J. Austral. Math. Soc. (*Series A*) **68** (2000), 301–311

GENERALIZED FUNCTIONS ASSOCIATED WITH SELF-ADJOINT OPERATORS

LIU SHANGPING

(Received 10 December 1998; revised 9 August 1999)

Communicated by P. G. Dodds

Abstract

In this paper, from several commutative self-adjoint operators on a Hilbert space, we define a class of spaces of fundamental functions and generalized functions, which are characterized completely by self-adjoint operators. Specially, using the common eigenvectors of these self-adjoint operators, we give the general form of expansion in series of generalized functions

2000 *Mathematics subject classification*: primary 46F05, 47B25.
Keywords and phrases: generalized function, self-adjoint operator, completeness, nuclearity.

1. Fundamental space and generalized space

Let H be a Hilbert space and $A_1, A_2, \dots, A_n$ be commutative unbounded self-adjoint operators on H. For every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{Z}_+)^n$ we denote by $A^\alpha = A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}$, where $\mathbb{Z}_+ = \{\alpha; \alpha \in \mathbb{Z}, \alpha \geq 0\}$. Suppose that $\mathscr{D}_{A^\alpha}$ is the domain of A^α, and m is a non-negative integer. Let $\Phi_m = \bigcap_{0 \leq |\alpha| \leq m} \mathscr{D}_{A^\alpha}$, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, for every $\alpha \in (\mathbb{Z}_+)^n$. Define an inner product in Φ_m as follows:

$$(x, y)_m = \sum_{0 \leq |\alpha| \leq m} (A^\alpha x, A^\alpha y), \quad \text{for every } x, y \in \Phi_m,$$

where $(\cdot, \cdot)$ is the inner product of H. Since H is complete and A^α is closed, we can see that Φ_m is a Hilbert space with the inner product $(\cdot, \cdot)_m$. Obviously, $H = \Phi_0 \supset \Phi_1 \supset \Phi_2 \supset \cdots$. Let $\Phi = \bigcap_{m=0}^{+\infty} \Phi_m$. We have the following proposition.

PROPOSITION 1.1. *Φ is a dense subset of Hilbert space Φ_m ($m \geq 0$).*

Project supported by the National Natural Science Foundation of China.

PROOF. Let $A_j = \int_{-\infty}^{+\infty} \lambda E^{(j)}(d\lambda)$ be the spectral decomposition of A_j, where $E^{(j)}$ is a spectral measure on the real line $\mathbb{R}$, $1 \le j \le n$. Because $A_1, \dots, A_n$ are commutative, we can define $E(d\lambda_1 d\lambda_2 \cdots d\lambda_n) = E^{(1)}(d\lambda_1)E^{(2)}(d\lambda_2)\cdots E^{(n)}(d\lambda_n)$, a spectral measure on $\mathbb{R}^n$. For $N \in \mathbb{N}$, let $P_N = \int_{|\lambda|\le N} E(d\lambda)$, where $\mathbb{N}$ is the set of all positive integers, $d\lambda = d\lambda_1 d\lambda_2 \cdots d\lambda_n$, $|\lambda| = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$. Obviously, P_N is a projection on H, and $P_N H \subset \Phi_m$, for any $m \in \mathbb{Z}_+$, $N \in \mathbb{N}$, so $\bigcup_{N=1}^{+\infty} P_N H \subset \Phi$. Suppose that $x \in \Phi_m$, $m \in \mathbb{Z}_+$. We have

$$\|P_N x - x\|_m^2 = \sum_{0\le|\alpha|\le m} \|A^\alpha P_N x - A^\alpha x\|^2 = \sum_{0\le|\alpha|\le m} \int_{|\lambda|\ge N} |\lambda^\alpha|^2 \|E(d\lambda)x\|^2 \xrightarrow{N} 0,$$

where $\lambda^\alpha = \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}$. So $\bigcup_{N=1}^{+\infty} P_N H$ is dense in Hilbert space Φ_m. □

PROPOSITION 1.2. *With the countable inner products* $\{(\cdot,\cdot)_m, m \in \mathbb{Z}_+\}$, Φ *is a countable Hilbert space.*

PROOF. For the definition of a countable Hilbert space, see [2]. Let $\phi \in \Phi$. We have $\|\phi\|_0 \le \|\phi\|_1 \le \|\phi\|_2 \le \cdots$. Now we show that these countable norms are compatible. If $\|\phi_k\|_m \to 0$, and ϕ_k is a Cauchy sequence in the norm $\| \; \|_{m+l}$, l being some positive integer, then by the completeness of H there exists $x^{(\alpha)} \in H$ for each $\alpha \in (\mathbb{Z}_+)^n$, $|\alpha| \le m + l$, such that $\|A^\alpha\phi_k - x^{(\alpha)}\| \to 0$. Since A^α is a closed operator, we have $x^{(\alpha)} = A^\alpha x^{(0)}$, $|\alpha| \le m + l$. From $\|\phi_k\|_m \to 0$, it follows that $x^{(0)} = 0$, and $\|\phi_k\|_{m+l} \to 0$.

By Proposition 1.1, the completion of $\{\Phi, (\cdot,\cdot)_m\}$ is Φ_m. Since $\Phi = \bigcap_{m=0}^{+\infty} \Phi_m$, it follows that Φ is a countable Hilbert space ([2]). □

DEFINITION 1.1. The space Φ defined as above is called the *fundamental space* associated with self-adjoint operators $\{A_j \mid 1 \le j \le n\}$. The dual space Φ' is called the *generalized space* associated with $\{A_j \mid 1 \le j \le n\}$.

Later we see that if a Hilbert space H consists of functions, then Φ is a fundamental space of functions, and $A_1, A_2, \dots, A_n$ can operate on Φ infinitely, and Φ' is the corresponding generalized space of functions.

From [2], $\Phi' = \bigcup_{m=0}^{+\infty} \Phi'_m$, where for each m Φ'_m is the dual space of Φ_m and it is also a Hilbert space. We have $H = \Phi'_0 \subset \Phi'_1 \subset \Phi'_2 \subset \cdots$. The weak $*$ topology $\sigma(\Phi', \Phi)$ of Φ' is defined as follows. The fundamental system of neighborhoods of zero consists of

$$U(0;\phi_1,\dots,\phi_l;\varepsilon) = \{f \mid f \in \Phi', |\langle f, \phi_k\rangle| < \varepsilon,\ 1 \le k \le l\},$$

where $\phi_1, \phi_2, \dots, \phi_l \in \Phi$, l is some positive integer, $\varepsilon > 0$. Later the topologies of Φ' are all $\sigma(\Phi', \Phi)$.

PROPOSITION 1.3. *Φ'_m $(m \in \mathbb{Z}_+)$ is a dense subset of Φ'.*

PROOF. Since $H = \Phi'_0 \subset \Phi'_1 \subset \cdots$, it is sufficient to prove that H is dense in Φ'. In the proof of Proposition 1.1, we introduced a sequence of projections $\{P_N; N \in \mathbb{N}\}$, such that $P_N H \subset \Phi$, $N \in \mathbb{N}$. Let $x \in H$. Then we have

$$\|P_N x\|_m^2 = \sum_{0\le|\alpha|\le m} \|A^\alpha P_N x\|^2 = \sum_{0\le|\alpha|\le m} \int_{|\lambda|\le N} |\lambda^\alpha|^2 \|E(d\lambda)x\|^2$$
$$\le \left(\sum_{0\le|\alpha|\le m} N^{2|\alpha|}\right)\|P_N x\|^2 \le \left(\sum_{0\le|\alpha|\le m} N^{2|\alpha|}\right)\|x\|^2,\ m \in \mathbb{Z}_+.$$

Hence P_N is a continuous linear operator from H to Φ. Its adjoint operator P'_N : $\Phi' \to H$ is defined by $(P'_N f, x) = \langle f, P_N x\rangle$, for every $x \in H, f \in \Phi'$. Then P'_N is a continuous linear operator from Φ' to H and $P'_N f \to f$, for every $f \in \Phi'$. □

PROPOSITION 1.4. *For each $j \in \{1, 2, \ldots, n\}$, A_j is a continuous linear operator on Φ, and its adjoint operator A'_j is a continuous linear operator on Φ'. Moreover, A'_j is an extension of A_j from Φ to Φ', and it can operate on Φ' infinitely, $1 \le j \le n$.*

PROOF. Let $\phi \in \Phi$. We have $\|A_j\phi\|_m \le \|\phi\|_{m+1}$, $m \in \mathbb{Z}_+$, so $A_1, A_2, \ldots, A_n$ are continuous linear operators on Φ. The other conclusions are obvious. □

EXAMPLE 1.1. Let $H = L^2(\mathbb{R}^n)$, $A_j = -iD_j$, $D_j = \partial/\partial t_j$, $1 \le j \le n$. Then

$$\Phi = \{x(t) \mid x(t) \in C^\infty(\mathbb{R}^n),\ D^\alpha x \in L^2(\mathbb{R}^n),\ \text{for every } \alpha \in (\mathbb{Z}_+)^n\},$$
$$\Phi_m = \{x(t) \mid D^\alpha x \in L^2(\mathbb{R}^n),\ |\alpha| \le m\},$$

where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, for every $\alpha \in (\mathbb{Z}_+)^n$. We see that $\Phi = \mathscr{D}_{L^2}$, $\Phi' = \mathscr{D}'_{L^2}$, where $\mathscr{D}_{L^2}$, and $\mathscr{D}'_{L^2}$ are defined in [6].

EXAMPLE 1.2. Let $H = L^2([0, 2\pi]^n)$, $A_j = -iD_j$, $\mathscr{D}_{A_j} = \{x(t) \mid x(t), D_j x(t) \in L^2([0, 2\pi]^n), x(t)|_{t_j=2\pi} = x(t)|_{t_j=0}\}$, $1 \le j \le n$. Then $\Phi = \{x(t) \mid x(t) \in C^\infty([0, 2\pi]^n), D^\alpha x(t)|_{t_j=2\pi} = D^\alpha x(t)|_{t_j=0}, 1 \le j \le n$, for every $\alpha \in (\mathbb{Z}_+)^n\}$. We have that $\Phi = \mathscr{D}_{2\pi}(\mathbb{R}^n) = \{x(t) \mid x(t) \in C^\infty(\mathbb{R}^n), x(t + 2\pi k) = x(t)$, for every $k \in \mathbb{Z}^n\}$, where $\mathscr{D}_{2\pi}(\mathbb{R}^n)$ is defined in [1]. Because the family of semi-norms $\{\|x\|_{\alpha,2} = \|D^\alpha x\|_2, \alpha \in (\mathbb{Z}_+)^n, x \in \Phi\}$ is equivalent to the family of semi-norms $\{\|x\|_{\alpha,\infty} = \|D^\alpha x\|_\infty, \alpha \in (\mathbb{Z}_+)^n, x \in \Phi\}$, we have $\Phi' = \mathscr{D}'_{2\pi}(\mathbb{R}^n)$.

EXAMPLE 1.3. Let $H = L^2(\mathbb{R}^n)$, $A_j = 2^{-1}(t_j^2 - 1 - D_j^2)$, $1 \le j \le n$. We have

$$\Phi = \{x(t) | x(t) \in C^\infty(\mathbb{R}^n),\ \|A^\alpha x\|_2 < +\infty,\ \text{for every } \alpha \in (\mathbb{Z}_+)^n\}.$$

Because the family of semi-norms $\{\|x\|_\alpha = \|A^\alpha x\|_2, \alpha \in (\mathbb{Z}_+)^n\}$ is equivalent to the family of semi-norms $\{\|x\|_{\beta,\gamma,2} = \|t^\beta D^\gamma x\|_2, \beta, \gamma \in (\mathbb{Z}_+)^n\}$ ([4]), we have

$$\Phi = S(\mathbb{R}^n) = \{x(t) | x(t) \in C^\infty(\mathbb{R}^n), \|t^\beta D^\gamma x\|_2 < +\infty, \text{ for every } \beta, \gamma \in (\mathbb{Z}_+)^n\},$$

where $S(\mathbb{R}^n)$ is the set of all rapid descent C^∞ functions on $\mathbb{R}^n$. The topology of Φ is equivalent to the well-known topology of $S(\mathbb{R}^n)$, thus $\Phi' = S'(\mathbb{R}^n)$, the set of all slow growth generalized functions.

2. The criterion for the completeness and nuclearity of a fundamental space

How do we decide about the completeness and nuclearity of a fundamental space associated with self-adjoint operators? In this section we give a complete answer to this problem.

The completeness and nuclearity of a countable Hilbert space are defined in [2]. Now let Hilbert space H, commutative self-adjoint operators $A_1, A_2, \ldots, A_n$ and the associated basic space Φ be as in Section 1.

LEMMA 2.1. *$U_m = \left(\sum_{0\le|\alpha|\le m} A^{2\alpha}\right)^{1/2}$ is a unitary operator from Hilbert space Φ_m onto H, $m \in \mathbb{Z}_+$.*

PROOF. In Proposition 1.1, we have defined the spectral measure $E(d\lambda) = E^{(1)}(d\lambda_1) E^{(2)}(d\lambda_2) \cdots E^{(n)}(d\lambda_n)$ on $\mathbb{R}^n$. Now we can set up the functional calculus for the spectral measure as follows. Suppose $f(\lambda)$ is a complex Borel measurable function on $\mathbb{R}^n$. We define a linear operator $T_f = \int_{\mathbb{R}^n} f(\lambda) E(d\lambda)$ on H as follows

$$(T_f x, y) = \int_{\mathbb{R}^n} f(\lambda)(E(d\lambda)x, y), \quad \text{for every } x \in \mathscr{D}_{T_f}, y \in H,$$

$$\mathscr{D}_{T_f} = \left\{x \mid x \in H, \int_{\mathbb{R}^n} |f(\lambda)|^2 \|E(d\lambda)x\|^2 < +\infty\right\},$$

and we have $\|T_f x\|^2 = \int_{\mathbb{R}^n} |f(\lambda)|^2 \|E(d\lambda)x\|^2$, $x \in \mathscr{D}_{T_f}$ (see [5] about the functional calculus).

Let $p_m(\lambda) = \left(\sum_{0\le|\lambda|\le m} \lambda^{2\alpha}\right)^{1/2}$, where $2\alpha = (2\alpha_1, 2\alpha_2, \cdots, 2\alpha_n)$. Obviously $U_m = T_{p_m}$, $\mathscr{D}_{U_m} = \bigcap_{0\le|\alpha|\le m} \mathscr{D}_{A^\alpha} = \Phi_m$. Since U_m is a self-adjoint operator on H, and $\|U_m x\|^2 = \|x\|_m^2 \ge \|x\|^2$, so zero is a regular point of U_m, and U_m^{-1} is defined on the whole H. Therefore, U_m is an operator from Φ_m onto H and U_m is unitary. □

LEMMA 2.2. *Suppose that the imbedding operator from Φ_{m+k} to Φ_m is denoted by I_m^{m+k}, $m \ge 0$, $k \ge 1$. Then we have the equalities of operators on H, that is*

$$U_{m+k}|I_m^{m+k}|U_{m+k}^{-1} = U_m U_{m+k}^{-1}, \quad \textit{for every } m \ge 0,\ k \ge 1,$$

where $|I_m^{m+k}| = [(I_m^{m+k})^* I_m^{m+k}]^{1/2}$ is a non-negative self-adjoint operator on Φ_{m+k}.

https://doi.org/10.1017/S1446788700001397 Published online by Cambridge University Press

PROOF. Because p_{m+k}^{-1} is a bounded continuous function on $\mathbb{R}^n$, we have

$$T_{p_m p_{m+k}^{-1}} = T_{p_m} T_{p_{m+k}^{-1}} = U_m U_{m+k}^{-1},$$

by the functional calculus for the spectral measure $E(d\lambda)$. Since $p_m p_{m+k}^{-1}$ is also bounded, it follows that $U_m U_{m+k}^{-1}$ is a bounded self-adjoint operator on H. For any $x, y \in H$, we have

$$\begin{aligned}\left(U_{m+k}\left(I_m^{m+k}\right)^* I_m^{m+k} U_{m+k}^{-1}x, y\right) &= \left(\left(I_m^{m+k}\right)^* I_m^{m+k} U_{m+k}^{-1}x, U_{m+k}^{-1}y\right)_{m+k} \\ &= \left(I_m^{m+k} U_{m+k}^{-1}x, I_m^{m+k} U_{m+k}^{-1}y\right)_m = \left(U_{m+k}^{-1}x, U_{m+k}^{-1}y\right)_m \\ &= \left(U_m U_{m+k}^{-1}x, U_m U_{m+k}^{-1}y\right) = \left(\left(U_m U_{m+k}^{-1}\right)^2 x, y\right).\end{aligned}$$

Since the square root of a non-negative self-adjoint operator is unique, it follows that

$$U_{m+k}\left|I_m^{m+k}\right| U_{m+k}^{-1} = U_m U_{m+k}^{-1}. \qquad \square$$

LEMMA 2.3. *Φ is a complete space (or nuclear space) if and only if there exists some positive integer k, such that U_k^{-1} is a compact operator (or nuclear operator) on H.*

PROOF. First we show necessary condition. Suppose Φ is complete (or nuclear). Then there exists some positive integer k_m for each $m \geq 0$, such that $I_m^{m+k_m}$ is a compact (or nuclear) operator. In particular, $I_0^{k_0}$ is a compact (or nuclear) operator from Φ_{k_0} to H. Therefore, $|I_0^{k_0}|$ is a compact (or nuclear) operator on Φ_{k_0}. By Lemma 2.2, it follows that $U_{k_0}^{-1}$ is a compact (or nuclear) operator on H.

Next we show sufficiency condition. Suppose that U_k^{-1} is a compact (or nuclear) operator on H, where k is some positive integer. Because $p_m p_{m+k}^{-1} = p_k^{-1} p_k p_m p_{m+k}^{-1}$, and $p_k p_m p_{m+k}^{-1}$ is a bounded continuous function for each $m \in \mathbb{Z}_+$, then $U_m U_{m+k}^{-1} = U_k^{-1} T_{p_k p_m p_{m+k}^{-1}}$, and $T_{p_k p_m p_{m+k}^{-1}}$ is a bounded linear operator on H by the functional calculus for the spectral measure $E(d\lambda)$. Thus $U_m U_{m+k}^{-1}$ is a compact (or nuclear) operator on H.

From Lemma 2.2, it follows that $|I_m^{m+k}|$ is a compact (or nuclear) operator on Φ_{m+k}, and I_m^{m+k} is a compact (or nuclear) operator from Φ_{m+k} to Φ_m for every $m \in \mathbb{Z}_+$. So Φ is a complete (or nuclear) space. $\square$

THEOREM 2.1. *Φ is a complete (or nuclear) space if and only if there exists some positive integer k, such that $(I + R)^{-k}$ is a compact (or nuclear) operator on H, where $R = \sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}$.*

PROOF. Note that $(I + R)^{-k} = T_{q_k^{-1}}$, where $q_k(\lambda) = (1 + r(\lambda))^k$, and $r(\lambda) = \sqrt{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2}$. Since $p_k^{-1} = q_k^{-1} q_k p_k^{-1}$, $q_k^{-1} = p_k^{-1} p_k q_k^{-1}$, if both $q_k p_k^{-1}$ and

https://doi.org/10.1017/S1446788700001397 Published online by Cambridge University Press

$p_k q_k^{-1}$ are bounded continuous functions on $\mathbb{R}^n$, then $T_{p_k^{-1}}$ and $T_{q_k^{-1}}$ are compact (or nuclear) simultaneously.

Since

$$\max_{1\le i\le l}|a_i| \le \left(\sum_{i=1}^{l} a_i^2\right)^{1/2} \le |a_1|+|a_2|+\cdots+|a_l| \le l\left(\sum_{i=1}^{l} a_i^2\right)^{1/2},$$

we have that

$$\begin{aligned} q_k(\lambda) &\le (1+|\lambda_1|+|\lambda_2|+\cdots+|\lambda_n|)^k = \sum_{0\le|\alpha|\le k}\frac{k!}{(k-|\alpha|)!\alpha_1!\alpha_2!\cdots\alpha_n!}|\lambda^\alpha| \\ &\le \left(\max_{0\le|\alpha|\le k}\frac{k!}{(k-|\alpha|)!\alpha_1!\alpha_2!\cdots\alpha_n!}\right)\sum_{0\le|\alpha|\le k}|\lambda^\alpha| \\ &\le \left(\max_{0\le|\alpha|\le k}\frac{k!}{(k-|\alpha|)!\alpha_1!\alpha_2!\cdots\alpha_n!}\right) d_k\left(\sum_{0\le|\alpha|\le k}|\lambda^{2\alpha}|\right)^{1/2}, \end{aligned}$$

where d_k is the number of elements of the set $\{\alpha \mid \alpha\in(\mathbb{Z}_+)^n, 0\le|\alpha|\le k\}$. So $q_k p_k^{-1}$ is a bounded continuous function on $\mathbb{R}^n$. Moreover,

$$\begin{aligned} p_k(\lambda) &\le \sum_{0\le|\alpha|\le k}|\lambda^\alpha| \le \sum_{0\le|\alpha|\le k}\frac{k!}{(k-|\alpha|)!\alpha_1!\alpha_2!\cdots\alpha_n!}|\lambda^\alpha| \\ &= (1+|\lambda_1|+|\lambda_2|+\cdots+|\lambda_n|)^k \le (1+nr(\lambda))^k. \end{aligned}$$

Thus

$$p_k(\lambda)q_k^{-1}(\lambda) \le \left(\frac{1+nr(\lambda)}{1+r(\lambda)}\right)^k \xrightarrow{r\to+\infty} n^k,$$

and $p_k q_k^{-1}$ is a bounded continuous function on $\mathbb{R}^n$. Use Lemma 2.3 to finish the proof. □

DEFINITION 2.1. R is called the *absolute value operator* of the commutative self-adjoint operators $\{A_j \mid 1\le j\le n\}$.

DEFINITION 2.2. Let B be a self-adjoint operator on Hilbert space H, $\sigma(B)$ be the spectrum of B, $P_\sigma(B)$ be the point spectrum of B. Suppose that $\sigma(B) = P_\sigma(B) = \{\lambda_m\}$, $|\lambda_m| \uparrow +\infty$, and also the multiplicity of each eigenvalue in $P_\sigma(B)$ is finite and is exactly the number of times the eigenvalue is repeated in the sequence $\{\lambda_m\}$. Then we say that B *has spectral property C*. In addition, if there exists some positive integer k, such that $\sum_{\lambda_m\ne0}|\lambda_m|^{-k} < +\infty$, then we say that B *has spectral property N*.

THEOREM 2.2. *Φ is a complete space (or nuclear space) if and only if the absolute value operator R has spectral property C (or N).*

PROOF. From Theorem 2.1, it suffices to show the compactness (or nuclearity) of $(I+R)^{-k}$. Since $(I+R)^{-1}$ is a bounded self-adjoint operator on H, it follows that $(I+R)^{-k} (k \geq 1)$ is compact if and only if $(I+R)^{-1}$ is compact. By the spectral decomposition of a self-adjoint operator, it is clear that the compactness of $(I+R)^{-1}$ means that R has spectral property C. Furthermore, the nuclearity of $(I+R)^{-k}$ means that R has spectral property N. □

THEOREM 2.3. *Suppose $\mathscr{H}$ is a Hilbert space, A is a self-adjoint operator on $\mathscr{H}$. Let $\mathscr{H}_1 = \mathscr{H}_2 = \cdots = \mathscr{H}_n = \mathscr{H}$, $H = \otimes_{j=1}^{n} \mathscr{H}_j$, where $\otimes$ denotes the tensor product. For each $1 \leq j \leq n$ let $A_j = I \otimes I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I$, where A is on the jth position. Then the fundamental space Φ associated with $\{A_j \mid 1 \leq j \leq n\}$ is complete (or nuclear) if and only if A has spectral property C (or N).*

PROOF. By Theorem 2.2, we want to prove that R has spectral property C (or N) if and only if A has spectral property C (or N). Without loss of generality we assume that $n = 2$.

We show sufficiency first. Suppose that A has spectral property C, $\sigma(A) = P_\sigma(A) = \{\lambda_m\}$ is as in Definition 2.2. From the spectral decomposition of A, we can take e_m as an eigenvector corresponding to an eigenvalue λ_m, such that $\{e_m; m \in \mathbb{N}\}$ is an orthonormal basis of $\mathscr{H}$. Then $\{e_m \otimes e_l; m, l \in \mathbb{N}\}$ is an orthonormal basis of $H = \mathscr{H} \otimes \mathscr{H}$.

In addition, $R^2 = A^2 \otimes I + I \otimes A^2$ is a diagonal operator: $R^2(e_m \otimes e_l) = (\lambda_m^2 + \lambda_l^2)(e_m \otimes e_l)$, $m, l \in \mathbb{N}$. It is obvious that $\sigma(R^2) = P_\sigma(R^2) = \{\lambda_m^2 + \lambda_l^2;\ m, l \in \mathbb{N}\}$ has unique cluster point ∞. Therefore, $\sigma(R) = \left\{\sqrt{\lambda_m^2 + \lambda_l^2};\ m, l \in \mathbb{N}\right\}$, and R has spectral property C. Suppose that A has spectral property N, that is $\sum |\lambda_m|^{-k} < +\infty$ for some positive integer k. Since

$$\sum \left(\sqrt{\lambda_m^2 + \lambda_l^2}\right)^{-2k} \leq \sum \left(2|\lambda_m||\lambda_l|\right)^{-k} = 2^{-k}\left(\sum |\lambda_m|^{-k}\right)^2,$$

it follows that R has spectral property N.

Next we show necessity. If R has spectral property C (or N), then R^2 has spectral property C (or N). Suppose $\sigma(R^2) = \{\gamma_m\}$ is as in Definition 2.2, and $0 \leq \gamma_m \uparrow +\infty$. Because $B_1 = A^2 \otimes I$, $B_2 = I \otimes A^2$ and R^2 are commutative, so in each finite dimensional eigensubspace of R^2, we can find an orthogonal basis, such that B_1, B_2 are both diagonal in this subspace. Then there exists an orthonormal basis $\{u_m|_{m=1}^{+\infty}\}$ of $\mathscr{H} \otimes \mathscr{H}$, such that $R^2 u_m = \gamma_m u_m$, $B_1 u_m = \alpha_m u_m$, $B_2 u_m = \beta_m u_m$, $\gamma_m = \alpha_m + \beta_m$, $m \in \mathbb{N}$. Since B_1, B_2 are both diagonal in the basis $\{u_m\}$, it follows that $P_\sigma(B_1) = \{\alpha_m|_{m=1}^{+\infty}\}$, $P_\sigma(B_2) = \{\beta_m|_{m=1}^{+\infty}\}$ and $\sigma(B_1) = \overline{P_\sigma(B_1)}$, $\sigma(B_2) = \overline{P_\sigma(B_2)}$ (see [3]). Let $A^2 = \int_0^{+\infty} \mu F(d\mu)$ be the spectral decomposition of A^2 on $\mathscr{H}$. Then $B_1 = \int_0^{+\infty} \mu(F(d\mu) \otimes I)$, $B_2 = \int_0^{+\infty} \mu(I \otimes F(d\mu))$ on $\mathscr{H} \otimes \mathscr{H}$. Therefore, $\sigma(B_1) = \sigma(A^2) = \sigma(B_2)$, $P_\sigma(B_1) = P_\sigma(A^2) = P_\sigma(B_2)$, and $\sigma(A^2) = \overline{P_\sigma(A^2)}$, $P_\sigma(A^2) = \{\alpha_m|_{m=1}^{+\infty}\} = \{\beta_m|_{m=1}^{+\infty}\}$.

If α is an eigenvalue of A^2, then there exists $e \in \mathscr{H}$, $e \neq 0$, such that $A^2 e = \alpha e$. Clearly, $R^2(e \otimes e) = 2\alpha(e \otimes e)$ and $2\alpha \in P_\sigma(R^2)$. Since $2P_\sigma(A^2) \subset P_\sigma(R^2)$, we have that ∞ is the unique cluster point of $P_\sigma(A^2)$, and $\sigma(A^2) = P_\sigma(A^2)$. If α is an eigenvalue with infinite multiplicity of A^2, then 2α is an eigenvalue with infinite multiplicity of R^2. This contradicts the spectral property C of R^2. Hence the multiplicity of each eigenvalue of A^2 is finite, so A^2 has spectral property C. It is immediate that A has spectral property C. If R^2 has spectral property N, then there exists some positive integer k, such that $\sum |\gamma_m|^{-k} < +\infty$. By $\{2\alpha_m|_{m=1}^{+\infty}\} \subset \{\gamma_m|_{m=1}^{+\infty}\}$, we have $\sum |\alpha_m|^{-k} < +\infty$. Then A^2 has spectral property N, and so does A. □

Now we use Theorem 2.2 to analyse the three examples from Section 1.

In Example 1.1, we take $\mathscr{H} = L^2(\mathbb{R}^1)$, $A = -iD_t$, $D_t = d/dt$. It is known that $\sigma(A) = C_\sigma(A) = \mathbb{R}^1$, so $\Phi = \mathscr{D}_{L^2}$ is not a complete space.

In Example 1.2, we take $\mathscr{H} = L^2[0, 2\pi]$, $A = -iD_t$, $\mathscr{D}_A = \{x(t) \mid x, x' \in L^2[0, 2\pi], x(0) = x(2\pi)\}$. It is known that $\sigma(A) = \mathbb{Z}$, $Ae^{ikt} = ke^{ikt}$, $k \in \mathbb{Z}$, and $\dim\ker(A - k) = 1$. Since $\sum_{k\in\mathbb{Z}, k\neq 0} k^{-2} < +\infty$, we have that $\Phi = \mathscr{D}_{2\pi}(R^n)$ is a nuclear space.

In Example 1.3, we take $\mathscr{H} = L^2(\mathbb{R}^1)$, $A = 2^{-1}(t^2 - 1 - D_t^2)$. It is known that $\sigma(A) = \mathbb{Z}_+$, $A\phi_k = k\phi_k$, $k \in \mathbb{Z}_+$, $\phi_k(t) = (2^k k! \sqrt{\pi})^{-1/2} e^{-t^2/2} H_k(t)$, where $H_k(t) = (-1)^k e^{t^2} D_t^k e^{-t^2}$ is a Hermite polynomial. Since $\dim\ker(A - k) = 1$, $\sum_{k=1}^{+\infty} k^{-2} < +\infty$, so $\Phi = S(\mathbb{R}^n)$ is a nuclear space.

Below we will give a sufficient condition for the completeness and nuclearity of a fundamental space.

THEOREM 2.4. *If some A_j $(1 \leq j \leq n)$ has spectral property C (or N), then Φ is a complete (or nuclear) space.*

PROOF. Without loss of generality suppose A_1 has spectral property C (or N). This is equivalent to the fact that $(1 + |A_1|)^{-k}$ is compact (or nuclear), where k is some positive integer, and $|A_1| = \sqrt{A_1^2}$. Obviously, $(1 + |A_1|)^{-k} = T_s$, where $s(\lambda) = (1 + |\lambda_1|)^{-k}$. Since $q_k^{-1}(\lambda) = q_k^{-1}s^{-1}s$, it follows, by the functional calculus, that $(I + R)^{-k} = T_{q_k^{-1}} = T_{q_k^{-1}s^{-1}}(1 + |A_1|)^{-k}$. Since

$$q_k^{-1}s^{-1} = \left(\frac{1 + |\lambda_1|}{1 + r(\lambda)}\right)^k \leq 1,$$

https://doi.org/10.1017/S1446788700001397 Published online by Cambridge University Press

we have that $T_{q_k^{-1}s^{-1}}$ is a bounded linear operator, and $(I+R)^{-k}$ is a compact (or nuclear) operator on H. We apply Theorem 2.1 to finish the proof. □

3. Construction of generalized functions and their expansion in series

We continue to study the absolute value operator R. Since R is a self-adjoint operator on H, we have $\mathscr{D}_R = \{x \mid x \in H, \int_{\mathbb{R}^n} r^2(\lambda)\|E(d\lambda)x\|^2 < +\infty\}$. Noticing $p_1^2(\lambda) = 1 + r^2(\lambda)$, $T_{p_1} = U_1$, we get

$$\mathscr{D}_R = \left\{x \mid x \in H, \int_{\mathbb{R}^n} p_1^2(\lambda)\|E(d\lambda)x\|^2 < +\infty\right\} = \mathscr{D}_{U_1} = \Phi_1.$$

On the other hand, we have

$$\begin{aligned}\|Rx\|_m^2 = \|U_m Rx\|^2 &= \int_{\mathbb{R}^n} |p_m(\lambda)r(\lambda)|^2\|E(d\lambda)x\|^2 \\ &\le \int_{\mathbb{R}^n} |p_{m+1}(\lambda)|^2\|E(d\lambda)x\|^2 = \|x\|_{m+1}^2,\end{aligned}$$

for every $x \in \Phi_{m+1}$, so R is a bounded linear operator from Hilbert space Φ_{m+1} to Φ_m, $m \in \mathbb{Z}_+$. Then its adjoint operator R' is a bounded linear operator from Φ'_m to Φ'_{m+1}. Furthermore, R is also a continuous linear operator from Φ to Φ, and R' is a continuous linear operator from Φ' to Φ'. Moreover, R' is an extension of R from Φ to Φ', and there is an operator equality $R'^2 = A_1'^2 + A_2'^2 + \cdots + A_n'^2$.

THEOREM 3.1. *For any $f \in \Phi'$, there exists a unique element $z \in H$, such that $f = (I+R'^k)z = \lim_{N\to+\infty}(I+R^k)P_N z$, where k is some positive integer, P_N $(N \in \mathbb{N})$ is defined in the proof of Proposition* 1.1, *and the limit is taken for the weak $*$ topology of Φ'.*

PROOF. For each $m \in \mathbb{Z}_+$, $I + R^m$ is a self-adjoint operator on H, and zero is its regular point. In addition, $\mathscr{D}_{I+R^m} = \Phi_m$, so $I + R^m$ is a one-to-one bounded linear operator from Hilbert space Φ_m onto Hilbert space H. Therefore, $I + R'^m$ is a one-to-one bounded linear operator from H onto Φ'_m.

For each $f \in \Phi'$, there exists some positive integer k, such that $f \in \Phi'_k$. Then there exists a unique $z \in H$, such that $f = (I + R'^k)z$. Since $z = \lim_{N\to+\infty} P_N z$, the limit being in H, it follows that the limit equality is also true in Φ'. Hence $f = \lim_{N\to+\infty}(I + R^k)P_N z$. □

REMARK. If $H = L^2(R^n)$, and $A_1, A_2, \ldots, A_n$ are partial differential operators, then Theorem 3.1 shows that every generalized function of Φ' is a finite sum of some partial derivatives of some L^2 function.

THEOREM 3.2. *Suppose Φ is a complete space, $\sigma(R) = P_\sigma(R) = \{\lambda_m\}$ is as in Definition 2.2, and $0 \leq \lambda_m \uparrow +\infty$. Then there exists an orthonormal basis $\{\phi_m;\ m \in \mathbb{N}\}$ in H, such that $R\phi_m = \lambda_m\phi_m$, $A_j\phi_m = \mu_{jm}\phi_m$, $1 \leq j \leq n$, $\sum_{j=1}^n \mu_{jm}^2 = \lambda_m^2$, $m \in \mathbb{N}$, and*

$$\Phi = \left\{\phi \mid \phi = \sum_{m=1}^{+\infty} a_m\phi_m,\ a_m = (\phi, \phi_m),\ \{a_m\} \text{ is a } \{\lambda_m\}\text{-rapid descent sequence, the series is convergent in } \Phi\right\},$$

$$\Phi' = \left\{f \mid f = \sum_{m=1}^{+\infty} c_m\phi_m,\ c_m = \langle f, \phi_m\rangle,\ \{c_m\} \text{ is a } \{\lambda_m\}\text{-slow growth sequence, the series is weakly } * \text{ convergent in } \Phi'\right\},$$

where $\{\lambda_m\}$-rapid descent sequence means that $\{\lambda_m^k a_m\} \in l^2$, for every $k \in \mathbb{Z}_+$, and $\{\lambda_m\}$-slow growth sequence means that $\{c_m(1+\lambda_m^k)^{-1}\} \in l^2$ for some $k \in \mathbb{Z}_+$. Moreover, $A_j' f = \sum_{m=1}^{+\infty} c_m\mu_{jm}\phi_m$, for every $f \in \Phi'$, $1 \leq j \leq n$.

PROOF. Because $A_1, A_2, \dots, A_n$, and R are commutative, so in every finite dimensional eigensubspace of R, we can find an orthonormal basis, such that $A_1, A_2, \dots, A_n$ are all diagonal. By the spectral property C of R, there exists an orthonormal basis $\{\phi_m|_{m=1}^{+\infty}\}$ in H, such that $R\phi_m = \lambda_m\phi_m$, $A_j\phi_m = \mu_{jm}\phi_m$, $1 \leq j \leq n$, $\sum_{j=1}^n \mu_{jm}^2 = \lambda_m^2$, $m \in \mathbb{N}$.

Since $\Phi_k = \mathscr{D}_{R^k}$, it follows that $\phi = \sum_{m=1}^{+\infty} a_m\phi_m \in \Phi_k$ if and only if $\{\lambda_m^k a_m|_{m=1}^{+\infty}\} \in l^2$. Thus $\phi = \sum_{m=1}^{+\infty} a_m\phi_m \in \Phi$ if and only if $\{\lambda_m^k a_m|_{m=1}^{+\infty}\} \in l^2$, for every $k \in \mathbb{Z}_+$. This means that $\{a_m\}$ is a $\{\lambda_m\}$-rapid descent sequence. If $\phi = \sum_{m=1}^{+\infty} a_m\phi_m$ in H, $\phi \in \Phi$, then for each $k \in \mathbb{Z}_+$, we have $\|R^k\sum_1^N a_m\phi_m - R^k\phi\| \xrightarrow{N} 0$ in H by the closeness of R^k. Then $\|(I+R^k)(\sum_1^N a_m\phi_m - \phi)\| \xrightarrow{N} 0$, in H. It is clear that $U_k(I+R^k)^{-1}$ is a bounded linear operator on H, so $\|U_k(\sum_1^N a_m\phi_m - \phi)\| \xrightarrow{N} 0$ in H, that is $\sum_1^N a_m\phi_m \xrightarrow{N} \phi$ in Φ.

If $f \in \Phi'$, then there exist a unique $z \in H$ and some positive integer k, such that $f = (I + R'^k)z = \lim_{N\to+\infty}(I+R^k)P_N z$ by Theorem 3.1. Let $z = \sum_1^{+\infty} b_m\phi_m$, $\{b_m\} \in l^2$, and $P_N Z = \sum_1^N b_m\phi_m$. Then $(I+R^k)P_N Z = \sum_1^N(1+\lambda_m^k)b_m\phi_m$. Let $c_m = (1+\lambda_m^k)b_m$. We have $f = \lim_{N\to+\infty}\sum_1^N c_m\phi_m$, here the limit being taken for the weak $*$ topology of Φ'. We write $f = \sum_1^{+\infty} c_m\phi_m$, which means $\langle f, \phi\rangle = \lim_{N\to+\infty}\sum_1^N c_m\overline{(\phi_m, \phi)}$, for every $\phi \in \Phi$. Because $\{a_m = (\phi_m, \phi)\}$ is a $\{\lambda_m\}$-rapid descent sequence, and $\{c_m(1+\lambda_m^k)^{-1}\} \in l^2$, thus $\langle f, \phi\rangle = \sum_1^{+\infty} c_m\overline{a}_m$ is an absolutely convergent series. Moreover, we see that $c_m = \langle f, \phi_m\rangle$ and $\{c_m\}$ is a $\{\lambda_m\}$-slow growth sequence. Conversely, such a series $\sum_{m=1}^{+\infty} c_m\phi_m$ always converges to an element of Φ'. □

EXAMPLE 3.1. In Example 1.2, $H = L^2([0, 2\pi]^n)$ has an orthonormal basis

$$\left\{(2\pi)^{-n/2}e^{ikt};\ k \in \mathbb{Z}^n,\ kt = \sum_{j=1}^{n} k_j t_j\right\},$$

such that $A_j e^{ikt} = k_j e^{ikt}$, $Re^{ikt} = r(k)e^{ikt}$, $r(k) = \sqrt{k_1^2 + k_2^2 + \cdots + k_n^2}$. Therefore,

$$\mathscr{D}_{2\pi} = \left\{\phi(t) \mid \phi(t) = \sum_{k\in\mathbb{Z}^n} a_k e^{ikt},\ \{a_k\} \text{ is an } \{r(k); k \in \mathbb{Z}^n\}\text{-rapid descent sequence}\right\},$$

$$\mathscr{D}'_{2\pi} = \left\{f(t) \mid f(t) = \sum_{k\in\mathbb{Z}^n} c_k e^{ik\cdot t},\ \{c_k\} \text{ is an } \{r(k); k \in \mathbb{Z}^n\}\text{-slow growth sequence}\right\}.$$

This result agrees with that in [1].

In Example 1.3, $H = L^2(\mathbb{R}^n)$ has an orthonormal basis $\{\psi_k(t) = \Pi_{j=1}^n \phi_{k_j}(t_j);\ k \in (\mathbb{Z}_+)^n\}$, where $\phi_k(t)$ satisfies the following equation

$$\frac{1}{2}\left(t^2 - 1 - \frac{d^2}{dt^2}\right)\phi_k(t) = k\phi_k(t), \quad k \in \mathbb{Z}_+,\ t \in \mathbb{R}^1,$$

and $\phi_k(t) = (2^k k!\sqrt{\pi})^{-1/2} e^{-t^2/2} H_k(t)$, here $H_k(t) = (-1)^k e^{t^2} D_t^k e^{-t^2}$ being a Hermite polynomial. Then we have $A_j\psi_k = k_j\psi_k$, $R\psi_k = r(k)\psi_k$, $k \in (\mathbb{Z}_+)^n$. So $S(\mathbb{R}^n)$ is equivalent to the set of all $\{r(k);\ k \in (\mathbb{Z}_+)^n\}$-rapid descent sequences. $S'(\mathbb{R}^n)$ is equivalent to the set of all $\{r(k);\ k \in (\mathbb{Z}_+)^n\}$-slow growth sequences. This result agrees with that in [4].

References

[1] Chou Chin-Cheng, *Séries de Fourier et théorie des distributions* (Éditions Scientifiques, Beijing, 1983).

[2] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions. IV. Applications of harmonic analysis* (Academic Press, New York, 1964).

[3] P. R. Halmos, *A Hilber space problem book*, Graduate Texts in Math. 19 (Springer, New York, 1982).

[4] M. Reed and B. Simon, *Methods of modern physics. I. Functional analysis* (Academic Press, New York, 1980).

[5] F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, 4th edition (Akad. Kiadó, Budapest, 1965).

[6] L. Schwartz, *Théorie des distributions* (Hermann, Paris, 1966).

Institute of Mathematics
Chinese Academy of Sciences
Beijing 100080
China
e-mail: libr@public2.east.net.cn