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A NOTE ON WIELANDT'S THEOREM

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Abstract

Let π be a set of primes. We say that a group G satisfies D_π if G possesses a Hall π-subgroup H and every π-subgroup of G is contained in a conjugate of H. We give a new D_π-criterion following Wielandt's idea, which is a generalisation of Wielandt's and Rusakov's results.

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1. Introduction

All groups considered here are finite. Let G be a group and let π be a set of primes. Recall that a subgroup H of a group G is called a π-subgroup of G if all primes dividing the order of H lie in π. Moreover, a subgroup H is called a Hall π-subgroup of G if it is a π-subgroup and its index is not divisible by the elements of π.

According to Hall [2], a group G is said to satisfy E_π (or $G \in E_\pi$ for short) if there exists a Hall π-subgroup of G. If $G \in E_\pi$ and all π-Hall subgroups are conjugate then we say that G satisfies C_π ($G \in C_\pi$). If $G \in E_\pi$ and every π-subgroup of G is contained in a conjugate of a π-Hall subgroup of G then we say that G satisfies D_π.

In 1954, Wielandt [5] proved the classical result that a group G possessing a nilpotent π-Hall subgroup satisfies D_π. After several years, one of the earliest generalisations of Wielandt's theorem was obtained by Wielandt himself in [6]. Suppose that a set of primes π is a union of disjoint subsets σ and τ, and a group G possesses a Hall π-subgroup $H = H_\sigma \times H_\tau$, where H_σ is a nilpotent σ-subgroup of H and H_τ is a τ-subgroup of H. If G satisfies D_τ, then G satisfies D_π. In the same paper, Wielandt conjectured that one can replace 'the nilpotency of H_σ' with the weaker condition that 'G satisfies D_σ' in the above theorem. This conjecture was completely confirmed by Guo *et al.* [1] by using the classification of finite simple groups. It is worth pointing out another inspiring result due to Rusakov [4] that if a

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group G possesses a Hall π-subgroup H whose Sylow subgroups are all cyclic, then G satisfies D_π.

In this paper, we try to weaken the 'direct product relation' between H_σ and H_τ in Wielandt's theorem, replacing it with a special semidirect product. Our main theorem is the following result.

THEOREM 1.1. *Let a set π of primes be a union of disjoint subsets σ and τ. Let a group G possess a Hall π-subgroup $H = H_\sigma H_\tau$ such that:*

(1) *H_σ is a normal σ-subgroup of H;*
(2) *H_τ is a τ-subgroup of H with all Sylow subgroups cyclic;*
(3) *H_τ normalises every subgroup of H_σ.*

If G satisfies D_σ, then G satisfies D_π.

It is worth mentioning that the proof of Theorem 1.1 does not depend on the classification of finite simple groups. In Theorem 1.1, neither of the hypotheses (2) or (3) can be removed. In fact, we can see this in the projective linear group $G = \mathrm{PSL}(2, 11)$. Let $\pi = \{2, 3\}$ and notice that Hall π-subgroups of G are isomorphic to the alternating group A_4 or the dihedral group D_{12}. Hence $G \notin D_\pi$.

If $H \cong D_{12}$, then we can set $\sigma = \{3\}$ and $\tau = \{2\}$. Hence $H_\sigma \cong \mathrm{C}_3$ and $H_\tau \cong \mathrm{C}_2 \times \mathrm{C}_2$ are the Sylow 3-subgroup and the Sylow 2-subgroup of H, respectively. Clearly $G \in D_{\{3\}}$, $H_\sigma \trianglelefteq H$ and H_τ normalises every subgroup of H_σ, which means that all hypotheses in Theorem 1.1 hold except hypothesis (2) because H_τ is not cyclic.

If $H \cong A_4$, we may assume that $\sigma = \{2\}$ and $\tau = \{3\}$. Then $H_\sigma \cong \mathrm{C}_2 \times \mathrm{C}_2$ and $H_\tau \cong \mathrm{C}_3$ are the Sylow 2-subgroup and the Sylow 3-subgroup of H, respectively. We see that $G \in D_{\{2\}}, H_\sigma \trianglelefteq H$ and H_τ is cyclic but H_τ does not normalise all subgroups of H_σ.

The following two corollaries both follow directly from Theorem 1.1. The first unifies Wielandt's and Rusakov's results.

COROLLARY 1.2. *Let a set π of primes be a union of disjoint subsets σ and τ. Let a group G possess a Hall π-subgroup $H = H_\sigma H_\tau$ such that:*

(1) *H_σ is a normal nilpotent σ-subgroup of H;*
(2) *H_τ is a τ-subgroup with all Sylow subgroups cyclic;*
(3) *H_τ normalises every subgroup of H_σ.*

Then G satisfies D_π.

The second corollary is a generalisation of Rusakov's result following Wielandt's idea; it is also a direct corollary of the result of Guo *et al.* [1].

COROLLARY 1.3. *Let a set π of primes be a union of disjoint subsets σ and τ. Let a group G possess a Hall π-subgroup $H = H_\sigma \times H_\tau$ such that:*

(1) *H_σ is a σ-subgroup of H;*
(2) *H_τ is a τ-subgroup with all Sylow subgroups cyclic.*

If G satisfies D_σ, then G satisfies D_π.

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2. Lemmas

In this section we list some lemmas, most of which are well known. The first is a direct consequence of Burnside's p-nilpotency criterion.

LEMMA 2.1 [3, IV, Theorem 2.8]. *Let G be a group and let p be the smallest prime dividing the order of G. If G possesses a cyclic Sylow p-subgroup, then G is p-nilpotent.*

LEMMA 2.2. *Let G be a group with all Sylow subgroups cyclic. Then G has two cyclic subgroups N_1, N_2 such that $G = N_1N_2, N_1 \trianglelefteq G$ and $(|N_1|, |N_2|) = 1$.*

PROOF. This is a consequence of [3, IV, Theorem 2.11]. □

The following lemma gives a property of groups with all Sylow subgroups cyclic and will be useful in the proof of our main theorem.

LEMMA 2.3. *Let G be a group with all Sylow subgroups cyclic and assume that p is the smallest prime dividing the order of G. If K is a p'-subgroup of G, then there exists a Sylow p-subgroup P of G such that $P \leq \mathrm{N}_G(K)$.*

PROOF. By Lemma 2.1, G is p-nilpotent. Denote by N the normal p-complement of G. Clearly $K \leq N$. Notice that all Sylow subgroups of N are also cyclic. By Lemma 2.2, we can assume that $N = N_1N_2$, where N_1, N_2 are cyclic, $(|N_1|, |N_2|) = 1$ and $N_1 \trianglelefteq N$.

Let $K_1 = K \cap N_1$. As N_1 is a normal Hall subgroup of N, it is easy to see that K_1 is a normal Hall subgroup of K. By the Schur–Zassenhaus theorem, we may assume that $K = K_1K_2$, where K_2 is a complement of K_1 in K. Since $N/N_1 \cong N_2$ is cyclic, it follows that K_2N_1/N_1 is a characteristic subgroup of N/N_1. Since N_1 is a characteristic subgroup of N, it is not difficult to check, by definition, that K_2N_1 is a characteristic subgroup of N. As $N \trianglelefteq G$, it follows that $K_2N_1 \trianglelefteq G$. By a Frattini argument, $G = \mathrm{N}_G(K_2)N_1$. Since the index of $\mathrm{N}_G(K_2)$ in G is a p'-number, there exists a Sylow p-subgroup P of G such that $P \leq \mathrm{N}_G(K_2)$.

On the other hand, it is easy to see that K_1 is a characteristic subgroup of N_1 as N_1 is cyclic. Since N_1 is a characteristic subgroup of G, we deduce that $K_1 \trianglelefteq G$. It follows that P normalises $K = K_1K_2$. Hence $P \leq \mathrm{N}_G(K)$, as desired. □

Recall that a group G is called minimal non-p-nilpotent if G is not p-nilpotent but every proper subgroup of G is p-nilpotent. The structure of minimal non-p-nilpotent groups is well known, due to N. Itô.

LEMMA 2.4. *Let p be a prime and G be a minimal non-p-nilpotent group. Then G possesses a normal Sylow p-subgroup P and a cyclic Sylow q-subgroup $Q \neq 1$ for some $q \neq p$ such that $G = PQ$.*

PROOF. See [3, IV, Theorem 5.4]. □

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3. Proof of Theorem 1.1

Suppose that the theorem is false so that there exists a group G possessing a Hall π-subgroup $H = H_\sigma H_\tau$ such that G, H_σ, H_τ satisfy the hypotheses of Theorem 1.1 but $G \notin D_\pi$, which means that there exists a π-subgroup K of G such that $K^g \nleq H$ for each $g \in G$. Choose the counterexample triple (G, H, K) with $|G| + |H| + |K|$ minimal and, without loss of generality, assume that $\pi \subseteq \pi(G)$.

By hypothesis, if τ is empty, the result is trivial. Now let p be the smallest prime in τ. Since all Sylow subgroups of H_τ are cyclic, H_τ is p-nilpotent. As $H_\sigma \trianglelefteq H$, H is also p-nilpotent. Set $\widetilde{\pi} = \pi - \{p\}$ and $\widetilde{\tau} = \tau - \{p\}$. Then we will derive a contradiction from the following three steps.

Step 1: $p \in \pi(K)$.

If $p \notin \pi(K)$, then K is a $\widetilde{\pi}$-subgroup of G. Let $\widetilde{H} = H_\sigma H_{\widetilde{\tau}}$, where $H_{\widetilde{\tau}}$ is the normal p-complement of H_τ. It is obvious that $\widetilde{H}$ is a $\widetilde{\pi}$-Hall subgroup of G and $\widetilde{H} \le H$. Considering the triple $(G, \widetilde{H}, K)$, by minimality, K is contained in a conjugate of $\widetilde{H}$ and also in a conjugate of H, contrary to the choice of K.

Step 2: K is minimal non-p-nilpotent.

For any proper subgroup T of K, minimality implies that T is contained in a conjugate of H. As H is p-nilpotent, so is T. Now we will show that K is not p-nilpotent.

Assume that K is p-nilpotent and let $K = K_p K_{\widetilde{\pi}}$, where K_p and $K_{\widetilde{\pi}}$ are the Sylow p-subgroup and the normal p-complement of K, respectively. Since $p \in \pi(K)$ by Step 1, $K_p \ne 1$ and $K_{\widetilde{\pi}} < K$. By minimality, $K_{\widetilde{\pi}}$ is contained in a conjugate of H. Without loss of generality, we can assume that $K_{\widetilde{\pi}} \le H$. Write $K_\sigma = H_\sigma \cap K_{\widetilde{\pi}}$. Since H_σ is a normal Hall σ-subgroup of H, it follows that K_σ is a normal Hall σ-subgroup of $K_{\widetilde{\pi}}$. By the Schur–Zassenhaus theorem, we may assume that $K_{\widetilde{\tau}}$ is the complement of K_σ in $K_{\widetilde{\pi}}$, which is also a Hall τ-subgroup of K.

Since H has a Hall τ-subgroup H_τ with all Sylow subgroups cyclic, by Rusakov's theorem, $H \in D_\tau$. As $K_{\widetilde{\tau}}$ is a τ-subgroup of H, $K_{\widetilde{\tau}} \le H_\tau^h$ for some $h \in H$.

Moreover, as $K_{\widetilde{\tau}}$ is a p'-subgroup, it follows from Lemma 2.3 that there exists a Sylow p-subgroup P of H_τ^h such that P normalises $K_{\widetilde{\tau}}$. On the other hand, by hypothesis, $P^{h^{-1}} \le H_\tau$ normalises every subgroup of H_σ. Since $h \in H$ and $H_\sigma \trianglelefteq H$, $K_\sigma^{h^{-1}} \le H_\sigma^{h^{-1}} = H_\sigma$. Hence $P^{h^{-1}}$ normalises $K_\sigma^{h^{-1}}$, and so P normalises K_σ. Thus P normalises $K_{\widetilde{\pi}} = K_\sigma K_{\widetilde{\tau}}$, that is, $P \in \mathrm{N}_G(K_{\widetilde{\pi}})$.

Notice that P is also a Sylow p-subgroup of G and also of $\mathrm{N}_G(K_{\widetilde{\pi}})$, and $K_p \le K \le \mathrm{N}_G(K_{\widetilde{\pi}})$ since $K_{\widetilde{\pi}} \trianglelefteq K$. By Sylow's theorem, there exists an element $x \in \mathrm{N}_G(K_{\widetilde{\pi}})$ such that $K_p^x \le P$. Thus

$$K^x = K_p^x K_{\widetilde{\pi}}^x \le K_p^x K_{\widetilde{\pi}} \le PH \le H_\tau^h H = H,$$

which is a contradiction. Hence K is minimal non-p-nilpotent.

Step 3: The final contradiction.

Since K is minimal non-p-nilpotent, it follows from Lemma 2.4 that $K = K_p K_q$, where K_p is the normal Sylow p-subgroup of K and $K_q \ne 1$ is a cyclic Sylow

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q-subgroup of K for some prime $q \neq p$ and $q \in \pi$. Notice that $K_q \neq 1$ and so $K_p < K$. Minimality implies that K_p is contained in a conjugate of H_τ. Since all Sylow subgroups of H_τ are cyclic, K_p is cyclic. Hence, by the N/C theorem,

$$K/\,\mathrm{C}_K(K_p) \lesssim \mathrm{Aut}(K_p).$$

Note that $K \neq \mathrm{C}_K(K_p)$ otherwise K_p is in the centre of K and K is nilpotent, contrary to Step 2. Hence the order of $K/\,\mathrm{C}_K(K_p)$ is a positive power of q, which implies that q divides $|\mathrm{Aut}(K_p)| = p^a(p-1)$ for some nonnegative integer a. As $p \neq q$, q divides $p-1$. Since p is the smallest prime in τ, it follows that $q \notin \tau$. As $q \in \pi = \sigma \cup \tau$, this forces $q \in \sigma$.

By Rusakov's theorem, G satisfies D_τ. Hence we can assume that $K_p \leq H_\tau^g$ for some $g \in G$. Let Q be a Sylow q-subgroup of H_σ^g. By hypothesis, H_τ^g normalises every subgroup of H_σ^g. This implies that K_p normalises every subgroup of Q. For each cyclic subgroup Q_1 of Q, Q_1 is also K_p-invariant and so $|K_p/\,\mathrm{C}_{K_p}(Q_1)|$ divides $|\mathrm{Aut}(Q_1)| = q^{n-1}(q-1)$, for some positive integer n. As $q < p$, it follows that $K_p = \mathrm{C}_{Q_1}(K_p)$ and thus K_p acts trivially on each cyclic subgroup Q_1 and also on Q. In particular, $Q \leq \mathrm{N}_G(K_p)$. Since Q is a Sylow q-subgroup of $\mathrm{N}_G(K_p)$ and $K_q \leq K \leq \mathrm{N}_G(K_p)$, there exists an element $y \in \mathrm{N}_G(K_p)$ such that $K_q^y \leq Q$. Now we see that

$$K^y = K_p^y K_q^y \leq K_p K_q^y \leq H_\tau^g Q \leq H_\tau^g H_\sigma^g = H^g,$$

which is the final contradiction. The proof is complete.

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