

## DIMENSIONAL GROUPS AND FIELDS

FRANK O. WAGNER

**Abstract.** We shall define a general notion of dimension, and study groups and rings whose interpretable sets carry such a dimension. In particular, we deduce chain conditions for groups, definability results for fields and domains, and show that a pseudofinite  $\mathfrak{M}_c$ -group of finite positive dimension contains a finite-by-abelian subgroup of positive dimension, and a pseudofinite group of dimension 2 contains a soluble subgroup of dimension 2.

**Introduction.** In this paper, we shall define a general notion of dimension, and study groups and rings whose interpretable sets carry such a dimension. The aim is to unify results from stability and simplicity theory,  $o$ -minimality, and the study of pseudofinite structures (with dimensions induced by Lascar or SU-rank,  $o$ -minimal dimension, or the logarithm of the counting measure, see Example 1.5). In particular, we deduce chain conditions for groups, definability results for fields and domains, and show that pseudofinite groups contain big finite-by-abelian subgroups, and pseudofinite groups of dimension 2 contain big soluble subgroups, generalizing a result of Elwes and Ryten [6].

### §1. Dimension.

**DEFINITION 1.1.** A theory  $T$  is *dimensional* if there is a dimension function  $\dim$  from the collection of all interpretable sets in models of  $T$  to an ordered abelian group  $\Gamma$  together with  $\pm\infty$ , satisfying for a formula  $\varphi(x, y)$  and interpretable sets  $X$  and  $Y$ :

- *Invariance:* If  $a \equiv a'$  then  $\dim(\varphi(x, a)) = \dim(\varphi(x, a'))$ .
- *Algebraicity:* If  $X$  is finite nonempty then  $\dim(X) = 0$ , and  $\dim(\emptyset) = -\infty$ .
- *Union:*  $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$ .
- *Fibration:* If  $f: X \rightarrow Y$  is an interpretable map such that  $\dim(f^{-1}(y)) \geq d$  for all  $y \in Y$ , then  $\dim(X) \geq \dim(Y) + d$ .

A dimension is *real* (or *archimedean*) if  $\Gamma \leq \mathbb{R}$ , *discrete* if  $\Gamma$  is discrete, and *integer* if it is discrete and real (thus  $\Gamma \cong \mathbb{Z}$ ). Note that in a real or integer-dimensional theory sets of dimension  $\infty$  are still allowed; a set (or a structure)  $X$  is *finite-dimensional* if the dimension is integer and  $\dim(X) < \infty$ .

---

Received September 17, 2018.

2020 *Mathematics Subject Classification.* 03C45, 03C20, 03C60, 12L12, 20F24.

*Key words and phrases.* dimension, chain condition, group, domain, skew field, definability, pseudofinite group of small dimension.

© 2020, Association for Symbolic Logic

0022-4812/20/8503-0003

DOI:10.1017/jsl.2020.48

For a partial type  $\pi$  put  $\dim(\pi) = \inf\{\dim(\varphi) : \pi \vdash \varphi\}$ , where the infimum is considered in a completion  $\bar{\Gamma}$  of the ordered set  $\Gamma \cup \{\infty\}$ . Note that unless the dimension is real, there will be nonrealized cuts in  $\Gamma$  (for instance the sup of a proper convex subgroup), and the extension of the group-operation by continuity is only well-defined on the semigroup  $\bar{\Gamma}^+ = \{\gamma \in \bar{\Gamma} : \gamma \geq 0\}$ . We write  $\dim(a/B)$  for  $\dim(\text{tp}(a/B))$ , and  $\dim(a)$  for  $\dim(a/\emptyset)$ . Then for types dimension is invariant, algebraic and satisfies union, but need not satisfy fibration. By the union condition any partial type can be completed to a type of the same dimension. In particular, for any nonempty type-definable  $X$  and any set  $A$  of parameters over which  $X$  is type-definable, there is  $b \in X$  with  $\dim(b/A) = \dim(X)$ .

REMARK 1.2. There are some additional axioms and variants one might also consider:

- *Finesse*: If  $\dim(X) = 0$  then  $X$  is finite.
- *Product*:  $\dim(X \times Y) = \dim(X) + \dim(Y)$ .
- *Strong fibration*: If  $f : X \rightarrow Y$  is a interpretable map such that  $\dim(f^{-1}(y)) = d$  for all  $y \in Y$ , then  $\dim(X) = \dim(Y) + d$ .
- *Weak fibration*: If  $f : X \rightarrow Y$  is a interpretable map such that  $\dim(f^{-1}(y)) = d$  for all  $y \in Y$ , then  $\dim(X) \geq \dim(Y) + d$ .
- *Lower fibration*: If  $f : X \rightarrow Y$  is a interpretable map such that  $\dim(f^{-1}(y)) \leq d$  for all  $y \in Y$ , then  $\dim(X) \leq \dim(Y) + d$ .
- *Definability*: If  $\dim(\varphi(x, a)) = d$  then there is a formula  $\theta_{\varphi, d} \in \text{tp}(a)$  such that  $\dim(\varphi(x, a')) = d$  for all  $a' \models \theta_{\varphi, d}$ .
- *Semidefinability*: If  $\dim(\varphi(x, a)) > d$  then there is a formula  $\theta_{\varphi, d} \in \text{tp}(a)$  such that  $\dim(\varphi(x, a')) > d$  for all  $a' \models \theta_{\varphi, d}$ .
- *Additivity*:  $\dim(a, b/A) = \dim(a/A, b) + \dim(b/A)$ .
- *Semiadditivity*:  $\dim(a, b/A) \geq \dim(a/A, b) + \dim(b/A)$ .

We call a dimension *coarse* if it is not fine, that is, does not satisfy finesse.

Clearly both fibration and strong fibration imply weak fibration, strong fibration implies product, fibration and lower fibration imply strong fibration, any kind of definability implies invariance, definability plus strong fibration imply additivity, and semidefinability plus fibration imply semiadditivity.

REMARK 1.3.

- Invariance is equivalent to type-definability, that is, definability where  $\theta_{\varphi, d}$  is a partial type (for instance  $\text{tp}(a)$  itself will do).
- We have chosen weak inequalities for (weak) fibration, as this behaves better under limits. On the other hand, we took strict inequalities for semidefinability, as this seems easier to achieve in examples.
- One might also define lower semidefinability where the inequalities are reversed. This seems less useful, though.
- Note that fibration yields the inequality  $\dim(X \times Y) \geq \dim(X) + \dim(Y)$ .
- If definability holds, then by compactness  $\dim$  only takes finitely many values in any uniformly definable family of sets.

REMARK 1.4. Additivity implies fibration and lower fibration, whence strong fibration.

PROOF. Let  $f : X \rightarrow Y$  be a definable map; adding parameters to the language we may assume that  $X, Y,$  and  $f$  are  $\emptyset$ -definable. Note that  $\dim(f(x)/x) = 0$  for all  $x \in X,$  whence

$$\dim(x) = \dim(x) + \dim(f(x)/x) = \dim(x, f(x)) = \dim(x/f(x)) + \dim(f(x)).$$

Suppose first that  $\dim(f^{-1}(y)) \geq d$  for all  $y \in Y.$  Take  $b \in Y$  with  $\dim(b) = \dim(Y),$  and choose  $a \in f^{-1}(b)$  with  $\dim(a/b) \geq d.$  Then

$$\dim(X) \geq \dim(a) = \dim(a/b) + \dim(b) \geq d + \dim(Y).$$

Suppose now that  $\dim(f^{-1}(y)) \leq d$  for all  $y \in Y.$  Take  $a \in X$  with  $\dim(a) = \dim(X),$  and put  $b = f(a).$  Then

$$\dim(X) = \dim(a) = \dim(a/b) + \dim(b) \leq d + \dim(Y). \quad \dashv$$

EXAMPLE 1.5. Examples for integer dimensions with lower and strong fibration include:

1. Finite Lascar rank,  $SU$ -rank or  $U^b$ -rank on *formulas*, possibly localized at some  $\emptyset$ -invariant family of types;
2. For any ordinal  $\alpha,$  the coefficient of  $\omega^\alpha$  in one of the ordinal-valued ranks in (1.5) above (when written on Cantor normal form).

In these examples, for an interpretable set  $X$  we let  $\dim(X)$  be the maximum of the finite ranks (Lascar,  $SU$  ou  $U^b$ ) of the types extending  $x \in X,$  and  $\dim(X) = \infty$  if no such maximum exists. In particular, in the case of finite Lascar rank, the dimension of a formula is just equal to its Shelah rank. Note that in general the dimension of a type need not be equal to the rank of the type even when the dimension is finite, as witnessed by the standard example where Lascar and Shelah rank are finite and different.<sup>1</sup> Nevertheless, the Lascar inequalities for types are sufficient to show that fibration and lower fibration hold (whence strong fibration as well). Note that in example (1), not localized, the dimension is fine; if the rank is finite, so is the dimension. (See e.g. [15] for details.)

3.  $\mathcal{o}$ -minimal dimension on a densely ordered set. This dimension is finite, and satisfies all additional properties of Remark 1.2.

An example for a real additive dimension is

4. coarse pseudofinite dimension (in some expansion by cardinality comparison quantifiers), defined for a definable set  $X = \prod_i X_i/\mathcal{U}$  in a pseudofinite structure  $\prod_i \mathfrak{M}_i/\mathcal{U}$  as  $\log(\prod |X_i|/\mathcal{U}) + C \in (\prod_i \mathbb{R}/\mathcal{U})/C,$  where  $C$  is a convex additive subgroup containing the integers in the nonstandard real field  $\prod_i \mathbb{R}/\mathcal{U},$  and  $|X_i|$  is the cardinality of  $X_i,$  see [8–10]. If  $C$  is the convex hull of the integers, this is *fine* pseudofinite dimension.

Here additivity holds by [8, Lemma 2.10], and in fact all additional properties of Remark 1.2 hold except for definability and finesse. Clearly fine pseudofinite dimension is fine in the sense of Remark 1.2.

---

<sup>1</sup>A structure with disjoint unary predicates  $\{P_i : i < \omega\},$  and on every  $P_i$  an equivalence relation  $E_i$  with infinitely many infinite classes. The type  $\{\neg P_i : i < \omega\}$  has Lascar rank 1 but Shelah rank 2.

REMARK 1.6. If  $\dim$  is a  $\Gamma$ -valued dimension and  $\gamma \in \Gamma$ , put

$$\Gamma_\gamma = \{\gamma' \in \Gamma : |\gamma'| \leq n|\gamma| \text{ for some } n < \omega\}, \quad \text{and}$$

$$\Gamma_{\gamma^-} = \{\gamma' \in \Gamma : n|\gamma'| < |\gamma| \text{ for all } n < \omega\}.$$

Then  $\Gamma_{\gamma^-} < \Gamma_\gamma \leq \Gamma$  are subgroups, and there is a unique additive monomorphism  $\sigma : \Gamma_\gamma / \Gamma_{\gamma^-} \rightarrow \mathbb{R}$  with  $\sigma(\gamma) = 1$ . The function  $\dim_\gamma$  defined by

$$\dim_\gamma(X) = \begin{cases} \sigma(\dim(X)) & \text{if } \dim(X) \in \Gamma_\gamma, \\ \infty & \text{otherwise} \end{cases}$$

is a real dimension, the *localization* of  $\dim$  at  $\gamma$ . If  $\gamma = \dim(Y)$  for some definable set  $Y$ , we also write  $\dim_Y$  instead of  $\dim_\gamma$ .

LEMMA 1.7. *Just assuming weak fibration, dimension is invariant under definable bijections. Under strong fibration, dimension is invariant under finite-to-finite definable correspondences.*

PROOF. Let  $f : X \rightarrow Y$  be a definable bijection. Then  $\dim(f^{-1}(y)) = 0$  for all  $y \in Y$ , so  $\dim(X) \geq \dim(Y)$ . Considering the definable bijection  $f^{-1} : Y \rightarrow X$ , we obtain  $\dim(Y) \geq \dim(X)$ , whence equality.

Now assume strong fibration, and let  $R \subset X \times Y$  be a finite-to-finite correspondence with nonempty fibres of size at most  $n$ . For  $x \in X$  put  $Y_x = \{y \in Y : (x, y) \in R\}$ , and consider  $Z = \{Y_x : x \in X\}$ . Then the map  $x \mapsto Y_x$  from  $X$  to  $Z$  has finite fibres bounded by  $n$ , so  $\dim(X) = \dim(Z)$  by strong fibration. Next, consider  $Z' = \{(y_1, \dots, y_n) \in Y^n : \{y_1, \dots, y_n\} \in Z\}$ . Then the map  $(y_1, \dots, y_n) \mapsto \{y_1, \dots, y_n\}$  from  $Z'$  to  $Z$  has nonempty fibres of size at most  $n!$ , so  $\dim(Z') = \dim(Z)$  by strong fibration. Finally, the projection to the first coordinate from  $Z'$  to  $Y$  has nonempty fibres of size at most  $n^n$ : for any  $y_1 \in Y$  there are at most  $n$  different  $x \in X$  with  $(x, y) \in R$ , and for each  $x$  there are at most  $n^{n-1}$  choices for the coordinates  $y_2, \dots, y_n$ . Thus  $\dim(Z') = \dim(Y)$ . -1

DEFINITION 1.8. Let  $Y \subseteq X$  be type-definable sets. We say that  $X$  is *broad* if  $0 < \dim(X) < \infty$ . If  $X$  is broad, we say that  $Y$  is *X-broad* if  $\frac{1}{n} \dim(X) \leq \dim(Y) \leq n \dim(X)$  for some  $n < \omega$ .

$Y$  is *wide* in  $X$  if for every definable superset  $\tilde{Y} \supseteq Y$  there is a definable superset  $\tilde{X} \supseteq X$  with  $\dim(\tilde{Y}) \geq \dim(\tilde{X})$ ; in particular for definable  $X$  and  $Y$  we have that  $Y$  is wide in  $X$  iff  $\dim(Y) = \dim(X)$ . We say that  $Y$  is *wide* if  $Y$  is wide in its sort.

A definable set  $X$  is *negligible* if  $\dim(X) = 0$ ; a type-definable set is *negligible* if it is contained in some negligible definable set.

An element  $x$  is *wide/broad/negligible* over some parameters  $A$  if  $\text{tp}(x/A)$  is.

REMARK 1.9.

- $X$  is  $Y$ -broad iff  $Y$  is  $X$ -broad.
- If  $X$  is broad,  $Y$  is  $X$ -broad iff  $Y$  is broad for the localized dimension  $\dim_X$ .
- If the dimension is real and  $X$  is broad, then  $Y$  is broad iff it is  $X$ -broad.
- In fine dimension, a negligible type-definable set is finite.
- In fine, discrete dimension, a type-definable set of dimension 0 is finite.

**§2. Dimensional groups.**

LEMMA 2.1. *Let  $G$  be a type-definable group over  $A$  in a dimensional theory, and  $g, h \in G$ . If  $h$  is wide in  $G$  over  $A, g$ , then  $gh$  and  $hg$  are wide in  $G$  over  $A, g$ .*

PROOF. Let  $X$  be an  $A, g$ -definable set containing  $gh$ . Since  $G$  is a type-definable group, restricting  $X$  we may assume that  $x \mapsto g^{-1}x$  is a bijection between  $X$  and  $g^{-1}X$ . As dimension is invariant under definable bijections,  $\dim(X) = \dim(g^{-1}X)$ . But  $h \in g^{-1}X$ , so  $gh$  is wide over  $A, g$ . The proof for  $hg$  is similar. ⊢

In a dimensional theory, we need not have fibration for definable maps between type-definable sets. The situation is different for group homomorphisms.

LEMMA 2.2. *In a dimensional theory, let  $G$  and  $H$  be type-definable groups and  $f : G \rightarrow H$  a definable surjective homomorphism. Then  $\dim(G) \geq \dim(\ker f) + \dim(H)$ .*

PROOF. If  $\dim(G) = \infty$  this is clear. Otherwise, consider a definable  $X_0 \supseteq G$ . Reducing  $X_0$ , if necessary, we may assume by compactness that there is a definable map  $\tilde{f}$  extending  $f$  with domain  $X_0$ . Again by compactness there is a definable  $X_1 = X_1^{-1} \supseteq G$  with  $X_1^2 \subseteq X_0$ , and such that  $\tilde{f}(xx') = \tilde{f}(x)\tilde{f}(x')$  for all  $x, x' \in X_1$ . Put  $Y = \tilde{f}(X_1)$  and  $X = \tilde{f}^{-1}(Y) \subseteq \text{dom}(\tilde{f}) = X_0$ . Then for  $y \in Y$  we have

$$\tilde{f}^{-1}(y) \supseteq (\tilde{f}^{-1}(y) \cap X_1) \ker f,$$

so  $\dim(\tilde{f}^{-1}(y)) \geq \dim(\ker f)$ . Since  $H \subseteq Y$  it follows that

$$\dim(X_0) \geq \dim(X) \geq \dim(\ker f) + \dim(Y) \geq \dim(\ker f) + \dim(H).$$

Thus  $\dim(G) = \inf\{\dim(X_0) : X_0 \supseteq G \text{ definable}\} \geq \dim(\ker f) + \dim(H)$ . ⊢

We now turn to chain conditions.

PROPOSITION 2.3. *In a fine dimensional theory, let  $H$  be a definable subgroup of infinite index in a definable group  $G$  with  $\dim(G) < \infty$ . Then  $\dim(H) < \dim(G)$ .*

PROOF. Since  $G$  and  $H$  are definable, the quotient space  $G/H$  is interpretable. Thus  $\dim(G/H)$  is well-defined, and strictly positive by finesse. The map  $G \rightarrow G/H$  has fibres of dimension  $\dim(H)$ . Hence by fibration,

$$\dim(G) \geq \dim(H) + \dim(G/H) > \dim(H). \quad \text{⊢}$$

This immediately yields:

COROLLARY 2.4. *In a fine integer-dimensional theory there is no infinite descending chain  $(G_i : i < \omega)$  of definable groups of finite dimension, each of infinite index in its predecessor.*

Since we have not required the dimension to be defined on quotients by type-definable equivalence relations, Proposition 2.3 and Corollary 2.4 may fail for type-definable subgroups. This happens for instance in  $o$ -minimal theories: the additive subgroup of infinitesimals has the same dimension as the ambient nonstandard real field, but infinite, and even unbounded, index. In fact, we can have arbitrarily long

(infinite) chains of more and more infinitesimal type-definable subgroups. However, we shall show next that this does not happen for a particular kind of type-definable groups.

DEFINITION 2.5. We call a group (relatively)  $\wedge$ -definable if it is an intersection of (relatively) definable groups. A ring is (relatively)  $\wedge$ -definable if it is so as an additive group.

As we have not defined dimension on arbitrary hyperimaginaries (quotients modulo type-definable equivalence relations), we do not have a dimension on the quotient of two type-definable groups. However, for a quotient of a type-definable group  $G = \bigwedge_{i \in I} X_i$  by a relatively  $\wedge$ -definable subgroup  $H = \bigwedge_{j \in J} H_j$  (where the  $X_i$  are definable sets, the  $H_j$  are definable subgroups, and we take both to be closed under finite intersections for ease of notation), we can put

$$\dim(G/H) = \sup_{j \in J} \dim(G/H_j) = \sup_{j \in J} \inf_{i \in I} \dim(X_i/H_j) \in \bar{\Gamma}.$$

REMARK 2.6.

- In an  $\omega$ -stable theory (and in particular in a theory of finite Morley rank), definable,  $\wedge$ -definable and type-definable groups coincide.
- In a stable or supersimple theory, type-definable groups are  $\wedge$ -definable.
- In an  $o$ -minimal theory (as in any theory with descending chain condition on definable groups),  $\wedge$ -definable subgroups are definable.
- Even in the fine integer-dimensional context, the three classes need not coincide. For instance:
  - The connected component of  $\mathbb{Z}$  (in a saturated model) is a  $\wedge$ -definable group which is not definable, in a superstable theory of Lascar rank 1.
  - In a nonstandard real field the infinitesimals form a type-definable additive group which is not  $\wedge$ -definable, in an  $o$ -minimal theory.
- If  $(G_i : i < \omega)$  is a descending chain of definable groups with  $\dim(G_i/G_{i+1}) = 1$  for all  $i < \omega$ , our definition yields  $\dim(G/G) = 0$  as expected. Had we exchanged the limits, we would get  $\inf_i \sup_j \dim(G_i/G_j) = \infty$ , which is clearly wrong.

We first check that  $\wedge$ -definability behaves well with respect to relative definability.

LEMMA 2.7. Let  $G$  be a  $\wedge$ -definable group, and  $H$  a relatively definable subgroup. Then  $H$  is  $\wedge$ -definable. In fact, there is a definable group  $H^*$  with  $H = G \cap H^*$ .

PROOF. Suppose  $G = \bigwedge_{i \in I} G_i$ , where the  $G_i$  are definable groups, and  $H = G \cap X$ , where  $X$  is a definable set. Then  $x, y \in G \cap X$  implies  $x^{-1}y \in X$ . By compactness there is some finite  $I_0 \subseteq I$  such that  $x, y \in \bigcap_{i \in I_0} G_i \cap X$  implies  $x^{-1}y \in X$ . As the  $G_i$  are groups, we also have  $x^{-1}y \in \bigcap_{i \in I_0} G_i$ , and  $Y = \bigcap_{i \in I_0} G_i \cap X$  is a definable group with  $H = G \cap Y$ . ⊔

PROPOSITION 2.8. In a dimensional theory, let  $H$  be a  $\wedge$ -definable subgroup in a type-definable group  $G$ . Then  $\dim(G) \geq \dim(H) + \dim(G/H)$ . In particular, if the dimension is fine and discrete, or if the dimension is fine and  $G$  is definable, then  $\dim(H) < \dim(G)$  iff  $H$  has unbounded index in  $G$ .

PROOF. Suppose  $H = \bigwedge_i H_i$ , where each  $H_i$  is a relatively definable subgroup of  $G$  and the system  $(H_i)_i$  is closed under finite intersection. Then for every  $i$  the projection  $G \rightarrow G/H_i$  has fibres of dimension  $\dim H_i$ , whence by Lemma 2.2

$$\dim(G) \geq \dim(H_i) + \dim(G/H_i) \geq \dim(H) + \dim(G/H_i).$$

Therefore

$$\begin{aligned} \dim(G) &\geq \sup_i [\dim(H) + \dim(G/H_i)] \\ &= \dim(H) + \sup_i \dim(G/H_i) = \dim(H) + \dim(G/H). \end{aligned}$$

In fine dimension, if the dimension is discrete or  $G$  is definable, then

$$\begin{aligned} G/H \text{ is unbounded} &\Leftrightarrow G/H_i \text{ infinite for some } i \\ &\Leftrightarrow \dim(G/H_i) > 0 \text{ for some } i \Leftrightarrow \dim(G/H) > 0. \end{aligned}$$

Thus under either condition,  $\dim(G) > \dim(H)$  iff  $G/H$  is unbounded. ⊢

COROLLARY 2.9. *In a fine integer-dimensional theory there is no infinite descending chain of relatively  $\wedge$ -definable groups of finite dimension, each of unbounded index in its predecessor.*

PROOF. Let  $(G_i : i < \omega)$  be such a chain. As the dimension is finite and discrete,  $\dim(G_i) > \dim(G_{i+1})$  by Proposition 2.8. But there is no infinite descending sequence of positive integers. ⊢

REMARK 2.10. In a coarse integer-dimensional theory we can still conclude that there is no infinite descending chain of relatively  $\wedge$ -definable groups of finite dimension, with non-negligible successive quotients.

### §3. Fields and domains.

**3.1. Skew fields.** Let us note first that fields have better definability properties than groups. Our first result generalizes the well-known fact that a supersimple type-definable field is in fact definable.

PROPOSITION 3.1. *In a real-dimensional theory, a type-definable broad skew field  $K$  is definable.*

PROOF. Suppose  $K = \bigcap_{i \in I} X_i$ , where  $(X_i : i \in I)$  is a system of definable sets closed under finite intersections. As  $0 < \dim(K) < \infty$  we may also assume  $\dim(X_i) < 2 \dim(K)$  for all  $i \in I$ . By compactness, we may further suppose that there is a minimal element  $0 \in I$  such that addition and multiplication are defined, commutative, associative and distributive on  $X_0$  (but may take values outside), and for every  $i > 0$  all nonzero elements in  $X_i$  have an additive and a multiplicative inverse in  $X_i$ .

By compactness there is  $i \in I$  such that  $X_i \cdot (X_i - X_i) + (X_i - X_i) \subseteq X_0$ . But then for any  $X_j \subseteq X_i$  and  $g \in X_j$  we have  $gX_j + X_j \subseteq X_0$ . Moreover, if  $g(X_j - X_j) \cap (X_j - X_j) = \{0\}$ , then the map from  $X_j^2 \rightarrow X_0$  given by  $(x, y) \mapsto gx + y$  is injective, contradicting

$$\dim(X_j^2) \geq 2 \dim(X_j) \geq 2 \dim(K) > \dim(X_0).$$

Thus  $g \in (X_j - X_j) \cdot ((X_j - X_j) \setminus \{0\})^{-1}$ , and

$$X_i \subseteq \bigcap_{X_j \subseteq X_i} (X_j - X_j) \cdot ((X_j - X_j) \setminus \{0\})^{-1}.$$

But by compactness for every  $k \in I$  there is some  $j \in I$  such that

$$(X_j - X_j) \cdot ((X_j - X_j) \setminus \{0\})^{-1} \subseteq X_k.$$

Thus

$$K \subseteq X_i \subseteq \bigcap_j (X_j - X_j) \cdot ((X_j - X_j) \setminus \{0\})^{-1} \subseteq \bigcap_k X_k \subseteq K,$$

and  $K = X_i$  is definable. ⊣

REMARK 3.2. Note that the hypothesis of Proposition 3.1 requires that  $\dim(K)$  is bounded away from 0.

PROPOSITION 3.3. *In a real-dimensional theory, let  $K$  be a non-negligible definable skew field and  $V$  a nontrivial type-definable  $K$ -vector space with  $\dim(V) < \infty$ . Then  $\dim(V) \geq \text{lin. dim}_K(V) \dim(K)$ . In particular  $K$  and  $V$  are broad,  $\text{lin. dim}_K(V)$  is finite and  $V$  is definable.*

PROOF. Clearly  $\dim(K^n) \geq n \dim(K)$ . As dimension is preserved under definable bijection,  $\dim(K) > 0$  and  $\dim(V) < \infty$ , we get

$$\infty > \dim(V) \geq \text{lin. dim}_K V \dim(K) \geq \dim(K) > 0.$$

So  $K$  is broad,  $\text{lin. dim}_K V$  is finite, and  $V$  is definable as  $\sum_i K e_i$ , for a  $K$ -basis  $(e_i)_i$  of  $V$ . ⊣

REMARK 3.4. If the dimension satisfies *product*, then  $\dim V = \text{lin. dim}_K V \dim K$ .

PROPOSITION 3.5. *Let  $K$  be a definable broad skew field in a fine real-dimensional theory. Then  $K$  has finite dimension over its center.*

PROOF. If  $K^\times$  has finite exponent, then  $K$  satisfies a polynomial identity and has finite dimension over its center by Kaplansky’s PI-Theorem [10]. In fact, the center is a field of finite exponent, whence finite, and so is  $K$ .

Otherwise, there is an element  $a \in K$  of infinite order, and  $Z(C_K(a))$  is an infinite definable commutative subfield. But then  $K$  has finite dimension over the infinite definable subfield  $Z(C_K(a))$ , whence finite dimension over its center. ⊣

**3.2. Domains.** We now move to domains, generalizing both [11, Theorem 2.2] and [8, Exercises 3.5 and 3.6]. Recall that localizing a noncommutative domain is not always possible: not only may right and left fractions differ, it is not a priori possible to multiply say two right fractions in a consistent way.

The right/left *Ore condition* in a domain  $R$  asks that  $rR \cap r'R \neq \{0\}$  for all nonzero  $r, r' \in R$  (or  $Rr \cap Rr' \neq \{0\}$ , respectively). If it holds then there is a right (resp., left) fraction skew field; if both hold they give rise to the same skew field. A domain is a left/right *Ore domain* if it satisfies the left/right Ore condition.



**PROPOSITION 3.6.** *In a real-dimensional theory, let  $R$  be an invariant domain, and suppose that there is a constant  $d < \infty$  such that  $\dim(X) \leq d$  for all type-definable  $X \subseteq R$ , and  $\dim(X) > 0$  for some such  $X$ . Then  $R$  is right and left Ore, and its skew field  $K$  of (right or left) fractions is definable.*

**PROOF.** As  $R$  is invariant, it is a union of type-definable sets (over some set of parameters). The assumptions imply that there is some type-definable  $X \subseteq R$  with  $2\dim(X) > \dim(Y)$  for any type-definable  $Y \subseteq R$ . For any two nonzero  $r, r' \in R$  consider the map

$$\begin{aligned} X^2 &\rightarrow rX + r'X \\ (x, y) &\mapsto rx + r'y. \end{aligned}$$

Suppose it is injective. As  $rX + r'X$  is a type-definable subset of  $R$ , the choice of  $X$  implies  $\dim(rX + r'X) < 2\dim(X)$ . Hence there is a definable superset  $Y$  of  $X$  with  $\dim(rY + r'Y) < 2\dim(X)$ , and such that the map  $(x, y) \mapsto rx + r'y$  is bijective between  $Y^2$  and  $rY + r'Y$ . Then

$$2\dim(X) > \dim(rY + r'Y) = \dim(Y^2) \geq 2\dim(Y) \geq 2\dim(X)$$

(where the equality holds by Lemma 1.7 and the first weak inequality by Remark 1.2), a contradiction.

Thus there are  $(y, y') \neq (z, z')$  with  $ry + r'y' = rz + r'z'$ . Then  $r(y - z) = r'(z' - y') \neq 0$ , so  $R$  is right Ore. Similarly,  $R$  is left Ore. It follows that its field of left fractions is equal to its field of right fractions, and equal to  $(X - X)/(X - X)^\times$ , which is type-definable. Hence  $K$  is definable by Proposition 3.1. +

**PROPOSITION 3.7.** *Let  $K$  be a definable skew field in a fine dimensional theory satisfying product. Then  $K$  has no proper broad  $\wedge$ -definable (noncommutative, nonunitary) left or right Ore sub-domain  $R$  whose fraction field is  $K$ .*

**PROOF.** We may assume that  $R$  is infinite by Wedderburn's Theorem.

Suppose  $\dim(R) < \dim(K)$ . Then  $2\dim(R) < \dim(K) + \dim(R)$ , so by  $\wedge$ -definability of  $R$  we can choose a definable additive subgroup  $A$  with  $R \leq A \leq K$  such that  $2\dim(A) < \dim(K) + \dim(R)$ . Consider a definable additive group  $B$  with  $R \leq B \leq A$  and  $B + B^2 \subseteq A$ .

Consider the map  $f : (x, y) \mapsto xy^{-1}$  on  $A \times A^\times$ , put  $Y = BB^{\times-1} \supseteq K$  and  $X = f^{-1}(Y) \subseteq A \times A^\times$ . For every  $y \in Y$  we have

$$\dim(f^{-1}(y)) \geq \dim((b, b')B^\times) = \dim(B)$$

for any  $(b, b') \in B \times B^\times$  with  $y = bb'^{-1}$ . Then product and fibration for  $f$  yield

$$2\dim(A) = \dim(A^2) \geq \dim(X) \geq \dim(B) + \dim(Y) \geq \dim(R) + \dim(K) > 2\dim(A),$$

a contradiction. It follows that  $\dim(R) = \dim(B) = \dim(K)$ .

As the dimension is fine and  $K$  definable,  $B$  has finite index in  $(K, +)$  by Proposition 2.3. Hence for any  $r \in R^\times$  there are natural numbers  $m, n$  with  $n + 1 < m$  such that  $r^{-m}$  and  $r^{-n}$  lie in the same coset modulo  $B$ . Thus there is  $b \in B$  with  $r^{-m} = r^{-n} + b$ , and

$$r^{-1} = r^{m-1}r^{-m} = r^{m-1}(r^{-n} + b) = r^{m-n-1} + r^{m-1}b \in R + RB \subseteq B + B^2 \subseteq A.$$

As this also holds for any definable subgroup of  $A^*$  with  $R \leq A^* \leq A$ , we get  $r^{-1} \in R$  by  $\wedge$ -definability, whence  $R = K$ .  $\dashv$

**COROLLARY 3.8.** *In a fine real-dimensional theory, a  $\wedge$ -definable broad (noncommutative, nonunitary) domain is a definable skew field.*

**REMARK 3.9.** The infinitesimals in a nonstandard real closed field show that Proposition 3.7 and Corollary 3.8 may fail for type-definable domains in an  $\omega$ -minimal (and hence fine integer-dimensional) theory.

**COROLLARY 3.10.** *In a real-dimensional theory, let  $A$  be a type-definable abelian group with  $\dim(A) < \infty$ . Suppose that there is an invariant set  $X$  generating a domain  $R$  of definable automorphisms of  $A$ , and such that  $\dim(Y) > 0$  for some type-definable  $Y \subseteq X$ . Then the skew field of fractions  $K$  of  $R$  exists and is definable;  $A$  is a definable  $K$ -vector space of finite linear dimension.*

**PROOF.** Fix any nonzero  $a \in A$ . Since  $R$  acts by automorphisms, the evaluation map  $R \rightarrow R \cdot a$  is injective, which bounds  $\dim(Z) = \dim(Z \cdot a)$  by  $\dim(A)$  for any type-definable subset  $Z \subseteq R$ . Now apply Proposition 3.6, and note that the fraction field  $K$  acts on  $A$  since  $R^\times$  acts by automorphisms. We finish by Proposition 3.3.  $\dashv$

Notice that if  $R$  is commutative, so is  $K$ .

**REMARK 3.11.** In Corollary 3.10 the hypothesis that  $\dim(Y) > 0$  for some type-definable  $Y \subseteq X$  is necessary: Just consider an infinite field  $K$  in the language of modules consisting of addition and unary functions  $\lambda_r$  for scalar multiplication by  $r \in K$ , for every  $r \in K$ . This is an abelian structure which does not interpret an infinite field.

**3.3. Automorphisms.** As opposed to the stable case, an infinite field with integer fine dimension need not be  $\emptyset$ -connected and may have  $\emptyset$ -definable additive or multiplicative subgroups of finite index: any pseudo-finite field will serve as an example.

**LEMMA 3.12.** *In a real-dimensional theory, a definable endomorphism  $\varphi$  of a definable broad skew field  $K$  is either 0 or a genuine skew field automorphism.*

**PROOF.** If  $\varphi$  is not zero, it is injective, and  $0 < \dim(\varphi(K)) = \dim(K) < \infty$ . Hence the degree  $[K : \varphi(K)] = 1$  and  $K = \varphi(K)$ .  $\dashv$

It follows that a dimensional broad (commutative) field is perfect.

Next, we show that a  $\emptyset$ -connected definable broad field in a fine real-dimensional theory does not admit an infinite type-definable family of automorphisms. This generalizes [1, Theorem 8.3]. Recall that an element of a field is *absolutely algebraic* if it is the root of a nonzero polynomial with coefficients in  $\mathbb{Z}$ ; the set of absolutely algebraic elements forms a subfield containing the prime field.

**LEMMA 3.13.** *In a fine dimensional theory, a  $\emptyset$ -definable broad additively or multiplicatively  $\emptyset$ -connected field  $K$  contains infinitely many absolutely algebraic elements.*

**PROOF.** In characteristic zero this is clear. So suppose the field  $K$  has characteristic  $p > 0$  and is additively  $\emptyset$ -connected; consider the  $\emptyset$ -definable additive endomorphism

$\phi : x \mapsto x^p - x$  with finite kernel  $\mathbb{F}_p$ . If there are only finitely many absolutely algebraic elements, then there is some  $n < \omega$  such that  $\phi^n(K) \cap \mathbb{F}_p = \{0\}$ . So  $\phi : \phi^n(K) \rightarrow \phi^{n+1}(K)$  is injective, whence  $\dim(\phi^{n+1}(K)) = \dim(\phi^n(K))$ ; as the dimension is fine,  $\phi^{n+1}(K)$  has finite index in  $\phi^n(K)$ . But  $\emptyset$ -connectivity of  $K$  implies that of  $\phi^n(K)$ , and  $\phi^n(K) = \phi^{n+1}(K) = \phi^{2n}(K)$ . But then  $K = \phi^n(K) \oplus \ker \phi^n$ , so  $\phi^n(K)$  has finite index in  $K$  as an additive subgroup. Again by connectivity  $\phi$  is surjective. Hence  $K$  has no Artin–Schreier extension. But the subfield of absolutely algebraic elements is relatively algebraically closed in  $K$ ; if it were finite, it would have an extension of degree  $p$ , which would yield an Artin–Schreier extension of  $K$ .

If  $K$  is multiplicatively  $\emptyset$ -connected, we use the multiplicative endomorphism  $\phi : x \mapsto x^q$  for big prime  $q$  such that  $K$  has no primitive  $q$ th root of unity, and argue with Kummer extensions. –

**PROPOSITION 3.14.** *In a fine real-dimensional theory, let  $K$  be a definable broad field with infinitely many absolutely algebraic elements. Then there is no infinite type-definable family of definable automorphisms of  $K$ .*

**PROOF.** Suppose  $\Phi$  is such a family. As  $K$  is definable, we may assume that  $\Phi$  is also definable (being an automorphism of  $K$  is a definable property). If  $\text{char}(K) = 0$ , put  $K_0 = \bigcap_{\sigma \in \Phi} \text{Fix}(\sigma)$ . Then  $K_0$  is definable and infinite, whence broad, and  $[K : K_0]$  is finite by Proposition 3.3. Thus  $\Phi$  is a subfamily of  $\text{Gal}(K/K_0)$ , which is finite.

Now suppose  $\text{char}(K) = p > 0$ ; we put

$$\Psi = \{\tau^{-1}\sigma : \sigma, \tau \in \Phi\}.$$

If  $a \in K$  is absolutely algebraic, it has only finitely many images under  $\Phi$ , and there are infinitely many automorphisms in  $\Psi$  fixing  $a$ . By compactness there is some  $a$  of infinite multiplicative order fixed by infinitely many automorphisms of  $\Psi$ . Put  $\Psi_a = \{\sigma \in \Psi : \sigma(a) = a\}$  and

$$K_0 = \bigcap_{\sigma \in \Psi_a} \text{Fix}(\sigma).$$

Then  $K_0$  is definable and infinite, whence broad, and  $\Psi_a$  injects into  $\text{Gal}(K/K_0)$  which is finite.

In both cases we obtain a contradiction, so no infinite type-definable family of automorphisms of  $K$  can exist. –

**COROLLARY 3.15.** *In a fine real-dimensional theory, a definable automorphism of a  $\emptyset$ -definable, broad, additively or multiplicatively  $\emptyset$ -connected field is  $\text{acl}^{eq}(\emptyset)$ -definable.*

**PROOF.** The field has infinitely many absolutely algebraic elements by Lemma 3.13. If there were an  $a$ -definable automorphism  $\sigma_a$  not definable over  $\text{acl}^{eq}(\emptyset)$ , there would be an infinite type-definable family  $\{\sigma_{a'} : a' \models \text{tp}(a)\}$  of definable automorphisms, contradicting Proposition 3.14. –

**§4. Pseudofinite dimensional groups.**

**PROPOSITION 4.1.** *Let  $G$  be a broad pseudofinite dimensional group with strong fibration. Then there is an element  $g \in G \setminus \{1\}$  with  $G$ -broad centralizer. More precisely, there is  $g \in G \setminus \{1\}$  with  $\dim(C_G(g)) \geq \frac{1}{3} \dim(G)$ .*

PROOF. Suppose first that  $G$  has no involution. If  $G \equiv \prod_I G_i/\mathcal{U}$  for some family  $(G_i)_I$  of finite groups and some nonprincipal ultrafilter  $\mathcal{U}$ , then  $G_i$  has no involution for almost all  $i \in I$ , and is soluble by the Feit–Thompson theorem. So for almost all  $i \in I$  there is  $g_i \in G_i \setminus \{1\}$  such that  $\langle g_i^{G_i} \rangle$  is commutative. Put  $g = [g_i]_I \in G \setminus \{1\}$ . Then  $\langle g^G \rangle$  is commutative and  $g^G \subseteq C_G(g)$ . As  $g^G$  is in definable bijection with  $G/C_G(g)$ , we have

$$\dim(C_G(g)) \geq \dim(g^G) = \dim(G/C_G(g)) = \dim(G) - \dim(C_G(g)).$$

In particular  $\dim(C_G(g)) \geq \frac{1}{2} \dim(G)$ .

Now suppose  $G$  has an involution  $i$ , but all centralizers of nontrivial elements have dimension  $< \frac{1}{3} \dim(G)$ . Then

$$\dim(i^G) = \dim(G/C_G(i)) = \dim(G) - \dim(C_G(i)) > \frac{2}{3} \dim(G).$$

For  $h \in G \setminus \{1\}$  put  $H_h = \{x \in G : h^x = h^{\pm 1}\}$ . Then  $H_h$  is an  $h$ -definable subgroup of  $G$ , and  $C_G(h)$  has index two in  $H_h$ , so  $\dim(H_h) = \dim(C_G(h)) < \frac{1}{3} \dim(G)$ . Moreover, if  $j \in i^G$  and  $h \in j^G j$ , then  $j \in H_h$ . Now by strong fibration

$$\dim(\{(j, h) \in G \times G : j \in H_h\}) = \dim(G) + \dim(H_h) < \frac{4}{3} \dim(G).$$

On the other hand by fibration

$$\begin{aligned} \dim(\{(j, h) \in G \times G : j \in H_h\}) &\geq \dim(\{(j, h) \in i^G \times G : h \in j^G j\}) \\ &\geq 2 \dim(i^G) > \frac{4}{3} \dim(G), \end{aligned}$$

as  $\dim(j^G j) = \dim(j^G) = \dim(i^G) > \frac{2}{3} \dim(G)$  for all  $j \in i^G$ . This contradiction finishes the proof. ⊥

We recall from [6] the definition of  $\tilde{\mathfrak{M}}_c$ , the centralizer condition up to finite index.

DEFINITION 4.2. A group  $G$  satisfies the  $\tilde{\mathfrak{M}}_c$ -condition if there is  $n < \omega$  such that there are no  $(g_i : i < n)$  in  $G$  such that  $|C_G(g_j : j < i) : C_G(g_j : j \leq i)| \geq n$  for all  $i < n$ . In other words, in a saturated model there is no infinite chain of centralizers  $C_G(g_j : j < i)$  for  $i < \omega$ , each of infinite index in its predecessor.

Examples for  $\tilde{\mathfrak{M}}_c$ -groups include all groups definable in a simple theory. Note that a subgroup of an  $\tilde{\mathfrak{M}}_c$ -group is again  $\tilde{\mathfrak{M}}_c$ .

LEMMA 4.3. Let  $G$  be an  $\tilde{\mathfrak{M}}_c$ -group and  $Z$  a finite central subgroup. Then  $G/Z$  is an  $\tilde{\mathfrak{M}}_c$ -group.

PROOF. As  $Z$  is central,  $x \mapsto [g, x]$  is a homomorphism from  $C_G(g/Z)$  to  $Z$ , whose kernel is  $C_G(g)$ . It follows that  $C_G(g)$  is a subgroup of  $C_G(g/Z)$  of index at most  $|Z|$ . The lemma follows. ⊥

LEMMA 4.4. Let  $G$  be an  $\tilde{\mathfrak{M}}_c$ -group. Then for any subgroup  $H$  there is a bound for finite indices of the form  $|H : C_H(g)|$ .

PROOF. Suppose not. Let  $n$  be given by the  $\tilde{\mathfrak{M}}_c$ -condition, and choose a maximal chain

$$H > C_H(g_0) > \dots > C_H(g_0, \dots, g_m)$$

with every group of finite index at least  $n$  in its predecessor. Then  $m < n$ . However, if  $\infty > |H : C_H(g)| > n |H : C_H(g_0, \dots, g_m)|$ , then

$$|C_H(g_0, \dots, g_m) : C_H(g_0, \dots, g_m, g)| > n,$$

contradicting maximality of  $m$ . ⊥

DEFINITION 4.5. Let  $G$  be a group, and  $H, K$  subgroups.

- We say that  $H$  is *almost contained* in  $K$ , written  $H \lesssim K$ , if  $H \cap K$  has bounded index in  $H$ . Clearly,  $\lesssim$  is transitive. If  $H \lesssim K$  and  $K \lesssim H$ , then  $H$  and  $K$  are *commensurable*, denoted  $H \sim K$ .
- The *almost centralizer* of  $H$  in  $K$  is the subgroup

$$\tilde{C}_K(H) = \{g \in K : H \lesssim C_G(g)\}.$$

The *almost center*  $\tilde{Z}(G)$  of  $G$  is the characteristic subgroup  $\tilde{Z}(G) = \tilde{C}_G(G)$ .

FACT 4.6.

1. If  $K$  is definable and  $H$  is type-definable, then by compactness  $H \lesssim K$  if and only if  $H \cap K$  has finite index in  $H$ .
2. [6, Proposition 2.23] In an  $\tilde{\mathfrak{M}}_c$ -group the almost centralizer of a definable subgroup is definable by Lemma 4.4, and the almost center is finite-by-abelian, as its conjugacy classes must be uniformly finite [12].

FACT 4.7 ([6, Theorem 2.10]). If  $H$  and  $K$  are type-definable, then  $H \lesssim \tilde{C}_G(K)$  if and only if  $K \lesssim \tilde{C}_G(H)$ .

FACT 4.8 ([6, Theorem 2.18]). Let  $G$  be a group,  $H$  and  $K$  subgroups, and suppose

$$H \leq N_G(K), \quad H \leq \tilde{C}_G(K), \quad \text{and} \quad K \leq \tilde{C}_G(H) \text{ uniformly}$$

(meaning that there is  $n < \omega$  such that  $|H : C_H(k)| \leq n$  for all  $k \in K$ ). Then  $[H, K]$  is finite.

COROLLARY 4.9. Let  $G$  be an  $\tilde{\mathfrak{M}}_c$ -group, and  $M, N$  normal subgroups of  $G$ . Then  $[\tilde{C}_M(N), \tilde{C}_N(M)]$  is finite.

REMARK 4.10. It follows in particular that  $F = [\tilde{C}_G(\tilde{Z}(G)), \tilde{Z}(G)]$  is finite in an  $\tilde{\mathfrak{M}}_c$ -group  $G$ , as  $\tilde{Z}(G) = \tilde{C}_{\tilde{Z}(G)}(G)$ . But  $G \lesssim \tilde{C}_G(\tilde{Z}(G)) = \tilde{C}_G(\tilde{C}_G(G))$  by Fact 4.7, and  $F \leq \tilde{Z}(G)$ , so  $G_1 = \tilde{C}_G(\tilde{Z}(G)) \cap C_G(F)$  has finite index in  $G$ . Moreover,  $F_1 = G_1 \cap F$  is finite central in  $G_1$ , and  $(G_1 \cap \tilde{Z}(G))/F_1$  is central in  $G_1/F_1$ .

Recall that a group is *virtually P* if it has a subgroup of finite index which is  $P$ .

THEOREM 4.11. Let  $G$  be a broad pseudofinite dimensional  $\tilde{\mathfrak{M}}_c$ -group satisfying strong fibration. Then  $G$  has a  $G$ -broad definable finite-by-abelian subgroup  $C$ . More precisely, any  $G$ -broad minimal centralizer (up to finite index) of a finite tuple is

virtually finite-by-abelian, and has a  $G$ -broad finite-by-abelian centralizer (of some bigger finite tuple) of finite index.

PROOF. By the  $\tilde{\mathfrak{M}}_e$ -condition, there is a  $G$ -broad centralizer  $C$  of some finite tuple, such that  $C_C(g)$  is not  $G$ -broad for any  $g \in C \setminus \tilde{Z}(C)$ . Put  $Z = \tilde{Z}(C)$ , a finite-by-abelian normal subgroup of  $C$  which is definable by Fact 4.6.

We claim that  $Z$  is  $G$ -broad. Otherwise  $\dim_G(Z) = 0$ , and

$$\dim_G(C/Z) = \dim_G(C) - \dim_G(Z) = \dim_G(C) > 0.$$

For  $g \in C \setminus Z$  we have  $\dim_G(C_C(g)) = 0$ , whence for  $\bar{g} = gZ$  we have

$$\begin{aligned} \dim_G(\bar{g}^{C/Z}) &= \dim_G(g^C Z/Z) = \dim_G(g^C Z) - \dim_G(Z) \geq \dim_G(g^C) - \dim_G(Z) \\ &= \dim_G(C) - \dim_G(C_C(g)) - \dim_G(Z) = \dim_G(C/Z). \end{aligned}$$

Hence for all  $\bar{g} \in (C/Z) \setminus \{\bar{1}\}$  we have

$$\dim_G(C_{C/Z}(\bar{g})) = \dim_G(C/Z) - \dim_G(\bar{g}^{C/Z}) = 0.$$

As  $C$  and  $Z$  are definable,  $C/Z$  is again pseudofinite, contradicting Proposition 4.1. This shows that  $\dim_G(Z) > 0$ , proving the claim.

Now for  $g \in \tilde{C}_C(Z)$  the index  $|Z : C_Z(g)|$  is finite. In particular  $\dim(C_C(g)) \geq \dim(Z) > 0$  and  $C_C(g)$  is  $G$ -broad. By minimality,  $g \in \tilde{Z}(C)$ . It follows that  $\tilde{Z}(C) = \tilde{C}_C(Z)$ , which has finite index in  $C$  since  $Z \lesssim \tilde{C}_C(C)$  implies  $C \lesssim \tilde{C}_C(Z)$  by Fact 4.7. Moreover  $\tilde{Z}(C)$  is finite-by-abelian, and  $C$  is virtually finite-by-abelian.

By compactness (or Lemma 4.4) there is a bound  $n$  on the index  $|C : C_C(c)|$  for  $c \in \tilde{Z}(C)$ . For any  $c' \in C \setminus \tilde{Z}(C)$ , the index  $|\tilde{Z}(C) : C_{\tilde{Z}(C)}(c')|$  is infinite. Fix such  $c'$ , let  $c_0, \dots, c_n$  in  $\tilde{Z}(C)$  lie in different cosets modulo  $C_{\tilde{Z}(C)}(c')$ , and put  $C' = C_C(c_i : i \leq n)$ , a subgroup of finite index in  $C$ .

We claim that  $|C : \tilde{Z}(C)| > |C' : \tilde{Z}(C')|$ . So suppose first that  $c' \in C' \tilde{Z}(C)$ , say  $c' = c''c$  with  $c'' \in C'$  and  $c \in \tilde{Z}(C)$ . Then there are  $i < j$  such that  $c_i^{-1}c_j \in C_C(c)$ , whence  $c_i^{-1}c_j \in C_C(c''c) = C_C(c')$ , a contradiction to the choice of the  $c_i$ .

It follows that  $C' \tilde{Z}(C)$  is a proper subgroup of finite index in  $C$ . Then  $\tilde{Z}(C' \tilde{Z}(C)) = \tilde{Z}(C)$  and  $\tilde{Z}(C') = \tilde{Z}(C) \cap C'$ , whence

$$|C : \tilde{Z}(C)| > |C' \tilde{Z}(C) : \tilde{Z}(C)| = |C' : \tilde{Z}(C')|,$$

proving the claim.

Inductively, we find a finite tuple  $\bar{c}$  in  $\tilde{Z}(C)$  such that  $C_C(\bar{c}) = \tilde{Z}(C_C(\bar{c}))$ , a  $G$ -broad finite-by-abelian centralizer of finite index in  $C$ . ⊣

Theorem 4.11 holds in particular for any pseudofinite  $\tilde{\mathfrak{M}}_e$ -group with coarse pseudofinite dimension (see Example 1.5(4)). Note that the  $\tilde{\mathfrak{M}}_e$ -condition is just used in  $G$ , not in the section  $C/Z$ .

COROLLARY 4.12. *A superrosy pseudofinite group with  $U^b(G) \geq \omega^\alpha$  has a definable finite-by-abelian subgroup  $A$  with  $U^b(A) \geq \omega^\alpha$ .*

PROOF. By [3, Proposition 1.4] a superrosy group is  $\tilde{\mathfrak{M}}_e$ . If  $\alpha$  is minimal with  $U^b(G) < \omega^{\alpha+1}$ , put

$$\dim(X) \geq n \quad \text{if} \quad U^b(X) \geq \omega^\alpha \cdot n,$$

then  $\dim$  is an integer dimension with  $0 < \dim(G) < \infty$  and strong fibration (Example 1.5(2)). The assertion now follows from Theorem 4.11.  $\dashv$

**COROLLARY 4.13.** *For any  $d, d' < \omega$  there is  $n = n(d, d')$  such that if  $G$  is a finite group without elements  $(g_i : i \leq d')$  such that*

$$|C_G(g_i : i < j) : C_G(g_i : i \leq j)| \geq d$$

*for all  $j \leq d'$ , then  $G$  has a subgroup  $A$  with  $|A'| \leq n$  and  $n|A|^n \geq |G|$ .*

**PROOF.** If the assertion were false, then given  $d, d'$ , there would be a sequence  $(G_i : i < \omega)$  of finite groups satisfying the condition, such that  $G_i$  has no subgroup  $A_i$  with  $|A'_i| \leq i$  and  $i|A_i|^i \geq |G_i|$ . But any nonprincipal ultraproduct  $G = \prod G_n/\mathcal{U}$  is a pseudofinite  $\mathfrak{M}_c$ -group; by Proposition 4.11 there is a definable subgroup  $A$  with  $A'$  finite and pseudofinite dimension  $\dim(A) \geq \frac{1}{n} \dim(G)$  for some  $n < \omega$ . Hence  $\log |A_i| \geq \frac{1}{n} \log |G_i| - m$  for some  $m \in \mathbb{N}$  and almost all  $i < \omega$ , whence  $e^{nm} |A_i|^n \geq |G_i|$ . For  $i \geq \max\{n, |A'|, e^{nm}\}$  this yields a contradiction.  $\dashv$

If  $G$  is a definable group, a definable subgroup  $H$  is *definably characteristic* in  $G$  if it is invariant under all definable automorphisms of  $G$ .

**COROLLARY 4.14.** *Let  $G$  be a pseudofinite  $\mathfrak{M}_c$ -group of integer dimension 1 with strong fibration. Then  $G$  has a definably characteristic wide finite-by-abelian subgroup, which is a finite extension of the centralizer of a finite tuple (and hence quantifier-free definable).*

**PROOF.** By Theorem 4.11 there is a broad finite-by-abelian centralizer  $C$  of a finite tuple. As  $\dim(G) = 1$  and  $C$  is broad,  $\dim(C) = 1$  and  $\dim(G/C) = 0$ .

For any definable automorphism  $\gamma$  of  $G$  the image  $C^\gamma$  is still definable as the centralizer of a finite tuple, and  $\dim(G/C^\gamma) = 0$ . As  $G/(C \cap C^\gamma)$  definably embeds into  $G/C \times G/C^\gamma$ , we have

$$\dim(G/(C \cap C^\gamma)) \leq \dim(G/C) + \dim(G/C^\gamma) = 0,$$

whence  $\dim(C \cap C^\gamma) = 1$ . By Lemma 4.4 there is a bound on the index of  $C \cap C^\gamma$  in  $C$  and in  $C^\gamma$ . Schlichting's Theorem now yields a definably characteristic subgroup  $N$  commensurable with  $C$ , which is a finite extension of a finite intersection of conjugates of  $C$  under definable automorphisms, and thus a finite extension of the centralizer of a finite tuple.

Now  $\tilde{Z}(N)$  is finite-by-abelian, and characteristic of finite index in  $N$ . It therefore contains all finite-by-abelian subgroups of finite index in  $N$ , and in particular  $N \cap C$ . So  $\tilde{Z}(N)$  is a finite extension of  $N \cap C$ , and thus definable as a finite extension of the centralizer of a finite tuple; clearly it is definably characteristic in  $G$ .  $\dashv$

**§5. Pseudofinite  $\mathfrak{M}_c$ -groups of integer dimension 2.**

**THEOREM 5.1.** *Let  $G$  be a pseudofinite integer-dimensional  $\mathfrak{M}_c$ -group with strong fibration. If  $\dim(G) = 2$ , then  $G$  has a broad definable finite-by-abelian subgroup with wide normalizer.*

**PROOF.** Note that by Corollary 4.14 we are done as soon as we find a definable subgroup of dimension 1 with wide normalizer. By the  $\mathfrak{M}_c$ -condition, there is a

minimal wide centralizer of a finite tuple, up to finite index, and we can assume that this is already  $G$ . Thus

$$\tilde{Z}(G) = \{g \in G : \dim(C_G(g)) = 2\}.$$

If  $\dim(\tilde{Z}(G)) \geq 1$  we are done, since  $\tilde{Z}(G)$  will be as required. Otherwise  $\tilde{Z}(G)$  contains any subgroup  $H$  of dimension 0 with wide normalizer  $N_G(H)$ , since if  $g \in H$  then  $\dim(g^{N_G(H)}) = 0$  and  $\dim(C_{N_G(H)}(g)) = 2$ , whence  $g \in \tilde{Z}(G)$ . By Remark 4.10 there is a definable subgroup  $G_1$  of finite index in  $G$  and a finite central subgroup  $F_1$  of  $G_1$  such that  $(G_1 \cap \tilde{Z}(G))/F_1$  is central in  $G_1/F_1$ . Now  $G_1/F_1$  is still  $\mathfrak{M}_c$  by Lemma 4.3; replacing  $G$  by  $G_1/F_1$  we can suppose  $\tilde{Z}(G) = Z(G)$ .

We may assume  $G = \prod_{i \in \mathcal{U}} G_i$ , where the  $G_i$  are finite groups. For a subgroup  $H \leq G$  we shall put  $\tilde{H} = (HZ(G))/Z(G)$ ; similarly we shall put  $\tilde{H}_i = (H_i Z(G_i))/Z(G_i)$  for a subgroup  $H_i \leq G_i$ .

CLAIM. *There is some  $k < \omega$  such that for almost all  $i$  there is no direct product of simple nonabelian groups of length  $k$  in  $\tilde{G}_i$ .* −

PROOF OF CLAIM. Suppose otherwise. Fix a decreasing chain  $(I_k : k < \omega)$  of sets in  $\mathcal{U}$  such that for  $i \in I_k$  there is a direct product  $\prod_{j < k} \tilde{N}_{ij} \leq \tilde{G}_i$  of nonabelian simple groups. Fix such a product of length  $k$  for all  $i \in I_k \setminus I_{k+1}$ , and note that  $\bigcap_{k < \omega} I_k = \emptyset$ , as  $i \in I_k$  implies  $k \leq |G_i|$ . Put  $M_{ij} = N'_{ij}$ . Then  $M_{ij}Z(G_i)/Z(G_i) = \tilde{N}_{ij}$ , and  $M_{ij}$  is perfect. Recall that the three subgroup Lemma states that for three normal subgroups  $K, L, M$  of a group we have  $[[K, L], M] \leq [[L, M], K][[M, K], L]$ . Hence, for  $j' \neq j$

$$[M_{ij}, M_{ij'}] = [M'_{ij}, M'_{ij'}] \leq [[M_{ij}, M_{ij'}], M_{ij}][[M_{ij'}, M_{ij}], M_{ij}] = \{1\},$$

since  $[M_{ij}, M_{ij'}] \leq Z(G_i)$ .

For  $j < \omega$  let  $M_j = \prod_{i \in \mathcal{U}} M_{ij}$ ; note that this is well-defined, and the  $M_j$  are nonabelian and commute with one another. Put  $H = \prod_{j < \omega} M_j$ . Then if  $m_j \in M_j \setminus Z(M_j)$  and  $g_k = \prod_{j < k} m_j$ , we have

$$|C_H(g_0, \dots, g_\ell) : C_H(g_0, \dots, g_\ell, g_{\ell+n})| \geq 2^n$$

for all  $\ell, n < \omega$ , contradicting the  $\tilde{\mathfrak{M}}_c$ -condition. −

CLAIM. *We may assume that almost all  $\tilde{G}_i$  are semisimple, that is, have no nontrivial abelian normal subgroup.*

PROOF. Suppose otherwise, and let  $\bar{A}_i$  be a nontrivial abelian normal subgroup of  $\tilde{G}_i$ . Then for  $a_i \in A_i \setminus Z(G_i)$  the element  $a = [a_i]$  is not in  $Z(G)$ , so  $\dim(C_G(a)) \leq 1$  and  $\dim(a^G) \geq 1$ . Consider  $\bar{A} = Z(C_{\tilde{G}}(\bar{a}^{\tilde{G}}))$ , an abelian normal subgroup of  $\tilde{G}$  containing  $\bar{a}^{\tilde{G}}$ . Then its preimage  $A$  is a definable normal subgroup of  $G$  with  $\dim(A) \geq \dim(a^G) \geq 1$ . Note that for  $a \in A$  we have  $a^{-1}a^A \subseteq Z(G)$  since  $\bar{A} = A/Z(G)$  is abelian, so  $\dim(a^A) = 0$  and  $\dim(C_A(a)) = \dim(A)$ .

It follows that if  $C$  is a minimal wide centralizer in  $A$  of some finite tuple up to finite index (which exists by the  $\tilde{\mathfrak{M}}_c$ -condition), then  $C \sim C^g$  for all  $g \in G$ , whence  $\tilde{C}_G(C) = \tilde{C}_G(C^g)$  is normal in  $G$ . Therefore  $\tilde{Z}(\tilde{C}_G(C))$  is a definable finite-by-abelian normal subgroup containing  $C$ , whence of dimension  $\dim(A) \geq 1$ : we are done. −



Suppose  $\bar{G}_i$  has at least two distinct minimal normal subgroups  $\bar{N}_i$  and  $\bar{N}_i^*$  for almost all  $i$ . If  $n_i \in N_i \setminus Z(G_i)$  for almost  $i$ , then  $n = [n_i] \notin Z(G)$ , so  $\dim(n^G) \geq 1$ . The same holds for  $n^* = [n_i^*]$  with  $n_i^* \in N_i^* \setminus Z(G_i)$ . We put

$$\bar{N}_0 = C_{\bar{G}}(\bar{n}^{*\bar{G}}) \supseteq \bar{n}^{\bar{G}} \quad \text{and} \quad \bar{N}_1 = C_{\bar{G}}(\bar{N}_0) \supseteq \bar{n}^{*\bar{G}}.$$

Then  $\bar{N}_0$  and  $\bar{N}_1$  are definable commuting normal subgroups of  $\bar{G}$  of dimension at least 1; their intersection must be trivial, as  $\bar{G}$  is definably semisimple. Since  $\dim(\bar{G}) = \dim(G) = 2$  we must have  $\dim(\bar{N}_j) = 1$  for  $j = 0, 1$ . Now the preimages  $N_j$  are definable normal subgroups of  $G$  with  $\dim(N_j) = \dim(\bar{N}_j) = 1$ ; we are done again by Corollary 4.14.

So we may assume that almost each  $\bar{G}_i$  has a unique minimal normal subgroup  $\bar{N}_i$ . Then  $\bar{N}_i$  is a finite direct product of (finite) simple groups  $\bar{N}_i^j$  for  $j < k_i$ , which are permuted transitively by  $G_i$ . By the first claim, almost all  $k_i$  take the same value  $k$ .

Now every finite simple group is the product of two conjugacy classes [5, Theorem 1.4], so there are elements  $\bar{n}_i, \bar{n}_i^*$  with

$$\bar{N}_i^0 = \{\bar{1}\} \cup \bar{n}_i^{\bar{N}_i^0} \bar{n}_i^{*\bar{N}_i^0}.$$

Note that if  $\bar{N}_i^j = (\bar{N}_i^0)^{\bar{g}_{ij}}$  for some  $\bar{g}_{ij} \in \bar{G}_i$ , then

$$\bar{N}_i^j = \{\bar{1}\} \cup \bar{n}_i^{\bar{N}_i^0 \bar{g}_{ij}} \bar{n}_i^{*\bar{N}_i^0 \bar{g}_{ij}}.$$

Put  $\bar{X}_i = \{\bar{1}\} \cup \bar{n}_i^{\bar{G}_i} \bar{n}_i^{*\bar{G}_i}$ . It follows from normality that  $\bigcup_{j < k} \bar{N}_i^j \subseteq \bar{X}_i \subseteq \bar{N}_i$ , so  $\bar{N}_i^0 = \{\bar{1}\} \cup \bar{n}_i^{\bar{X}_i} \bar{n}_i^{*\bar{X}_i}$  is uniformly definable, as is  $\bar{N}_i^j$  for  $j < k$ .

For  $j < k$  put  $\bar{N}^j = \prod_{\mathcal{U}} \bar{N}_i^j$ . Then  $\bar{N}$  is definable as the direct product of the  $\bar{N}^j$  for  $j < k$ . Since the  $\bar{N}^j$  are definably isomorphic and  $1 \leq \dim(\bar{N}) \leq 2$ , either  $k = 1$  and  $\bar{N} = \bar{N}^1$ , or  $k = 2 = \dim(\bar{N})$  and  $\dim(\bar{N}^0) = \dim(\bar{N}^1) = 1$ . If  $\dim(\bar{N}) = 1$  we are done by Corollary 4.14; if  $k = 2$  the normalizer  $N_G(\bar{N}^0) = N_G(\bar{N}^1)$  has index 2 in  $G$  and is wide, and we are done again.

So we may assume that  $\dim(\bar{N}) = 2$  and  $\bar{N}$  is an ultraproduct of finite simple groups  $\bar{N}_i$ . As  $\bar{N}$  is infinite, not almost all  $\bar{N}_i$  can be sporadic, and we may assume they are all alternating or Chevalley groups (possibly twisted) over a finite field. But their rank (where the rank of the alternating group  $A_k$  is  $k$ ) must be bounded, as otherwise the  $\bar{N}_i$  contain arbitrarily long direct products  $\bar{P}_i$  of  $A_4$  or  $\text{PSL}_2$ , contradicting the first claim.

It follows from [13] that  $\bar{N}$  must be a (possibly twisted) Chevalley group over a pseudofinite field. By results of Ryten [14] (see also [2]) the pure group  $\bar{N}$  is bi-interpretable with a pseudofinite (difference) field  $F$  of  $SU$ -rank 1. Now  $\bar{N}$  is internal in  $F$ , and  $F$  is internal in any generic definable subset  $X$  of  $F$ , whence

$$\dim(X) \geq \frac{1}{m} \dim(F) \geq \frac{1}{n} \dim(\bar{N})$$

for some positive integers  $m, n$ , and  $\dim(X) > 0$ . As the dimension is integer, this means  $\dim(X) \geq 1$  for any definable generic subset of  $F$ . But  $SU(\bar{N}) \geq 3$ , so a

generic element of  $\bar{N}$  (interpreted in  $F$ ) contains at least three independent generic coordinates from  $F$ ; as  $SU$ -rank is definable in  $F$ , we obtain a definable subset of  $\bar{N}$  of dimension 3, contradicting  $\dim(\bar{N}) \leq \dim(G) = 2$ .

**COROLLARY 5.2.** *Let  $G$  be a pseudofinite group whose definable sections are  $\tilde{\mathfrak{M}}_c$ , and  $\dim$  an integer dimension on  $G$  with strong fibration. If  $\dim(G) = 2$ , then  $G$  has a definable wide soluble subgroup.*

**PROOF.** By Theorem 5.1, there is a definable finite-by-abelian group  $N$  such that  $N_G(N)$  is wide. Replacing  $N$  by  $C_N(N')$ , we may assume that  $N$  is (finite central)-by-abelian. If  $\dim(N) = 2$  we are done. Otherwise  $\dim(N_G(N)/N) = 1$ ; by Corollary 4.14 there is a definable finite-by-abelian subgroup  $S/N$  with  $\dim(S/N) = 1$ . As above we may assume that  $S/N$  is (finite central)-by-abelian, so  $S$  is soluble. Moreover,

$$\dim(S) = \dim(N) + \dim(S/N) = 1 + 1 = 2,$$

so  $S$  is wide in  $G$ . ⊣

**COROLLARY 5.3.** *A pseudofinite superrosy group  $G$  with  $\omega^\alpha \cdot 2 \leq U^b(G) < \omega^\alpha \cdot 3$  has a definable soluble subgroup  $S$  with  $U^b(S) \geq \omega^\alpha \cdot 2$ .*

**PROOF.** Superrosiness implies that all definable sections of  $G$  are  $\tilde{\mathfrak{M}}_c$ . We put

$$\dim(X) = n \iff \omega^\alpha \cdot n \leq U^b(X) < \omega^\alpha \cdot (n + 1).$$

This defines an integer dimension with strong fibration, and  $\dim(G) = 2$ . The result now follows from Corollary 5.2. ⊣

**Acknowledgments.** Partially supported by ANR-13-BS01-0006 ValCoMo and ANR-17-CE40-0026 Agrume. The author would like to thank the referee for his careful reading.

REFERENCES

[1] A. BOROVIK and A. NESIN, *Groups of Finite Morley rank*, Oxford Logic Guides, vol. 26, The Clarendon Press—Oxford University Press, New York, NY, 1994.

[2] R. ELWES and D. MACPHERSON, *A survey of asymptotic classes and measurable structures*, *Model Theory with Applications to Algebra and Analysis*, vol. 2 (Z. Chatzidakis, D. Macpherson, A Pillay, and A. Wilkie, editors), London Mathematical Society Lecture Note Series, vol. 350, Cambridge University Press, Cambridge, 2008, pp. 125–160.

[3] C. EALY, K. KRUPINSKI, and A. PILLAY, *Superrosy dependent groups having finitely satisfiable generics*, *Annals of Pure and Applied Logic*, vol. 151 (2008), pp. 1–21.

[4] R. ELWES and M. RYTEN, *Measurable groups of low dimension*, *Mathematical Logic Quarterly*, vol. 54 (2008), no. 4, pp. 374–386.

[5] R. GURALNICK and G. MALLE, *Products of conjugacy classes and fixed point spaces*, preprint, arXiv:1005.3756.

[6] N. HEMPEL, *Almost group theory*, preprint, 2016. Available at hal.archives-ouvertes.fr/hal-01206954.

[7] E. HRUSHOVSKI, *Stable group theory and approximate subgroups*, *Journal of the American Mathematical Society*, vol. 25 (2012), pp. 189–243.

[8] ———, *On pseudo-finite dimensions*, *Notre Dame Journal of Formal Logic*, vol. 54 (2013), no. 3–4, pp. 463–495.

- [9] E. HRUSHOVSKI and F. O. WAGNER, *Counting and dimensions*, *Model Theory with Applications to Algebra and Analysis*, vol. 2 (Z. Chatzidakis, D. Macpherson, A. Pillay, and A. Wilkie, editors), London Mathematical Society Lecture Note Series, vol. 350, Cambridge University Press, Cambridge, 2008, pp. 161–176.
- [10] I. KAPLANSKY, *Rings with a polynomial identity*. *Bulletin of the American Mathematical Society*, vol. 54 (1948), pp. 575–580.
- [11] K. KRUPINSKI, *Fields definable in rosy theories*. *Israel Journal of Mathematics*, vol. 175 (2010), no. 1, pp. 421–444.
- [12] B. H. NEUMANN, *Groups with finite classes of conjugate elements*. *Proceedings of the London Mathematical Society*, vol. 1 (1951), pp. 178–187.
- [13] F. POINT, *Ultraproducts and Chevalley groups*. *Archive for Mathematical Logic*, vol. 38 (1999), no. 6, pp. 355–372.
- [14] M. RYTEN, *Model theory of finite difference fields and simple groups*, PhD thesis, University of Leeds, 2007. Available at <http://www1.maths.leeds.ac.uk/pmthdm/ryten1.pdf>.
- [15] F. O. WAGNER, *Simple Theories*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

UNIVERSITÉ LYON 1, CNRS  
INSTITUT CAMILLE JORDAN UMR 5208  
21 AVENUE CLAUDE BERNARD  
69622 VILLEURBANNE CEDEX, FRANCE

E-mail: [wagner@math.univ-lyon1.fr](mailto:wagner@math.univ-lyon1.fr)