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A NOTE ON THE INSURANCE RISK PROBLEM

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We give a new characterization of the ruin probability of the classical insurance risk model and use it to obtain bounds on and a computational approach for evaluating this probability.

1. THE PROBLEM

Consider the classical insurance ruin problem in which claims are made to an insurance company according to a Poisson process with rate λ; the successive claim amounts $Y_1, Y_2, \ldots$ are independent random variables with a common distribution function F having mean μ, and are independent of the claim arrival times; the company starts with an initial capital x and receives income at a constant rate c per unit time. We are interested in the probability that the company's net capital ever becomes negative; that is, in

$$p(x) = P\left\{\sum_{i=1}^{N(t)} Y_i > x + ct \text{ for some } t \geq 0\right\},$$

where $N(t)$ is the number of claims made by time t. We give a new characterization of $p(x)$ and then use it to obtain bounds, an efficient simulation procedure, and a numerical procedure for determining $p(x)$.

2. A CHARACTERIZATION OF $p(x)$

Let $\rho = \lambda\mu/c$ and assume that $\rho < 1$ (for otherwise $p(x) = 1$). Also, let $\bar{F}(x) = 1 - F(x)$. We start with the well-known characterization (see [5] or [2])

$$p(x) = P\left\{\sum_{i=1}^{T} X_i > x\right\}, \tag{1}$$

where $T, X_1, X_2, \ldots$ are independent, with the X_i having the equilibrium distribution F_e whose density function is

$$F_e'(x) = \frac{\bar{F}(x)}{\mu}, \qquad x \geq 0,$$

and

$$P\{T = n\} = \rho^n(1-\rho), \qquad n \geq 0.$$

Now, let $N(t)$, $t \geq 0$ be the renewal process having $X_1, X_2, \ldots$ as its interarrival times, and condition on $N(x)$ to obtain

$$p(x) = E[\rho^{N(x)+1}]. \tag{2}$$

Remarks:

1. If F is exponential with mean μ, then so is F_e. Hence, $N(x)$ is Poisson with mean x/μ, giving the result

$$p(x) = \sum_n \rho^{n+1} e^{-x/\mu}(x/\mu)^n/n! = \rho e^{-x(1-\rho)/\mu}. \tag{3}$$

2. The failure rate function of X (call it $r_e(t)$) is given by

$$r_e(t) = \bar{F}(t) \Big/ \int_t^\infty \bar{F}(s)\, ds.$$

Therefore, if F is a new better than used (NBU) distribution—meaning that, for $s > t$, $\bar{F}(s)/\bar{F}(t) \leq \bar{F}(s-t)$—then

$$r_e(t) \geq \frac{1}{\int_t^\infty \bar{F}(s-t)\, ds} = \frac{1}{\mu}.$$

Consequently, X is failure rate ordered (and thus stochastically) smaller than an exponential with rate $1/\mu$. From (1), this implies that $p(x)$ is smaller than when the X_i are exponential with rate $1/\mu$. Hence, from (3), we see that when F is NBU,

$$p(x) \leq \rho e^{-x(1-\rho)/\mu}.$$

When F is new worse than used (NWU)—meaning that, for $s > t$, $\bar{F}(s)/\bar{F}(t) \geq \bar{F}(s-t)$—the preceding inequality is reversed. (The preceding strengthens a similar result by Gerber [1] that assumed an increasing (decreasing) failure rate function as opposed to our weaker NBU (NWU) assumption.)

3. It follows from (2) and Jensen's inequality that

$$p(x) \geq \rho^{E[N(x)]+1}. \tag{4}$$

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4. If $R(x)$ denotes the excess (or residual) life at x for the renewal process $N(t), t \geq 0$, then

$$E[X](E[N(x)] + 1) = x + E[R(x)]. \tag{5}$$

Suppose that F is an NBUE (new better than used in expectation) distribution (i.e., that $E[Y - s|Y > s] \leq \mu$). Because $R(x)$ is a remaining lifetime of a randomly aged item whose life distribution F_e is the (limiting) distribution of a remaining lifetime of a randomly aged item whose life distribution is F, it follows that $E[R(x)] \leq E[Y] = \mu$. Because $E[X] = E[Y^2]/2\mu$, this gives that

$$E[N(x)] \leq \frac{2\mu(x+\mu)}{E[Y^2]} - 1.$$

Thus, when F is NBUE, (4) yields the lower bound

$$p(x) \geq \rho^{2\mu(x+\mu)/E[Y^2]}.$$

If F has decreasing mean residual lifetime (i.e., if $E[Y - s|Y > s]$ is nonincreasing in s), then it can be shown (see [5]) that F_e has increasing failure rate, which implies that $E[R(x)] \leq E[X]$. Consequently, in this case, (5) and (4) yield the lower bound

$$p(x) \geq \rho^{2\mu x/E[Y^2]+1},$$

which is an improvement over the F NBUE bound because $E[Y^2] \leq 2\mu^2$ in this case.

5. If it is not too difficult to simulate from F_e, then (2) gives a very efficient way to use simulation to approximate $p(x)$. That is, generate random variables $X_1, X_2, \ldots$ until their sum exceeds x; if it requires m of them, then the value of the estimator from that run is ρ^m. A better procedure is to generate X_1 conditional on it being less than x and generate the others according to F_e, stopping when their sum exceeds x; the estimator in this case is

$$\text{Estimator} = \rho \bar{F}_e(x) + F_e(x) \times \text{Average value of } \rho^m.$$

3. A COMPUTATIONAL PROCEDURE FOR $p(x)$

Again let $N(t), t \geq 0$ be the renewal process having $X_1, X_2, \ldots$ as its interarrival times, where X_i has distribution F_e. Fix a positive integer r and let $Z_1, \ldots, Z_r$ be independent exponential random variables with rate $\alpha = r/x$ that are also independent of the renewal process. Let

$$m(k) = E[\rho^{N(Z_1 + \cdots + Z_k)}], \qquad k = 1, \ldots, r.$$

We propose using $\rho m(r)$ to estimate $p(x)$. To determine $m(k)$, first condition on X_1:

$$m(k) = \int_0^\infty E[\rho^{N(Z_1 + \ldots + Z_k)} | X_1 = s] \frac{\bar{F}(s)}{\mu} \, ds.$$

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Now, for a given s, let R denote the number of partial sums $\sum_{i=1}^{j} Z_i, j = 1, \ldots, k$, that are less than or equal to s. Noting that

$$E[\rho^{N(Z_1+\cdots+Z_k)} \mid X_1 = s, R = j] = \begin{cases} \rho m(k-j) & \text{if } j < k \\ 1 & \text{if } j = k \end{cases}$$

and that

$$P\{R = j \mid X_1 = s\} = \begin{cases} e^{-\alpha s} \dfrac{(\alpha s)^j}{j!} & \text{if } j < k \\ 1 - \displaystyle\sum_{i=0}^{k-1} e^{-\alpha s} \dfrac{(\alpha s)^i}{i!} & \text{if } j = k, \end{cases}$$

we obtain that

$$m(k) = \int_0^\infty \left[\rho \sum_{j=0}^{k-1} m(k-j) e^{-\alpha s} \frac{(\alpha s)^j}{j!} + \left(1 - \sum_{i=0}^{k-1} e^{-\alpha s} \frac{(\alpha s)^i}{i!} \right) \right] \frac{\bar{F}(s)}{\mu} \, ds.$$

Thus, with

$$a_j = \frac{1}{\mu} \int_0^\infty e^{-\alpha s} \frac{(\alpha s)^j}{j!} \bar{F}(s) \, ds, \qquad j = 0, 1, \ldots, r,$$

we obtain that

$$m(k) = \frac{1 - a_0 + \displaystyle\sum_{j=1}^{k-1} (\rho m(k-j) - 1) a_j}{1 - \rho a_0}, \qquad k = 1, \ldots, r.$$

That is, after computing the $r+1$ one-dimensional integrals $a_j, j = 0, \ldots, r$, we can recursively compute $m(1)$, then $m(2)$, up to $m(r)$. Because $Z_1 + \cdots + Z_r$ has mean x and variance x^2/r, by making r sufficiently large, $m(r)$ should be a good approximation to $p(x)/\rho$.

Remarks: A similar computational procedure was developed in [3] to approximate the renewal function. For an analytic approximation to $p(x)$, see [4].

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