcal{R}_x).$$

Set

$$\mathcal{S}(x,r)=\begin{cases}\{d(x)\geqslant\tau\} & \text{if } r\geqslant 2/\varepsilon,\\ \{d(x)\geqslant\alpha-r\} & \text{if } r<2/\varepsilon,\end{cases}$$

and notice that $\mathcal{S}(x,r)\supseteq\mathcal{Q}(x,r)$. (For $r\geqslant 2/\varepsilon$ this is contained in the definition of $\mathcal{Q}(x,r)$ (which promises $d_{\mathcal{C}}(x)\geqslant\tau$), and for smaller r it is given by $d(x)\geqslant d_{\mathcal{C}}(x)=|\mathcal{C}|-r\geqslant\alpha-r$.) Thus we have

$$\mathbb{P}(\mathcal{Q}(x,r)\wedge\mathcal{R}_x)=\mathbb{P}(\mathcal{Q}(x,r)\wedge\mathcal{S}(x,r)\wedge\mathcal{R}_x)\leqslant\mathbb{P}(\mathcal{S}(x,r))\,\mathbb{P}(\mathcal{Q}(x,r)\mid\mathcal{R}_x\wedge\mathcal{S}(x,r)).\tag{5.8}$$

For all but quite small r, a bound on the second factor in (5.8) will suffice for our purposes. To bound this factor, we condition on values $\mathcal{H}_x=\{A_1,\ldots,A_d\}$ and $|\mathcal{H}_{\overline{x}}|=t$ satisfying $\mathcal{S}(x,r)\wedge\mathcal{R}_x$ (in particular $d\leqslant\beta$ and $t\leqslant m_0$); thus $\mathcal{H}_{\overline{x}}$ is a uniform t-subset, say $\{B_1,\ldots,B_t\}$, of $\mathcal{K}_{\overline{x}}$. If $\mathcal{Q}(x,r)$ holds under this conditioning, then there are $I\subseteq[d]$ of size at least τ and $J\subseteq[t]$ of size r such that the families $\{A_i:i\in I\}$ and $\{B_j:j\in J\}$ are cross-intersecting (namely each of the r members of $\mathcal{C}_{\overline{x}}$ meets each of the $d_{\mathcal{C}}(x)\geqslant\tau$ members of $\mathcal{C}_x$).

The probability that this happens for a fixed I and J as above (note the remaining randomization is in the choice of B_j) is, by Corollary 5.2, less than $(1+o(1))^r q^{\tau r}$, and it follows that the probability of $\mathcal{Q}(x,r)$ under the present conditioning – so also under conditioning on $\mathcal{S}(x,r)\wedge\mathcal{R}_x$ – is less than

$$\binom{d}{\leqslant r}\binom{m_0}{r}(1+o(1))^r q^{\tau r}<[(1+o(1))\beta m_0 n^{-(1-\varepsilon)}]^r<n^{-(\varepsilon-o(1))r}.\tag{5.9}$$

Here the first factor on the left bounds the number of possibilities for the $d-d_{\mathcal{C}}(x)\leqslant r$ members of $[d]\setminus I$; the first inequality uses $d\leqslant\beta$ and (4.11), and the second uses $\beta m_0<(1+o(1))(\varphi^*)^2 n/k<n^{1-2\varepsilon+o(1)}$ (see (4.12), (4.4)).

Thus, as suggested above, the second factor on the right-hand side of (5.8) is enough for us unless r is very small, namely

$$\sum_{r>2/\varepsilon}\mathbb{P}(\mathcal{Q}(x,r)\mid\mathcal{R}_x\wedge\mathcal{S}(x,r))=o\left(\frac{1}{n}\right).\tag{5.10}$$

For smaller r we must use the factor $\mathbb{P}(\mathcal{S}(x,r))$ from (5.8) (together with (5.9)). Here (4.18) gives $\mathbb{P}(d_v=t)/\mathbb{P}(d_v=t+1)<n^{o(1)}$ for $t\in[\alpha-r,\alpha]$, which, since $r<O(1)$, implies

$$\mathbb{P}(\mathcal{S}(x,r))<n^{o(1)}\mathbb{P}(d_x\geqslant\alpha+1)<n^{-1+o(1)}.$$

(Recall from the definitions preceding Proposition 4.3 that $\mathbb{P}(d_x\geqslant\alpha+1)\leqslant\mathbb{P}(d_x\geqslant\alpha_1+1)<\psi/n$.) Finally, recalling (5.9), we find that (for $r\leqslant 2/\varepsilon$) the right-hand side of (5.8) is less than $n^{-1+o(1)}n^{-(\varepsilon-o(1))r}=n^{-(1+r\varepsilon-o(1))}$, yielding

$$\sum_{r=1}^{\lfloor 2/\varepsilon\rfloor}\mathbb{P}(\mathcal{Q}(x,r)\wedge\mathcal{R}_x)<\sum_{r\geqslant 1}n^{-(1+r\varepsilon-o(1))}=o\left(\frac{1}{n}\right),$$

and combining this with (5.10) gives (5.7). □

6. Proof of Lemma 2.2

We recall the statement.

Lemma 2.2. *A.s. $\mathcal{H}$ does not contain a clique with two vertices of degree at least λ.*

We prove the following equivalent form.

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Lemma 6.1. *A.s. there do not exist $x, y \in V$ and $\mathcal{F} \subseteq \mathcal{H}_x$, $\mathcal{G} \subseteq \mathcal{H}_y$ with $|\mathcal{F}| = |\mathcal{G}| = \lambda$ and $\mathcal{F}$, $\mathcal{G}$ cross-intersecting.*

Proof. Let $\mathcal{Q}(x, y)$ be the event described in Lemma 6.1 and $\mathcal{Q} = \cup\mathcal{Q}(x, y)$. We want $\mathbb{P}(\mathcal{Q}) = o(1)$, for which it is enough to show that (for any x, y)

$$\mathbb{P}(\mathcal{Q}(x, y) \wedge \mathcal{R}) < o(n^{-2}). \tag{6.1}$$

For the proof of (6.1) we condition on values of m satisfying (4.5) (so we may think of $\mathcal{H}$ as $\{A_i : i \in [m]\}$),

$$I_x := \{i \in [m] : x \in A_i\}, \quad I_y := \{i \in [m] : y \in A_i\},$$

with $|I_x|, |I_y| \leqslant \beta$ and $|I_x \cap I_y| \leqslant 8$ (see (4.9), (4.20)), and a value of $(A_i : i \in I_x)$ for which

$$|\{z \in V \setminus \{x\} : |\{i : v \in A_i\}| \geqslant 2\}| < \mathsf{w}$$

(see (4.21)). If $\mathcal{Q}(x, y)$ holds (under this conditioning), then there are $J_x \subseteq I_x \setminus I_y$ and $J_y \subseteq I_y \setminus I_x$, each of size $\lambda - 8$, with the families $\{A_i : i \in J_x\}$ and $\{A_j : j \in J_y\}$ cross-intersecting.

The probability that this happens for a given J_x, $(A_i : i \in J_x)$ and J_y is, by Corollary 5.2, at most

$$[(1 + o(1))q^{\lambda-8}]^{\lambda-8} = q^{(1-o(1))\lambda^2},$$

whence

$$\begin{aligned} \mathbb{P}(\mathcal{Q}(x, y) \wedge \mathcal{R}) &< \binom{\beta}{\lambda}^2 q^{(1-o(1))\lambda^2} \\ &< \exp\left[\lambda\left(2\log\left(\frac{e\beta}{\lambda}\right) - (1 - o(1))\lambda\log\left(\frac{1}{q}\right)\right)\right] \\ &< \exp\left[(1 - o(1))\lambda^2\log\left(\frac{1}{q}\right)\right] < o(n^{-3}), \end{aligned} \tag{6.2}$$

where the third inequality uses $\beta < (1 + o(1))\varphi^*$ and $\lambda \geqslant \sqrt{\log n / \log(1/q)}$ (the first of these from (4.12)) to say $\log(e\beta/\lambda) = O(\log\log n)$, and the last uses $\lambda \geqslant 2\sqrt{\log n / \log(1/q)}$.

7. Proof of Lemma 2.3

We again recall the statement (and that γ, τ, λ were defined in (2.5)–(2.7)).

Lemma 2.3. *A.s. $\mathcal{H}$ does not contain a clique of size γ with at most one vertex of degree greater than λ and maximum degree less than τ.*

Of the lemmas of Section 2 this is the one requiring the most work. We will try to say a little about its proof after some minor reformulating.

We again condition on a value of m satisfying (4.5) (so $\mathcal{H}$ is chosen uniformly from the m-subsets of $\mathcal{K}$), and then, rather than dealing directly with $\mathcal{H}$, find it easier to work with sets chosen *independently* from $\mathcal{K}$, which makes essentially no difference since m is so small compared to $|\mathcal{K}|$. Precisely, if m satisfies (4.5), $B_1, \ldots, B_m$ is a uniform m-subset of $\mathcal{K}$, $A_1, \ldots, A_m$ are chosen uniformly and independently from $\mathcal{K}$, and we set $\mathcal{D} = \{A_1, \ldots, A_m$ are distinct$\}$, then for any event $\mathcal{B}$ we have

$$\begin{aligned} \mathbb{P}(A_1, \ldots, A_m \models \mathcal{B}) &\geqslant \mathbb{P}(\mathcal{D})\,\mathbb{P}(A_1, \ldots, A_m \models \mathcal{B} \mid \mathcal{D}) \\ &= \mathbb{P}(\mathcal{D})\,\mathbb{P}(B_1, \ldots, B_m \models \mathcal{B}), \end{aligned}$$

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whence

$$\begin{aligned}\mathbb{P}(B_1,\ldots,B_m\models\mathcal{B})&\leqslant\mathbb{P}(A_1,\ldots,A_m\models\mathcal{B})/\mathbb{P}(\mathcal{D})\\&\leqslant\left[1-m^2\Big/\binom{n}{k}\right]^{-1}\mathbb{P}(A_1,\ldots,A_m\models\mathcal{B})\\&=(1+o(1))\,\mathbb{P}(A_1,\ldots,A_m\models\mathcal{B}).\end{aligned}$$

It is thus enough to prove the following statement.

Lemma 7.1. *Suppose $A_1,\ldots,A_\gamma$ are drawn uniformly and independently from $\mathcal{K}$, and let $\mathcal{Q}$ be the event that $\{A_1,\ldots,A_\gamma\}$ is a clique with at most one vertex of degree greater than λ and none of degree greater than τ. Then*

$$\mathbb{P}(\mathcal{Q})=o\left(\binom{m}{\gamma}^{-1}\right).$$

Sketch. The basic idea here is not so bad, but implementation is delicate and, as promised, we now try to say a little about what is involved (for which, as suggested in Section 2, one should think of $\gamma=\alpha$).

We regard the A_i as chosen sequentially, so we are interested in the probability that A_i meets each of $A_1,\ldots,A_{i-1}$. Each of these events occurs with probability q, so if they were independent then

$$\mathbb{P}(A_i\cap A_j\text{ for all }i,j)=\mathsf{q}^{\binom{\gamma}{2}}=o\left(\binom{m}{\gamma}^{-1}\right)\tag{7.1}$$

would follow from (4.8). (As noted earlier, this is the source of (1.5); note also that, by (4.19), passage from m to m does not affect (7.1). The calculation would still be suitable even without independence if each A_i were required to meet the earlier edges in distinct points; this can be seen using Proposition 4.1.)

The problem is that A_i may contain vertices lying in more than one of $A_1,\ldots,A_{i-1}$, thus losing some of the factors q in (7.1). What saves us here is that in the present regime – this is where our argument runs into trouble for slightly larger k – such 'heavy' vertices are typically rare, so containing them also imposes a cost. Thus, roughly speaking, the proof of Lemma 2.3 is a matter of understanding the trade-off between probabilities of containing heavy vertices and the corresponding decrease in the number of (*distinct*) vertices each A_i must contain to meet its intersection obligations. This is implemented in Lemma 7.3, which owes some of its length to the need to separate the possibilities that there is or is not a vertex lying in at least λ of the A_i. (As noted earlier it is here – see (7.20) – that we need the full power of (1.5).)

Given $A=(A_1,\ldots,A_\gamma)\in\mathcal{K}^\gamma$ we define several related quantities. Write $d_i(v)$ for the degree of v in the multiset $\{A_1,\ldots,A_i\}$ and set $d_v=d_\gamma(v)$. (We no longer default to $d_v=d_{\mathcal{H}}(v)$, since $\mathcal{H}$ plays no further role in this section.) Note that we regard A as given and sometimes (not always) suppress it in our notation; for example $d_i(v)$ could also be written (say) $d_{A,i}(v)$.

We will need to distinguish two possibilities, depending on whether there is or is not an x with $d_A(x)>\lambda$. We treat these in parallel, the analysis in the second case eventually being more or less contained in that for the first. To this end we let $V'=V\setminus\{x\}$ if we have specified such a high-degree x and $V'=V$ otherwise.

Set

$$W_i=\{v\in V':d_i(v)=2\},\quad Z_i=\{v\in V':d_i(v)\geqslant 3\},\quad U_i=W_i\cup Z_i,$$
$$W=W_\gamma,\quad Z=Z_\gamma\quad\text{and}\quad U=U_\gamma\quad(=W\cup Z).$$

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In addition – now, for reasons which will appear below (see (7.10)–(7.13)), retaining A in the notation – set

$$s_i(A) = |A_i \cap W_{i-1}|, \quad r_i(A) = |A_i \cap Z_{i-1}| \quad \text{for } i \in [\gamma]$$

(with $W_0 = Z_0 = \emptyset$),

$$\sigma(A) = (s_1(A), \ldots, s_\gamma(A)), \quad \rho(A) = (r_1(A), \ldots, r_\gamma(A)),$$
$$s(A) = \sum s_i(A) \quad \text{and} \quad r(A) = \sum r_i(A).$$

Notice that

$$s(A) = |Z| \quad \text{and} \quad r(A) = \sum_{v \in Z} (d_v - 3). \tag{7.2}$$

Finally, set

$$\Psi = \sum_{v \in Z} \left[\binom{d_v}{2} - 1 \right] \tag{7.3}$$

and notice that

$$\Psi = 2|Z| + \sum_{v \in Z} \left[\binom{d_v}{2} - 3 \right] = 2|Z| + \frac{1}{2} \sum_{v \in Z} (d_v - 3)(d_v + 2).$$

We will only use this when $d_v \leqslant \lambda$ for all $v \in V'$, in which case, in view of (7.2), we have

$$\Psi \leqslant 2s(A) + (\lambda + 2) r(A)/2. \tag{7.4}$$

From this point we take $A = (A_1, \ldots, A_\gamma)$ with the A_i as in Lemma 7.1 (so chosen uniformly and independently from $\mathcal{K}$); thus the quantities defined above ($d_i(v)$ to Ψ) become random variables determined by A. We retain $q = (1 + 2k^2\mathsf{w}/(\mathsf{q}n^2))\mathsf{q}$ from (5.2) and, recycling, set $\mathsf{w} = \max\{\varphi^2 k^2/n, \log^6 n\}$. (This slightly modifies the w of (5.1); its role here is similar to that of the earlier one.)

Proposition 7.2. *With probability* $1 - o\left(\binom{m}{\gamma}^{-1}\right)$,

(a) $|U| < \mathsf{w}$ *and*

(b) $|Z| < \gamma/\varepsilon =: \mathsf{z}$.

Proof. Notice first that

$$\binom{m}{\gamma} < \exp\left[\gamma \log\left(\frac{em}{\gamma}\right)\right] < \exp\left[\left(\frac{1}{2} + o(1)\right)\gamma \log n\right], \tag{7.5}$$

since $m/\gamma \leqslant 3m/\varphi \sim 3n/k < n^{1/2+o(1)}$.

Since each d_v has the binomial distribution $\mathrm{Bin}(\gamma, k/n)$, we have (for all v, ℓ) $\mathbb{P}(d_v \geqslant \ell) < (k\gamma/n)^\ell/\ell!$, whence $\mathbb{E}|U| < k^2\gamma^2/(2n)$ and $\mathbb{E}|Z| < (k\gamma)^3/(6n^2) < n^{-(2\varepsilon - o(1))}\gamma$ (see (4.12)).

On the other hand, by Propositions 3.1 and 3.2, the events $\{d_v \geqslant \ell\}$ are negatively associated for any ℓ; so the probabilities in question may be bounded using Corollary 3.5. For (b), we have

$$\mathbb{P}(|Z| > \mathsf{z}) < n^{-(2\varepsilon - o(1))\gamma/\varepsilon} < n^{-\gamma} = o\left(\binom{m}{\gamma}^{-1}\right). \tag{7.6}$$

The calculations for (a) are more annoying. Here we set $k = \sqrt{\zeta n}$ and $\mu = k^2\gamma^2/(2n) = \zeta\gamma^2/2$ (our upper bound on $\mathbb{E}|U|$). The desired inequality is

$$\mathbb{P}(|U| \geqslant \mathsf{w}) = o\left(\binom{m}{\gamma}^{-1}\right).$$

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We first observe that this is true provided

$$\gamma > 3\log n/\zeta, \tag{7.7}$$

since then (using (3.1) with $\lambda = \mu$) we have (see (7.5))

$$\mathbb{P}(|U| \geqslant \mathsf{w}) \leqslant \mathbb{P}(|U| \geqslant 2\mu) < \exp\left[-\frac{3\mu}{8}\right] = \exp\left[-\frac{3\zeta\gamma^2}{16}\right] < \exp\left[-\frac{9}{16}\gamma\log n\right].$$

In particular, (7.7) holds if (for example) $\zeta \geqslant 2$, since then (according to (4.11)) we have

$$\gamma > (1 - o(1))\frac{\log n}{-\log(1 - e^{-\zeta})} > 3\log n/\zeta.$$

So we may assume

$$\gamma \leqslant 3\log n/\zeta \quad \text{and} \quad \zeta \leqslant 2.$$

We then have $\log^6 n > 2\mu$, since $\log^6 n \leqslant 2\mu = \zeta\gamma^2 \leqslant 9\log^2 n/\zeta$ implies $\zeta < o(1)$, yielding $\log(1/q) = \omega(1)$ and $2\mu = \zeta\gamma^2 = o((\varphi^*)^2) = o(\log^6 n)$, a contradiction. Thus, again using Corollary 3.5, we have

$$\mathbb{P}(|U| > \mathsf{w}) \leqslant \mathbb{P}(|U| > \log^6 n) < \exp\left[-\Omega(\log^6 n)\right] < o\left(\binom{m}{\gamma}^{-1}\right)$$

(the last inequality holding since $\zeta \leqslant 2$ implies $\gamma\ (< \varphi^*) = O(\log^3 n)$; see (2.5)). □

Set

$$\mathcal{S} = \{|W| \leqslant \mathsf{w},\ |Z| \leqslant \mathsf{z}\}.$$

By Proposition 7.2, Lemma 7.1 will follow from

$$\mathbb{P}(\mathcal{Q} \wedge \mathcal{S}) = o\left(\binom{m}{\gamma}^{-1}\right). \tag{7.8}$$

For the proof of (7.8) we will bound the probabilities of various events whose union contains $\mathcal{Q} \wedge \mathcal{S}$. Set $\theta = \lfloor (n^{\varepsilon}\log(1/q))^{-1} \rfloor$ and

$$\mathcal{A} = \{\{A_1, \ldots, A_\gamma\} \text{ is a clique}\}.$$

Note that θ need not be large – *e.g.* it will be zero for k less than about $\sqrt{\varepsilon n \log n}$ – so for once we do need the floor symbols. The parts of the following argument involving θ could be avoided when θ is small, but there seems no point in treating this separately.

For

$$x \in V, \quad d \in (\lambda, \tau] \quad \text{and} \quad \sigma, \rho \in \mathbb{N}^\gamma, \tag{7.9}$$

let

$$\mathcal{A}(x, d, \rho, \sigma)) = \mathcal{A} \wedge \{d_x = d; d_v \leqslant \lambda \text{ for all } v \neq x; \rho(A) = \rho; \sigma(A) = \sigma\}, \tag{7.10}$$
$$\mathcal{A}(x, d, \rho)) = \mathcal{A} \wedge \{d_x = d; d_v \leqslant \lambda \text{ for all } v \neq x; \rho(A) = \rho; s(A) \leqslant \theta\}, \tag{7.11}$$
$$\mathcal{A}(\rho, \sigma)) = \mathcal{A} \wedge \{d_v \leqslant \lambda \text{ for all } v; \rho(A) = \rho; \sigma(A) = \sigma\}, \quad \text{and} \tag{7.12}$$
$$\mathcal{A}(\rho) = \mathcal{A} \wedge \{d_v \leqslant \lambda \text{ for all } v; \rho(A) = \rho; s(A) \leqslant \theta\}. \tag{7.13}$$

For $r, s \in \mathbb{N}$, let $X(r, s) = (\lambda + 2)r/2 + 2s$ (the value in (7.4)), and, for $\rho = (\rho_1, \ldots, \rho_\gamma)$, set $|\rho| = \sum \rho_i$.

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Lemma 7.3. *For any x, d, ρ, σ as in* (7.9) *with* $|\rho| = r$ *and* $|\sigma| = s$,

$$\mathbb{P}(\mathcal{A}(x,d,\rho,\sigma)\wedge\mathcal{S}) < \binom{\gamma}{d}\left(\frac{k}{n}\right)^d\left(\frac{\mathsf{z}k}{n}\right)^r\left(\frac{\mathsf{w}k}{n}\right)^s q^{\binom{\gamma}{2}-\binom{d}{2}-X(r,s)} \tag{7.14}$$

and

$$\mathbb{P}(\mathcal{A}(\rho,\sigma)\wedge\mathcal{S}) < \left(\frac{\mathsf{z}k}{n}\right)^r\left(\frac{\mathsf{w}k}{n}\right)^s q^{\binom{\gamma}{2}-X(r,s)}. \tag{7.15}$$

For any x, d, ρ as above with $|\rho| = r$,

$$\mathbb{P}(\mathcal{A}(x,d,\rho)\wedge\mathcal{S}) < \binom{\gamma}{d}\left(\frac{k}{n}\right)^d\left(\frac{\mathsf{z}k}{n}\right)^r q^{\binom{\gamma}{2}-\binom{d}{2}-X(r,\theta)} \tag{7.16}$$

and

$$\mathbb{P}(\mathcal{A}(\rho)\wedge\mathcal{S}) < \left(\frac{\mathsf{z}k}{n}\right)^r q^{\binom{\gamma}{2}-X(r,\theta)}. \tag{7.17}$$

We will only use (7.14) and (7.15) with $s > \theta$.

Before proving Lemma 7.3 we show that it implies (7.8). Notice that $\mathcal{Q}$ is the (disjoint) union of the events

$$\mathcal{A}(x,d,\rho,\sigma),\quad \mathcal{A}(x,d,\rho),\quad \mathcal{A}(\rho,\sigma)\quad\text{and}\quad\mathcal{A}(\rho), \tag{7.18}$$

where $x \in V$, $d \in (\lambda,\tau]$, $\rho \in \mathbb{N}^\gamma$ and $\sigma \in \{(s_1,\dots,s_\gamma)\in\mathbb{N}^\gamma : \sum s_i > \theta\}$. Thus

$$\mathbb{P}(\mathcal{Q}\wedge\mathcal{S}) \leqslant \sum \mathbb{P}(\mathcal{E}\wedge\mathcal{S}), \tag{7.19}$$

where $\mathcal{E}$ ranges over the events in (7.18).

It is now convenient to separate the contributions involving x, ρ and σ. Set

$$\begin{aligned} f(d) &= n\binom{\gamma}{d}\left(\frac{k}{n}\right)^d q^{-\binom{d}{2}},\\ g(r) &= \binom{\gamma+r-1}{r}\left(\frac{\mathsf{z}k}{n}\right)^r q^{-(\lambda+2)r/2},\\ h(s) &= \binom{\gamma+s-1}{s}\left(\frac{\mathsf{w}k}{n}\right)^s q^{-2s},\quad\text{and}\\ h^* &= q^{-2\theta}. \end{aligned}$$

Then, noting that (for example) $|\{\rho\in\mathbb{N}^\gamma : |\rho| = r\}| = \binom{\gamma+r-1}{r}$ and using (7.14)–(7.17), we find that $\mathbb{P}(\mathcal{Q}\wedge\mathcal{S})$ (or the right-hand side of (7.19)) is less than

$$q^{\binom{\gamma}{2}}\left[\sum_{d,r,s} f(d)g(r)h(s) + h^*\sum_{d,r} f(d)g(r) + \sum_{r,s} g(r)h(s) + h^*\sum_r g(r)\right],$$

where d, r and s range over $(\lambda,\tau]$, $\mathbb{N}$ and (θ,∞) respectively. Thus, since

$$q^{\binom{\gamma}{2}} = o\left(\binom{m}{\gamma}^{-1}\right) \tag{7.20}$$

(by (4.8), (4.19) and (5.3) if $\gamma = \alpha$, and with plenty of room if $\gamma = \varphi^*/3$), it is enough to show that each of

$$\sum_{r\geqslant 0} g(r),\quad \sum_{s>\theta} h(s)\quad\text{and}\quad h^*$$

is $O(1)$ and that, with $F = \sum_{d\in(\lambda,\tau]} f(d)$,

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$$q^{\binom{\gamma}{2}} F \binom{m}{\gamma} \quad (= \Lambda(\gamma)F) \;\; = o(1). \tag{7.21}$$

These are all easy calculations, as follows.

First,

$$g(r) \leqslant \left[e\gamma \left(\frac{\mathsf{z}k}{n} \right) n^{o(1)} \right]^r < [\gamma^2 n^{-1/2+o(1)}]^r < n^{-(2\varepsilon - o(1))r},$$

where the first inequality uses

$$k > n^{1/2-o(1)} \Rightarrow q > n^{-o(1)} \Rightarrow \log\left(\frac{1}{q}\right) = o(\log n) \Rightarrow \lambda \log\left(\frac{1}{q}\right) = o(\log n)$$

and the third uses $\gamma \leqslant \alpha \leqslant \beta$ and (4.12). This implies $\sum_{r \geqslant 0} g(r) = 1 + o(1)$.

Second, since

$$\binom{\gamma + s - 1}{s}^{1/s} < \frac{e(\gamma + s)}{s} < \frac{e(\gamma + \theta)}{\theta} < n^{\varepsilon + o(1)}$$

(for $s > \theta$), $\mathsf{w}k/n < n^{-2\varepsilon + o(1)}$ and $q = n^{-o(1)}$, we have

$$\sum_{s > \theta} h(s) < \sum_{s > \theta} n^{-(\varepsilon - o(1))s} = o(1).$$

Third, $h^* = 1 + o(1)$ is immediate from our choice of θ.

The calculation for (7.21) requires a little more care. Notice first that

$$f(d) < n \left(\left(\frac{e\gamma}{d} \right) \left(\frac{k}{n} \right) \right)^d q^{-\binom{d}{2}} < n \cdot n^{-(1/2 - o(1))d} q^{-\binom{d}{2}} < n \cdot [n^{-(1-o(1))} q^{-d}]^{d/2} \tag{7.22}$$

(where the second inequality uses $\gamma/d < (1 + o(1))\varphi^*/\lambda < n^{o(1)}$; again see (4.12)). Here we may confine ourselves to

$$d > (1 - o(1)) \frac{\log n}{\log(1/q)}, \tag{7.23}$$

since for smaller d the expression in square brackets in (7.22) is less than $n^{-\Omega(1)}$ (and the exponent $d/2$ is at least $\lambda/2 = \omega(1)$), so that the contribution of such d to F is $o(1)$.

For d as in (7.23) the bound in (7.22) is (rapidly) increasing in d (passing from d to $d+1$ multiplies it by roughly $n^{-1/2} q^{-d}$, so at least about $\sqrt{n}$); so the contribution of such d to $\Lambda(\gamma)F$ is dominated by that of $d = \tau$. For this term we have

$$\gamma = (1 - \varepsilon)^{-1} \tau = (1 - \varepsilon)^{-1} d > (1 + \varepsilon) \frac{\log n}{\log(1/q)} \quad (= \tilde{\Omega}(n^{1/4 - \varepsilon})), \tag{7.24}$$

whence

$$\begin{aligned} \Lambda(\gamma) f(\tau) &< n^{-(1/2 - o(1))\tau + \gamma/2} q^{(\gamma - \tau)(\gamma + \tau - 1)/2} \\ &< n^{[\varepsilon/2 - (1+\varepsilon)\varepsilon(1 - \varepsilon/2) + o(1)]\gamma} \\ &= n^{-(\varepsilon/2 + \varepsilon^2/2 - \varepsilon^3/2 - o(1))\gamma}. \end{aligned}$$

In more detail: in the first line the q term combines the $q^{-\binom{d}{2}}$ of (7.22) and the $q^{\binom{\gamma}{2}}$ from $\Lambda(\gamma)$, while (7.24) and $m < n^{3/4 - \varepsilon + o(1)}$ (implied by (4.12)) bound the $\binom{m}{\gamma}$ of $\Lambda(\gamma)$ by $n^{(1+o(1))\gamma}$; and the second line uses $\tau = (1 - \varepsilon)\gamma$ and $q^\gamma < n^{(1+\varepsilon)}$ (again from (7.24)). Thus we have (7.21). □

For the proof of Lemma 7.3, we need the following easy observation.

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Proposition 7.4. *Let $Y_1, \ldots, Y_\ell$ be random variables (not necessarily real-valued) and write y_i for a possible value of Y_i. Let $\mathcal{Z}$ be a set of ('bad') prefixes $(y_1, \ldots, y_i)$ closed under extension (i.e. $i < \ell$ and $(y_1, \ldots, y_i) \in \mathcal{Z}$ imply $(y_1, \ldots, y_i, y_{i+1}) \in \mathcal{Z}$ for every choice of y_{i+1}). Set*

$$\mathbb{P}((Y_1, \ldots, Y_i) \in \mathcal{Z} \mid y_1, \ldots, y_{i-1}) = 1 - \xi(y_1, \ldots, y_{i-1}),$$

where the conditioning has the obvious meaning and when $i = 1$ the left-hand side is $\mathbb{P}((Y_1) \in \mathcal{Z})$. Then

$$\mathbb{P}((Y_1, \ldots, Y_\ell) \notin \mathcal{Z}) \leqslant \max_{(y_1,\ldots,y_\ell)\notin\mathcal{Z}} \prod_{i=1}^{\ell} \xi(y_1, \ldots, y_{i-1}) =: \xi.$$

Proof. Define an auxiliary sequence $(X_0, \ldots, X_\ell)$ with $X_0 \equiv 1$ and, for $i \in [\ell]$,

$$X_i = \begin{cases} 0 & \text{if } (Y_1, \ldots, Y_i) \in \mathcal{Z}, \\ \xi(Y_1, \ldots, Y_{i-1})^{-1} X_{i-1} & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}X_\ell = X_0 = 1$ (since $(X_0, \ldots, X_\ell)$ is a martingale), while $X_\ell \geqslant \xi^{-1}$ whenever $(Y_1, \ldots, Y_\ell) \notin \mathcal{Z}$ (using the fact that $\mathcal{Z}$ is closed under extensions). The conclusion follows. □

We now turn to the proof of Lemma 7.3, beginning with the simpler (7.15) and (7.17); the arguments for (7.14) and (7.16) are similar, and when we come to these we will mainly just point out the necessary modifications.

For both (7.15) and (7.17) we will apply Proposition 7.4 to the sequence $(Y_1, \ldots, Y_{2\gamma})$, where

$$Y_{2j-1} = A_j \cap U_{j-1} \quad \text{and} \quad Y_{2j} = A_j \setminus U_{j-1}. \tag{7.25}$$

We first prove (7.15) and then discuss the changes needed for (7.17).

Proof of (7.15). Here we say $(Y_1, \ldots, Y_i) \in \mathcal{Z}$ (recall this is the set of 'bad' prefixes) if the associated A_j (or parts of A_j) satisfy at least one of:

$$\{A_1, \ldots, A_{\lfloor i/2 \rfloor}\} \text{ is not a clique,} \tag{7.26}$$

$$\text{for some } j \leqslant \lceil i/2 \rceil,\ |A_j \cap Z_{j-1}| \neq r_j \text{ or } |A_j \cap W_{j-1}| \neq s_j, \tag{7.27}$$

$$|Z_{\lceil i/2 \rceil}| > \mathsf{z},\ |W_{\lceil i/2 \rceil}| > \mathsf{w} \text{ or } d_{\lceil i/2 \rceil}(v) > \lambda \text{ for some } v. \tag{7.28}$$

Then $\mathcal{A}(\rho, \sigma) \wedge \mathcal{S} = \{(Y_1, \ldots, Y_{2\gamma}) \notin \mathcal{Z}\}$.

We next need to say something about the quantities

$$\xi(y_1, \ldots, y_{i-1}) = \mathbb{P}(Y_1, \ldots, Y_i \notin \mathcal{Z} \mid y_1, \ldots, y_{i-1})$$

appearing in Proposition 7.4, where (we may assume) $(y_1, \ldots, y_{i-1}) \notin \mathcal{Z}$.

If $i = 2j - 1$ then

$$\begin{aligned} \xi(y_1, \ldots, y_{i-1}) &\leqslant \mathbb{P}(|A_j \cap Z_{j-1}| \geqslant r_j, |A_j \cap W_{j-1}| \geqslant s_j \mid y_1, \ldots, y_{i-1}) \\ &\leqslant \binom{\mathsf{z}}{r_j}\binom{\mathsf{w}}{s_j}\left(\frac{k}{n}\right)^{r_j}\left(\frac{k}{n}\right)^{s_j} \\ &\leqslant \left(\frac{\mathsf{z}k}{n}\right)^{r_j}\left(\frac{\mathsf{w}k}{n}\right)^{s_j}. \end{aligned} \tag{7.29}$$

Here (7.29) is a rather trivial use of Propositions 3.1–3.3, which give negative correlation of the events $\{|A_j \cap Z_{j-1}| \geqslant r_j\}$ and $\{|A_j \cap W_{j-1}| \geqslant s_j\}$ (and we use $|Z_{j-1}| \leqslant \mathsf{z}$ and $|W_{j-1}| \leqslant \mathsf{w}$, as implied by $(y_1, \ldots, y_{i-1}) \notin \mathcal{Z}$).

The case $i = 2j$ is more interesting. Here, conditioning on the event

$$\{(Y_1, \ldots, Y_{i-1}) = (y_1, \ldots, y_{i-1})\},$$

we set

$$\beta_j = \sum \{d_{j-1}(v) : v \in A_j \cap U_{j-1}\}. \tag{7.30}$$

(Notice that this is determined by $(y_1, \ldots, y_{i-1})$, which includes specification of $Y_{2j-1} = A_j \cap U_{j-1}$.) We will show

$$\xi(y_1, \ldots, y_{i-1}) \leqslant q^{j-1-\beta_j}. \tag{7.31}$$

Here we only consider (7.26); that is, we ignore the requirements in (7.28) (those in (7.27) are not affected by Y_i) and show that (given our conditioning) the right-hand side of (7.31) bounds the probability that A_j meets all of $A_1, \ldots, A_{j-1}$. Now A_j meets at most β_j members of $\{A_1, \ldots, A_{j-1}\}$ in U_{j-1}, so to avoid (7.26) must meet the $j-1-\beta_j$ or more remaining members – say those indexed by I – in $V \setminus U_{j-1}$, where they are pairwise disjoint. This gives (7.31) since the events $Q_h = \{A_j \cap (A_h \setminus U_{j-1}) \neq \emptyset\}$ $(h \in I)$ satisfy $\mathbb{P}(Q_h) < q$ (by Proposition 4.1; note q here is as in (5.2)) and are NA (by Propositions 3.1 and 3.2), so by Proposition 3.3 we have

$$\mathbb{P}(\cap_{h \in I} Q_h) \leqslant \prod_{h \in I} \mathbb{P}(Q_h) < q^{j-1-\beta_j}.$$

The last thing to notice here is that provided $d_\gamma(v) \leqslant \lambda$ for all v – which in particular is true whenever $(Y_1, \ldots, Y_{2\gamma}) \notin \mathcal{Z}$; see (7.28)) – we have

$$\sum \beta_j = \Psi \leqslant X(r, s) \tag{7.32}$$

(see (7.3) for Ψ and (7.4) for the inequality). Finally, combining (7.29), (7.31) and (7.32) (and $\sum_{j \in [\gamma]} (j-1) = \binom{\gamma}{2}$) and applying Proposition 7.4 gives (7.15). □

Proof of (7.17). We now take $(Y_1, \ldots, Y_i) \in \mathcal{Z}$ if the associated A_j satisfy at least one of:

$$\{A_1, \ldots, A_{\lfloor i/2 \rfloor}\} \text{ is not a clique,} \tag{7.33}$$

$$\textstyle\sum_{j \leqslant \lceil i/2 \rceil} s_j(A) > \theta, \text{ or for some } j \leqslant \lceil i/2 \rceil, |A_j \cap Z_{j-1}| \neq r_j, \tag{7.34}$$

$$|Z_{\lceil i/2 \rceil}| > \mathsf{z},\ |W_{\lceil i/2 \rceil}| > \mathsf{w} \text{ or } d_{\lceil i/2 \rceil}(v) > \lambda \text{ for some } v. \tag{7.35}$$

Then $\mathcal{A}(\rho) \wedge \mathcal{S} \subseteq \{(Y_1, \ldots, Y_{2\gamma}) \notin \mathcal{Z}\}$.

The arguments bounding the quantities

$$\xi(y_1, \ldots, y_{i-1}) = \mathbb{P}(Y_1, \ldots, Y_i \notin \mathcal{Z} \mid y_1, \ldots, y_{i-1})$$

(again, for $(y_1, \ldots, y_{i-1}) \notin \mathcal{Z}$) are essentially identical to those above. For $i = 2j-1$ the bound

$$\xi(y_1, \ldots, y_{i-1}) \leqslant \mathbb{P}(|A_i \cap Z_{i-1}| \geqslant r_i \mid y_1, \ldots, y_{i-1}) \leqslant \left(\frac{\mathsf{z}k}{n}\right)^{r_i} \tag{7.36}$$

is justified in the same way as (7.29). For $i = 2j$ we again define β_j as in (7.30), and (7.31) follows as before. (Note that our only reason for retaining the constraint on $|W_{\lceil i/2 \rceil}|$ in (7.35) is to enforce $\mathbb{P}(A_j \cap (A_h \setminus U_{j-1}) \neq \emptyset) < q$ in the proof of (7.31).)

Finally, (7.32) again holds provided $(Y_1, \ldots, Y_{2\gamma}) \notin \mathcal{Z}$ (this is where we use the first condition in (7.34)), and combining this with (7.36) and (7.31) we obtain (7.17) via Proposition 7.4. □

We now turn to the parts of Lemma 7.3 involving x. For $D \in \binom{[\gamma]}{d}$ let

$$\mathcal{A}(x, D, \rho, \sigma) = \mathcal{A} \wedge \{x \in A_i \Leftrightarrow i \in D;\ d_v \leqslant \lambda \text{ for all } v \neq x;\ \rho(A) = \rho;\ \sigma(A) = \sigma\},$$
$$\mathcal{A}(x, D, \rho) = \mathcal{A} \wedge \{x \in A_i \Leftrightarrow i \in D;\ d_v \leqslant \lambda \text{ for all } v \neq x;\ \rho(A) = \rho;\ s(A) \leqslant \theta\}.$$

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Since $\mathbb{P}(\mathcal{A}(x,d,\rho,\sigma)$ is the sum of the $\mathbb{P}(\mathcal{A}(x,D,\rho,\sigma))$ (and similarly for $\mathbb{P}(\mathcal{A}(x,d,\rho))$, (7.14) and (7.16) will follow from (respectively)

$$\mathbb{P}(\mathcal{A}(x,D,\rho,\sigma)) < \left(\frac{k}{n}\right)^d \left(\frac{\mathsf{z}k}{n}\right)^r \left(\frac{\mathsf{w}k}{n}\right)^s q^{\binom{t}{2}-\binom{d}{2}-X(r,s)} \tag{7.37}$$

and

$$\mathbb{P}(\mathcal{A}(x,D,\rho)) < \left(\frac{k}{n}\right)^d \left(\frac{\mathsf{z}k}{n}\right)^r q^{\binom{t}{2}-\binom{d}{2}-X(r,\theta)}. \tag{7.38}$$

As the proofs of these closely track those of (7.15) and (7.17) (respectively), with exactly the same modifications, we confine ourselves to indicating what changes to the proof of (7.15) are needed for (7.37).

We again apply Proposition 7.4, in this case to the sequence $(Y_1,\dots,Y_{2\gamma})$ given by

$$Y_{2j-1} = A_j \cap (U_{j-1}\cup\{x\}) \quad \text{and} \quad Y_{2j} = A_j \setminus (U_{j-1}\cup\{x\})$$

(which differs from (7.25) in the addition of $\{x\}$ to the U_{j-1}). We say $(Y_1,\dots,Y_i)\in\mathcal{Z}$ if the associated A_j satisfy at least one of (7.26), (7.27),

$$|Z_{\lceil i/2\rceil} > \mathsf{z},\ |W_{\lceil i/2\rceil}| > \mathsf{w} \text{ or } d_{\lceil i/2\rceil}(v) > \lambda \text{ for some } v\neq x \tag{7.39}$$

(which differs from (7.28) in the stipulation $v\neq x$) and

$$\text{for some } j\leqslant \lceil i/2\rceil, \text{ either } j\in D \text{ and } x\notin A_j \text{ or } j\notin D \text{ and } x\in A_j. \tag{7.40}$$

Then $\mathcal{A}(x,D,\rho,\sigma)\wedge\mathcal{S} = \{(Y_1,\dots,Y_{2\gamma})\notin\mathcal{Z}\}$.

The bounds on the quantities

$$\xi(y_1,\dots,y_{i-1}) = \mathbb{P}(Y_1,\dots,Y_i\notin\mathcal{Z}\mid y_1,\dots,y_{i-1})$$

(again, for $(y_1,\dots,y_{i-1})\notin\mathcal{Z}$) are modified as follows. For $i=2j-1$ we use

$$\xi(y_1,\dots,y_{i-1}) \leqslant \begin{cases} \left(\frac{k}{n}\right)\left(\frac{\mathsf{z}k}{n}\right)^{r_i}\left(\frac{\mathsf{w}k}{n}\right)^{s_i} & \text{if } j\in D,\\ \left(\frac{\mathsf{z}k}{n}\right)^{r_i}\left(\frac{\mathsf{w}k}{n}\right)^{s_i} & \text{otherwise.}\end{cases} \tag{7.41}$$

This is justified (via Propositions 3.1 and 3.3) in the same way as (7.29).

For $i=2j$ we define β_j as before ($\beta_j = \sum\{d_{j-1}(v): v\in A_j\cap U_{j-1}\}$) and set $\mathsf{c}_j = [j-1]\setminus D$ (again, a function of $(y_1,\dots,y_{i-1})$). We then have

$$\xi(y_1,\dots,y_{i-1}) \leqslant \begin{cases} q^{\mathsf{c}_j-\beta_j} & \text{if } j\in D,\\ q^{j-1-\beta_j} & \text{otherwise.}\end{cases} \tag{7.42}$$

The proof is essentially the same as that for (7.31), the only difference being that when $j\in D$, there is no requirement that A_j meet those earlier A_l for which $l\in D$. (On the other hand, the second bound in (7.42) uses the fact that $x\notin A_j$ (for $j\notin D$), which follows from $(y_1,\dots,y_{i-1})\notin\mathcal{Z}$; see (7.40).)

Finally, applying Proposition 7.4 with the combination of (7.41), (7.42) and $\sum\beta_j = \Psi\leqslant X(r,s)$ (noted earlier in (7.32)) gives (7.37), once we observe that

$$\sum_{j\notin D}(j-1) + \sum_{j\in D}\mathsf{c}_j = \sum_j (j-1) - \sum_{j\in D}|[j-1]\cap D| = \binom{\gamma}{2} - \binom{d}{2}. \qquad \square$$

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8. Large φ

Here we complete the proof of Theorem 1.2 for k as in (1.9) by showing

$$\text{for } \varphi > \varphi^*, \mathcal{H} \text{ satisfies EKR a.s.} \tag{8.1}$$

As already mentioned, this is mostly a matter of reducing to φ^* and applying Lemmas 2.1–2.3. (While there ought to be other ways to handle this, our main argument runs into difficulties when φ is large, since the sets W_x, W, Z used in the proofs of Lemmas 2.1–2.3 are no longer small.)

From now on we assume $\varphi > \varphi^*$ ($> \log^2 n$). We use the following natural reduction (coupling). Setting $\rho = \varphi^*/\varphi$, we take $\mathcal{H} = \mathcal{H}_k(n, p)$ as usual and let $\mathcal{G}$ be the random sub-hypergraph of $\mathcal{H}$ obtained by retaining edges independently, each with probability ρ; thus $\mathcal{G} \sim \mathcal{H}_k(n, p^*)$, with $p^* = \varphi^*/M$.

We would *like* to say that if EKR fails for $\mathcal{H}$, say at the non-trivial clique $\mathcal{C}$, then there is a decent chance that the clique $\mathcal{D} := \mathcal{C} \cap \mathcal{G}$ fits one of the unlikely scenarios described in Lemmas 2.1–2.3; but this is not always true, since if $\mathcal{C}$ is too close to trivial then $\mathcal{D}$ is likely to actually *be* trivial. This special situation is handled by the perhaps not uninteresting Lemma 8.1, and in other cases the desired reduction is given by the routine Proposition 8.2.

Set $r_0 = \xi\varphi$ with $\xi = \log(1/q)/(2 \log n)$ (as elsewhere, just a convenient value).

Lemma 8.1. *A.s. there do not exist (in $\mathcal{H}$) a non-trivial clique $\mathcal{C}$ and vertex x such that $|\mathcal{C}| \geqslant \max\{\varphi/2, d(x)\}$ and $|\mathcal{C}_{\overline{x}}| \leqslant r_0$.*

Proposition 8.2. *Suppose $\mathcal{C}$ is a non-trivial clique of $\mathcal{H}$ with $|\mathcal{C}| \geqslant \Delta \geqslant \varphi/2$ and $\Delta_{\mathcal{C}} \leqslant |\mathcal{C}| - r_0$, and let x be a maximum degree vertex of $\mathcal{C}$. Then, with probability at least $1/2 - o(1)$, $\mathcal{D} := \mathcal{C} \cap \mathcal{G}$ satisfies:*

(a) $|\mathcal{D}| \geqslant \max\{d_{\mathcal{G}}(x), \gamma\}$,

(b) $|\mathcal{D}_{\overline{x}}| > 2/\varepsilon$,

(c) *either* $\Delta_{\mathcal{D}} < \tau$ *or* $d_{\mathcal{D}}(x) > \lambda$.

Recall that γ, τ and λ were defined in (2.5)–(2.7), and note that in the present situation we have $\gamma = \varphi^*/3$.

Before proving these assertions we show that they (with Lemmas 2.1–2.3) give (8.1). Since $\Delta \geqslant \varphi/2$ a.s. (really, $d_v \geqslant \varphi/2$ for all v a.s. by Theorem 3.4), Lemma 8.1 says it is enough to show that $\mathcal{H}$ is unlikely to contain a non-trivial clique $\mathcal{C}$ with $|\mathcal{C}| \geqslant \Delta \geqslant \varphi/2$ and $\Delta_{\mathcal{C}} < |\mathcal{C}| - r_0$. So we suppose this does happen. Let x be some maximum degree vertex of $\mathcal{C}$, and observe that $\mathcal{D}$ and x are then fairly likely (*i.e.* with probability at least $1/2 - o(1)$) to exhibit one of the improbable behaviours described in Lemmas 2.1–2.3. Namely, this is true if the conclusions of Proposition 8.2 hold:

(i) if $\mathcal{D}$ has at least two vertices of degree at least λ, then Lemma 2.2 applies; otherwise,

(ii) if $\Delta_{\mathcal{D}} < \tau$ then we are in the situation of Lemma 2.3 (since Proposition 8.2(a) gives $|\mathcal{D}| \geqslant \gamma$ and we assume $\mathcal{D}$ has at most one vertex of degree at least λ);

(iii) if $\Delta_{\mathcal{D}} \geqslant \tau$ then in fact $d_{\mathcal{D}}(x) \geqslant \tau$ (by (c), since we assume $\mathcal{D}$ has at most one vertex of degree at least λ ($< \tau$)), so in view of (a) and (b) we are in the situation described in Lemma 2.1. □

Proof of Lemma 8.1. We need one preliminary observation. For given x and $\mathcal{B} \subseteq \mathcal{K}_x$, let $g(\mathcal{B})$ be the probability that A chosen uniformly from $\mathcal{K}_{\overline{x}}$ meets all members of $\mathcal{B}$.

Suppose that, for some s, $\mathcal{B}$ is a uniform s-subset of $\mathcal{K}_x$ and A is uniform from $\mathcal{K}_{\overline{x}}$ (these choices made independently). Then

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$$\mathbb{E}g(\mathcal{B}) = \mathbb{P}(A \cap B \neq \emptyset \text{ for all } B \in \mathcal{B}) < \mathsf{q}^s, \tag{8.2}$$

the inequality holding because (i) $\mathbb{P}(A \cap B \neq \emptyset) < \mathsf{q}$ for A and B uniform from $\mathcal{K}_{\overline{x}}$ and $\mathcal{K}_x$ respectively, and (ii) the probability in (8.2) is (obviously) no more than it would be if the members of $\mathcal{B}$ were chosen independently. Markov's inequality thus gives (for any $a \leqslant s$)

$$\mathbb{P}(g(\mathcal{B}) > \mathsf{q}^a) < \mathsf{q}^{s-a}. \tag{8.3}$$

Now let $\mathcal{S} = \mathcal{P} \wedge \{d(x) \geqslant \varphi/2 \text{ for all } x\}$ (recall $\mathcal{P}$ was defined in the paragraph containing (4.13)), noting that (by (4.13) and Theorem 3.4) $\mathbb{P}(\overline{\mathcal{S}}) = o(1)$. Let

$$\mathcal{Q}(x) = \{\exists \mathcal{B} \subseteq \mathcal{H}_x : |\mathcal{B}| = d(x) - r_0 \text{ and } g(\mathcal{B}) > \mathsf{q}^{\varphi/4}\}$$

and $\mathcal{Q} = \cup \mathcal{Q}(x)$. Then, using $\varphi/2 \leqslant d(x) \leqslant \beta$ for all x, as follows from $\mathcal{P}$, and applying (8.3) with $s = d(x) - r_0$ and $a = \varphi/4$,

$$\mathbb{P}(\mathcal{Q} \wedge \mathcal{S}) < n\binom{\beta}{r_0}\mathsf{q}^{\varphi/4 - r_0} < n\exp\left[r_0 \log\left(\frac{e\beta}{r_0}\right) - \left(\frac{\varphi}{4} - r_0\right)\log\left(\frac{1}{\mathsf{q}}\right)\right]. \tag{8.4}$$

Recalling that $\varphi \sim \beta$ (see following Proposition 4.3), we have

$$r_0 \log\left(\frac{e\beta}{r_0}\right) \sim \xi\varphi \log\left(\frac{1}{\xi}\right) < \left(\frac{1}{8}\right)\varphi \log\left(\frac{1}{q}\right)$$

(since $\log(1/q) > n^{-1/4+\Omega(1)}$ implies $\log(1/\xi) < (1/4)\log n$); so, noting that $q > n^{-o(1)}$ implies $r_0 = o(\varphi)$ and recalling that $\varphi > \varphi^*$, we find that the right-hand side of (8.4) is $o(1)$.

Thus, with $\mathcal{T}$ the event in Lemma 8.1, the lemma will follow from

$$\mathbb{P}(\mathcal{T} \wedge \overline{\mathcal{Q}} \wedge \mathcal{S}) = o(1). \tag{8.5}$$

We show that

$$\mathbb{P}(\mathcal{T} \wedge \overline{\mathcal{Q}} \wedge \mathcal{S}) \leqslant n\sum_{r=1}^{r_0} (\beta m_0 \mathsf{q}^{\varphi/4})^r \tag{8.6}$$

(and then observe that the right-hand side is small). □

Proof of (8.6) and (8.5). We consider occurrence of $\mathcal{T}$ at a given x, writing $\mathcal{T}(x)$ for this event. Since

$$\mathbb{P}(\mathcal{T} \wedge \overline{\mathcal{Q}} \wedge \mathcal{S}) \leqslant \mathbb{P}(\mathcal{T} \mid \overline{\mathcal{Q}} \wedge \{d(x) \leqslant \beta,\ m \leqslant m_0\})$$

(the conditioning event contains $\overline{\mathcal{Q}} \wedge \mathcal{S}$), it is enough to show that

$$\mathbb{P}^*(\mathcal{T}(x)) < (\beta m_0 \mathsf{q}^{\varphi/4})^r,$$

where $\mathbb{P}^*$ denotes probability under conditioning on some $\mathcal{H}_x$ of size at most β satisfying $\overline{\mathcal{Q}}(x)$, together with a value $m \leqslant m_0$ of $|\mathcal{H}|$.

If, under this conditioning, $\mathcal{T}(x)$ occurs at $\mathcal{C}$ with $|\mathcal{C} \setminus \mathcal{H}_x| = r$ $(\in [1, r_0])$, then, since $d_{\mathcal{C}}(x) = |\mathcal{C}| - r \geqslant \varphi/2 - r$ and $|\mathcal{H}_x \setminus \mathcal{C}| \leqslant r$, there are $\mathcal{B} \subseteq \mathcal{H}_x$ and $\mathcal{D} \subseteq \mathcal{H}_{\overline{x}}$ (namely $\mathcal{B} = \mathcal{C}_x$, $\mathcal{D} = \mathcal{C}_{\overline{x}}$) with

$$|\mathcal{B}| = |\mathcal{C}| - r \geqslant \max\left\{d(x) - r,\ \frac{\varphi}{2} - r\right\},$$

$|\mathcal{D}| = r$, $\mathcal{B}$ and $\mathcal{D}$ cross-intersecting, and $g(\mathcal{B}) \leqslant \mathsf{q}^{\varphi/4}$ (the last property implied by $\overline{\mathcal{Q}}(x)$; of course if $|\mathcal{B}| \geqslant d(x) - r$ and $g(\mathcal{B}) > \mathsf{q}^{\varphi/4}$, then $g(\mathcal{B}') > \mathsf{q}^{\varphi/4}$ for any $(d(x) - r_0)$-subset $\mathcal{B}'$ of $\mathcal{B}$). But the probability that this occurs given $\mathcal{H}_x$ and m as above is at most

$$\binom{d(x)}{\leqslant r}\binom{m - d(x)}{r}\mathsf{q}^{\varphi r/4} < (\beta m_0\, \mathsf{q}^{\varphi/4})^r,$$

which gives (8.6).

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Finally (now for (8.5)), we have $\beta m_0 \mathsf{q}^{\varphi/4} < \varphi^2 n^{1/2+o(1)}\mathsf{q}^{\varphi/4} = o(1/n)$, where the first inequality uses $\beta \sim \varphi$ and $m_0 \sim \varphi n/k$, and the second holds because $\varphi^2\mathsf{q}^{\varphi/4}$ is decreasing in $\varphi > \varphi^*$ and is $n^{-\omega(1)}$ when $\varphi = \varphi^*$. □

Proof of Proposition 8.2. Of course, $\mathbb{P}(|\mathcal{D}| \geqslant d_{\mathcal{G}}(x)) \geqslant 1/2$, so it is enough to show that each of the other requirements (namely $|\mathcal{D}| \geqslant \gamma$ and those in (b) and (c)) holds a.s. These are all routine applications of Theorem 3.4 (or Corollary 3.5). First, $|\mathcal{D}|$ is binomial with mean $|\mathcal{C}|\rho \geqslant (\varphi/2)\rho = \varphi^*/2 = 3\gamma/2$, implying $\mathbb{P}(|\mathcal{D}| < \gamma) < \exp[-\Omega(\gamma)]$. Second, $\mathbb{E}|\mathcal{D}_{\overline{x}}| \geqslant r_0\rho = \xi\varphi^* = \omega(1)$, so $\mathbb{P}(|\mathcal{D}_{\overline{x}}| < 2/\varepsilon) < \exp[-\omega(1)]$. Third, since $\tau \gg \lambda$ we have either $\Delta_{\mathcal{C}}\rho\ (= \mathbb{E}d_{\mathcal{D}}(x)) > 2\lambda$, implying $\mathbb{P}(d_{\mathcal{D}}(x) < \lambda) = o(1)$, or $\Delta_{\mathcal{C}}\rho < \tau/2$, implying $\mathbb{P}(\Delta_{\mathcal{D}} \geqslant \tau) < n\exp[-\Omega(\tau)] = o(1)$; thus (c) also holds a.s. □

9. Small *k*

Finally, we turn to the proof of Theorem 1.2 for $k < n^{1/2-\Omega(1)}$, say

$$k \leqslant n^{1/2-\varepsilon} \tag{9.1}$$

with $\varepsilon > 0$ fixed (note that this is not the ε of Sections 2–8). As noted earlier, this is easier than what we have already done, one reason being the absence of the issue discussed following (2.4): there will now always be an α such that $\Delta \geqslant \alpha$ a.s. and there is a.s. no non-trivial clique of size at least α. This will mean that here we only need Proposition 4.4 (which for k as in (9.1) and *fixed* α was proved in [3]) and a simplified Lemma 2.3. Since most of this consists of easier versions of earlier arguments, parts of the discussion will be a bit sketchy.

It will be helpful to think of three regimes: (i) $\varphi < n^{-\Omega(1)}$, (ii) $n^{-o(1)} < \varphi \ll 1$ and (iii) $\varphi = \Omega(1)$. The last of these is treated in [3, Theorem 1.1(iv)], so we may concentrate on the first two.

We first need to specify α. If we are in regime (ii) then we take α as in Section 4 (recall that this assumed $\varphi > n^{-o(1)}$ but not (1.9)), noting that, in addition to $\Delta \geqslant \alpha$ a.s. (see (4.9)) and $\Lambda(\alpha) = o(1)$ (see (4.8)), we have $\alpha = \omega(1)$. (Note that here $\alpha = \alpha_1$.)

In regime (i) we may assume (possibly passing to a subsequence of n) that there is a (positive) integer c such that $n^{-1/c} \ll \varphi = O(n^{-1/(c+1)})$; we then take $\alpha = c$, but will sometimes use c to remind ourselves that the value is a constant. Here again we have $\Delta \geqslant \alpha$ a.s. (by Proposition 4.4 since $\mathbb{P}(d_v \geqslant \alpha) \asymp \varphi^\alpha$), as well as $\Lambda(\alpha) = o(1)$, which is given by (1.5) once we observe that, by Harris's inequality [13],

$$\mathbb{P}(\Delta \leqslant \alpha) = \mathbb{P}(d_v \leqslant \alpha \text{ for all } v) \geqslant \prod_v \mathbb{P}(d_v \leqslant \alpha) = \left(1 - O\left(\frac{1}{n}\right)\right)^n = \Omega(1). \tag{9.2}$$

(Of course, if $\varphi \ll n^{-1/(\alpha+1)}$, then $\Delta = \alpha$ a.s., and it is not hard to see that if $\varphi \asymp n^{-1/(\alpha+1)}$, then $\Delta \in \{\alpha, \alpha+1\}$ a.s. and each possibility occurs with probability $\Omega(1)$.) Note also that we may assume $c\ (= \alpha)\ \geqslant 3$, since if $c \leqslant 2$ then $\varphi^2 n \asymp \Lambda(2) = o(1)$ gives $\varphi \ll n^{-1/2}$ and $\Delta_{\mathcal{H}} \leqslant 1$ a.s. (We may also note that if $c = 3$ then $\Lambda(3) = o(1)$ implies $k \ll n^{1/3}$; it is shown in [3] that for such k EKR holds a.s. for *any* φ.)

In either regime we just need to show that $\mathcal{H}$ is unlikely to contain a non-trivial α-clique. The arguments for the two regimes are similar and we treat them in parallel. In each case we will avoid some complications by first disposing of cliques with very large degrees (see Lemma 2.1).

If $\mathcal{H}$ contains a non-trivial clique of maximum degree at least d then it contains a 'Hilton–Milner' family of size $d+1$; that is, $B_0, \ldots, B_d$ such that $\cap_{i=1}^d B_i \setminus B_0 \neq \emptyset$ and $B_i \cap B_0 \neq 0$ for all $i \in [d]$. The probability that this occurs is, by Proposition 4.2, within $o(1)$ of the probability that it occurs for $A_1, \ldots, A_m$ chosen independently from $\mathcal{K}$ (with m chosen as usual). The latter probability is less than

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$$\mathbb{P}(m \not\models (4.5)) + \binom{m_0}{d+1}(d+1)n\left(\frac{k}{n}\right)^d \mathsf{q}^d < o(1) + \varphi^{d+1}k^{2d-1}n^{-(d-2)}. \tag{9.3}$$

Here the factor n on the left-hand side is for a choice of $x \in \cap_{i=1}^d B_i \setminus B_0$ and the inequality uses $m_0 \sim \mathsf{m} = \varphi n/k$ (with '$\sim$' holding since, as already noted, we may assume $\varphi = \Omega(n^{-1/2})$ and therefore $\mathsf{m} > n^{\Omega(1)}$). We then need to show that the right-hand side of (9.3) is $o(1)$ for suitable d.

For regime (i) we take $d = c-1$. We have

$$\Lambda(c) \asymp \left(\frac{\varphi n}{k}\right)^c \left(\frac{k^2}{n}\right)^{\binom{c}{2}} = [\varphi k^{c-2}n^{-(c-3)/2}]^c,$$

so $\Lambda(c) = o(1)$ implies $k^{c-2} \ll n^{(c-3)/2}/\varphi$. Thus (for typographical reasons considering the $(c-2)$nd power of the right-hand side of (9.3)),

$$\begin{aligned}
[\varphi^c k^{2c-3} n^{-(c-3)}]^{c-2} &\ll \frac{\varphi^{c(c-2)} n^{(2c-3)(c-3)/2}}{n^{(c-2)(c-3)} \varphi^{2c-3}} \\
&= [\varphi^{c-1} n^{1/2}]^{c-3} \\
&= O(n^{-(c-1)/(c+1)+1/2})^{c-3} \\
&= O(n^{-(c-3)^2/(2(c+1))}) \\
&= o(1),
\end{aligned}$$

where we used $\varphi = O(n^{-1/(c+1)})$ in the third step and $c \geqslant 3$ in the fourth. Thus the right-hand side of (9.3) is $o(1)$.

For regime (ii) we take $d = \lfloor \alpha/2 \rfloor$ (say) and find that, since $k < n^{1/2-\varepsilon}$, the right-hand side of (9.3) is less than $n^{-\Omega(\alpha)}$.

So (in either case) we just need to show that $\mathcal{H}$ is unlikely to contain a non-trivial α-clique with maximum degree at most $d-1$ (d as above). The reduction to independent A_i preceding Lemma 7.1 of course remains valid here, so the following analogue of Lemma 2.3 completes the argument.

Lemma 9.1. *Let α be as above, suppose $A_1, \ldots, A_\alpha$ are drawn uniformly and independently from $\mathcal{K}$, and let $\mathcal{Q}$ be the event that the multiset $\mathcal{C} := \{A_1, \ldots, A_\alpha\}$ is a non-trivial clique with $\Delta_{\mathcal{C}} \leqslant d-1$. Then*

$$\mathbb{P}(\mathcal{Q}) = o\left(\binom{m_0}{\alpha}^{-1}\right).$$

Proof. This is a (much) simpler version of the proof of Lemma 7.3. We retain the definitions of $d_i(v)$ and d_v from that argument, but now set $W_i = \{v : d_i(v) \geqslant 2\}$, $W = W_\alpha$, $s_i(A) = |A_i \cap W_{i-1}|$, $\sigma(A) = (s_1(A), \ldots, s_\alpha(A))$,

$$s(A) = \sum s_i(A) = \sum_{v \in W} (d_v - 2)$$

and

$$\Psi = \sum_{v \in W}\left[\binom{d_v}{2} - 1\right] = \frac{1}{2}\sum_{v \in W}(d_v+1)(d_v-2),$$

noting that if all d_v are at most d then

$$\Psi \leqslant (d+1)s(A)/2. \tag{9.4}$$

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For a counterpart of Proposition 7.2, with $\mathsf{w} = (\alpha/\varepsilon)$ (ε as in (9.1)) and $\mathcal{S} = \{|W| \leqslant \mathsf{w}\}$, we have

$$\mathbb{P}(\overline{\mathcal{S}}) = o(m_0^{-\alpha})$$

(since $\mathbb{E}|W| < (\varphi k)^2/n < n^{-2\varepsilon}$ implies $\mathbb{P}(|W| \geqslant \mathsf{w}) < n^{-2\varepsilon\mathsf{w}} = n^{-2\alpha} \ll m_0^{-\alpha}$). So we need

$$\mathbb{P}(\mathcal{Q} \wedge \mathcal{S}) = o\left(\binom{m_0}{\alpha}^{-1}\right).$$

We again let $\mathcal{A} = \{\mathcal{C} \text{ is a clique}\}$ and for $\sigma = (s_1, \ldots, s_\alpha) \in \mathbb{N}^\alpha$ set

$$\mathcal{A}(\sigma) = \mathcal{A} \wedge \{\Delta_{\mathcal{C}} \leqslant d-1\} \wedge \mathcal{S} \wedge \{\sigma(A) = \sigma\}.$$

We have $\mathcal{Q} \wedge \mathcal{S} = \cup_\sigma \mathcal{A}(\sigma)$ so, finally, just need to show

$$\sum_\sigma \mathbb{P}(\mathcal{A}(\sigma)) < o\left(\binom{m_0}{\alpha}^{-1}\right). \tag{9.5}$$

With q as in (5.2) (with the present w), this will follow from the next result.

Lemma 9.2. *For any σ as above with $\sum s_i = s$,*

$$\mathbb{P}(\mathcal{A}(\sigma)) \leqslant \min\left\{\left(\frac{\mathsf{w}k}{n}\right)^s q^{\binom{\alpha}{2}-(d+1)s/2}, \left(\frac{\mathsf{w}k}{n}\right)^s\right\}. \tag{9.6}$$

Before sketching the proof of this, we show that it implies (9.5), beginning with regime (i) (so $\alpha = c$, $d = c-2$ and $\binom{m_0}{\alpha} \asymp \mathsf{m}^c$). We use the first bound in (9.6) for $s := |\sigma| < c$ and the second for $s \geqslant c$. For the latter we find that the contribution to $\mathsf{m}^c \sum_{|\sigma| \geqslant c} \mathbb{P}(\mathcal{A}(\sigma))$ is at most

$$\sum_{s \geqslant c} \binom{s+c-1}{c-1}\left(\frac{\varphi n}{k}\right)^c \left(\frac{\mathsf{w}k}{n}\right)^s < \sum_{s \geqslant c} ((s+c)\varphi\mathsf{w})^c \left(\frac{\mathsf{w}k}{n}\right)^{s-c} = o(1).$$

For the former, the product of m^c and the first bound in (9.6) is

$$\left(\frac{\varphi n}{k}\right)^c \left(\frac{\mathsf{w}k}{n}\right)^s q^{(c-1)(c-s)/2} \sim \left(\frac{\varphi n}{k}\right)^c \left(\frac{\mathsf{w}k}{n}\right)^s \left(\frac{k^2}{n}\right)^{(c-1)(c-s)/2}$$
$$= \varphi^c \mathsf{w}^s \left[\left(\frac{n}{k}\right)\left(\frac{k^2}{n}\right)^{(c-1)/2}\right]^{c-s}.$$

If the expression in brackets is at most 1, then we have

$$\mathsf{m}^c \sum_{|\sigma| < c} \mathbb{P}(\mathcal{A}(\sigma)) = O(\varphi^c) \tag{9.7}$$

(since w and $\binom{s+c-1}{c-1}$ are $O(1)$, as is the number of terms in the sum), and otherwise the sum in (9.7) is on the order of

$$\varphi^c \left[\left(\frac{n}{k}\right)\left(\frac{k^2}{n}\right)^{(c-1)/2}\right]^c \asymp \Lambda(c) = o(1)$$

(see (4.8)).

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For regime (ii), we use the second bound in (9.6) for $s \geqslant 3\alpha/2$, yielding

$$\binom{m_0}{\alpha} \sum_{|\sigma| \geqslant 3\alpha/2} \mathbb{P}(\mathcal{A}(\sigma)) < \sum_{s \geqslant 3\alpha/2} \binom{s+\alpha-1}{\alpha-1} \left(\frac{\varphi n}{k}\right)^{\alpha} \left(\frac{\mathsf{w}k}{n}\right)^{s}$$
$$< \sum_{s \geqslant 3\alpha/2} ((s+\alpha)\varphi \mathsf{w})^{\alpha} \left(\frac{\mathsf{w}k}{n}\right)^{s-\alpha} = o(1)$$

(using $\mathsf{w} = O(\alpha)$ and $\alpha \leqslant (1+o(1))\varphi^* < n^{o(1)}$; see (4.12), (2.1)). On the other hand, the first bound in (9.6) gives

$$\binom{m_0}{\alpha} \sum_{|\sigma| < 3\alpha/2} \mathbb{P}(\mathcal{A}(\sigma)) < \binom{m_0}{\alpha} \sum_{s < 3\alpha/2} \binom{s+\alpha-1}{\alpha-1} \left(\frac{\mathsf{w}k}{n}\right)^{s} q^{\binom{\alpha}{2}-(d+1)s/2} = o(1)$$

(as each of $\binom{m_0}{\alpha}$, $\binom{s+\alpha-1}{\alpha-1}$ is at most $\exp[O(\alpha \log n)]$, the q-term is less than $\exp[-\Omega(\alpha^2 \log n)]$, since $q < n^{-\Omega(1)}$, and, as noted above, $\alpha = \omega(1)$). □

Proof of Lemma 9.2. This is similar to the proof of Lemma 7.3 and we just indicate the little changes. For the first bound in (9.6) we follow the proof of (7.15) (beginning with the paragraph containing (7.25)), with changes: replace the γ with c and the U with W; in (7.27) and (7.28) omit the condition involving Z and replace λ with $d-1$ in (7.28); omit the first factor in (7.29) (the proof does not change); and replace $X(r,s)$ in (7.32) with $(d+1)s(A)/2$ (see (9.4)).

For the second bound we use the same modifications and simply sacrifice the contributions of the terms with $i = 2j$ (so for these we can just say $\xi(y_1, \ldots, y_{i-1}) \leqslant 1$; thus the clique condition (7.26) could be omitted here). □

10. Necessity

Our main job in this section is to sketch the proof that the condition

$$\Lambda'(\Delta) < o(1) \text{ a.s.} \tag{10.1}$$

in (1.7) is necessary for EKR to hold a.s., but before doing so we say why Theorem 1.2 implies that it is sufficient. Since $\mathcal{H}$ is a.s. EKR if either (1.5) holds or $\Delta \leqslant 2$ a.s. (the first by Theorem 1.2, the second since in this case the probability of a triangle is trivially $o(1)$), we only need to consider what happens when neither of these alternatives holds but (10.1) does. This means that each of $\{\Delta \leqslant 2\}$ and $\{\Delta \geqslant 3, \ \Lambda(\Delta) < o(1)\}$ occurs with probability $\Omega(1)$ and their union occurs a.s., implying $\varphi \asymp n^{-1/3}$ and $\Lambda(3) < o(1)$. But then $\Lambda(3) \asymp (\varphi n/k)^3 \mathsf{q}^3$ implies $\mathsf{q} = o(1)$, so $k \ll \sqrt{n}$, $\mathsf{q} \sim k^2/n$ and $\Lambda(3) \asymp k^3/n$. Thus $k \ll n^{1/3}$ (again using $\Lambda(3) \to 0$), in which case EKR holds a.s. regardless of φ (as mentioned in Section 9, this was shown in [3]; of course it can also be extracted from the discussion in that section).

We now turn to necessity. We believe this actually holds for general k (*i.e.* without assuming (1.4)), but our proof does not give this. Of course, in view of the discussion preceding Conjecture 1.4, the assertion seems less interesting for k above about $\sqrt{(1/2)n \log n}$.

The proof of necessity becomes easier (still not immediate) if we retreat to, say, $k = O(\sqrt{n})$. Here we give only a sketch of the argument, restricting to k as in (1.9) to avoid some annoyances, with details – such as they are – mostly restricted to the more interesting points. (Some instances of failure of EKR for smaller k are given in [3].)

Note first of all that failure of (10.1) means that there is some fixed $\delta > 0$ such that, for infinitely many n, $\mathbb{P}(\Lambda'(\Delta) > \delta) > \delta$, whence also

$$\mathbb{P}(\Lambda(\Delta) > \delta) > \delta; \tag{10.2}$$

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so it is enough to show that (10.2) implies that EKR fails with probability at least some $\eta_\delta > 0$. (In what follows we just use $\Omega(1)$.)

Set $\alpha = \max\{t \in \mathbb{N} : \Lambda(t) > \delta\}$ and $\mathcal{A} = \{\Delta \leqslant \alpha\}$; thus (10.2) is

$$\mathbb{P}(\mathcal{A}) > \delta, \tag{10.3}$$

which we assume henceforth.

It is easy to see (*cf.* (4.11)) that

$$\alpha \sim \frac{\log n}{\log(1/\mathsf{q})}, \tag{10.4}$$

and we observe that (for any v)

$$\mathbb{P}(d_v > \alpha) = O\left(\frac{1}{n}\right), \tag{10.5}$$

since otherwise Proposition 4.4 gives $\Delta > \alpha$ a.s., contradicting (10.3).

Here we do (finally) need some concrete notion of a 'generic' clique: taking $\mathsf{z} = \alpha/\varepsilon$ (with $\varepsilon = 1/4 - c$ as in Sections 2–8), say a clique – possibly with repeated edges – is *generic* if it has maximum degree at most 3 and at most z vertices with degree equal to 3. Then with

$$\mathcal{B} = \{\mathcal{H} \text{ contains a generic clique of size } \alpha\},$$

we will be done if we show

$$\mathbb{P}(\mathcal{AB}) = \Omega(1). \tag{10.6}$$

The negative results of [3] are achieved by showing (probable) existence of Δ-cliques of maximum degree 2.

Here again Proposition 4.2 allows us to work with independent A_i; namely it implies that (10.6) will follow from the next result.

Lemma 10.1. *For any m satisfying (4.5) and $\mathcal{H} = \{A_1, \ldots, A_m\}$, with the A_i chosen uniformly and independently from $\mathcal{K}$,*

$$\mathbb{P}(\mathcal{AB}) = \Omega(1). \tag{10.7}$$

So we are using '$\mathbb{P}$' for probabilities in this model. Note that $\mathcal{H}$ may now – in principle, though in reality essentially never – have repeated edges.

We first assert that

$$\mathbb{P}(\mathcal{A}) = \Omega(1). \tag{10.8}$$

This actually requires a little argument, but we just point out the difficulty. That (10.5) implies the corresponding $\mathbb{P}(d_v > \alpha) = O(1/n)$ is easy, the change in the distribution of d_v from $\mathrm{Bin}(M, p)$ to $\mathrm{Bin}(m, k/n)$ having almost no effect. But getting from this to (10.8) – an implication which for $\mathcal{H}_k(n,p)$ is given by Harris's inequality; see (9.2) – is no longer immediate, since negative association now works against us.

One way to handle this is to compare the present $\mathcal{H}$ with $\mathcal{H}' = \mathcal{H}_k(n,p')$, with $p' > p$ chosen so that, writing $\mathbb{P}'$ for the corresponding probabilities, we have $\mathbb{P}'(d_v > \alpha) = O(1/n)$ and $|\mathcal{H}'| \geqslant m$ a.s. (We can then couple so that $\mathcal{H}' \supset \mathcal{H}$ a.s. – note that $\mathcal{H}$ a.s. avoids repeats – yielding $\mathbb{P}(\Delta \leqslant \alpha) > \mathbb{P}'(\Delta \leqslant \alpha) - o(1) = \Omega(1)$. Of course one must show there is such a p', but we omit this easy arithmetic.)

For the proof of (10.7) we use the second moment method. Set $N = [m]$ and $\mathcal{S} = \binom{N}{\alpha}$. We now use $\mathcal{G}$ for the set of generic α-cliques (again, with repeated edges allowed). For $S \subseteq N$ write A_S for the multiset $\{A_i : i \in S\}$ and Δ_S for Δ_{A_S} (so $\Delta = \Delta_N$). In addition, set $\mathcal{B}_S = \{A_S \in \mathcal{G}\}$, $X_S = \mathbf{1}_{\mathcal{B}_S}$ (these are only of interest if $S \in \mathcal{S}$) and $X = \sum_{S \in \mathcal{S}} X_S$.

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We actually need estimates for the quantities $\mathbb{E}X_S$ and $\mathbb{E}X_SX_T$ (for $S, T \in \mathcal{S}$) *conditioned on* $\mathcal{A}$, but will get these by first dealing with the unconditional versions and then showing – the most interesting point – that the conditioning has little effect. Thus we show (for any $S, T \in \mathcal{S}$)

$$\mathbb{E}X_S \sim \mathsf{q}^{\binom{\alpha}{2}}, \tag{10.9}$$

$$\mathbb{E}X_SX_T < (1+o(1))\mathsf{q}^{2\binom{\alpha}{2}-\binom{|S\cap T|}{2}}, \tag{10.10}$$

$$\mathbb{E}[X_S \mid \mathcal{A}] \sim \mathbb{E}X_S \quad \text{and} \quad \mathbb{E}[X_SX_T \mid \mathcal{A}] \sim \mathbb{E}X_SX_T. \tag{10.11}$$

We will say a little about the proofs of these main points below. Once they are established we have, setting $\tilde{\mathbb{E}}[\,\cdot\,] = \mathbb{E}[\cdot \mid \mathcal{A}]$,

$$\mu := \tilde{\mathbb{E}}X \sim \binom{m}{\alpha}\mathsf{q}^{\binom{\alpha}{2}} \sim \Lambda(\alpha) = \Omega(1)$$

(using (4.19) for '$\sim$') and an easy calculation gives

$$\tilde{\mathbb{E}}X^2 = \sum_S\sum_T \tilde{\mathbb{E}}X_SX_T < (1+o(1))\binom{m}{\alpha}\mathsf{q}^{2\binom{\alpha}{2}}\sum_{i=0}^{\alpha}\binom{\alpha}{i}\binom{m-\alpha}{\alpha-i}\mathsf{q}^{-\binom{i}{2}} \sim \mu^2+\mu,$$

whence

$$\mathbb{P}(X \neq 0) \geqslant \frac{\mu^2}{\tilde{\mathbb{E}}X^2} = \Omega(1),$$

which is what we want.

The proofs of (10.10) and $\mathbb{E}X_S < (1+o(1))\mathsf{q}^{\binom{\alpha}{2}}$ (for (10.9)) are similar to (easier than) that of Lemma 7.1 and we will not pursue them here. The proof of the reverse inequality in (10.9) is also similar in spirit, but less so in details. We again think of choosing $A_1, \ldots, A_\alpha$ in order and use d_i for degrees in $\{A_1, \ldots, A_i\}$. Set $Z_i = \{v : d_i(v) \geqslant 3\}$, $\mathcal{Q}_i = \{|Z_i| \leqslant \mathsf{z}\}$, $\mathcal{R}_i = \{A_i \cap Z_{i-1} = \emptyset\}$, $\mathcal{T}_i = \{A_i \cap A_j \neq \emptyset \text{ for all } j \in [i-1]\}$ and

$$\mathcal{B}_i = \{\{A_1, \ldots, A_i\} \text{ is a generic clique}\}.$$

Then $\mathcal{B}_i = \mathcal{B}_{i-1}\mathcal{R}_i\mathcal{T}_i\mathcal{Q}_i$ and

$$\mathbb{P}(\mathcal{B}_i) \geqslant \mathbb{P}(B_{i-1})\,\mathbb{P}(\mathcal{R}_i\mathcal{T}_i \mid B_{i-1}) - \mathbb{P}(\overline{\mathcal{Q}}_i). \tag{10.12}$$

We show by induction on i (with $i = 1$ trivial)

$$\mathbb{P}(\mathcal{B}_i) \geqslant (1-\delta_i)\mathsf{q}^{\binom{i}{2}}, \tag{10.13}$$

for some $\delta_i < in^{-1/4+o(1)}$. (This suffices because of (10.4), since $(\log(1/q))^{-1} < (1+o(1))n^c$; see (4.2).)

The relevant probabilities are bounded as follows. First, the proof of Proposition 7.2(b) (see (7.6)) gives

$$\mathbb{P}(\overline{\mathcal{Q}}_i) < \eta \tag{10.14}$$

for some $\eta < n^{-(2-o(1))\alpha}$. Second, trivially,

$$\mathbb{P}(\mathcal{R}_i \mid \mathcal{B}_{i-1}) \geqslant 1 - \frac{\mathsf{z}k}{n} \tag{10.15}$$

(this just uses $\mathcal{B}_{i-1} \subseteq \mathcal{Q}_{i-1}$).

Third,

$$\mathbb{P}(\mathcal{T}_i \mid \mathcal{R}_i\mathcal{B}_{i-1}) \geqslant (1-\theta)\mathsf{q}^{i-1}, \tag{10.16}$$

where $\theta < n^{-1/4+o(1)}$. This one is less trivial than the first two. We need the following general observation.

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Proposition 10.2. *If $C_1, \ldots, C_s, D_1, \ldots, D_s$ are subsets of V with $|C_i| = |D_i|$ for all i and the D_i pairwise disjoint, and A is uniform from $\binom{V}{k}$, then*

$$\mathbb{P}(A \cap C_i \neq \emptyset \text{ for all } i) \geqslant \mathbb{P}(A \cap D_i \neq \emptyset \text{ for all } i). \tag{10.17}$$

This follows via induction from the fact – an easy coupling argument – that (10.17) holds when $x \in C_i \cap C_j$ $(i \neq j)$, $D_i = C_i \setminus \{x\} \cup \{y\}$ for some $y \in V \setminus \cup C_\ell$, and $D_\ell = C_\ell$ for $\ell \neq i$.

By Proposition 10.2, the left-hand side of (10.16) is at least

$$\mathbb{P}(A \cap A_j \neq \emptyset \text{ for all } j \in [i-1]),$$

where $A_1, \ldots, A_{i-1}$ are (fixed) disjoint $(k - \mathsf{z})$-subsets of $U \in \binom{V}{n-\mathsf{z}}$ and A is uniform from $\binom{U}{k}$. Say $Y \subseteq U$ is *good* if $Y \cap A_j \neq \emptyset$ for all $j \in [i-1]$; so we want

$$\mathbb{P}(A \text{ is good}) \geqslant (1-\theta)\mathsf{q}^{i-1}. \tag{10.18}$$

One way – there ought to be an easier one – to show this goes as follows. Let $X \subseteq U$ be random with each member of U contained in X with probability $\rho = (k - 2\sqrt{k \ln n})/n$, these choices made independently. Then

$$\mathbb{P}(A \text{ is good}) \geqslant \mathbb{P}(X \text{ is good}) - \mathbb{P}(|X| > k) > \mathbb{P}(X \text{ is good}) - n^{-2},$$

where the first inequality holds because we can couple so that $A \supseteq X$ whenever $|X| \leqslant k$, and the second is given by Theorem 3.4. Thus, since $q^{i-1} > q^{\alpha} > n^{-1-o(1)}$, (10.18) will follow from

$$\mathbb{P}(X \text{ is good}) \geqslant (1 - n^{-1/4+o(1)})\mathsf{q}^{i-1}. \tag{10.19}$$

For verification of (10.19), set $\ell = k - \mathsf{z}$. Since $\mathbb{P}(X \text{ is good}) = [1 - (1-\rho)^{\ell}]^{i-1}$ and $i < \alpha$, it is enough to show that

$$[(1 - (1-\rho)^{\ell})/\mathsf{q}]^{\alpha} > 1 - n^{-1/4+o(1)}. \tag{10.20}$$

As before (see Section 4), set

$$\vartheta = 1 - \mathsf{q} = \frac{(n-k)_k}{(n)_k} \sim e^{-k^2/n}$$

and define γ by $(1-\rho)^{\ell} = (1+\gamma)\vartheta$. Then $(1 - (1-\rho)^{\ell})/\mathsf{q} = 1 - (\gamma\vartheta/(1-\vartheta))$, so for (10.20) we need $\alpha\gamma\vartheta/(1-\vartheta) < n^{-1/4+o(1)}$. But it is easy to see that we always have $\alpha\vartheta < O(\log n)$ (using (10.4)) and $1 - \vartheta$ $(= q) > n^{-o(1)}$ (since we assume (1.9)); so we really just need

$$\gamma < n^{-1/4+o(1)}. \tag{10.21}$$

Here we may expand

$$1 + \gamma = \frac{(1-\rho)^{\ell}}{\vartheta} = \frac{(n-k)^k}{(n-k)_k}\,\frac{(n)_k}{n^k}\left(1 - \frac{k}{n}\right)^{-\mathsf{z}}\left(\frac{1-\rho}{1-k/n}\right)^{\ell}.$$

The last two factors are at most $1 + k\mathsf{z}/n + O(k^2\mathsf{z}^2/n^2) < 1 + n^{-1/4-\varepsilon+o(1)}$ and

$$1 + O(n^{-1}k^{3/2}\sqrt{\log n}) < 1 + n^{-1/4+o(1)}$$

(respectively), while a little rearranging shows the product of the first two to be

$$\prod_{j=0}^{k-1}\left(1 + \frac{jk}{n(n-k-j)}\right) < 1 + \frac{k^3}{n^2} < 1 + n^{-1/2+o(1)}.$$

This proves (10.21) and finally establishes (10.16). □

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By (10.14)–(10.16) and (10.13) for $i-1$, the right-hand side of (10.12) is at least

$$(1-\delta_{i-1})\mathsf{q}^{\binom{i-1}{2}}\left[\left(1-\frac{\mathsf{z}k}{n}\right)(1-\theta)-\eta'\right]\mathsf{q}^{i-1} > (1-\delta_{i-1})\left[1-\left\{\frac{\mathsf{z}k}{n}+\theta+\eta'\right\}\right]\mathsf{q}^{\binom{i}{2}},$$

where we set $\eta' = \eta[(1-\delta_{i-1})\mathsf{q}^{\binom{i}{2}}]^{-1}$. This gives (10.13) since the expression in the { } is less than $n^{-1/4+o(1)}$. □

Finally we turn to (10.11), for which we need the following observation.

Proposition 10.3. *Let $s \in [m]$ and $t = m - s$. Suppose $S \in \binom{N}{s}$ and $\mathcal{D}$ is an s-multisubset of $\mathcal{K}$ with $\Delta_{\mathcal{D}} \leqslant C$. If $\mathbb{P}(B(t,k/n) \geqslant \alpha - C) = \rho/n$ then*

$$|\mathbb{P}(\Delta \leqslant \alpha \mid A_S = \mathcal{D}) - \mathbb{P}(\Delta \leqslant \alpha)| \leqslant \frac{sk\rho}{n} + n\left(\frac{sk}{n}\right)^{C+1}. \tag{10.22}$$

Proof. Let $B_S = \{B_i : i \in S\}$, where the B_i are chosen uniformly and independently (of each other and the A_j) from $\mathcal{K}$, and write Δ^* for the maximum degree of $A_T \cup B_S$. Set $V(X) = \{v : d_X(v) > 0\}$ (for X a multisubset of $\mathcal{K}$). On $\{A_S = \mathcal{D}\}$ we have

$$\{\Delta^* \leqslant \alpha\} \setminus \{\Delta \leqslant \alpha\} \subseteq \{\exists v \in V(\mathcal{D})\ d_T(v) > \alpha - C\} \tag{10.23}$$

and

$$\{\Delta \leqslant \alpha\} \setminus \{\Delta^* \leqslant \alpha\} \subseteq \{\Delta_{B_S} > C\} \cup \{\exists v \in V(B_S)\ d_T(v) > \alpha - C\} \tag{10.24}$$

The probabilities of the event on the right-hand side of (10.23) and the second event on the right-hand side of (10.24) are at most $sk\rho/n$, and the probability of the first event on the right-hand side of (10.24) is less than $n(sk/n)^{C+1}$ (since $\mathbb{E}d_{B_S}(v) = sk/n$). The proposition follows. □

The arguments for the two statements in (10.11) are nearly the same and we speak mainly of the first. This is equivalent to $\mathbb{P}(\mathcal{A} \mid \mathcal{B}_S) \sim \mathbb{P}(\mathcal{A})$ or, in view of (10.8), $\mathbb{P}(\mathcal{A} \mid \mathcal{B}_S) = \mathbb{P}(\mathcal{A}) \pm o(1)$, which will follow if we show that, for any generic α-clique $\mathcal{D}$,

$$\mathbb{P}(\mathcal{A} \mid A_S = \mathcal{D}) = \mathbb{P}(\mathcal{A}) \pm o(1).$$

This is, of course, an instance of Proposition 10.3, for which we just have to make sure that, with $s = \alpha$ and $C = 3$, each part of the bound in (10.22) is $o(1)$. For the second part this is given by $\alpha k/n < n^{-1/4-\varepsilon+o(1)}$. For the first, with $\xi = B(t,k/n)$, we have $\mathbb{P}(\xi > \alpha) \leqslant \mathbb{P}(B(m,k/n) > \alpha) = O(1/n)$ (see (10.5)) and, for $u \sim \alpha$,

$$\frac{\mathbb{P}(\xi = u-1)}{\mathbb{P}(\xi = u)} = \frac{u(1-k/n)}{(t-u+1)k/n} \sim \frac{u}{tk/n} \sim \frac{\alpha}{\varphi} < n^{o(1)}$$

(with the inequality given by (4.10)), whence $\rho < n^{o(1)}$ (and $sk\rho/n < n^{-1/4+o(1)}$).

For the second part of (10.11) we would have $s \in [\alpha, 2\alpha]$ and $C = 6$. □

Glossary

$\mathcal{K} = \binom{V}{k}$, page 881

$\mathcal{H}_x = \{E \in \mathcal{H} : x \in E\}$, page 882

$\varphi = p\binom{n-1}{k-1}$, page 882

$\mathsf{m} = \mathbb{E}|\mathcal{H}| = \frac{\varphi n}{k}$, page 882

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$\mathsf{q} = \mathbb{P}(A \cap B \neq \emptyset)$, see (1.3) on page 882

$\Lambda(t) = \binom{\mathsf{m}}{t} \mathsf{q}^{\binom{t}{2}}$, see (1.6) on page 883

$\Lambda'(t) = 0$ if $t \leqslant 2$; $= \Lambda(t)$ otherwise, page 883

$\mathcal{G}_{\overline{x}} = \mathcal{G} \setminus \mathcal{G}_x$, page 885

$\varphi^* = \dfrac{\log^3 n}{\log(1/\mathsf{q})}$, see (2.1) on page 885

α, β : defined before (2.3) (for key properties see (2.3), (2.4), and the paragraphs leading up to Proposition 4.3); page 885

$\gamma = \min\left\{\alpha, \dfrac{\varphi^*}{3}\right\}$, see (2.5) on page 886

$\tau = (1-\varepsilon)\gamma$, see (2.6) on page 886

$\lambda = \max\left\{\dfrac{\sqrt{\log n}}{\log(1/\mathsf{q})}, 2\sqrt{\dfrac{\log n}{\log(1/\mathsf{q})}}\right\}$, see (2.7) on page 886

NA : negatively associated, page 887

ψ : a slowly growing function of n (say $\psi = \log n$), page 889

$m_0 = \mathsf{m} + \psi\sqrt{\mathsf{m}}$, page 889

$\mathcal{P} = \{m$ satisfies (4.5)$\} \wedge \{\Delta \leqslant \beta\}$, page 890

$\mathcal{R}$: the intersection of $\mathcal{P}$, $\{\Delta \geqslant \alpha\}$, and the events in (4.20), (4.21), page 892

$\mathsf{w} = \max\left\{\dfrac{\varphi^2 k^2}{n}, 6\log n\right\}$, see (5.1) on page 893

$q = \left(1 + \dfrac{2k^2\mathsf{w}}{\mathsf{q}n^2}\right)\mathsf{q}$, see (5.2) on page 893

$\Psi = \sum_{v \in Z}\left[\binom{d_v}{2} - 1\right]$, see (7.3) on page 898

$\mathsf{z} = \dfrac{\gamma}{\varepsilon}$, see Proposition 7.2 (b) on page 898

$X(r,s) = \dfrac{(\lambda+2)r}{2} + 2s$ (for r and s see (7.2)), page 899

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