

# THE DESIGN OF AN OPTIMAL RETROSPECTIVE RATING PLAN

BY

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## ABSTRACT

A retrospective rating plan, whose insurance premium depends upon an insured's actual loss during the policy period, is a special insurance agreement widely used in liability insurance. In this paper, the design of an optimal retrospective rating plan is analyzed from the perspective of the insured who seeks to minimize its risk exposure in the sense of convex order. In order to reduce the moral hazard, we assume that both the insured and the insurer are obligated to pay more for a larger realization of the loss. Under the further assumptions that the minimum premium is zero, the maximum premium is proportional to the expected indemnity, and the basic premium is the only free parameter in the formula for retrospective premium given by Meyers (2004) and that the basic premium is determined in such a way that the expected retrospective premium equates to the expected indemnity with a positive safety loading, we formally establish the relationship that the insured will suffer more risk for a larger loss conversion factor or a higher maximum premium. These findings suggest that the insured prefers an insurance policy with the expected value premium principle, which is a special retrospective premium principle with zero loss conversion factor. In addition, we show that any admissible retrospective rating plan is dominated by a stop-loss insurance policy. Finally, the optimal retention of a stop-loss insurance is derived numerically under the criterion of minimizing the risk-adjusted value of the insured's liability where the liability valuation is carried out using the cost-of-capital approach based on the conditional value at risk.

## KEYWORDS

Conditional value at risk, convex order, moral hazard, optimal retrospective rating plan, stop-loss insurance.

## 1. INTRODUCTION

Insurance has become an indispensable tool for an individual or a corporation to manage its risk. An income earner who wishes to protect his/her surviving

family's income in the event of his/her death can achieve the aim by purchasing a life insurance policy. Similarly, car insurance can be used to recover some of the losses incurred to the driver in the event of a car accident. Other insurance policies such as unemployment insurance, health and dental insurance, and product liability, are typically available to a corporation to provide some form of protection to its employees and the corporation itself.

By transferring all or part of its risk to an insurance company (i.e. the insurer), the individual or the corporation (i.e. the insured) incurs an additional cost in the form of the insurance premium. The insurance premium is expected to increase with higher expected risk that is ceded to the insurer. This implies that the insured can transfer more of its risk to the insurer at the expense of a higher insurance premium. If the insured were to reduce the cost of insurance, its risk exposure will be much higher. This classical tradeoff implies that there exists an optimal strategy between risk retaining and risk transferring. By formulating as an optimization problem with an appropriate objective and constraints, this allows the insured to optimally determine the best strategy to insure its risk.

The pioneering work on optimal insurance is attributed to Borch (1960) who shows that the stop-loss insurance is optimal under the criterion of minimizing the variance of the insured's retained risk when the insurance premium is calculated by the expected value premium principle. By maximizing the expected utility of the final wealth of a risk-averse insured, Arrow (1963) derives a similar result justifying the optimality of stop-loss insurance. These classical results have been extended in a number of interesting directions. One generalization is to consider more complicated premium principles, as opposed to the standard expected value premium principle. These results can be found in Raviv (1979), Young (1999), Kaluszka (2001), Gajek and Zagrodny (2004), Kaluszka and Okolewski (2008), Bernard and Tian (2009), and references therein. Due to the popularity of risk measures such as value at risk (VaR) and conditional value at risk (CVaR) in quantifying the financial and insurance risks, the risk measure based optimal insurance/reinsurance problems have been studied by many researchers in the past ten years. See, for example, Cai and Tan (2007), Cai *et al.* (2008), Balbás *et al.* (2009), Cheung (2010), Chi and Tan (2011), Chi (2012), Asimit *et al.* (2013), Cong and Tan (2014), and references therein. In the afore-mentioned studies, most of the optimization criteria have a common property of preserving the convex order, which enables us to use a unified approach to tackle optimal insurance problems. For unified treatments of the problems, we refer to Van Heerwaarden *et al.* (1989), Gollier and Schlesinger (1996) and Chi and Lin (2014).

The existing literature typically assumes that the insurance premium is known at the inception of the insurance contract and is calculated based on the distribution of indemnity about which the insured and the insurer have symmetric information. However, in some insurance practice, not only the indemnity distribution but also its actual realization are used to compute the insurance premium. A special example is the retrospective rating plan which is widely

used in liability insurance and is summarized in Meyers (2004). More precisely, let  $\pi(\cdot)$  denote a premium principle of a retrospective rating plan, then for any non-negative risk  $Y$ ,

$$\pi(Y) = \min \{ \max \{ (B(Y) + L(Y) \times Y) T, G(Y) \}, H(Y) \}, \quad (1.1)$$

where  $B(Y) \geq 0$  is the basic premium,  $L(Y) \geq 0$  is the loss conversion factor that covers the loss adjustment expenses,  $T > 1$  is the tax multiplier including premium tax, and  $G(Y) \geq 0$  and  $H(Y) \geq 0$  represent the minimum premium and the maximum premium, respectively. Compared to other well-known premium principles such as the expected value premium principle and the standard deviation premium principle, the retrospective rating plan based premium principle (1.1) is considerably more complicated. First, it requires five parameters in order to fully specify the above premium principle. Second, while the parameters  $B$ ,  $L$ ,  $G$ ,  $H$  depend explicitly on the non-negative risk  $Y$  as indicated in (1.1), in practice some of these parameters could be related to  $Y$  implicitly. The complexity of the premium principle also implies that some simplifying assumptions are typically imposed to fully specify the premium principle. For example, as discussed in Meyers (2004), in practice four of the parameters are agreed upon between the insured and the insurer at the inception of the insurance agreement. In this case, the agreed upon parameters may depend explicitly on, implicitly on, or even be independent of  $Y$ . Once these four parameters are well-specified, the remaining “free” fifth parameter is then implied by a desired expected retrospective premium so that the fifth parameter depends explicitly on  $Y$ . The parameter  $L(Y)$  or  $B(Y)$  is often set to be the free parameter. Consequently, all these five parameters have deterministic values when the insurance contract is in effect. Another distinctive feature of the above retrospective rating plan is that the resulting insurance premium depends on the insured’s actual loss. This implies that the actual insurance premium is not known until the insurance policy has matured. This feature is also well suited for the above mentioned optimal insurance/reinsurance models since the underlying problems are typically formulated as one-period optimization problems. For these reasons, it is very interesting to develop the optimal strategy for the above retrospective rating plan. However, to the best of our knowledge, there is hardly any paper that studies its optimality.

In this paper, the design of an optimal retrospective rating plan is studied from the perspective of an insured who seeks to minimize its risk exposure in the sense of convex order, where the insurance premium is calculated by a retrospective premium principle (1.1). For simplicity, the minimum premium and the maximum premium are assumed to be

$$G(Y) = 0 \quad \text{and} \quad H(Y) = (1 + \vartheta)\mathbb{E}[Y] \quad (1.2)$$

for some loading coefficient  $\vartheta > 0$ . Moreover, we assume that the basic premium  $B(Y)$  is the only free parameter which is the minimal solution to the following equation:

$$\mathbb{E}[\pi(Y)] = (1 + \rho)\mathbb{E}[Y], \quad (1.3)$$

where  $\rho \in (0, \vartheta)$  is the safety loading coefficient. In order to reduce the moral hazard, we further assume that both the insured and the insurer are obligated to pay more for a larger realization of loss. The design of an optimal retrospective rating plan is analyzed from two aspects. On one hand, we analyze the effects of the changes in the parameters of retrospective premium principle on the insured's risk exposure. More specifically, by a comparative static analysis, we find that the insured will suffer more risk for a larger loss conversion factor or a higher  $\vartheta$ . These findings suggest that the insured will prefer an insurance contract with the expected value principle which is a special retrospective premium principle with zero loss conversion factor. On the other hand, by fixing the retrospective premium principle we discuss the optimal ceded strategies. More precisely, we show, via a constructive approach, any admissible retrospective rating plan is dominated by a stop-loss insurance contract. Noting that the expected value premium principle is a special case, our result generalizes Theorem 6.1 of Van Heerwaarden *et al.* (1989). Finally, to illustrate the applicability of our results, we derive numerically the optimal retention of stop-loss insurance under the criterion of minimizing the risk-adjusted value of the insured's liability where the liability valuation is carried out using a cost-of-capital approach based on CVaR, and analyze the effects of loss conversion factor and the loading coefficient  $\vartheta$  on the optimal retention.

The rest of this paper is organized as follows. In Section 2, we describe a retrospective rating plan. Under such a plan, the effects of loss conversion factor and the loading coefficient in the maximum premium on the insured's total risk exposure are analyzed in Section 3. Section 4 shows that any admissible insurance policy is dominated by a stop-loss insurance contract. To illustrate the applicability of the results established in Section 4, we derive the optimal retention of stop-loss insurance under the criterion of minimizing the risk-adjusted value of an insured's liability in Section 5. Finally, some concluding remarks are provided in Section 6.

## 2. A RETROSPECTIVE RATING PLAN

Suppose  $X$  denotes the amount of loss an insured is facing over a given time period. We assume  $X$  is a non-negative random variable defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with cumulative distribution function (cdf)  $F_X(x) = \mathbb{P}(X \leq x)$ ,  $x \geq 0$  and  $0 < \mathbb{E}[X] < \infty$ . A retrospective rating plan is concerned with a partition of  $X$  into two parts:  $f(X)$  and  $R_f(X)$ , where  $f(X)$ , satisfying  $0 \leq f(X) \leq X$ , represents the portion of the loss that is ceded to an insurer, while  $R_f(X) = X - f(X)$  represents the loss retained by the insured. The functions  $f(x)$  and  $R_f(x)$  are known as the ceded and retained loss functions, respectively. As pointed out earlier, we assume both the insured and the insurer are obligated to pay more for a larger realization of loss in order to reduce the moral hazard. In other words, both the ceded loss function  $f(x)$  and the retained loss function  $R_f(x)$  should be increasing. As shown in Chi and Tan (2011), it is equivalent to

that the ceded loss function is increasing and Lipschitz continuous, i.e.

$$0 \leq f(x_2) - f(x_1) \leq x_2 - x_1, \quad \forall 0 \leq x_1 \leq x_2. \tag{2.1}$$

Thus, the set of admissible ceded loss functions is given by

$$\mathfrak{C} \triangleq \{0 \leq f(x) \leq x : \text{both } f(x) \text{ and } R_f(x) \text{ are increasing functions}\}. \tag{2.2}$$

Under a retrospective rating plan, the insurance premium is calculated according to (1.1). However, it was also pointed out that the general form of the retrospective premium principle can be very complicated and hence some simplifying assumptions are often made to ensure tractability and ease of analysis. The assumptions are collected in Assumption 2.1:

**Assumption 2.1.** (1) The maximum premium  $H(Y)$  and the minimum premium  $G(Y)$  are set as in (1.2); (2) the loss conversion factor  $L(Y)$  is independent of  $Y$  and is rewritten by  $L$  for brevity; (3) the basic premium  $B(Y)$  is the only free parameter determined by (1.3).

Under the above assumptions, the retrospective premium principle  $\pi(\cdot)$  can be rewritten by

$$\pi(Y) = \min \{(B(Y) + L \times Y)T, (1 + \vartheta)\mathbb{E}[Y]\} \tag{2.3}$$

for any non-negative random variable  $Y$ . It is easy to see from the above equation and (1.3) that

$$(1 + \vartheta)\mathbb{E}[Y] > B(Y) \times T \quad \text{if and only if} \quad \mathbb{E}[Y] > 0. \tag{2.4}$$

In particular, when  $L = 0$ , it follows from the above equation that

$$\pi(Y) = (1 + \rho)\mathbb{E}[Y], \tag{2.5}$$

which is exactly the expected value principle. Moreover, both (1.3) and (2.3) ensure the free parameter  $B(Y)$  can be easily derived from the following equation:

$$(\vartheta - \rho)\mathbb{E}[Y] = \mathbb{E} [((1 + \vartheta)\mathbb{E}[Y] - (B(Y) + LY)T)_+], \tag{2.6}$$

where  $(x)_+ = \max(x, 0)$ .

In the presence of a retrospective rating plan  $f(x)$ , the risk the insured is facing is no longer  $X$ ; it is the total risk exposure  $T_f(X)$  which is defined as

$$T_f(X) = R_f(X) + \pi(f(X)). \tag{2.7}$$

Recall that the insurance premium depends on the actual loss. Thus, in order to mitigate the moral hazard, we should assume  $T'_f(x) \leq 1$ , a.s. for any  $f \in \mathfrak{C}$ , which is equivalent to

$$TL \leq 1. \tag{2.8}$$

To quantify the insured's total risk exposure  $T_f(X)$ , a number of risk measures, including those that are coherent and convex, can be used. For a detailed

discussion of risk measures, see Artzner *et al.* (1999) and Föllmer and Schied (2004). It is well-known that most risk measures, including the CVaR, have a common property of preserving the convex order. To ensure that our results are as general as possible so that our results are applicable to a wide range of insured's risk preferences, in this paper we only assume that the adopted risk measure preserves the convex order. Specifically, let  $\Psi(\cdot)$ , a mapping from the set of non-negative random variables to  $\mathbb{R}$ , be the risk measure the insured uses, then

$$\Psi(Y) \leq \Psi(Z), \quad \text{if } Y \leq_{cx} Z.^1$$

The assumption of convex order preservation is quite reasonable as the convex order is frequently used to compare the risks in finance and insurance. Moreover, such an assumption is quite general, as it is not only satisfied by the well-known coherent risk measures that are law-invariant and continuous from above but also followed by the expected utility of the final wealth of a risk-averse insured.

Note that  $T_f(X)$  depends on the retrospective premium principle  $\pi(\cdot)$  and the ceded loss function  $f(x)$ . We will analyze the effects of these two factors on the insured's risk measure  $\Psi(T_f(X))$  in the next two sections respectively.

### 3. THE EFFECTS OF RETROSPECTIVE PREMIUM PRINCIPLE

Recall that the retrospective premium principle defined in this paper has four non-free parameters, namely the loss conversion factor  $L$ , the loading coefficient  $\vartheta$  for the maximum premium, the loading coefficient  $\rho$  in (1.3) and the tax multiplier  $T$ . In this section, we will study the effects of these parameters on the insured's risk exposure  $T_f(X)$ . In particular, we will only focus on  $L$  and  $\vartheta$ , since the tax rate is usually set by the government and is beyond our control and we also fix  $\rho$  such that the expected retrospective premium is a constant. The main result of this section is obtained in the following theorem.

**Theorem 3.1.** *Under Assumption 2.1, given a ceded strategy  $f(x) \in \mathfrak{C}$ , the insured's total risk exposure  $T_f(X)$  is increasing in the loss conversion factor  $L$  and the loading coefficient  $\vartheta$  in the sense of convex order.*

**Proof.** As  $T_f(X)$  is a function of the loss conversion factor  $L$ , we rewrite  $T_f(X)$  by  $T_f(X; L)$  to emphasize this dependence. We first show that

$$T_f(X; L_1) \leq_{cx} T_f(X; L_2), \quad \forall 0 \leq L_1 < L_2 \leq \frac{1}{T}. \tag{3.1}$$

Specifically, we rewrite the basic premium by  $B_f^i$  for  $L_i$ , then it follows from (1.3) and (2.3) that

$$T_f(X; L_i) = R_f(X) + (1 + \vartheta)\mathbb{E}[f(X)] - \psi_i(X), \quad i = 1, 2, \tag{3.2}$$

where

$$\psi_i(x) \triangleq ((1 + \vartheta)\mathbb{E}[f(X)] - B_f^i T - L_i T f(x))_+, \quad x \geq 0.$$

The following proof is divided into two cases:  $\mathbb{E}[f(X)] = 0$  and  $\mathbb{E}[f(X)] > 0$ .

- If  $\mathbb{E}[f(X)] = 0$ , we have  $f(X) = 0, a.s.$  and  $\psi_i(x) = 0$  for  $i = 1, 2$ , then we can see from (3.2) that  $T_f(X; L_i) = X, a.s.$ . Thus, the result is obtained.
- Otherwise, if  $\mathbb{E}[f(X)] > 0$ , it follows from (2.6) that

$$\mathbb{E}[\psi_i(X)] = (\vartheta - \rho)\mathbb{E}[f(X)], \tag{3.3}$$

then we have  $B_f^1 \geq B_f^2$ . By defining

$$x_i \triangleq \sup \{x \geq 0 : \psi_i(x) > 0\}, \quad i = 1, 2, \tag{3.4}$$

it follows from (2.1) and (2.4) that  $x_i > 0$ . Moreover, we must have  $x_2 \leq x_1$ ; otherwise, if  $x_2 > x_1$ , we have

$$\psi_2(x) - \psi_1(x) = \begin{cases} (B_f^1 - B_f^2)T + (L_1 - L_2)Tf(x), & x \in [0, x_1]; \\ \psi_2(x) > 0, & x \in (x_1, x_2); \\ 0, & x \geq x_2. \end{cases}$$

From the above equation, it is easy to see that  $\psi_2(x) - \psi_1(x)$  is decreasing and continuous over  $[0, x_1]$ , then it is positive over  $[0, x_2)$  and is equal to zero for  $x \geq x_2$ . Since it is assumed that  $\mathbb{E}[f(X)] > 0$ , then using (3.3) and (3.4), we must have  $\mathbb{E}[\psi_2(X)] > \mathbb{E}[\psi_1(X)]$ , which contradicts (3.3).

Now, using the similar arguments, it is easy to see that over  $[0, x_2]$ ,  $\psi_2(x) - \psi_1(x)$  is a decreasing continuous function with  $\psi_2(0) - \psi_1(0) = (B_f^1 - B_f^2)T \geq 0$ . Moreover, for any  $x > x_2$ , we have  $\psi_2(x) - \psi_1(x) = -\psi_1(x)$  which is a non-positive increasing function. As a consequence, we get that  $\psi_1(x)$  up-crosses  $\psi_2(x)$ .<sup>2</sup> Using Lemma 3 in Ohlin (1969) and (3.3), we get

$$(1 + \vartheta)\mathbb{E}[f(X)] - \psi_1(X) \leq_{cx} (1 + \vartheta)\mathbb{E}[f(X)] - \psi_2(X).$$

Moreover, it is easy to see from (2.1) that the retained loss function  $R_f(x)$  is comonotonic with  $(1 + \vartheta)\mathbb{E}[f(X)] - \psi_i(x)$ , then (3.1) can be obtained by using Corollary 1 in Dhaene *et al.* (2002).

Next, we similarly rewrite  $T_f(X)$  by  $T_f^\vartheta(X)$  and  $B(f(X))$  by  $B_f^\vartheta$  to emphasize their dependence on the loading coefficient  $\vartheta$ . We demonstrate that

$$T_f^{\vartheta_1}(X) \leq_{cx} T_f^{\vartheta_2}(X), \quad \forall \rho < \vartheta_1 < \vartheta_2.$$

More specifically, if  $\mathbb{E}[f(X)] = 0$ , the similar analysis leads to  $T_f^{\vartheta_i}(X) = X, a.s.$  so that the above equation holds. Otherwise, if  $\mathbb{E}[f(X)] > 0$ , as in (3.2), we have

$$T_f^{\vartheta_i}(X) = R_f(X) + \phi_i(X), \quad i = 1, 2,$$

where

$$\phi_i(x) \triangleq \min \left\{ B_f^{\vartheta_i} T + LTf(x), (1 + \vartheta_i)\mathbb{E}[f(X)] \right\}, \quad x \geq 0.$$

The definition of  $\phi_i(x)$  together with (1.3) implies  $B_f^{\vartheta_1} \geq B_f^{\vartheta_2}$  and

$$\mathbb{E}[\phi_i(X)] = (1 + \rho)\mathbb{E}[f(X)], \quad i = 1, 2.$$

By defining

$$\tilde{x}_i \triangleq \sup \left\{ x \geq 0 : f(x) \leq \frac{(1 + \vartheta_i)\mathbb{E}[f(X)] - B_f^{\vartheta_i} T}{LT} \right\},$$

we obviously have  $0 < \tilde{x}_1 \leq \tilde{x}_2$  and

$$\phi_1(x) - \phi_2(x) = \begin{cases} (B_f^{\vartheta_1} - B_f^{\vartheta_2})T \geq 0, & 0 \leq x \leq \tilde{x}_1; \\ (1 + \vartheta_1)\mathbb{E}[f(X)] - B_f^{\vartheta_2} T - LTf(x) \\ \quad \text{is decreasing,} & \tilde{x}_1 < x \leq \tilde{x}_2; \\ (\vartheta_1 - \vartheta_2)\mathbb{E}[f(X)] < 0, & x > \tilde{x}_2. \end{cases}$$

Thus,  $\phi_2(x)$  up-crosses  $\phi_1(x)$ . Using Lemma 3 in Ohlin (1969), we have  $\phi_1(X) \leq_{cx} \phi_2(X)$ . Moreover, it follows from (2.1) that  $R_f(x)$  and  $\phi_i(x)$  are comonotonic, then using Corollary 1 in Dhaene *et al.* (2002) again, we get the final result. The proof is thus complete. ■

From the above theorem, we know that the insured who buys a retrospective rating plan pays less basic premium but suffers more risk for the premium principle with the larger loss conversion factor or the larger loading coefficient for maximum premium in the sense of convex order. As a result, a risk-averse insured will choose a retrospective rating plan with the premium principle including smaller loss conversion factor or lower loading coefficient. In the extreme case the insured will prefer the insurance contract with expected value premium principle as it is a special retrospective premium principle with zero loss conversion factor. Moreover, these findings are available for all the insureds whose risk preferences preserve the convex order. In the next section, we will fix the retrospective premium principle and study an optimal ceded strategy.

#### 4. AN OPTIMAL CEDED STRATEGY

In this section, we will investigate optimal retrospective rating plans for an insured under a retrospective premium principle. More specifically, we attempt to solve the following optimal insurance problem:

$$\min_{f \in \mathcal{C}} \Psi(T_f(X)). \tag{4.1}$$



Recall that the basic premium  $B(f(X))$  is an implicit functional of the ceded loss  $f(X)$ . It becomes very challenging to solve the above infinite-dimensional minimization problem. However, using a constructive approach, we obtain the main result of this section in the following theorem.

**Theorem 4.1.** *Under the retrospective premium principle  $\pi(\cdot)$  in (2.3), any admissible insurance contract is dominated by a stop-loss insurance policy. More specifically, for any  $f \in \mathfrak{C}$ , there exists a stop-loss insurance  $f_d(x) = (x - d)_+$  for some  $d \geq 0$  such that*

$$\mathbb{E}[f_d(X)] = \mathbb{E}[f(X)] \quad \text{and} \quad T_{f_d}(X) \leq_{cx} T_f(X). \tag{4.2}$$

As a result, the optimal insurance model (4.1) is equivalent to

$$\min_{d \geq 0} \Psi(T_{f_d}(X)). \tag{4.3}$$

**Proof.** When  $L = 0$ ,  $\pi(\cdot)$  is the expected value principle and the result has been obtained by Theorem 6.1 in Van Heerwaarden *et al.* (1989). Therefore, we will assume  $L > 0$  in the following proof.

For any  $f \in \mathfrak{C}$ , if  $\mathbb{E}[f(X)] = 0$ , we have  $f(X) = 0, a.s.$  and hence  $T_f(X) = X, a.s.$ , then the result is trivial. Thus, we assume  $\mathbb{E}[f(X)] > 0$ . To proceed, we define

$$x_f \triangleq \sup \left\{ x \geq 0 : f(x) \leq ((1 + \vartheta)\mathbb{E}[f(X)]/T - B_f) / L \right\}, \tag{4.4}$$

where  $B_f$  is an abbreviation of  $B(f(X))$ . It is easy to see from (2.4) that  $x_f > 0$ .

We first show that  $f(x)$  is dominated by a two-layer insurance policy, and the proof is divided into two cases:  $x_f = \infty$  and  $x_f < \infty$ .

(i) For the case  $x_f = \infty$ , we denote

$$w(t) \triangleq \mathbb{E}[\mathcal{L}_{(t, t + ((1 + \vartheta)\mathbb{E}[f(X)]/T - B_f)/L)}(X)], \quad \forall t \geq 0,$$

where  $\mathcal{L}_{(a,b]}(Y)$  represents a layer  $(a, b]$  of a non-negative risk  $Y$  and is formally defined as

$$\mathcal{L}_{(a,b]}(Y) \triangleq \min \{ (Y - a)_+, b - a \}, \quad 0 \leq a \leq b.$$

It is easy to see that  $w(t)$  is a decreasing continuous function with  $w(\infty) = 0$ . Moreover, we have

$$\begin{aligned} w(0) &= \mathbb{E} \left[ \min \{ X, ((1 + \vartheta)\mathbb{E}[f(X)]/T - B_f) / L \} \right] \\ &\geq \mathbb{E} \left[ \min \{ f(X), ((1 + \vartheta)\mathbb{E}[f(X)]/T - B_f) / L \} \right] = \mathbb{E}[f(X)], \end{aligned}$$

where the last equality is derived by (4.4) and the assumption of  $x_f = \infty$ . Consequently, there must exist a  $t_0 \in [0, \infty)$  such that  $w(t_0) = \mathbb{E}[f(X)]$ .

Further, since it is assumed that  $x_f = \infty$ , we have

$$\pi(f(X)) = T \times (B_f + L \times f(X)),$$

then it follows from (1.3) that

$$B_f \times T = (1 + \rho - LT)\mathbb{E}[f(X)],$$

which in turn implies

$$T_f(X) = LTX + (1 - LT)R_f(X) + (1 + \rho - LT)\mathbb{E}[f(X)].$$

Now, we introduce a layer insurance policy with ceded loss function

$$\tilde{f}(x) \triangleq \mathcal{L}_{(t_0, t_0 + ((1+\vartheta)\mathbb{E}[f(X)]/T - B_f)/L]}(x), \quad x \geq 0. \tag{4.5}$$

Then we have

$$\mathbb{E}[\tilde{f}(X)] = w(t_0) = \mathbb{E}[f(X)].$$

Furthermore, it follows from (2.6) that

$$\begin{aligned} \frac{(\vartheta - \rho)}{LT} w(t_0) &= \mathbb{E} \left[ \left( (1 + \vartheta)w(t_0) - B_{\tilde{f}}T / TL - \tilde{f}(X) \right)_+ \right] \\ &= ((1 + \vartheta)w(t_0) - B_{\tilde{f}}T) / TL \\ &\quad - \int_0^{((1+\vartheta)w(t_0) - B_{\tilde{f}}T) / TL} S_{\tilde{f}(X)}(x) dx, \end{aligned} \tag{4.6}$$

where  $S_Y(t) = 1 - F_Y(t)$  is the survival function of random variable  $Y$ . Similarly, we have

$$\begin{aligned} \frac{\vartheta - \rho}{LT} w(t_0) &= ((1 + \vartheta)w(t_0) - B_fT) / TL - \int_0^{((1+\vartheta)w(t_0) - B_fT) / TL} S_{f(X)}(x) dx \\ &= ((1 + \vartheta)w(t_0) - B_fT) / TL - w(t_0) \\ &= ((1 + \vartheta)w(t_0) - B_fT) / TL - \int_0^{((1+\vartheta)w(t_0) - B_fT) / TL} \tilde{f}_{f(X)}(x) dx, \end{aligned}$$

where the second equality follows from the assumption  $x_f = \infty$  and the last equality is derived by the fact  $\tilde{f}(x) \leq ((1 + \vartheta)w(t_0) - B_fT) / TL$  for any  $x \geq 0$ . By comparing the above equation with (4.6), since it is assumed that  $\mathbb{E}[f(X)] > 0$ , then it is easy to get

$$B_{\tilde{f}} = B_f,$$

which in turn implies  $x_{\tilde{f}} = \infty$ . As a result, we have

$$T_{\tilde{f}}(X) = LTX + (1 - LT)R_{\tilde{f}}(X) + (1 + \rho - LT)w(t_0).$$

From (2.1), we know that the retained loss function is increasing and Lipschitz continuous, then using (2.8), it is easy to show that the function  $LTx + (1 - LT)R_f(x)$  up-crosses  $LTx + (1 - LT)R_{\hat{f}}(x)$ . Further, note that  $\mathbb{E}[T_f(X)] = \mathbb{E}[T_{\hat{f}}(X)]$ , then it follows from Lemma 3 in Ohlin (1969) that  $T_{\hat{f}}(X) \leq_{cx} T_f(X)$ .

(ii) Now, we consider the case  $0 < x_f < \infty$ . Building upon  $f$ , we define

$$f_1(x) \triangleq \begin{cases} f(x), & 0 \leq x \leq x_f; \\ f(x_f) + (x - d_f)_+, & x > x_f, \end{cases}$$

where  $d_f \geq x_f$  is determined by  $\mathbb{E}[f_1(X)] = \mathbb{E}[f(X)]$ . Using (2.6), we obtain

$$\begin{aligned} (\vartheta - \rho)\mathbb{E}[f(X)] &= \mathbb{E} \left[ ((1 + \vartheta)\mathbb{E}[f(X)] - (B_f + Lf(X))T) 1_{\{X \leq x_f\}} \right] \\ &= \mathbb{E} \left[ ((1 + \vartheta)\mathbb{E}[f_1(X)] - (B_f + Lf_1(X))T)_+ \right], \end{aligned}$$

where  $1_A$  is an indicator function of the event  $A$ . By comparing the above equation with (2.6) for  $Y = f_1(X)$ , it is easy to get

$$B_{f_1} = B_f. \tag{4.7}$$

Moreover, according to the definition of  $T_f(X)$ , we have

$$T_f(x) = \begin{cases} B_f T + x - (1 - LT)f(x), & 0 \leq x \leq x_f; \\ R_f(x) + (1 + \vartheta)\mathbb{E}[f(X)], & x > x_f. \end{cases} \tag{4.8}$$

By (2.1) and (4.7), it is easy to see that  $T_f(x)$  up-crosses  $T_{f_1}(x)$ . Moreover, we can get from (1.3) that

$$\mathbb{E}[T_f(X)] = \mathbb{E}[R_f(X)] + \mathbb{E}[\pi(f(X))] = \mathbb{E}[X] + \rho\mathbb{E}[f(X)] = \mathbb{E}[T_{f_1}(X)]. \tag{4.9}$$

Consequently, using Lemma 3 in Ohlin (1969) again, we get

$$T_{f_1}(X) \leq_{cx} T_f(X).$$

Further, building upon  $f_1$ , it follows from (2.1) that there must exist an  $a$  satisfying  $0 \leq a \leq x_f - f(x_f)$  such that  $\mathbb{E}[f_1(X)] = \mathbb{E}[f_2(X)]$ , where  $f_2(x)$  is a ceded loss function defined as

$$f_2(x) = \mathcal{L}_{(a, a+f(x_f))}(x) + (x - d_f)_+, \quad x \geq 0.$$

Using (2.1) again, it is easy to verify that  $f_2(x)$  up-crosses  $f_1(x)$ , then it follows from Lemma 3 in Ohlin (1969) that  $f_1(X) \leq_{cx} f_2(X)$ , which together with (2.6) implies

$$B_{f_1} \leq B_{f_2}.$$

If  $B_{f_1} = B_{f_2}$ , let

$$\hat{f}(x) = f_2(x), \quad x \geq 0, \tag{4.10}$$

then we have  $x_{\hat{f}} = x_{f_1} = x_{f_2} = d_f$  and  $\hat{f}(x_{\hat{f}}) = f(x_f)$ . For this case, it is easy to see from (4.8) that  $T_{\hat{f}_1}(x)$  up-crosses  $T_f(x)$ . Moreover, (4.9) implies  $\mathbb{E}[T_{f_i}(X)] = \mathbb{E}[X] + \rho\mathbb{E}[f(X)]$  for  $i = 1, 2$ , then it follows from Lemma 3 in Ohlin (1969) that

$$T_{\hat{f}}(X) \leq_{cx} T_{f_1}(X) \leq_{cx} T_f(X).$$

Otherwise, if  $B_{f_1} < B_{f_2}$ , then  $f_2(x_{f_2}) < f(x_f)$  such that

$$a < x_{f_2} \leq a + f(x_f) < x_f \leq d_f = x_{f_1}.$$

For any  $0 \leq x \leq a$ , we have  $f_2(x) = 0$ , then it follows from the Lipschitz-continuous property of  $f_1(x)$  and (4.8) that

$$T_{f_1}(x) - T_{f_2}(x) \leq 0.$$

For any  $x \in (a, a + f(x_f)]$ , (4.8) implies  $T'_{f_2}(x) = 0$  or  $LT$  but  $T'_{f_1}(x) \geq LT$ . Thus, over this interval,  $T_{f_1}(x) - T_{f_2}(x)$  is increasing. Over the interval  $(a + f(x_f), d_f]$ , we have

$$T_{f_1}(x) - T_{f_2}(x) = B_{f_1}T - (1 + \vartheta)\mathbb{E}[f(X)] + f(x_f) - (1 - LT)f_1(x),$$

which is decreasing in  $x$  and is equal to 0 when  $x = d_f$ . Moreover, it is easy to see from (4.8) that  $T_{f_1}(x) - T_{f_2}(x) = 0$  for any  $x > d_f$ . Collecting the above arguments yields that  $T_{f_1}(x)$  up-crosses  $T_{f_2}(x)$ . Thus, Lemma 3 in Ohlin (1969) leads to

$$T_{f_2}(X) \leq_{cx} T_{f_1}(X).$$

Similar to the construction of  $f_1$  from  $f$  and based on  $f_2$ , we can construct a ceded loss function of the form

$$\hat{f}(x) = \mathcal{L}_{(a, a+f_2(x_{f_2}))}(x) + (x - d_{f_2})_+, \tag{4.11}$$

for some  $x_{f_2} \leq d_{f_2} \leq d_f$  such that

$$\mathbb{E}[f_2(X)] = \mathbb{E}[\hat{f}(X)], \quad x_f = d_{f_2}, \quad \hat{f}(x_f) = f_2(x_{f_2}) \quad \text{and} \quad T_f(X) \leq_{cx} T_{f_2}(X) \leq_{cx} T_{\hat{f}}(X).$$

In summary, any admissible insurance contract  $f(x)$  is dominated by a layer insurance policy  $\tilde{f}(x)$  or a two-layer insurance policy  $\hat{f}(x)$ . Moreover, from (4.5), (4.10) and (4.11), we find that  $\tilde{f}(x)$  and  $\hat{f}(x)$  have a similar structure and can be written in a unified form as

$$h(x) = \mathcal{L}_{(a, a+((1+\vartheta)\mathbb{E}[h(X)] - B_h T) / TL)}(x) + (x - x_h)_+.$$

Furthermore, for any ceded loss function  $h$  with the above structure, there must exist a stop-loss insurance policy  $f_d(x)$  with  $a \leq d \leq x_h - ((1 + \vartheta)\mathbb{E}[h(X)] - B_h T) / TL$  such that  $\mathbb{E}[f_d(X)] = \mathbb{E}[h(X)]$ . It is easy to see from (2.1) that  $f_d(x)$  up-crosses  $h(x)$ , then we have  $h(X) \leq_{cx} f_d(X)$ , which together with (2.6) imply  $B_h \leq B_{f_d}$ . As a consequence, we can see from Figure 1 that

$$d < x_{f_d} \leq d + ((1 + \vartheta)\mathbb{E}[h(X)] - B_h T) / TL \leq x_h.$$

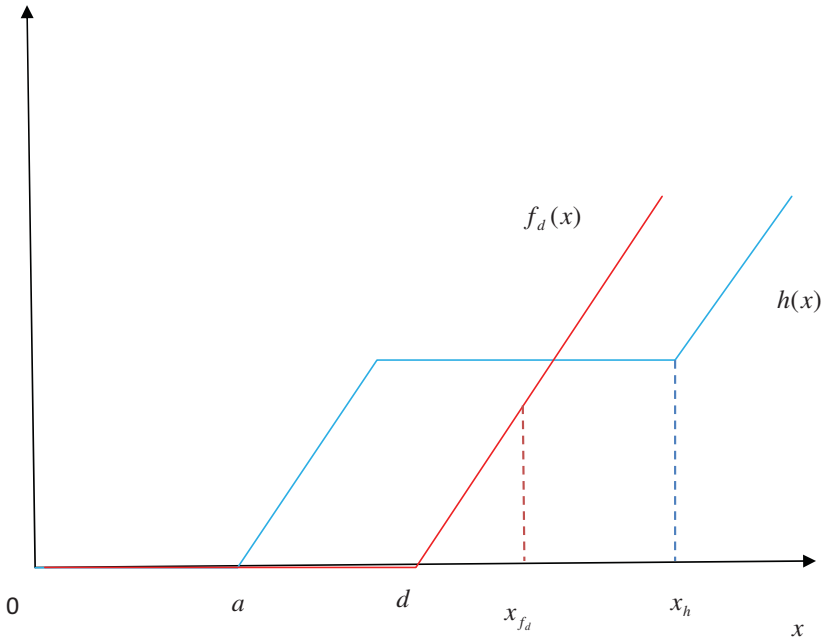


FIGURE 1: The ceded loss functions  $h(x)$  and  $f_d(x)$ . (Color online)

We now demonstrate that  $T_h(x)$  up-crosses  $T_{f_d}(x)$ . Specifically, for any  $x \geq x_{f_d}$ , it is easy to see that  $T_{f_d}(x) = d + (1 + \vartheta)\mathbb{E}[h(X)]$  and that  $T_h(x)$  is increasing with  $T_h(x_h) \geq T_{f_d}(x_h)$ . Furthermore, for any  $0 \leq x < x_{f_d}$ , it follows from (4.8) that

$$T_{f_d}(x) - T_h(x) = T \times (B_{f_d} - B_h) + (1 - LT)(h(x) - f_d(x)) \geq 0.$$

Collecting the above arguments leads to the result.

Finally, we know from (4.9) that  $\mathbb{E}[T_{f_d}(X)] = \mathbb{E}[X] + \rho\mathbb{E}[f_d(X)] = \mathbb{E}[T_h(X)]$ , then Lemma 3 in Ohlin (1969) implies  $T_{f_d}(X) \leq_{cx} T_h(X)$ . Hence, any admissible insurance policy is suboptimal to a stop-loss insurance contract and this completes the proof. ■

By the above theorem, we know that any admissible insurance policy is dominated by a stop-loss insurance contract under the retrospective premium principle (2.3), and hence the study of the infinite-dimensional optimal insurance model (4.1) is simplified to analyzing a minimization problem of just one variable (4.3), once, by hypothesis, the risk measure preserves the convex order. It seems impossible to solve this minimization problem without the specification of the objective functional  $\Psi(\cdot)$ . In the next section, we will derive the optimal retention of stop-loss insurance under the criterion of minimizing the risk-adjusted value of an insured’s liability where the liability valuation is carried out by using a cost-of-capital approach based on the CVaR risk measure.

## 5. MINIMIZING THE INSURED'S RISK-ADJUSTED LIABILITY

In this section, we will investigate the optimal insurance model (4.1) under the criterion of minimizing the risk-adjusted value of an insured's liability. The liability valuation is carried out using a cost-of-capital approach which was introduced by the Swiss insurance supervisor (see Swiss Federal Office of Private Insurance, 2006) and was accepted by the European Union insurance regulation (See European Commission, 2009). Under such an approach, the risk-adjusted value of the insured's liability is composed of two components: best estimate and risk margin. The best estimate is usually represented by the expected liability,  $\mathbb{E}[T_f(X)]$ . In addition, some capital is required to hold for partly covering the unexpected loss,  $T_f(X) - \mathbb{E}[T_f(X)]$ , the difference between the risk and its expectation. The unexpected loss is usually quantified by the VaR or CVaR risk measure, which can be defined formally as follows:

**Definition 5.1.** *The VaR of a random variable  $Z$  at a confidence level  $1 - \alpha$  where  $0 < \alpha < 1$  is defined as*

$$VaR_\alpha(Z) \triangleq \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq \alpha\}. \quad (5.1)$$

*Based upon the definition of VaR, CVaR of  $Z$  at a confidence level  $1 - \alpha$  is defined as*

$$CVaR_\alpha(Z) \triangleq \frac{1}{\alpha} \int_0^\alpha VaR_s(Z) ds. \quad (5.2)$$

From the above definition of  $VaR_\alpha(Z)$ , we have

$$VaR_\alpha(Z) \leq z \quad \text{if and only if} \quad S_Z(z) \leq \alpha \quad (5.3)$$

for any  $z \in \mathbb{R}$ . Moreover, for any increasing continuous function  $g(x)$ , we have (see Theorem 1 in Dhaene *et al.*, 2002)

$$VaR_\alpha(g(Z)) = g(VaR_\alpha(Z)). \quad (5.4)$$

It is well-known that VaR is not a coherent risk measure as it does not satisfy the sub-additive property. On the other hand, CVaR is a coherent risk measure and is recommended by Swiss Federal Office of Private Insurance (2006) to quantify insurance risk. We refer to Artzner *et al.* (1999) and Föllmer and Schied (2004) for more detailed discussions on the properties of VaR and CVaR.

Due to its desirable properties, in this paper we adopt CVaR to calculate the capital at risk, i.e.

$$CVaR_\alpha(T_f(X) - \mathbb{E}[T_f(X)]).$$

The return from a capital investment is much smaller than that required for shareholders in practice. We denote by  $\delta \in (0, 1)$  the return difference, which is known as the cost-of-capital rate. The risk margin is now set to be the product of the cost-of-capital rate and the capital at risk. Consequently, let  $\mathcal{L}(T_f(X))$

represent the risk-adjusted value of the insured’s liability, then we have

$$\mathcal{L}(T_f(X)) = \mathbb{E}[T_f(X)] + \delta \times CVaR_\alpha (T_f(X) - \mathbb{E}[T_f(X)]). \tag{5.5}$$

Thus, the optimal insurance model in this section is formulated by

$$\min_{f \in \mathcal{C}} \mathcal{L}(T_f(X)). \tag{5.6}$$

As pointed out earlier, the CVaR risk measure preserves the convex order; so does the objective functional  $\mathcal{L}(\cdot)$ . According to Theorem 4.1, we know that the stop-loss insurance is optimal, and hence the study of optimal insurance model (5.6) is simplified to analyzing the following one-parameter minimization problem

$$\min_{d \geq 0} \mathcal{L}(T_{f_d}(X)). \tag{5.7}$$

Recall that the basic premium is an implicit functional of the ceded loss. Consequently, the expression for  $\mathcal{L}(T_{f_d}(X))$  can be quite involved, as shown in the following proposition.

**Proposition 5.1.** *For any  $d \geq 0$ , we have*

$$\begin{aligned} & \mathcal{L}(T_{f_d}(X)) - (1 - \delta)\mathbb{E}[X] \\ = & \begin{cases} \delta d + (\delta + \rho + \delta(\vartheta - \rho)) \mathbb{E}[(X - d)_+], & \frac{v(d)}{LT} + d \leq VaR_\alpha(X); \\ \delta(1 - LT)d + \gamma \mathbb{E}[(X - d)_+] + \delta(\frac{1}{\alpha} - 1)v(d) \\ \quad + \delta LT \times CVaR_\alpha(X), & d \leq VaR_\alpha(X) < \frac{v(d)}{LT} + d; \\ \kappa \mathbb{E}[(X - d)_+] + \frac{\delta(1-\alpha)}{\alpha} v(d) + \delta CVaR_\alpha(X), & d > VaR_\alpha(X), \end{cases} \end{aligned}$$

where  $\gamma \triangleq (1 - \delta)\rho + \delta(1 + \vartheta) - \delta \frac{LT + \vartheta - \rho}{\alpha}$ ,  $\kappa \triangleq \rho - \delta(1/\alpha - 1)(1 + \vartheta - \rho)$  and

$$v(d) \triangleq (1 + \vartheta)\mathbb{E}[(X - d)_+] - B_{f_d}T. \tag{5.8}$$

**Proof.** The translation invariant property of CVaR implies that

$$\begin{aligned} \mathcal{L}(T_{f_d}(X)) &= (1 - \delta)\mathbb{E}[T_{f_d}(X)] + \delta CVaR_\alpha(T_{f_d}(X)) \\ &= (1 - \delta)\mathbb{E}[X] + \rho(1 - \delta)\mathbb{E}[(X - d)_+] + \delta CVaR_\alpha(T_{f_d}(X)) \end{aligned} \tag{5.9}$$

for any  $d \geq 0$ , where the last equality is derived by (4.9). Further, it follows from (2.3) and (2.7) that

$$T_{f_d}(x) = \min\{x, d\} + (1 + \vartheta)\mathbb{E}[(X - d)_+] - (v(d) - LT(x - d)_+)_+,$$

which is an increasing continuous function over  $\mathbb{R}_+$ . Thus, (5.4), together with the translation invariant property of CVaR risk measure, implies

$$\begin{aligned}
 CVaR_\alpha(T_{f_d}(X)) &= (1 + \vartheta)\mathbb{E}[(X - d)_+] \\
 &\quad + \frac{1}{\alpha} \int_0^\alpha \min\{VaR_s(X), d\} - (v(d) - LT(VaR_s(X) - d)_+)_+ ds.
 \end{aligned}
 \tag{5.10}$$

If  $d \geq VaR_\alpha(X)$ , it follows from the above equation that

$$\begin{aligned}
 CVaR_\alpha(T_{f_d}(X)) &= (1 + \vartheta)\mathbb{E}[(X - d)_+] + CVaR_\alpha(X) \\
 &\quad - \frac{1}{\alpha} \int_0^\alpha (VaR_s(X) - d)_+ + (v(d) - LT(VaR_s(X) - d)_+)_+ ds \\
 &= (1 + \vartheta)\mathbb{E}[(X - d)_+] - \frac{1}{\alpha} \int_0^1 (VaR_s(X) - d)_+ + (v(d) - LT(VaR_s(X) - d)_+)_+ ds \\
 &\quad + \frac{1 - \alpha}{\alpha} v(d) + CVaR_\alpha(X) \\
 &= (1 + \vartheta)\mathbb{E}[(X - d)_+] - \frac{1}{\alpha} \left( \mathbb{E}[(X - d)_+] + \mathbb{E}[(v(d) - LT(X - d)_+)_+] \right) \\
 &\quad + \frac{1 - \alpha}{\alpha} v(d) + CVaR_\alpha(X) \\
 &= \left( 1 + \vartheta - \frac{1 + \vartheta - \rho}{\alpha} \right) \mathbb{E}[(X - d)_+] + \frac{1 - \alpha}{\alpha} v(d) + CVaR_\alpha(X),
 \end{aligned}$$

where the third equality is derived by the fact that  $X$  and  $VaR_U(X)$  are equal in distribution and the last equality is derived by (2.6). Here,  $U$  is a random variable uniformly distributed on  $[0, 1]$ .

Otherwise, if  $d \leq VaR_\alpha(X)$ , we have

$$\frac{1}{\alpha} \int_0^\alpha \min\{VaR_s(X), d\} ds = d$$

and

$$\frac{1}{\alpha} \int_0^\alpha (v(d) - LT(VaR_s(X) - d)_+)_+ ds = \frac{LT}{\alpha} \int_0^\alpha (v(d)/LT + d - VaR_s(X))_+ ds.$$

Further, if  $v(d)/LT + d \leq VaR_\alpha(X)$ , we have

$$\frac{1}{\alpha} \int_0^\alpha (v(d) - LT(VaR_s(X) - d)_+)_+ ds = 0,$$

then by simple algebra, (5.10) can be rewritten by

$$CVaR_\alpha(T_{f_d}(X)) = d + (1 + \vartheta)\mathbb{E}[(X - d)_+].$$



Otherwise, if  $v(d)/LT + d > VaR_\alpha(X)$ , we have

$$\begin{aligned} & \frac{1}{\alpha} \int_0^\alpha (v(d) - LT(VaR_\alpha(X) - d))_+ ds \\ &= v(d) + dLT - LT \times CVaR_\alpha(X) + \frac{LT}{\alpha} \int_0^\alpha (VaR_\alpha(X) - (v(d)/LT + d))_+ ds \\ &= v(d) + dLT - LT \times CVaR_\alpha(X) + \frac{LT}{\alpha} \int_0^1 (VaR_\alpha(X) - (v(d)/LT + d))_+ ds \\ &= v(d) + dLT - LT \times CVaR_\alpha(X) + \frac{LT}{\alpha} \mathbb{E}[(X - (v(d)/LT + d))_+]. \end{aligned}$$

It also follows from (2.6) that

$$(\vartheta - \rho)\mathbb{E}[(X - d)_+] = LT \int_d^{\frac{v(d)}{LT} + d} F_X(t) dt, \tag{5.11}$$

then we have

$$\mathbb{E}[(X - (v(d)/LT + d))_+] = \left( \frac{\vartheta - \rho}{LT} + 1 \right) \mathbb{E}[(X - d)_+] - \frac{v(d)}{LT}.$$

As a consequence, by simple algebra,  $CVaR_\alpha(T_{f_d}(X))$  in (5.10) could be rewritten as

$$\begin{aligned} CVaR_\alpha(T_{f_d}(X)) &= (1 - LT)d + \left( 1 + \vartheta - \frac{\vartheta - \rho + LT}{\alpha} \right) \mathbb{E}[(X - d)_+] \\ &\quad + \left( \frac{1}{\alpha} - 1 \right) v(d) + LT \times CVaR_\alpha(X). \end{aligned}$$

Substituting the above expression into (5.9), we obtain the final result and this completes the proof. ■

In the above proposition,  $\mathcal{L}(T_{f_d}(X))$  is expressed in term of an auxiliary function  $v(d)$  which can be derived from (5.11). Before solving the minimization problem (5.7), we explore some properties of  $v(d)$  in the following proposition.

**Proposition 5.2.** *For  $v(d)$  defined in (5.8), we have  $v'(d) \leq 0$  and*

$$v'(d) \geq -LT \quad \text{if and only if} \quad d \geq VaR_{\frac{1}{1 + \frac{\vartheta - \rho}{LT}}}(X).$$

**Proof.** From (2.4), it is easy to see that  $v(d) \geq 0$  and that  $v(d) > 0$  if and only if  $\mathbb{E}[(X - d)_+] > 0$ . For any  $d \geq 0$  with  $\mathbb{E}[(X - d)_+] > 0$ , taking the derivatives of (5.11) with respect to  $d$  yields

$$v'(d)F_X\left(\frac{v(d)}{LT} + d\right) + LT\left(F_X\left(\frac{v(d)}{LT} + d\right) - F_X(d)\right) + (\vartheta - \rho)S_X(d) = 0.$$

Thus, we must have  $v'(d) \leq 0$ . Moreover, we can see from the above equation that

$$(v'(d)/LT + 1) F_X\left(\frac{v(d)}{LT} + d\right) = 1 - \left(1 + \frac{\vartheta - \rho}{LT}\right) S_X(d),$$

which together with (5.3) implies

$$v'(d)/LT + 1 \geq 0 \quad \text{if and only if} \quad d \geq VaR_{\frac{1}{1+\frac{\vartheta-\rho}{LT}}}(X).$$

The proof is thus complete. ■

By using Propositions 5.1 and 5.2, we are now ready to solve the optimal insurance problem (5.7). We illustrate this by resorting to two numerical examples so that the optimal retrospective rating plan can be identified explicitly and the effects of the loss conversion factor  $L$  and the loading coefficient  $\vartheta$  can be assessed in turn. As in Chi and Lin (2014), we consider two loss distributions with the same mean and variance. Example 5.1 examines the heavy tailed Pareto distribution while Example 5.2 analyses the light tailed Gamma distribution. In both cases, we assume the following values for the retrospective rating plan: the tax multiplier  $T = 1.025$ ,  $\delta = 3\%$ ,  $\alpha = 20\%$  and the safety loading coefficient  $\rho = 0.1$ .

**Example 5.1 (Pareto loss distribution).** *Suppose a loss random variable  $X$  follows a Pareto distribution with probability density function (pdf)*

$$p_1(x) = \frac{3 \times 10^6}{(x + 100)^4}, \quad x > 0, \tag{5.12}$$

then we have

$$S_X(t) = 10^6/(t + 100)^3, \quad \mathbb{E}[X] = 50, \quad \text{var}(X) = 7500, \quad VaR_\alpha(X) = 100(\alpha^{-1/3} - 1) = 71,$$

where  $\text{var}(X)$  is the variance of the random variable  $X$ , and

$$CVaR_\alpha(X) = 150\alpha^{-1/3} - 100, \quad \mathbb{E}[(X - d)_+] = 5 \times 10^5 \times (100 + d)^{-2}.$$

For this case, (5.11) is rewritten as

$$v(d) + 5 \times 10^5 \times LT \times \left(\frac{v(d)}{LT} + d + 100\right)^{-2} - 5 \times 10^5 \times (\vartheta - \rho + LT)(d + 100)^{-2} = 0.$$

Recall that Proposition 5.2 establishes  $v(d)$  is a decreasing function in  $d$ . For each  $d \geq 0$ , it is very easy to derive  $v(d)$  from the above equation and  $\mathcal{L}(T_{fd}(X))$  by using Proposition 5.1. Consequently, we can obtain the solution  $d^*$  by comparing the risk-adjusted value of the insured's liability with different retention levels.

First, we set the loading coefficient of maximum premium to  $\vartheta = 0.3$  and derive the optimal retention levels for loss conversion factor  $L = 0.1, 0.3, 0.5, 0.7, 0.9$ . The numerical results are reported in Table 1. It follows from the table that as  $L$

TABLE 1

OPTIMAL RETENTION LEVEL FOR DIFFERENT  $L$  UNDER PARETO LOSS DISTRIBUTION WHEN  $\vartheta = 0.3$ .

$L$	$d^*$	$\frac{v(d^*)}{LT} + d^*$	$\min_{d \geq 0} \mathcal{L}(T_{f_d}(X))$
0.1	66.54	108.21	52.9190
0.3	67.83	82.04	52.9399
0.5	67.71	76.35	52.9451
0.7	66.79	73.08	52.9472
0.9	66.03	71.00	52.9476

TABLE 2

OPTIMAL RETENTION LEVEL FOR DIFFERENT  $\vartheta$  UNDER PARETO LOSS DISTRIBUTION WHEN  $L = 0.5$ .

$\vartheta$	$d^*$	$\frac{v(d^*)}{LT} + d^*$	$\min_{d \geq 0} \mathcal{L}(T_{f_{d^*}}(X))$
0.15	63.66	66.00	52.8646
0.2	64.28	68.89	52.8925
0.25	65.39	72.15	52.9200
0.3	67.71	76.35	52.9451
0.35	69.69	80.11	52.9678
0.4	72.85	84.73	52.9883

increases, the minimal risk-adjusted value of the insured's liability also increases although the same property does not apply to the optimal retention level.

Next, we analyze the sensitivity of  $d^*$  and  $\mathcal{L}(T_{f_{d^*}}(X))$  with respect to the loading coefficient  $\vartheta$ . We set  $L = 0.5$  and choose  $\vartheta = 0.15, 0.2, 0.25, 0.3, 0.35, 0.4$ . The numerical results are depicted in Table 2. In this case, both  $\mathcal{L}(T_{f_{d^*}}(X))$  and  $d^*$  increase with  $\vartheta$ . This suggests that for a larger maximum premium, the insured would retain more risk and expose to a higher risk-adjusted liability. Exactly, this result could be explained by using Theorem 3.1. More precisely, it follows from Theorem 3.1 that as the maximum premium increases, the retrospective premium becomes more risky and the insured is facing more risk exposure. As the retrospective rating plan becomes less efficient to transfer risk, the insured would cede less risk for a larger  $\vartheta$ .

**Example 5.2 (Gamma loss distribution).** In this example we assume that the loss random variable  $X$  has a light tailed Gamma distribution with pdf

$$p_2(x) = \frac{1}{150^{\frac{1}{3}} \Gamma(\frac{1}{3})} x^{-\frac{2}{3}} e^{-\frac{x}{150}}, \quad x > 0,$$

where  $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx, t \geq 0$ . The parameter values for the above Gamma distribution are selected so that it has the same mean and variance as the Pareto distribution discussed in Example 5.1.

TABLE 3

OPTIMAL RETENTION LEVEL FOR DIFFERENT  $L$  UNDER GAMMA LOSS DISTRIBUTION WHEN  $\vartheta = 0.3$ .

$L$	$d^*$	$\frac{v(d^*)}{LT} + d^*$	$\min_{d \geq 0} \mathcal{L}(T_{f_d}(X))$
0.1	71.37	123.98	53.6748
0.3	72.62	90.57	53.6979
0.5	71.80	82.78	53.7029
0.7	70.46	78.47	53.70396
0.9	69.99	76.28	53.70403

The survival function and the stop-loss premium could be calculated via the function

$$G_{a,b}(t) = \int_0^t \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-\frac{x}{b}} dx, \quad t \geq 0$$

for  $a > 0, b > 0$ . Many softwares including Excel and Matlab have a package for  $G_{a,b}(t)$ . Then, we have  $VaR_\alpha(X) = G_{\frac{1}{3},150}^{-1}(1 - \alpha) = 78.47$  and

$$\mathbb{E}[(X - d)_+] = 50(1 - G_{\frac{4}{3},150}(d)) - d(1 - G_{\frac{1}{3},150}(d)).$$

For Gamma loss distribution, (5.11) can be written by

$$\begin{aligned} & (50 - d)(\vartheta - \rho) + d(\vartheta - \rho + LT)G_{\frac{1}{3},150}(d) + 50LT \times G_{\frac{4}{3},150}\left(d + \frac{v(d)}{LT}\right) \\ & = (v(d) + d \times LT)G_{\frac{1}{3},150}\left(d + \frac{v(d)}{LT}\right) + 50 \times (\vartheta - \rho + LT)G_{\frac{4}{3},150}(d). \end{aligned}$$

Similarly, for each  $d \geq 0$ , we can derive  $v(d)$  from the above equation and  $\mathcal{L}(T_{f_d}(X))$  with the help of Proposition 5.1. The optimal retention and other related quantities are listed in Tables 3 and 4.

Similar to the Pareto example, the risk-adjusted value of liability increases with the loss conversion factor  $L$  while both the risk-adjusted value of liability and the optimal retention level increase with the loading coefficient  $\vartheta$  for the Gamma loss distribution. It is also of interest to study the effect of the tail behavior of a loss distribution on the optimal insurance model. In particular,  $\mathcal{L}(T_{f_{d^*}}(X))$  and  $d^*$  for Gamma distribution are larger than the corresponding values for the heavy tailed Pareto distribution although both distributions have same mean and variance. This suggests that when an insured faces a risk that has a heavy tail, it would cede more risk to an insurer and result in a lower risk-adjusted value of liability.

### 6. CONCLUDING REMARKS

In this paper, we study the design of an optimal retrospective rating plan from the perspective of an insured under a criterion that preserves the convex order.

TABLE 4

OPTIMAL RETENTION LEVEL FOR DIFFERENT  $\vartheta$  UNDER GAMMA LOSS DISTRIBUTION WHEN  $L = 0.5$ .

$\vartheta$	$d^*$	$\frac{v(d^*)}{LT} + d^*$	$\min_{d \geq 0} \mathcal{L}(T_{fd}(X))$
0.15	67.15	70.10054	53.5999
0.2	68.1	73.90	53.6349
0.25	69.05	77.60851	53.6696
0.3	71.80	82.78	53.7029
0.35	74.71	87.91027	53.7331
0.4	77.29	92.60	53.7605

We find that the insured would suffer more risk exposure for a larger loss conversion factor or a higher loading coefficient in maximum premium. Moreover, it is shown that any admissible insurance contract is dominated by a stop-loss policy. Further, the optimal retention of stop-loss insurance is derived numerically under the criterion of minimizing the risk-adjusted value of the insured’s liability where the liability valuation is carried out by a cost-of-capital approach based on the CVaR risk measure.

To simplify the analysis, we make some rather stringent assumptions on the retrospective premium principle in this paper. For example, we set the minimum premium to be zero and the maximum premium to be proportional to the expected indemnity. Moreover, the basic premium is assumed to be a solution to (1.3). In practice, the retrospective rating plan may have a much more complicated pricing mechanism. Therefore, it will be of significant interest to study the optimal design of a retrospective rating plan with more general retrospective premium principle. We leave this for future research exploration.

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NOTES

1. Here,  $Y_{\leq cx}$   $Z$  is equivalent to

$$\mathbb{E}[Y] = \mathbb{E}[Z] \quad \text{and} \quad \mathbb{E}[(Y - d)_+] \leq \mathbb{E}[(Z - d)_+], \quad \forall d \in \mathbb{R}, \tag{2.9}$$

provided that the expectations exist. See Shaked and Shanthikumar (2007) for more details on the theory of stochastic orders.

2. A function  $g(x)$  is said to up-cross a function  $h(x)$ , if there exists an  $x_0 \in \bar{\mathbb{R}}$  such that

$$\begin{cases} g(x) \leq h(x), & x < x_0; \\ g(x) \geq h(x), & x > x_0. \end{cases}$$

## REFERENCES

- ARROW, K.J. (1963) Uncertainty and the welfare economics of medical care. *American Economic Review*, **53**(5), 941–973.
- ARTZNER, P., DELBAEN, F., EBER, J.M. and HEATH, D. (1999) Coherent measures of risk. *Mathematical Finance*, **9**(3), 203–228.
- ASIMIT, A.V., BADESCU, A.M. and VERDONCK, T. (2013) Optimal risk transfer under quantile-based risk measurers. *Insurance: Mathematics and Economics*, **53**(1), 252–265.
- BALBÁS, A., BALBÁS, B. and HERAS, A. (2009) Optimal reinsurance with general risk measures. *Insurance: Mathematics and Economics*, **44**(3), 374–384.
- BERNARD, C. and TIAN, W. (2009) Optimal reinsurance arrangements under tail risk measures. *The Journal of Risk and Insurance*, **76**(3), 709–725.
- BORCH, K. (1960) *An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance*. In: Transactions of the 16th International Congress of Actuaries, vol. I, pp. 597–610. Georges Thone, Brussels, Belgium.
- CAI, J. and TAN, K.S. (2007) Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures. *Astin Bulletin*, **37**(1), 93–112.
- CAI, J., TAN, K.S., WENG, C. and ZHANG, Y. (2008) Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics*, **43**(1), 185–196.
- CHEUNG, K.C. (2010) Optimal reinsurance revisited — a geometric approach. *Astin Bulletin*, **40**(1), 221–239.
- CHI, Y. (2012) Reinsurance arrangements minimizing the risk-adjusted value of an insurer's liability. *Astin Bulletin*, **42**(2), 529–557.
- CHI, Y. and LIN, X.S. (2014) Optimal reinsurance with limited ceded risk: A stochastic dominance approach. *Astin Bulletin*, **44**(1), 103–126.
- CHI, Y. and TAN, K.S. (2011) Optimal reinsurance under VaR and CVaR risk measures: A simplified approach. *Astin Bulletin*, **41**(2), 487–509.
- CONG, J. and TAN, K.S. (2014) *Optimal VaR-based Risk Management with Reinsurance*. Annals of Operations Research, in press. doi:10.1007/s10479-014-1584-8.
- DHAENE, J., DENUIT, M., GOOVAERTS, M.J., KAAS, R. and VYNCKE, D. (2002) The concept of comonotonicity in actuarial science and finance: Theory. *Insurance: Mathematics and Economics*, **31**(1), 3–33.
- EUROPEAN COMMISSION 2009. *Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the Taking-up and Pursuit of the Business of Insurance and Reinsurance (Solvency II)*. Official Journal of the European Union L335.
- FÖLLMER, H. and SCHIED, A., 2004. *Stochastic Finance: An Introduction in Discrete Time*, 2nd revised and extended edition. Berlin: Walter de Gruyter.
- GAJEK, L. and ZAGRODNY, D. (2004) Optimal reinsurance under general risk measures. *Insurance: Mathematics and Economics*, **34**(2), 227–240.
- GOLLIER, C. and SCHLESINGER, H. (1996) Arrow's theorem on the optimality of deductibles: A stochastic dominance approach. *Economic Theory*, **7**(2), 359–363.
- KALUSZKA, M. (2001) Optimal reinsurance under mean-variance premium principles. *Insurance: Mathematics and Economics*, **28**(1), 61–67.
- KALUSZKA, M. and OKOLEWSKI, A. (2008) An extension of Arrow's result on optimal reinsurance contract. *The Journal of Risk and Insurance*, **75**(2), 275–288.
- MEYERS, G.G. (2004) *Retrospective premium*. In Encyclopedia of Actuarial Science, vol 3, pp. 1452–1456 (ed. J. Teugels and B. Sundt) John Wiley & Sons, Chichester.

- OHLIN, J. (1969) On a class of measures of dispersion with application to optimal reinsurance. *Astin Bulletin*, **5**(2), 249–266.
- RAVIV, A. (1979) The design of an optimal insurance policy. *American Economic Review*, **69**(1), 84–96.
- SHAKED, M. and SHANTHIKUMAR, J.G. (2007) *Stochastic Orders*. New York: Springer.
- SWISS FEDERAL OFFICE OF PRIVATE INSURANCE, (2006) *Technical document on the Swiss Solvency Test*. FINMA.
- VAN HEERWAARDEN, A.E., KAAS, R. and GOOVAERTS, M.J. (1989) Optimal reinsurance in relation to ordering of risks. *Insurance: Mathematics and Economics*, **8**(1), 11–17.
- YOUNG, V.R. (1999) Optimal insurance under Wang's premium principle. *Insurance: Mathematics and Economics*, **25**(2), 109–122.

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