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# SOME SPECIAL CONJUGACY CLOSED LOOPS

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ABSTRACT. Some equationally defined classes of loops are identified and characterized among a class of loops which are isomorphic to all of their loop isotopes.

1. **Introduction.** In this paper we adopt the convention (see V. D. Belousov [1], Orin Chein and H. Pflugfelder [3], and the authors [5]) of calling a loop which is isomorphic to all of its loop isotopes a $G$-loop. Each of the following equationally defined classes of loops is known to be a class of $G$-loops. (Although, for the most part, the notation is standard and self-explanatory, the reader can consult section 2 for any clarification.)

Class A. A loop $(G, \cdot)$ is associative (i.e., is a group) provided that

$$xy \cdot z = x \cdot yz \tag{1.1}$$

for all $x, y, z \in G$.

Class B. A loop $(G, \cdot)$ is a Wilson loop provided that

$$x(xy)^{\rho} = (xz)(x \cdot yz)^{\rho} \tag{1.2}$$

for all $x, y, z \in G$ (see Eric L. Wilson [8]).

Class C. A loop $(G, \cdot)$ is an extra loop provided that

$$(xy \cdot z)x = x(y \cdot zx) \tag{1.3}$$

for all $x, y, z \in G$ (see Ferenc Fenyves [4]).

Class D. A loop $(G, \cdot)$ is a conjugacy closed loop provided that

$$g \cdot xy = (gx)R(g)^{-1} \cdot (gy) \tag{1.4}$$

and

$$xy \cdot f = (xf) \cdot (yf)L(f)^{-1} \tag{1.5}$$

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for all $x, y, f, g \in G$ (see the authors [5]).

Clearly, Class A is included in each of Classes B, C, and D. It is our purpose in this paper to show that Classes B and C are included in Class D and, more importantly, to determine precisely how the loops of Class B and the loops of Class C can be characterized or identified within Class D – thereby proving, among other things, a result already announced by the authors (see Remark 2.1 in [5]).

Specifically, three theorems are established.

THEOREM 1. *A loop* $(G, \cdot)$ *is a Wilson loop if and only if* $(G, \cdot)$ *is conjugacy closed and satisfies the weak inverse property*

$$y(xy)^\rho = x^\rho \tag{1.6}$$

*for all* $x, y \in G$. (See section 3 for a proof.)

THEOREM 2. *A loop* $(G, \cdot)$ *is an extra loop if and only if* $(G, \cdot)$ *is conjugacy closed and satisfies the flexible law*

$$xy \cdot x = x \cdot yx \tag{1.7}$$

*for all* $x, y \in G$. (See section 4 for a proof.)

We conclude with an interesting consequence of the two preceding theorems.

THEOREM 3. *If* $(G, \cdot)$ *is a Wilson loop with nucleus* $N$, *then* $N$ *is a normal subloop of* $(G, \cdot)$ *and the quotient loop* $G/N$ *is an extra loop*. (See section 5 for a proof.)

2. **Requisite information.** We recall here basic loop-theoretic notation, some of which has already been used in section 1, and also the loop-theoretic concept of autotopism, which is employed in sections 3 and 4.

Let $(G, \cdot)$ be a closed binary system. The translation maps $L(x)$ and $R(x)$ for $(G, \cdot)$ are defined by $yL(x) = xy$ and $yR(x) = yx$ for all $x, y \in G$. The system $(G, \cdot)$ is a quasigroup provided that $L(x)$ and $R(x)$ are permutations of $G$ (i.e., are one-to-one maps of $G$ onto $G$) for all $x \in G$ – thus making cancellation available. So for any quasigroup $(G, \cdot)$ the inverse maps $L(x)^{-1}$ and $R(x)^{-1}$, although not usually themselves translations of $(G, \cdot)$, are at least permutations of $G$. A loop is a quasigroup with a (unique) identity element, denoted by $e$ in this paper. Now if $(G, \cdot)$ is a loop and if $x \in G$ we define $x^\lambda$ and $x^\rho$ by $x^\lambda = eR(x)^{-1}$ and $x^\rho = eL(x)^{-1}$, that is, $x^\lambda$ and $x^\rho$ are those unique elements in $G$ corresponding to $x$ with the property that $xx^\rho = x^\lambda x = e$. In this paper it is convenient to let $\lambda$ and $\rho$ denote also the maps $\lambda : x \longrightarrow x^\lambda$ and $\rho : x \longrightarrow x^\rho$.

If $\alpha, \beta$, and $\gamma$ are one-to-one maps of $G$ onto $G$, then the triple $\langle \alpha, \beta, \gamma \rangle$ is an autotopism of a closed system $(G, \cdot)$ provided that

$$x\alpha \cdot y\beta = (xy)\gamma$$

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for all $x, y \in G$. Now the membership of a loop in Class B, C, or D can be readily reformulated in terms of autotopisms as follows.

RESULT 2.1. *Let* $(G, \cdot)$ *be a loop. Then identity* (1.4) *holds for all* $x, y, g \in G$ *if and only if*

$$\langle L(g)R(g)^{-1}, L(g), L(g)\rangle$$

*is an autotopism of* $(G, \cdot)$ *for each* $g \in G$*; and identity* (1.5) *holds for all* $x, y, f \in G$ *if and only if*

$$\langle R(f), R(f)L(f)^{-1}, R(f)\rangle$$

*is an autotopism of* $(G, \cdot)$ *for each* $f \in G$.

RESULT 2.2. *Let* $(G, \cdot)$ *be a loop. Then the following three statements are equivalent:*
(i) $(G, \cdot)$ *is extra, i.e., identity* (1.3) *holds for all* $x, y, z \in G$,
(ii) $\langle R(x), L(x)^{-1}R(x), R(x)\rangle$ *is an autotopism of* $(G, \cdot)$ *for all* $x \in G$,
(iii) $\langle R(x)^{-1}L(x), L(x), L(x)\rangle$ *is an autotopism of* $(G, \cdot)$ *for all* $x \in G$.

These two results are direct consequences of the definition of autotopism given above (in connection with Result 2.2 the reader may wish to see also Theorem 2 in F. Fenyves [4]), as is

RESULT 2.3. *If* $A_1 = \langle \alpha_1, \beta_1, \gamma_1\rangle$ *and* $A_2 = \langle \alpha_2, \beta_2, \gamma_2\rangle$ *are autotopisms of a loop* $(G, \cdot)$, *then so too are* $A_1^{-1} = \langle \alpha_1^{-1}, \beta_1^{-1}, \gamma_1^{-1}\rangle$ *and* $A_1A_2 = \langle \alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2\rangle$.

These three results on autotopisms provide us with a systematic means for dealing with various loop identities in sections 3 and 4 – a technique which has appeared in the work of R. H. Bruck (see, for instance, [2]) and others (see, for instance, [4], [5], [6]).

## 3. **A proof of Theorem 1.** Consider the following three results.

RESULT 3.1. *If* $(G, \cdot)$ *is a Wilson loop, then* $(G, \cdot)$ *satisfies the weak inverse property.*

PROOF. In (1.2) let $z = (xy)^\rho$ and then use left cancellation to get $(x\cdot(y\cdot(xy)^\rho))^\rho = e$. It follows that $x(y \cdot (xy)^\rho) = e$ and, in turn, that $y(xy)^\rho = x^\rho$ for all $x, y \in G$. Hence, (1.6) holds for all $x, y \in G$, and so $(G, \cdot)$ satisfies the weak inverse property. □

RESULT 3.2. *If* $(G, \cdot)$ *is a Wilson loop, then* $(G, \cdot)$ *is a conjugacy closed loop.*

PROOF. Let (1.2) hold for all $x, y \in G$. Then from Result 3.1 the loop $(G, \cdot)$ satisfies the weak inverse property, and so it follows (see J. Marshall Osborn [6]) that

$$(yz)^\lambda y = z^\lambda \tag{3.1}$$

for all $y, z \in G$, that $\langle \rho^2, \rho^2, \rho^2\rangle$ and $\langle \lambda^2, \lambda^2, \lambda^2\rangle$ are autotopisms of $(G, \cdot)$, and that $\langle \alpha, \beta, \gamma\rangle$ is an autotopism of $(G, \cdot)$ if and only if $\langle \beta, \lambda\gamma\rho, \lambda\alpha\rho\rangle$ is an autotopism of $(G, \cdot)$.

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From (1.2) it follows that

$$(x(xy)^{\rho})^{\lambda}(xz) = [(xz)(x \cdot yz)^{\rho}]^{\lambda}(xz)$$

for all $x, y, z \in G$. So we get

$$(x(xy)^{\rho})^{\lambda}(xz) = x \cdot yz$$

for all $x, y, z \in G$ when (3.1) is employed. It follows that

$$\langle L(x)\rho L(x)\lambda, L(x), L(x)\rangle$$

is an autotopism of $(G, \cdot)$ for all $x \in G$. But from (3.1) it also follows that

$$xy = (x(xy)^{\rho})^{\lambda}x \tag{3.2}$$

for all $x, y \in G$. From this it follows that

$$L(x)\rho L(x)\lambda R(x) = L(x)$$

for all $x \in G$. Thus,

$$A(x) = \langle L(x)R(x)^{-1}, L(x), L(x)\rangle$$

is an autotopism of $(G, \cdot)$ for all $x \in G$. So by Result 2.1 identity (1.4) holds for $(G, \cdot)$.

But from remarks at the beginning of the proof $A(x)$ being an autotopism for $(G, \cdot)$ implies that

$$B(x) = \langle L(x), \lambda L(x)\rho, \lambda L(x)R(x)^{-1}\rho\rangle$$

and, in turn,

$$C(x) = \langle \lambda L(x)\rho, \lambda^2 L(x)R(x)^{-1}\rho^2, \lambda L(x)\rho\rangle$$

are both autotopisms of $(G, \cdot)$ for all $x \in G$. From (3.2) it follows that $(xz^{\rho})^{\lambda}x = z$ for all $x, z \in G$, and so we get $\rho L(x)\lambda = R(x)^{-1}$ for all $x \in G$. Taking inverses, we get, for use below, $\rho L(x)^{-1}\lambda = R(x)$ for all $x \in G$. Then

$$D(x) = \langle \rho^2, \rho^2, \rho^2\rangle C(x)^{-1}\langle \lambda^2, \lambda^2, \lambda^2\rangle,$$

being the product of three autotopisms, is an autotopism of $(G, \cdot)$ by Result 2.3. But by direct calculation we get

$$\begin{aligned} D(x) &= \langle \rho L(x)^{-1}\lambda, R(x)L(x)^{-1}, \rho L(x)^{-1}\lambda\rangle \\ &= \langle R(x), R(x)L(x)^{-1}, R(x)\rangle \end{aligned}$$

for all $x \in G$. So by Result 2.1 identity (1.5) holds for $(G, \cdot)$. Now with both (1.4) and (1.5) holding for $(G, \cdot)$, we conclude that $(G, \cdot)$ is conjugacy closed. □

RESULT 3.3. *If $(G,\cdot)$ is a conjugacy closed loop which satisfies the weak inverse property, then $(G,\cdot)$ is a Wilson loop.*

PROOF. Let $(G,\cdot)$ be a conjugacy closed loop which satisfies the weak inverse property. Then we see that $\langle L(x)R(x)^{-1}, L(x), L(x)\rangle$ is an autotopism of $(G,\cdot)$ and $L(x)R(x)^{-1} = L(x)\rho L(x)\lambda$ for all $x \in G$. Thus, it follows that

$$\langle L(x)\rho L(x)\lambda, L(x), L(x)\rangle$$

is an autotopism of $(G,\cdot)$ for all $x \in G$, and so

$$(x(xy)^{\rho})^{\lambda} \cdot (xz) = x \cdot yz$$

for all $x, y, z \in G$. From this it follows that

$$(xz)[(x(xy)^{\rho})^{\lambda} \cdot (xz)]^{\rho} = (xz) \cdot (x \cdot yz)^{\rho}$$

for all $x, y, z \in G$, and now using the weak inverse property (1.6) to simplify the left hand side, we see that

$$x(xy)^{\rho} = (xz)(x \cdot yz)^{\rho}$$

for all $x, y, z \in G$. Thus, $(G,\cdot)$ is a Wilson loop. □

Clearly, Theorem 1 is a direct and immediate consequence of Results 3.1, 3.2, and 3.3.

4. **A proof of Theorem 2.** Results in section 2 (see Results 2.1 and 2.2) together with the observation that $L(x)R(x) = R(x)L(x)$, $L(x)^{-1}R(x) = R(x)L(x)^{-1}$, and $R(x)^{-1}L(x) = L(x)R(x)^{-1}$ for all $x \in G$ whenever $(G,\cdot)$ satisfies the flexible law afford us a direct proof of Theorem 2 as follows.

If $(G,\cdot)$ is a loop which is conjugacy closed and flexible, then

$$\langle R(x), L(x)^{-1}R(x), R(x)\rangle = \langle R(x), R(x)L(x)^{-1}, R(x)\rangle$$

is an autotopism of $(G,\cdot)$ for all $x \in G$ and $(G,\cdot)$ must then be an extra loop. Conversely, if $(G,\cdot)$ is an extra loop, then $(G,\cdot)$ satisfies the flexible law (merely set $z = e$ in (1.3)) and so

$$\langle R(x), R(x)L(x)^{-1}, R(x)\rangle = \langle R(x), L(x)^{-1}R(x), R(x)\rangle$$

and

$$\langle L(x)R(x)^{-1}, L(x), L(x)\rangle = \langle R(x)^{-1}L(x), L(x), L(x)\rangle$$

are autotopisms of $(G,\cdot)$ for all $x \in G$, forcing $(G,\cdot)$ to be conjugacy closed. This completes our proof of Theorem 2. □

5. **A proof of Theorem 3.** Let $(G, \cdot)$ be a Wilson loop. Then by Theorem 1 we note that $(G, \cdot)$ is conjugacy closed and satisfies the weak inverse property. But since $(G, \cdot)$ is conjugacy closed, its nucleus $N$ is normal in $(G, \cdot)$ and it is a $G$-loop (see the authors [5]). Hence, every loop isotopic to $(G, \cdot)$ is isomorphic to $(G, \cdot)$ and so must also satisfy the weak inverse property. So from a result of J. M. Osborn [6], the quotient loop $G/N$ is Moufang and so must be flexible. Since $(G, \cdot)$ is conjugacy closed, so too is $G/N$ (see the authors [5]). Since $G/N$ is conjugacy closed and flexible, we appeal to Theorem 2 and conclude that $G/N$ is an extra loop. □

6. **Questions for further investigation.** In view of Theorems 1 and 2 the following questions are of interest:

(1) Are there other equationally defined (and naturally characterized) classes of $G$-loops (like Classes B and C) which are contained in the Class D of all conjugacy closed loops? Are there others which contain Class D?

(2) Do those $G$-loops which have been constructed by ad hoc methods and which are not members of Class D (a notable example is that of one of the authors [7]) belong to some equationally defined class of loops?

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