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Why many theories of shock waves are necessary: kinetic relations for non-conservative systems

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For a class of non-conservative hyperbolic systems of partial differential equations endowed with a strictly convex mathematical entropy, we formulate the initial-value problem by supplementing the equations with a kinetic relation prescribing the rate of entropy dissipation across shock waves. Our condition can be regarded as a generalization to non-conservative systems of a similar concept introduced by Abeyaratne, Knowles and Truskinovsky for subsonic phase transitions and by LeFloch for non-classical undercompressive shocks to nonlinear hyperbolic systems. The proposed kinetic relation for non-conservative systems turns out to be equivalent, for the class of systems under consideration at least, to Dal Maso, LeFloch and Murat's definition based on a prescribed family of Lipschitz continuous paths. In agreement with previous theories, the kinetic relation should be derived from a phase-plane analysis of travelling-wave solutions associated with an augmented version of the non-conservative system. We illustrate with several examples that non-conservative systems arising in the applications fit in our framework, and for a typical model of turbulent fluid dynamics we provide a detailed analysis of the existence and properties of travelling waves which yields the corresponding kinetic function.

1. Introduction

Certain nonlinear hyperbolic models arising in continuum physics and, in particular, models describing complex fluid flows, do not take the standard form of conservation laws but, instead, are nonlinear hyperbolic systems in non-conservative form

$$\partial_t u + A(u)\partial_x u = 0, \quad x \in \mathbb{R}, \quad t \geqslant 0. \tag{1.1}$$

Here, $u = u(x,t) \in \Omega$ is an unknown field taking values in a (convex and open) domain $\Omega \subset \mathbb{R}^N$, while the matrix-valued field $A = A(u)$ is given and, for each state u, admits N real and distinct eigenvalues. It is well known that nonlinear hyperbolic

equations do not admit smooth solutions, since propagating discontinuities arise in finite time even from smooth initial data. For conservative systems, weak solutions in the sense of distributions are sought. However, for non-conservative systems (1.1), the distributional definition does not apply. A suitable notion of weak solutions was proposed by Dal Maso, LeFloch, and Murat, which, together with several nonlinear stability theorems, was presented in [20]. Non-conservative hyperbolic systems have been the subject of active research in the past 15 years. The theory covers the definition of weak solutions [20, 37–39, 43, 57], the existence of solutions to the Riemann problem [20, 37], the initial-value problem [19, 37, 45], the uniqueness of bounded variation solutions [6, 42] and their approximation via finite-difference schemes [14,31,46]. In addition, many non-conservative models arising in continuum mechanics have been systematically investigated, as such models play an important role in the modelling of multi-phase flows and turbulent fluid dynamics [3, 7–9, 15, 16].

Building upon the above works, our purpose in the present paper is to consider a restricted class of non-conservative systems of the form (1.1), characterized by the property that a large family of additional entropy functions (conservation laws) is also available. In other words, the systems to be considered below *formally* have a conservative form if nonlinear combinations of the given equations are allowed. However, the physical modelling dictates that non-conservative equations should be used, and it is precisely under these conditions that a 'kinetic relation', as we propose in the present paper, should enter into play.

The kinetic relations were initially introduced by LeFloch [41] for hyperbolic systems of conservation laws in order to handle non-classical undercompressive shocks, following earlier works by Abeyaratne and Knowles [1] and Truskinovsky [56] for subsonic phase transitions. (See [6, 27, 28, 40–44] for details.)

The concept of a kinetic relation for non-conservative systems discussed herein was actually first introduced by the authors in an unpublished manuscript. Later on, this concept was investigated numerically by Aubert *et al.* [3, 9, 16], and the control of the numerical dissipation of finite-difference schemes was extensively addressed. Our purpose in the present paper is to provide the required theoretical framework and demonstrate that the kinetic relation provides an efficient tool to handle complex fluids.

Recall that the design and the properties of difference schemes suitable for the numerical approximation of non-conservative systems (1.1) is very challenging. The main source of difficulty lies in the fact that shock waves to non-conservative systems are small-scale dependent and the dissipation terms induced by the numerical discretization tend to drive the propagation of the shocks. This phenomenon was rigorously analysed for scalar equations by Hou and LeFloch [31]. On the other hand, we emphasize that the Glimm scheme and front-tracking algorithms do not contain any numerical dissipation and, actually, have been proven to converge to the correct solutions [37, 42, 45]. The method based on the kinetic relation proposed in the present paper allows one to extend to non-conservative systems the conclusions made for non-classical shocks in [28, 46] (and the references therein).

We begin with a general discussion of non-conservative hyperbolic systems arising in continuum physics in order to motivate our general approach proposed in the next section and developed on selected examples in the rest of this paper. The models

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of interest here naturally stand in a non-conservative form, and this is a direct consequence of simplifying assumptions which are made in the derivation of these models; these assumptions are also necessary if a tractable model is to be found. Such assumptions typically originate in averaging procedures that intend to bypass the description of intricate mechanisms taking place at microscopic scales. The small-scale fluctuations that are thought to be of lesser interest induce dissipative and/or relaxation phenomena at the macroscopic level, and can also be accounted for as source terms.

Most (if not all) non-conservative hyperbolic models arising in the applications admit (several distinct) entropy balance laws which are consistent with the underlying dissipative and relaxation mechanisms. These additional balance equations, as we shall show, provide a natural approach to formulating additional generalized jump conditions built from entropy rate productions. Moreover, these entropy functions are sufficient in number to allow for a complete set of jump relations.

The objectives and results in this paper are as follows. First of all, as mentioned above, we restrict our attention to a class of non-conservative systems (defined in § 3) which encompasses, however, most of the models encountered in the applications. To motivate the definition of the class of systems studied in this paper, we observe that, in the applications we have in mind (e.g. multi-phase and multi-fluid models):

(i) all but one of the equations (1.1) can be rewritten in a conservative form and, moreover,

(ii) the system (1.1) is endowed with a mathematical entropy, i.e. a (strictly convex) nonlinear function $U = U(u)$ corresponding to an additional conservation law satisfied by all smooth solutions.

For such systems, the concept of weak solutions introduced by Dal Maso *et al.* [20] can be simplified. Therein, a family of Lipschitz continuous paths was necessary to uniquely define the non-conservative product $A(u)\partial_x u$ associated with the vector-valued field u. In contrast, for our particular class of non-conservative systems, one non-conservative product between scalar-valued functions only must be defined. This structure allows us to simply supplement the model (1.1) with an additional algebraic scalar equation which, for each shock wave, determines the *entropy dissipation rate* associated with the entropy U. We call this additional jump condition a *kinetic relation* and the entropy dissipation function a *kinetic function*. A precise definition is given in § 3. The main result of this section is a proof of the existence of a solution to the Riemann problem for (1.1) which satisfies the prescribed kinetic relation. Our proof is a generalization of an argument given in [20] in the setting of general families of paths.

It is remarkable that many models of interest arising in the applications take the form considered in § 3, and this will be illustrated in § 2. In § 4, we focus on a model of particular importance, which arises in turbulent fluid dynamics. Taking into account the dissipation terms induced by the physical modelling, the existence and properties of associated travelling waves are established. In § 5 we characterize the right-hand states of travelling waves with fixed left-hand states, which leads us to the desired kinetic relation. In turn, this provides us with the kinetic function needed to apply the general theory in § 3.

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2. Non-conservative systems in fluid dynamics

To show the structure of the non-conservative systems of interest, it is worth beginning with the *shallow water equations with topography*

$$\left.\begin{aligned} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + \tfrac{1}{2} g \rho^2) - g\rho \partial_x a = 0, \end{aligned}\right\} \tag{2.1}$$

where ρ and v are the mass density and the velocity of the fluid, respectively, and the (prescribed) topography function $a\colon \mathbb{R} \to \mathbb{R}_+$ depends on the spatial variable x and is assumed to be solely piecewise Lipschitz continuous. Here, g is the gravity constant. The product $g\rho\partial_x a$ is a non-conservative product which is not defined in a classical sense at points of discontinuity.

By setting $\boldsymbol{u} := (\rho, \rho v)$, weak solutions should obey the following entropy inequality:

$$\left.\begin{aligned} \partial_t \mathcal{U}(\boldsymbol{u}, a) + \partial_x \mathcal{F}(\boldsymbol{u}, a) \leqslant 0, \\ \mathcal{U}(\boldsymbol{u}, a) := \rho E(v) + \rho a, \qquad e'(\rho) = \frac{p(\rho)}{\rho^2}, \\ \mathcal{F}(\boldsymbol{u}, a) := \tfrac{1}{2}\rho v^3 + \rho e(\rho) v + p(\rho) v + \rho v a. \end{aligned}\right\} \tag{2.2}$$

Another model with a closely related structure is

$$\left.\begin{aligned} \partial_t(a\rho) + \partial_x(a\rho v) = 0, \\ \partial_t(a\rho v) + \partial_x(a\rho v^2 + a p(\rho)) - p(\rho)\partial_x a = 0, \end{aligned}\right\} \tag{2.3}$$

which describes *one-dimensional nozzle flows* as well as *compressible flows in porous media*. Again, the function $a\colon \mathbb{R} \to \mathbb{R}$ is solely piecewise Lipschitz continuous and denotes here the nozzle cross-section or the porosity function, respectively.

By setting $\boldsymbol{u} := (a\rho, a\rho v)$, weak solutions to (2.3) should obey the entropy inequality

$$\left.\begin{aligned} \partial_t \mathcal{U}(\boldsymbol{u}, a) + \partial_x \mathcal{F}(\boldsymbol{u}, a) \leqslant 0, \\ \mathcal{U}(\boldsymbol{u}, a) = \tfrac{1}{2} a^2 \rho v^2 + a\rho e(\rho), \\ \mathcal{F}(\boldsymbol{u}, a) = (\mathcal{U}(\boldsymbol{u}, a) + p(\rho)) v. \end{aligned}\right\} \tag{2.4}$$

The systems (2.1) and (2.3) and closely related models with source terms have received considerable attention over the past decade, from both analytical and numerical standpoints. We refer the reader to [38] (in connection with the theory of non-conservative systems), [4,11,14,21,25,26,33] (approximation by finite-difference or finite volume schemes) and [22,32,47–50] (construction of a Riemann solver). In particular, we refer the reader to [11] (and the references therein) for a comprehensive review. We also refer the reader to [12,13,34,53,54].

We observe here that both models (2.1) and (2.3) fall within the following class of non-conservative hyperbolic models with a singular source term:

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f}(\boldsymbol{u}, a) - \boldsymbol{g}(\boldsymbol{u}, a)\partial_x a = 0, \tag{2.5}$$

where a is a given (piecewise Lipschitz continuous) function of the spatial variable x and the unknown map $\boldsymbol{u}$ takes values in a convex and open domain $\Omega_{\boldsymbol{u}} \subset \mathbb{R}^N$, while $\boldsymbol{f}\colon \Omega_{\boldsymbol{u}} \times \mathbb{R} \to \mathbb{R}^N$ and $\boldsymbol{g}\colon \Omega_{\boldsymbol{u}} \times \mathbb{R} \to \mathbb{R}^N$ are given smooth mappings.

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Motivated by the structure of the above two examples, particularly the entropy inequalities (2.2) and (2.4), and in order to develop a general theory, we assume that the hyperbolic system (2.5) is endowed with a (sufficiently smooth) entropy function $\mathcal{U}\colon \Omega_{\boldsymbol{u}} \times \mathbb{R} \to \mathbb{R}$ and a corresponding entropy flux $\mathcal{F}\colon \Omega_{\boldsymbol{u}} \times \mathbb{R} \to \mathbb{R}$, so that solutions to (2.5) satisfy the entropy inequality

$$\partial_t \mathcal{U}(\boldsymbol{u}, a) + \partial_x \mathcal{F}(\boldsymbol{u}, a) \leqslant 0. \tag{2.6}$$

The principal examples of interest arising in the form (2.5) in the applications do admit such an entropy.

The above class is known to include, after the seminal work by Greenberg and Leroux [25] and Gosse [23], the class of hyperbolic systems with source terms

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{u}), \tag{2.7}$$

which, by introducing the (rather trivial) function $a(x) = x$, indeed take the form (cf. (2.5))

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{g}(\boldsymbol{u})\partial_x a = 0. \tag{2.8}$$

This non-conservative reformulation is useful for designing 'well-balanced schemes', which properly account for the competition between the source term and the differential hyperbolic operator in the large-time asymptotic $t \to +\infty$ [11,23–26]. The (somewhat artificial but useful) system (2.8) admits a conserved entropy in the scalar case $n = 1$, *provided* that the source g does not vanish, namely it suffices to define $\mathcal{U}'(u) := 1/g(u)$ and $\mathcal{F}'(u) := f'(u)/g(u)$.

As advocated by LeFloch [38] for the equations for nozzle flows (2.3), the prescribed function a, being independent of the time variable, can be regarded as an independent unknown of the following extended version of (2.5) in the extended variable $u := (\boldsymbol{u}, a)$:

$$\left.\begin{aligned} \partial_t \boldsymbol{u} + \partial_x \boldsymbol{f}(\boldsymbol{u}, a) - \boldsymbol{g}(\boldsymbol{u}, a)\partial_x a = 0, \\ \partial_t a = 0. \end{aligned}\right\} \tag{2.9}$$

Assuming from now on that the matrix $D_{\boldsymbol{u}} \boldsymbol{f}(\boldsymbol{u}, a)$ is diagonalizable for all $\boldsymbol{u} \in \Omega_{\boldsymbol{u}}$ and $a \in \mathbb{R}$ with real eigenvalues $\lambda_1(\boldsymbol{u}, a), \ldots, \lambda_n(\boldsymbol{u}, a)$ and a full set of eigenvectors $r_1(\boldsymbol{u}, a), \ldots, r_n(\boldsymbol{u}, a)$, it is obvious that (2.9) admits the same eigenvalues plus 0 (with multiplicity 1). Moreover, it admits a full set of eigenvectors *if and only if* $\lambda_j(\boldsymbol{u}, a) \neq 0$ for all $j = 1, \ldots, n$. In general, (2.9) is only weakly hyperbolic and, due to possible resonance effects, difficulties arise even in tackling the simplest initial-value problem, i.e. the Riemann problem; see the pioneering work of Isaacson and Temple [32], as well as [22] for a general Riemann solver. In the rest of this paper, our assumptions will explicitly exclude the resonance effect in solutions under consideration.

While a rigorous definition of non-conservative products can wait until the following section, we shall provide here some preliminary discussion, based on an observation by LeFloch [38] for the nozzle-flow equations and on the presentation in Bouchut [11] for more general systems.

With this non-resonance assumption enforced, we then observe from (2.9) that the variable a is a Riemann invariant associated with the eigenvalue $\lambda_{n+1}(\boldsymbol{u}, a) := 0$. In

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other words, a stays constant across waves associated with any other (non-vanishing) eigenvalue and, consequently, the non-conservative product $\boldsymbol{g}(\boldsymbol{u}, a)\partial_x a$ only needs to be defined for $(n+1)$-contact discontinuities.

The entropy inequality (2.6) should be satisfied as an equality in the sense of distributions across standing waves; hence,

$$\mathcal{F}(\boldsymbol{u}_+, a_+) - \mathcal{F}(\boldsymbol{u}_-, a_-) = 0. \tag{2.10}$$

From the physical viewpoint, we can further investigate the validity of (2.10), obtained as a direct consequence of the augmented form (2.9). To that purpose, we specialize (2.10) firstly to the case of the shallow water equations (2.1)

$$\frac{m^2}{2\rho_+^2} + e(\rho_+) + \frac{p(\rho_+)}{\rho_+} + a_+ = \frac{m^2}{2\rho_-^2} + e(\rho_-) + \frac{p(\rho_-)}{\rho_-} + a_-, \tag{2.11}$$

where $m =: \rho_- v_- = \rho_+ v_+$ denotes the mass, and secondly to the case of the nozzle-flow equations (2.3):

$$\frac{m^2}{2a_+^2\rho_+^2} + e(\rho_+) + \frac{p(\rho_+)}{\rho_+} = \frac{m^2}{2a_+^2\rho_-^2} + e(\rho_-) + \frac{p(\rho_-)}{\rho_-}, \tag{2.12}$$

in which $m := a_-\rho_- v_- = a_+\rho_+ v_+$. The above equations can be implicitly solved in ρ_+ away from resonance (see [11, 22, 47, 48] for details) and stand at the very basis of the design of well-balanced schemes.

Furthermore, in concrete experiments with fluid flows, for instance, in nozzles, it is observed that an abrupt change (modelled therefore by a discontinuity) in the topography function, the duct cross-section or the porosity function generally produces fine-scale features in solutions which may enter in competition with complex dissipation phenomena such as friction. To account for such dissipation mechanisms, the entropy law (2.2) or (2.4) can no longer be expressed as a conservation law across the standing wave. The associated entropy dissipation rates are the so-called 'singular loss of momentum' used by engineers and well-documented in the applied literature. It is necessary, on the grounds of physical experiments, to replace (2.12) by the more general condition

$$(a\rho v)_+ - (a\rho v)_- = 0, \qquad \mathcal{J} = -(a\rho v)_- \kappa(\boldsymbol{u}_-, a_-), \tag{2.13}$$

with

$$\mathcal{J} := (a\rho v)_+ \left(\tfrac{1}{2} v_+^2 + e(\rho_+) + \frac{p(\rho_+)}{\rho_+} \right) - (a\rho v)_- \left(\tfrac{1}{2} v_-^2 + e(\rho_-) + \frac{p(\rho_-)}{\rho_-} \right),$$

where the prescribed function $\kappa\colon \Omega_{\boldsymbol{u}} \times \mathbb{R} \to \mathbb{R}_+$ defines mathematically the singular loss of momentum. The extension relative to (2.11) is entirely analogous.

We continue this section with a more sophisticated model of compressible flows, describing *multi-fluid mixtures*, where the variable a introduced previously now stands for a fluid mass fraction. Following the celebrated review by Steward and Wendroff [55], (stratified) multi-fluid models may be regarded as two distinct fluids evolving with distinct velocities and distinct thermodynamic properties, each propagating within 'nozzles' whose cross-sections, denoted by $a \in (0,1)$ and $1 - a(x,t)$,

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respectively, depend on the spatial as well as the time variables (see, for example, [2]). For notational convenience, a is traditionally denoted by α_1 (void fraction of fluid 1) and $(1-a)$ by α_2 (void fraction of fluid 2), with

$$\alpha_1(x,t) + \alpha_2(x,t) = 1.$$

Furthermore, the evolution of a is now described either via an algebraic closure equation (based on the isobaric assumption; see [55] for details) or by considering it as an independent variable governed by a supplementary evolution equation. From the point of view of the present paper and in order to avoid instability issues due to lack of hyperbolicity of the model, we adopt the second strategy, following here [5, 52]. This approach has been investigated extensively in recent years (see [10, 21] and references therein).

In turn, the multi-fluid model under consideration takes the form

$$\left.\begin{aligned}
\partial_t\alpha_1 + V_{\mathrm{I}}(\boldsymbol{u})\partial_x\alpha_1 &= \lambda(p_2 - p_1),\\
\partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1 u_1) &= 0,\\
\partial_t(\alpha_1\rho_1 u_1) + \partial_x(\alpha_1\rho_1 u_1^2 + \alpha_1 p_1) - P_{\mathrm{I}}(\boldsymbol{u})\partial_x\alpha_1 &= \lambda(u_2 - u_1) + \epsilon\partial_x(\mu_1\partial_x u_1),\\
\partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2 u_2) &= 0,\\
\partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2 p_2) - P_{\mathrm{I}}(\boldsymbol{u})\partial_x\alpha_2 &= -\lambda(u_2 - u_1) + \epsilon\partial_x(\mu_2\partial_x u_2),
\end{aligned}\right\} \tag{2.14}$$

where $\boldsymbol{u} := (\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2)$ is the vector-valued unknown. Here, the barotropic pressure laws $p_i = p_i(\rho_i)$ are assumed to satisfy the monotonicity condition $p_i'(\rho_i) > 0$. The relaxation parameter $\lambda > 0$ may take arbitrarily large values, depending on the multi-fluid flow regime under consideration, while $\epsilon > 0$ (the inverse of the Reynolds number) is usually small. Moreover, the (smooth) functions $V_{\mathrm{I}}\colon \Omega_{\boldsymbol{u}} \to \mathbb{R}$ and $P_{\mathrm{I}}\colon \Omega_{\boldsymbol{u}} \to \mathbb{R}$ represent the interfacial velocity and interfacial pressure, respectively. Following the original work by Ransom and Hicks [52], one can set, for example,

$$V_{\mathrm{I}}(\boldsymbol{u}) := \tfrac{1}{2}(u_1 + u_2), \qquad P_{\mathrm{I}}(\boldsymbol{u}) := \tfrac{1}{2}(p_1 + p_2). \tag{2.15}$$

It turns out that, independently of the precise form of the constitutive equations, the system (2.14) admits five real eigenvalues, i.e.

$$V_{\mathrm{I}}(\boldsymbol{u}), \quad u_i \pm c_i(\rho_i),$$

where $c_i^2(\rho_i) := p'(\rho_i) > 0$, (as well as a basis of right eigenvectors) *if and only if*

$$|V_{\mathrm{I}}(\boldsymbol{u}) - u_i| \neq c_i(\rho_i), \quad i = 1, 2. \tag{2.16}$$

In other words, like the (much simpler) model (2.9), the principal (first-order) part of (2.14) is only weakly hyperbolic if (2.16) is violated. Here again, we tacitly assume that the solutions under consideration do not develop resonance phenomena.

One key constraint that arises in choosing the required closure laws for $V_{\mathrm{I}}(\boldsymbol{u})$ and $P_{\mathrm{I}}(\boldsymbol{u})$ is the existence of a mathematical entropy pair associated with (2.14). Interestingly, the total energy

$$\mathcal{U} := \alpha_1\rho_1 E_1(\boldsymbol{u}) + \alpha_2\rho_2 E_2(\boldsymbol{u})$$

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with $E_i(\boldsymbol{u}) := \frac{1}{2}u_i^2 + e_i(\rho_i)$ is an entropy for (2.14) if and only if the interfacial closure laws $V_{\rm I}(\boldsymbol{u})$ and $P_{\rm I}(\boldsymbol{u})$ satisfy the *interfacial compatibility condition*

$$V_{\rm I}(\boldsymbol{u})(p_2 - p_1) + P_{\rm I}(\boldsymbol{u})(u_2 - u_1) = p_2 u_1 - p_1 u_2 \tag{2.17}$$

for all states under consideration (see, for example, [17, 21]). Indeed, under the assumption (2.17), smooth solutions of (2.14) satisfy the entropy balance law

$$\left.\begin{aligned} \partial_t \mathcal{U}(\boldsymbol{u}) + \partial_x \mathcal{F}(\boldsymbol{u}) \ &= -\lambda(u_2 - u_1)^2 - \lambda(p_2 - p_1)^2 - \mathcal{D}, \\ \mathcal{U}(\boldsymbol{u}) &:= (\alpha_1\rho_1 E_1(\boldsymbol{u}) + \alpha_2\rho_2 E_2(\boldsymbol{u})), \\ \mathcal{F}(\boldsymbol{u}) &:= ((\alpha_1\rho_1 E_1(\boldsymbol{u}) + \alpha_1 p_1)u_1 + (\alpha_2\rho_2 E_2(\boldsymbol{u}) + \alpha_2 p_2)u_2), \\ \mathcal{D}(\boldsymbol{u}) &:= \epsilon\mu_1(\partial_x u_1)^2 + \epsilon\mu_2(\partial_x u_2)^2 - \epsilon\partial_x(\mu_1\alpha_1\partial_x u_1 + \mu_2\alpha_2\partial_x u_2). \end{aligned}\right\} \tag{2.18}$$

The dissipation $\mathcal{D}$ formally converges to a non-positive measure when $\epsilon \to 0$ and/or $\lambda \to +\infty$, so that in this limit we do have the entropy inequality

$$\partial_t \mathcal{U}(\boldsymbol{u}) + \partial_x \mathcal{F}(\boldsymbol{u}) \leqslant 0. \tag{2.19}$$

We conclude this section with another setting for complex compressible materials which naturally gives rise to hyperbolic equations with viscous perturbations in non-conserved form. The models under consideration may be regarded as natural extensions of the usual Navier–Stokes (NS) equations. Such extensions make use of N independent internal energies $(e_i)_{1\leqslant i\leqslant N}$ for governing N independent pressure laws $(p_i(\tau, e_i))_{1\leqslant i\leqslant N}$. These partial differential equation (PDE) models take the generic form:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + \sum_{i=1}^{N} p_i(\tau, e_i)\right) &= \epsilon\partial_x\left(\sum_{i=1}^{n} \mu_i(\tau, e_i)\partial_x u\right), \\ \partial_t(\rho e_i) + \partial_x(\rho u e_i) + p_i(\tau, e_i)\partial_x u &= \epsilon\mu_i(\tau, e_i)(\partial_x u)^2, \end{aligned} \tag{2.20}$$

where $\rho > 0$ denotes the density, $u \in \mathbb{R}$ is the velocity and $\tau = 1/\rho > 0$ is the specific volume. Here, $\epsilon > 0$ denotes the inverse of the Reynolds number.

Several models from physics enter the proposed framework and can be distinguished according to the precise definition of the constitutive closure laws for the pressures and the viscosities. Precise assumptions on the required state laws will be addressed in § 4, which is devoted to the analysis of the travelling-wave solutions of (2.20).

Models from plasma physics, where the temperature of the electron gases must be distinguished from the temperature of the other heavy species, typically take the form (2.20) with $N = 2$ [18]. Models from the physics of compressible turbulent flows can also be seen to fall within the frame of PDEs (2.20). We refer the reader to [7–9,15,16] for the mathematical and numerical analysis of several models ranging from two distinct internal energies (the so-called laminar and turbulent energies) to $N > 2$ different energies to account for a refined description of the turbulent energy cascade. We also emphasize that multi-fluid models, such as those studied in [10], enter the proposed framework.

In most if not all the applications to complex compressible materials, ϵ, the inverse of the Reynolds number modulating the strength of the viscous perturbation, is a small parameter. Solutions of interest therefore exhibit stiff transition zones, namely viscous shock layers and boundary layers. Viscous shocks cannot be properly resolved for mesh refinements of practical interest and we are thus led to study the limit $\epsilon \to 0^+$ in the system (2.20).

There exist several ways to tackle the system (2.20) in the limit $\epsilon \to 0^+$, depending on suitable change of variables. It can be seen that (2.20) can recast equivalently as

$$\mathcal{A}_0(\boldsymbol{w}^\epsilon)\partial_t \boldsymbol{w}^\epsilon + \mathcal{A}_1(\boldsymbol{w}^\epsilon)\partial_x \boldsymbol{w}^\epsilon = \epsilon \partial_x(\mathcal{D}(\boldsymbol{w}^\epsilon)\partial_x \boldsymbol{w}^\epsilon) \tag{2.21}$$

with $\mathcal{A}_0$ regular, or

$$\partial_t \boldsymbol{u}^\epsilon + \partial_x \mathcal{F}(\boldsymbol{u}^\epsilon) = \epsilon \mathcal{R}(\boldsymbol{u}^\epsilon, \partial_x \boldsymbol{u}^\epsilon, \partial_{xx} \boldsymbol{u}^\epsilon). \tag{2.22}$$

Namely, in (4.15), the diffusive operator is written in conservation form, while $\mathcal{A}_0(\boldsymbol{w})$ and $\mathcal{A}_1(\boldsymbol{w})$ are not Jacobian matrices of some flux function. By contrast, in (2.22) the first-order operator, but not the regularization terms, stand in conservation form.

The precise definitions of the change of unknown $\boldsymbol{w}$ and $\boldsymbol{u}$ are addressed in § 4. We just highlight at this stage that concerning the equivalent form (2.21), and provided that suitable estimates on the sequence of solutions $\boldsymbol{w}_\epsilon$ hold true, the right-hand side is expected to vanish in the limit $\epsilon \to 0^+$ in the usual sense of the distributions. By contrast, the left-hand side in non-conservation form may be handled due to the theory of family of paths introduced by LeFloch [38] and Dal Maso *et al*. [20]. As far as the next equivalent form (2.22) is concerned, the left-hand side now stands in conservation form and can be treated in the usual sense of the distributions. In contradiction, the right-hand side can no longer be expected to converge to 0, generally speaking, but is expected merely to converge to a bounded Borel measure concentrated on the shocks of the limit solutions. The next section provides a convenient framework for handling the required passage to the limit in the PDEs (2.22).

3. Kinetic relations for non-conservative systems

Having in mind the examples described in the previous section, we present one of the main contributions of the present paper, i.e. the *concept of kinetic relations for non-conservative systems*, which allows us to rigorously define certain non-conservative products arising in the applications.

Recall that weak solutions to non-conservative systems are defined in the class of functions with bounded variation (BV). By standard regularity theorems, such functions can be handled essentially as if they were piecewise Lipschitz continuous. Henceforth, for simplicity of presentation, we restrict our attention to piecewise Lipschitz continuous functions and refer the reader to [20] for details of the Dal Maso–LeFloch–Murat (DLM) theory.

For the simplicity of presentation, we restrict our attention to solutions defined in the neighbourhood of a constant state in $\mathbb{R}^N$ which can be normalized to be the origin. We denote by $\mathcal{B}_{\delta_0}$ the ball centred at the origin and of small radius $\delta_0 > 0$.

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Dal Maso *et al.*'s definition is based on prescribing a *family of Lipschitz continuous paths* $\phi = \phi(s; u_0, u_1) \in \mathcal{B}_{\delta_0}$ $(s \in [0,1])$, which allows one to connect any two points u_0, u_1 in $\mathcal{B}_{\delta_1}$ for some $\delta_1 \leqslant \delta_0$. In particular, it is assumed that

$$\phi(0; u_0, u_1) = u_0, \qquad \phi(1; u_0, u_1) = u_1. \tag{3.1}$$

(See [20, 43] for the precise conditions, omitted in this short review.) As proposed in [38], this family of paths should be determined from travelling-wave solutions of an augmented model.

Indeed, it has been recognized that weak solutions u of (1.1) depend on the effect of small scales that have been neglected at the hyperbolic level of modelling but are taken into account in the augmented version

$$\partial_t u + A(u)\partial_x u = R(u, \epsilon u_x, \epsilon^2 u_{xx}, \dots), \tag{3.2}$$

where $R^\epsilon = 0$ if $\epsilon = 0$. The family of paths determined by travelling-wave trajectories yields precisely the 'missing information' required to set up the hyperbolic theory.

A (piecewise Lipschitz continuous) function $u = u(x,t)$ is called a *weak solution* of the non-conservative system (1.1) if u satisfies the equations (1.1) in a classical sense in the regions where it is Lipschitz continuous and, additionally, the following generalization of the Rankine–Hugoniot jump relation holds along every curve of the discontinuity of u. More precisely, for any shock wave connecting two states u_0, u_1 at the speed $\bar{\Lambda} = \bar{\Lambda}(u_0, u_1)$,

$$u(x,t) = \begin{cases} u_0, & x < \bar{\Lambda} t, \\ u_1, & x > \bar{\Lambda} t, \end{cases} \tag{3.3}$$

we impose the *generalized jump relation*

$$-\bar{\Lambda}(u_1 - u_0) + \int_0^1 A(\phi(s; u_0, u_1))\partial_s \phi(s; u_0, u_1)\, \mathrm{d}s = 0. \tag{3.4}$$

Note that in the conservative case when $A(u) = Df(u)$ for some flux-function f this relation reduces to

$$-\bar{\Lambda}(u_1 - u_0) + f(u_1) - f(u_0) = 0,$$

which is *independent* of the paths ϕ and is the standard jump relation.

Based on the above definition, one can solve [20] the Riemann problem for (1.1), corresponding to the piecewise constant initial data

$$u(x,0) = \begin{cases} u_{\mathrm{L}}, & x < 0, \\ u_{\mathrm{R}}, & x > 0, \end{cases} \tag{3.5}$$

where u_{L}, u_{R} are constants in $\mathcal{B}_{\delta_2}$ with $\delta_2 \leqslant \delta_1$. This construction generalizes Lax's standard construction for conservative systems [35, 36]. Recall that (admissible) shock waves must be constrained by Lax shock inequalities (for some $j = 1, \dots, N$)

$$\lambda_j(u_0) > \bar{\Lambda} > \lambda_j(u_1). \tag{3.6}$$

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The Riemann solver can then be used to design numerical schemes for the approximation of the general initial-value problem, e.g. Glimm or front-tracking schemes.

In certain applications, it has been observed that the whole family of paths is not required in order to define the non-conservative products that arise in the hyperbolic system under consideration. It is precisely our purpose in the present paper to introduce, for a particular class of non-conservative systems, a new definition of weak solutions which imposes Rankine–Hugoniot jump relations in the form of 'kinetic relations' and does not require the knowledge of any 'internal structure' for shock waves.

We shall assume that the non-conservative system under consideration *formally* admits N conservation laws, so we consider the system

$$\partial_t u + \partial_x f(u) = 0, \tag{3.7}$$

which consists of conservation laws valid for *smooth* solutions only. Our goal is to describe singular limits (3.2), where $R = R^\epsilon$ is a *non-conservative regularization*. More precisely, we shall supplement (3.7) with N jump relations, referred to as 'kinetic relations', which determine the dynamics of shocks in weak solutions to (3.7).

We suppose that in an open and convex domain $\mathcal{U} \subset \mathbb{R}^N$ of the phase space, the system (1.1) is strictly hyperbolic, with eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$ and basis of eigenvectors $l_i(u)$, $r_i(u)$. Let $\mathcal{L} \subset \mathbb{R}$ be a compact set containing all speeds under consideration in the problem.

Definition 3.1. A *kinetic function* is a Lipschitz continuous map $\Phi\colon \mathcal{U} \times \mathcal{L} \to \mathbb{R}^N$ satisfying (for $j = 1, \ldots, N$)

$$\left.\begin{aligned} \Phi(u, \lambda_j(u)) &= 0, & u &\in \mathcal{U}, \\ |l_j(u) \cdot \partial_\Lambda \Phi(u, \Lambda)| &\leqslant c_1 |\Lambda - \lambda_j(u)|, & (u, \Lambda) &\in \mathcal{U} \times \mathcal{L}, \end{aligned}\right\} \tag{3.8}$$

for some constant $c_1 > 0$. Given a kinetic function Φ, a piecewise Lipschitz solution $u = u(x,t) \in \mathcal{U}$ is called a *Φ-admissible weak solution* to (1.1) if the differential equations (3.7) are satisfied in each region of continuity of u and, moreover, along any curve of discontinuity of u, connecting some values u_-, u_+ at the speed Λ. The following *kinetic relation* holds:

$$-\Lambda(u_+ - u_-) + f(u_+) - f(u_-) = \Phi(u_-, \Lambda). \tag{3.9}$$

In certain applications, it may be more convenient to express the kinetic functions in terms of the left- and right-hand states, that is, $\Phi = \Phi(u_-, u_+)$. In the applications, the kinetic function Φ should be determined from travelling-wave solutions of a specific system (3.2) and should be thought of as a 'correction' to the standard Rankine–Hugoniot relation.

By introducing the Borel measure (denoted by μ_u^Φ) that vanishes in the regions of continuity of u and has the mass $\Phi(u_-, \Lambda)$ along its curves of discontinuity, we easily see that definition 3.1 is equivalent to the requirement [40]

$$\partial_t u + \partial_x f(u) = \mu_u^\Phi, \tag{3.10}$$

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which is regarded as an equality between bounded measures. Note that we recover the usual conservative case by simply choosing both Φ and μ_u^Φ to vanish identically. In general, μ_u^Φ depends strongly on the function u.

In the rest of this section we study the case of genuinely nonlinear systems. This assumption allows us to use the shock speed as a regular parameter along the (generalized) Hugoniot curve.

THEOREM 3.2. (A Riemann problem for non-conservative systems with kinetic relations.) *Suppose that (3.7) is a strictly hyperbolic system in a neighbourhood $\mathcal{B}_{\delta_0}$ of the origin 0 and admits genuinely nonlinear characteristic fields only, i.e.*

$$(\nabla\lambda_j \cdot r_j)(0) > 0, \quad j = 1, \dots, N.$$

Let $\Phi = \Phi(u, \Lambda)$ be a (Lipschitz continuous) kinetic function defined in the neighbourhood $\mathcal{B}_{\delta_0} \times \mathcal{L}$ for some sufficiently small $\delta > 0$ by

$$\mathcal{L} := \bigcup_j \mathcal{L}_j, \qquad \mathcal{L}_j := (\lambda_j(0) - \delta, \lambda_j(0) + \delta).$$

Then, there exists $\delta_1 \leqslant \delta_0$ such that the Riemann problem (3.5), (3.7) with data $u_{\mathrm{L}}, u_{\mathrm{R}} \in \mathcal{B}_{\delta_1}$ admits a unique Φ-admissible weak solution in the class of piecewise smooth solutions consisting of a combination of rarefaction waves and shock waves satisfying the kinetic relation. Moreover, the corresponding wave curves are solely Lipschitz continuous.

Clearly, under the assumptions of the above theorem, (3.8) implies that, for $j = 1, \dots, N$,

$$|l_j(u) \cdot \Phi(u, \Lambda)| \leqslant c_2 |\Lambda - \lambda_j(u)|^2, \quad (\lambda, u) \in \mathcal{U} \times \mathcal{L}_j, \tag{3.11}$$

for some $c_2 > 0$.

Proof. We want to generalize the proof given in [20] for general families of paths; see also the related proof in [29] for non-classical shocks. We shall show that the given set of jump conditions (3.9) suffices to determine a (generalized) Hugoniot curve uniquely, and we shall investigate its tangency and regularity properties. The rest of the proof (selection of the admissible part of the Hugoniot curve, actual construction of the wave curves, Riemann solution) then follows as in [20] and will be omitted.

We denote by $\bar\lambda_j(u_0, u_1)$ and $\bar l_j(u_0, u_1)$ the eigenvalues and left-eigenvectors of the averaged matrix

$$A(u_0, u_1) := \int_0^1 Df(u_0 + m(u_1 - u_0))\, \mathrm{d}m.$$

In a neighbourhood of the point $(u_0, \lambda_j(u_0))$, we consider the kinetic relation

$$G(\Lambda, u_1) := -\Lambda(u_1 - u_0) + f(u_1) - f(u_0) - \Phi(u_0, \Lambda) = 0. \tag{3.12}$$

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Fix some index i and let us restrict our attention to the (nonlinear) cone-like region K determined by the two conditions on $u_1 \in \mathcal{B}_{\delta_1}$:

$$|(u_1 - u_0) \cdot l_i(u_0, u_1)| \geqslant C_* |\Lambda - \lambda_i(u_0)|, \qquad |u_1 - u_0| + |\Lambda - \lambda_i(u_0)| < \delta_2,$$

where a condition on $C_* > 0$ will be imposed below. Observe that $G(u_0, \lambda_i(u_0)) = 0$.

Multiplying the generalized jump relation (3.12) by $\bar{l}_i(u_0, u_1)$, we find

$$\begin{aligned} 0 &= \bar{l}_i(u_0, u_1) \cdot (A(u_0, u_1) - \Lambda)(u_1 - u_0) - \bar{l}_i(u_0, u_1) \cdot \Phi(u_0, \Lambda) \\ &= (\bar{\lambda}_i(u_0, u_1) - \Lambda)\bar{l}_i(u_0, u_1) \cdot (u_1 - u_0) - \bar{l}_i(u_0, u_1) \cdot \Phi(u_0, \Lambda). \end{aligned}$$

Therefore, we can express the shock speed $\Lambda = \bar{\Lambda}(u_0, u_1)$ in the form

$$0 = \bar{\Lambda} - \bar{\lambda}_i(u_0, u_1) + \frac{\bar{l}_i(u_0, u_1) \cdot \Phi(u_0, \bar{\Lambda})}{\bar{l}_i(u_0, u_1) \cdot (u_1 - u_0)} =: \Omega(u_1, \bar{\Lambda}). \tag{3.13}$$

Now, observe that the function Ω satisfies

$$\begin{aligned} \left| \frac{\partial \Omega}{\partial \lambda}(u_1, \bar{\Lambda}) - 1 \right| &= \frac{\bar{l}_i(u_0, u_1) \cdot \partial_\lambda \Phi(u_0, \bar{\Lambda})}{\bar{l}_i(u_0, u_1) \cdot (u_1 - u_0)} \\ &\geqslant -c_2 O(1) \frac{|u_1 - u_0| + |\bar{\Lambda} - \bar{\lambda}_i(u_0)|}{|\bar{l}_i(u_0, u_1) \cdot (u_1 - u_0)|}, \end{aligned}$$

where the constant $O(1)$ depends only on the flux. Hence, we have

$$\frac{\partial \Omega}{\partial \Lambda}(u_1, \bar{\Lambda}) = 1 + \frac{c_2}{C_*} O(1),$$

which is positive provided that c_1 is sufficiently small. As a consequence, the implicit function for Lipschitz continuous mappings applies and shows that the implicit equation (3.13) determines the shock speed $\bar{\Lambda} = \bar{\Lambda}(u_0, u_1)$ uniquely.

Next, we consider the remaining components, corresponding to $j \neq i$:

$$H(u_0, u_1) := l_j(u_0, u_1) \cdot (u_1 - u_0) - \frac{l_j(u_0, u_1) \cdot \Phi(u_0, \bar{\Lambda})}{\bar{\Lambda}(u_0, u_1) - \lambda_j(u_0, u_1)}.$$

Denoting by $L(u_0)$ the $N \times (N-1)$ matrix of vectors $l_j(u_0)$ for $j \neq i$, we can compute the differential of H as follows:

$$\begin{aligned} \frac{\mathrm{D}H}{\mathrm{D}u_1}(u_0, u_1) = L(u_0) &+ O(1)|u_1 - u_0| + O(1) C_1 \frac{|\bar{\Lambda}(u_0, u_1) - \lambda_i(u_0)|^2}{|\bar{\Lambda}(u_0, u_1) - \lambda_j(u_0)|} \\ &+ O(1) C_1 |\bar{\Lambda}(u_0, u_1) - \lambda_i(u_0)| \left| \frac{\partial \bar{\Lambda}}{\partial u_1}(u_0, u_1) \right| \\ &+ O(1) C_1 |\bar{\Lambda}(u_0, u_1) - \lambda_i(u_0)|^2 \left| \frac{\partial \bar{\Lambda}}{\partial u_1}(u_0, u_1) - \tfrac{1}{2} \nabla \lambda_i(u_0) \right|, \end{aligned}$$

where we have used that $\bar{\Lambda}(u_0, u_1) - \lambda_j(u_0, u_1)$ is bounded away from 0. Hence, we find

$$\frac{\partial H}{\partial u_1}(u_0, u_1) = L(u_0) + o(1) + o(1) \left| \frac{\partial \bar{\Lambda}}{\partial u_1}(u_0, u_1) \right|.$$

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On the other hand, the u_1-derivative of the shock speed satisfies

$$\begin{aligned}\frac{\partial \bar{\Lambda}}{\partial u_1}(u_0,u_1) = \tfrac{1}{2}\nabla\lambda_i(u_0) &+ O(1)\frac{|\lambda-\lambda_i(u_0)|^{b+1}}{|l_i(u_0,u_1)\cdot(u_1-u_0)|}\\ &+ O(1)\frac{|\bar{\Lambda}-\lambda_i(u_0)|^2}{|l_i(u_0,u_1)\cdot(u_1-u_0)|^2}\\ &+ O(1)\frac{|\bar{\Lambda}-\lambda_i(u_0)|}{|l_i(u_0,u_1)\cdot(u_1-u_0)|}\frac{\partial \bar{\Lambda}}{\partial u_1}(u_0,u_1),\end{aligned}$$

which shows that

$$\frac{\partial \bar{\Lambda}}{\partial u_1}(u_0,u_1) = \tfrac{1}{2}\nabla\lambda_i(u_0) + o(1).$$

In conclusion,

$$\frac{\partial H}{\partial u_1}(u_0,u_1) = L(u_0) + o(1),$$

and the implicit function theorem applies to the set of equations $H(u_0,u_1)=0$, which therefore determines a unique shock curve $s\mapsto u_1 = u_1(s;u_0)$, defined locally near u_0. Near the base point $u(0)=u_0$, the tangent of this curve is defined almost everywhere and, due to the smallness of the constant c_1 in (3.8), takes its values in a small neighbourhood of the eigenvector $r_i(u_0)$. □

We now introduce a class of non-conservative system to which the framework in the previous subsection can be applied. We assume that the first $N-p$ equations in (1.1) take a conservative form, while the remaining p equations are non-conservative. In other words, we set $u=(v,w)$ and we consider the non-conservative systems

$$\left.\begin{aligned}\partial_t v + \partial_x g(v,w) &= 0,\\ \partial_t w + B(v,w)\partial_x v + C(v,w)\partial_x w &= 0.\end{aligned}\right\} \tag{3.14}$$

Here $g=g(v,w)\in\mathbb{R}^{N-p}$ while $B=B(v,w), C=C(v,w)$ are $p\times(N-p)$ and $p\times p$ matrix-valued mappings, respectively.

It must be stressed that the assumption made here refers directly to the set of equations listed in (1.1) or to *linear* combinations of them. Of course, nonlinear functions of the original variable u cannot be considered at this level of the analysis, in general, since we seek discontinuous solutions.

Our second assumption is the existence of p mathematical entropy pairs. That is, we assume that there exist k strictly convex functions $U_k = U_k(v,w)$ together with their associated flux $F_k = F_k(v,w)$ such that

$$\partial_t U_k(v,w) + \partial_x F_k(v,w) = 0, \quad k=1,\dots,p, \tag{3.15}$$

holds for all *smooth* solutions to (3.8). We search for solutions satisfying the *entropy inequality*

$$\partial_t U_k(v,w) + \partial_x F_k(v,w) \leqslant 0, \quad k=1,\dots,p. \tag{3.16}$$

Many of the models of interest take the form (3.8)–(3.16).

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DEFINITION 3.3. Nonlinear hyperbolic systems in a non-conservative form that have the structure (3.14) admit at least p mathematical entropies and satisfy the non-degeneracy condition

$$\det(\nabla_w U_1(v,w),\dots,\nabla_w U_p(v,w)) \neq 0 \tag{3.17}$$

are called *non-conservative systems endowed with a full set of entropies.*

We now focus on the entropy dissipation associated with the entropies U_k. The basic idea is to replace the non-conservative equations in system (3.8) with conservative equations for the entropy dissipation, with the latter involving a measure source term. Of course, it is necessary for $(v,w)\mapsto(v,U(v,w))$ to define a change of variable, say $U_w\neq 0$.

Observe first that the inequality (3.16) implies a constraint on shock waves, i.e. with the notation introduced in (3.3),

$$E_k(\Lambda;u_0,u_1) := -\Lambda(U_k(u_1)-U_k(u_0))+F_k(u_1)-F_k(u_0)\leqslant 0 \tag{3.18}$$

for all $k=1,\dots,p$. On the other hand, the first $N-p$ equations in (3.8) yield $N-p$ jump relations in the fully explicit form

$$-\Lambda(v_1-v_0)+g(v_1,w_1)-g(v_0,w_0)=0. \tag{3.19}$$

Since p jump relations are 'missing', we supply them in the form

$$E_i(\Lambda;u_0,u_1)=\Phi_i(\Lambda;u_0)\leqslant 0,\quad i=1,\dots,p, \tag{3.20}$$

which we refer to as a kinetic relation and where Φ is a given 'constitutive' function, called a 'kinetic function', to be determined in a case-by-case manner in the examples.

DEFINITION 3.4. Let $\Phi=(0,\dots,0,\Phi_1,\dots\Phi_p)$ be a kinetic function. A piecewise Lipschitz continuous function $u=(v,w)$ is called a *Φ-admissible solution of the non-conservative system* (3.8) if it satisfies the equations in a classical sense in the regions of continuity and if each propagating discontinuity satisfies the $N-p$ jump relations (3.19) together with the kinetic relations (3.20).

We reformulate the main result in a slightly weaker form that is adapted to the present context, since it is natural to assume that the entropy dissipation is of cubic order near the base point.

COROLLARY 3.5. (Riemann problem for non-conservative systems endowed with a full set of entropies.) *Consider a non-conservative system endowed with a full set of entropies. Suppose that the system is strictly hyperbolic and genuinely nonlinear in the neighbourhood of some state $u_*=(v_*,w_*)$. Let $\Phi_i=\Phi_i(u_0,u_1)$ be a regular function defined in the neighbourhood of each speed $\lambda_j(u_*)$ for $j=1,\dots,N$ and satisfying, for all u_0, u_1,*

$$\left.\begin{aligned}\Phi_i(u_0,\lambda_i(u_0))&=0,\\ \partial_\Lambda\Phi_i(u_0,\Lambda)&=O(1)(\Lambda-\lambda_j(u_0))^2,\end{aligned}\right\} \tag{3.21}$$

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where $O(1)$ denotes a positive and bounded function. Then, the corresponding Riemann problem admits a unique admissible solution in the class of piecewise smooth solutions consisting of a combination of rarefaction waves and admissible shock waves.

The theory in this section applies to the examples listed in § 2, at least when the resonance effect is avoided. It is straightforward to include linearly degenerate characteristic fields, provided that the kinetic function is chosen to vanish identically for those fields.

4. Multi-pressure Navier–Stokes system

In this section, we establish the existence and uniqueness of the travelling-wave solutions of the multi-pressure NS equations introduced in § 2, under fairly general assumptions on the pressure and viscosity closure laws. The equations under consideration were stated in (2.20). Each smooth pressure law $p_i(\tau, e_i)$, $1 \leqslant i \leqslant N$, is assumed to obey the second law of thermodynamics, namely

$$T_i(\tau, e_i)\,\mathrm{d}s_i = \mathrm{d}e_i + p_i(\tau, e_i)\,\mathrm{d}\tau, \tag{4.1}$$

where $T_i(\tau, e_i) > 0$ is the corresponding temperature variable and $s_i > 0$ denotes the specific entropy.

The map $(\tau, s_i) \mapsto e_i(\tau, s_i)$ is thus well defined and is assumed to be strictly convex. In addition, the following asymptotic conditions are assumed:

$$\lim_{\tau\to 0^+} e_i(\tau, s_i) = +\infty, \qquad \lim_{s_i\to+\infty} e_i(\tau, s_i) = +\infty, \qquad \lim_{\tau\to+\infty} e_i(\tau, s_i) = 0. \tag{4.2}$$

It follows that

$$p_i(\tau, s_i) = -\frac{\partial e_i}{\partial \tau}(\tau, s_i) > 0, \qquad T_i(\tau, s_i) = \frac{\partial e_i}{\partial s_i}(\tau, s_i) > 0. \tag{4.3}$$

Furthermore, the following assumptions are introduced for any given $\tau > 0$:

$$\frac{\partial p_i}{\partial s_i}(\tau, s_i) > 0, \tag{4.4}$$

$$\sum_{i=1}^{N} \frac{\partial^2 p_i}{\partial \tau^2}(\tau, \boldsymbol{s}) > 0, \tag{4.5}$$

$$\lim_{\tau\to 0^+} \sum_{i=1}^{N} p_i(\tau, \boldsymbol{s}) = +\infty, \qquad \lim_{\tau\to+\infty} \sum_{i=1}^{N} p_i(\tau, \boldsymbol{s}) = 0, \tag{4.6}$$

$$\lim_{\tau\to 0^+} \sum_{i=1}^{N} \frac{\partial p_i}{\partial \tau}(\tau, \boldsymbol{s}) = -\infty, \qquad \lim_{\tau\to+\infty} \sum_{i=1}^{N} \frac{\partial p_i}{\partial \tau}(\tau, \boldsymbol{s}) = 0, \tag{4.7}$$

$$\lim_{s_i\to+\infty} \frac{\partial p_i}{\partial \tau}(\tau, s_i) = -\infty, \tag{4.8}$$

where $\boldsymbol{s} = (s_1, \ldots, s_N)$. We refer the reader to [51] for general properties of the fluid equations and the equation of state. Next, the viscosity laws are given smooth

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functions with

$$\mu_i(\tau, s_i) \geqslant 0, \quad 1 \leqslant i \leqslant N, \qquad \mu(\tau, s) := \sum_{i=1}^{N} \mu_i(\tau, s_i) > 0. \tag{4.9}$$

To abbreviate the notation, the PDE system (2.20) is given in the condensed form

$$\partial_t \boldsymbol{v}^\epsilon + \mathcal{A}(\boldsymbol{v}^\epsilon)\partial_x \boldsymbol{v}^\epsilon = \epsilon \mathcal{B}(\boldsymbol{v}^\epsilon, \partial_x \boldsymbol{v}^\epsilon, \partial_{xx} \boldsymbol{v}^\epsilon),$$

with $\boldsymbol{v}$ in the phase space

$$\Omega_{\boldsymbol{v}} = \{\boldsymbol{v} = (\rho, \rho u, (\rho e_i)_{1 \leqslant i \leqslant N}) \in \mathbb{R}^{N+2};\ \rho > 0,\ \rho u \in \mathbb{R},\ \rho e_i > 0,\ 1 \leqslant i \leqslant N\}.$$

The basic properties of (2.20) are summarized in the following statement.

LEMMA 4.1. *The underlying first-order part from (2.20) is hyperbolic in $\Omega_{\boldsymbol{v}}$ and admits three distinct eigenvalues:*

$$\lambda_1(\boldsymbol{v}) = u - c(\boldsymbol{v}),\ \lambda_2(\boldsymbol{v}) = \cdots = \lambda_{N+1}(\boldsymbol{v}) = u, \qquad \lambda_{N+2}(\boldsymbol{v}) = u - c(\boldsymbol{v}), \tag{4.10}$$

where we set

$$c^2(\boldsymbol{v}) = \sum_{i=1}^{N} -\tau^2 \frac{\partial p_i}{\partial \tau}(\tau, s_i). \tag{4.11}$$

The extreme fields are genuinely nonlinear, while the intermediate fields are linearly degenerate. Then, smooth solutions of (2.20) satisfy the additional conservation law

$$\partial_t(\rho E)^\epsilon + \partial_x\left(\{\rho E\}(\boldsymbol{v}^\epsilon) + \sum_{i=1}^{N} p_i(\tau^\epsilon, s_i^\epsilon)u^\epsilon\right) = \epsilon \partial_x\left(\sum_{i=1}^{N} \mu_i(\tau^\epsilon, s_i^\epsilon)u^\epsilon \partial_x u^\epsilon\right), \tag{4.12}$$

where the total energy reads

$$(\rho E) = \frac{(\rho u)^2}{2\rho} + \sum_{i=1}^{N} \rho e_i. \tag{4.13}$$

Finally, the smooth solutions of (2.20) obey the N balance equations

$$\partial_t(\rho s_i)^\epsilon + \partial_x((\rho s_i)^\epsilon u^\epsilon) = \epsilon \frac{\mu_i(\tau^\epsilon, s_i^\epsilon)}{T_i(\tau^\epsilon, s_i^\epsilon)}(\partial_x u^\epsilon)^2. \tag{4.14}$$

As previously claimed, changes of variables with distinctive features allow the recasting of (2.20) either in the equivalent form

$$\mathcal{A}_0(\boldsymbol{w}^\epsilon)\partial_t \boldsymbol{w}^\epsilon + \mathcal{A}_1(\boldsymbol{w}^\epsilon)\partial_x \boldsymbol{w}^\epsilon = \epsilon \partial_x(\mathcal{D}(\boldsymbol{w}^\epsilon)\partial_x \boldsymbol{w}^\epsilon), \tag{4.15}$$

with $\mathcal{A}_0$ regular, or in the form

$$\partial_t \boldsymbol{u}^\epsilon + \partial_x \mathcal{F}(\boldsymbol{u}^\epsilon) = \epsilon \mathcal{R}(\boldsymbol{u}^\epsilon, \partial_x \boldsymbol{u}^\epsilon, \partial_{xx} \boldsymbol{u}^\epsilon). \tag{4.16}$$

We briefly discuss the changes of variables involved in (4.15) and (4.16). Concerning (4.15), we first observe that summing the N governing equations for the internal

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energies yields

$$\partial_t \rho e + \partial_x \rho e u + \sum_{i=1}^{N} p_i \partial_x u = \epsilon \mu(\tau, \boldsymbol{s})(\partial_x u)^2,$$

so that the following identities are easily checked:

$$\mu(\tau,\boldsymbol{s})(\partial_t \rho e_i + \partial_x(\rho e_i u) + p_i(\tau, s_i)\partial_x u) - \mu_i(\tau, s_i)\left(\partial_t \rho e + \partial_x(\rho e u) + \sum_{i=1}^{N} p_i \partial_x u\right) = 0, \quad 1 \leqslant i \leqslant N-1. \quad (4.17)$$

Since $\rho e = \rho E - (\rho u)^2/2\rho$, the conservation laws for ρ, ρu and ρE supplemented by the $(N-1)$ balance equations in (4.17) can be seen to give the equivalent form stated in (4.15) when defining $\boldsymbol{w} = (\rho, \rho, \rho E, (\rho e_i)_{1\leqslant i\leqslant N-1})$. A direct calculation shows that $\det \mathcal{A}_0(\boldsymbol{w}) = \mu(\tau, \boldsymbol{s})^{n-1} > 0$.

Concerning system (4.16), several changes of variables can be used and we advocate in the following the change of variables $\boldsymbol{v} \in \Omega_{\boldsymbol{v}} \mapsto \boldsymbol{u}(\boldsymbol{v}) \in \Omega_{\boldsymbol{u}}$, with $\boldsymbol{u}(\boldsymbol{v}) = (\rho, \rho u, (\rho s_i)_{1\leqslant i\leqslant N})$.

As emphasized in § 3, both approaches rely on the study of the travelling-wave solutions of (2.20). Due to the frame invariance properties satisfied by the PDE model (2.20), it suffices to analyse travelling-wave solutions associated with the first extreme field. With this in mind, the main result of this section is as follows.

THEOREM 4.2. (Travelling-wave solutions to the multi-pressure NS system.) *Consider the multi-pressure Navier–Stokes system (2.20) when the pressure satisfies the positivity, convexity and asymptotic conditions (4.2)–(4.9). Let $\boldsymbol{u}_{\mathrm{L}} \in \Omega_{\boldsymbol{u}}$ and $\sigma \in \mathbb{R}$ be given such that*

$$\frac{u_{\mathrm{L}} - \sigma}{c(\boldsymbol{u}_{\mathrm{L}})} > 1, \quad c^2(\boldsymbol{u}_{\mathrm{L}}) = \sum_{i=1}^{N} -\tau_{\mathrm{L}}^2 \frac{\partial p}{\partial \tau}(\tau_{\mathrm{L}}, (s_i)_{\mathrm{L}}). \quad (4.18)$$

Then, there exists a unique travelling-wave solution to (2.20) issuing from the left-hand state $\boldsymbol{u}_{\mathrm{L}}$ and reaching some right-hand state $\boldsymbol{u}_{\mathrm{R}} \in \Omega_{\boldsymbol{u}}$ with

$$0 < \frac{u_{\mathrm{R}} - \sigma}{c(\boldsymbol{u}_{\mathrm{R}})} < 1. \quad (4.19)$$

The proof of this result will follow from the characterization of a positively invariant compact set of $\Omega_{\boldsymbol{u}}$. Then the LaSalle invariance principle applied in connection with a suitable Lyapunov function ensures the existence of a travelling wave. Uniqueness is obtained as a simple consequence of the centre manifold theorem.

We gather here some of the notation used repeatedly hereafter and give the precise form of the autonomous system which governs the viscous profiles we study for existence. Simple but useful geometrical properties induced by the corresponding vector field will be then put forward.

Due to Galilean invariance, it suffices to consider the case of a null velocity σ. The precise form of the PDE system governing the travelling-wave solutions then

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follows when restricting our attention to solutions which depend solely on x:

$$\left.\begin{aligned} (\rho u)_x &= 0, \\ (\rho u^2 + p(\tau, \boldsymbol{s}))_x &= (\mu(\tau, \boldsymbol{s}) u_x)_x, \\ T_i(\tau, s_i)(\rho s_i u)_x &= \mu_i(\tau, \boldsymbol{s})(u_x)^2, \quad 1 \leqslant i \leqslant N. \end{aligned}\right\} \tag{4.20}$$

The first equation in (4.20) implies that the relative mass flux ρu has a constant value denoted by $m = \rho_{\mathrm{L}} u_{\mathrm{L}}$. As already emphasized, we focus on travelling-wave solutions associated with the first genuinely nonlinear field; namely, we consider $m > 0$. Observe that the Lax condition (4.18) expressed for a null velocity σ reads

$$m > \rho_{\mathrm{L}} c_{\mathrm{L}}.$$

Next, by integrating once the second equation in (4.20), the identity $u = m\tau$ allows one to derive the following $(N+1)$-dimensional autonomous system:

$$\left.\begin{aligned} \dot{\tau} &= \frac{1}{\mu(\tau, \boldsymbol{s})}(p(\tau, \boldsymbol{s}) - p(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) + m^2(\tau - \tau_{\mathrm{L}})) := \frac{1}{\mu(\tau, \boldsymbol{s})}\mathcal{F}(\tau, \boldsymbol{s}), \\ \dot{s}_i &= \frac{\mu_i(\tau, s_i)}{\mu^2(\tau, \boldsymbol{s}) T_i(\tau, s_i)}\mathcal{F}^2(\tau, \boldsymbol{s}), \quad 1 \leqslant i \leqslant N, \end{aligned}\right\} \tag{4.21}$$

where dots denote differentiation with respect to the rescaled variable x/m that in the following we shall refer to, with slight abuse of notation, as a time.

This dynamical system is endowed with the following open subset of $\mathbb{R}^{N+1}$ which will serve as a natural phase space:

$$\Omega = \{\omega := (\tau, \boldsymbol{s}) \in \mathbb{R}^{N+1};\ \tau > 0\}. \tag{4.22}$$

To shorten the notation, a given function Ψ of τ and $\boldsymbol{s}$ is simply denoted hereafter by $\Psi(\omega)$.

Recall that the total viscosity $\mu(\omega)$ is assumed to stay strictly positive over Ω. Then, the regularity assumptions made on all the thermodynamic and viscosity mappings ensure that the vector field in (4.21) is continuously differentiable.

The unique non-extensible solution of (4.21) with initial data ω_0 in Ω is referred as to the flow $\omega_0 t$ for the times t in the maximal interval of existence $(t^-(\omega_0), t^+(\omega_0))$. The positive (respectively negative) semi-orbit $\gamma^+(\omega_0)$ (respectively, $\gamma^-(\omega_0)$) classically denotes the set of states $\omega_0 \cdot [0, t^+(\omega_0)) = \{\omega_0 \cdot t\colon 0 \leqslant t < t^+(\omega_0)\}$ (respectively, $\omega_0 \cdot (t^-(\omega_0), 0] = \{\omega_0 \cdot t\colon t^-(\omega_0) < t \leqslant 0\}$), the orbit then being defined as $\gamma(\omega_0) = \gamma^-(\omega_0) \cup \gamma^+(\omega_0)$. Finally, for each ω_0 in Ω, the positive limit set (the so-called ϖ-limit set in what follows) of ω_0 yields the definition $\varpi(\omega_0) := \bigcap_{t>0} \overline{\gamma^+(\omega_0 \cdot t)}$; such a set is thus empty as soon as $t^+(\omega_0)$ is finite.

Before we enter the central part of the analysis, let us emphasize that the $(N+1)$ constitutive variables of (4.21) lie, during the whole evolution, within an N-dimensional submanifold of Ω, the latter being entirely prescribed by the choice of the initial data $\omega_0 \in \Omega$. This is the argument of the following statement which essentially reflects the conservation property met by the total energy (4.13).

PROPOSITION 4.3. *Let ω_0 be a given state in Ω. Then the flow $\omega_0 \cdot t$ satisfies for all time in its maximal interval of existence:*

$$\mathcal{H}(\omega_0 \cdot t) = \mathcal{H}(\omega_0), \tag{4.23}$$

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where the regular mapping $\mathcal{H}\colon \Omega \to \mathbb{R}$ *is defined by*

$$\mathcal{H}(\omega) = e(\omega) - e(\omega_{\mathrm{L}}) - \tfrac{1}{2}m^2(\tau^2 - \tau_{\mathrm{L}}^2) + (m^2\tau_{\mathrm{L}} + p(\omega_{\mathrm{L}}))(\tau - \tau_{\mathrm{L}}).$$

Proof. All the flows under consideration are at least continuously differentiable in their maximal interval of existence. The additional conservation law (4.13) for the total energy therefore applies and its differential form reads

$$\{(E + \tau p(\omega))\rho u\}_x = \{\tfrac{1}{2}\mu(u^2)_x\}_x. \tag{4.24}$$

In view of the algebraic invariant $\rho u = m$, (4.24), once integrated for a prescribed ω_0 in Ω between time zero and a given time t in $(t^-(\omega_0), t^+(\omega_0))$, can be seen to read

$$\begin{aligned}\{E + \tau p(\omega)\rho u\}(t) - \{E + \tau p(\omega)\rho u\}(0) &= \{\tau(\mu\dot{\tau})\}(t) - \{\tau(\mu\dot{\tau})\}(0)\\ &= \{\tau\mathcal{F}(\omega)\}(t) - \{\tau\mathcal{F}(\omega)\}(0).\end{aligned}$$

Since E denotes $\frac{1}{2}m^2\tau^2 + \epsilon(\omega)$, the definition of $\mathcal{F}$ given in (4.21) easily yields the required identity (4.23) after some rearrangements of the terms while, for convenience, subtracting from both sides the constant $\epsilon_{\mathrm{L}} + \frac{1}{2}m^2\tau_{\mathrm{L}}^2 + \tau_{\mathrm{L}}p_{\mathrm{L}}$. □

The above statement clearly implies that all the possible heteroclinic orbits of (4.21) which connect the critical point ω_{L} in the past are only made up of states ω such that

$$\mathcal{H}(\omega) = \mathcal{H}(\omega_{\mathrm{L}}) = 0. \tag{4.25}$$

To conclude these preliminary remarks, we point out an elementary but useful geometrical property of the flows associated with (4.21), which will restrict possible right-connecting states.

LEMMA 4.4. *Let* ω_0 *be given in* Ω*. Then the subset of* Ω *defined by*

$$\Omega(\omega_0) = \{\omega \in \Omega;\ \boldsymbol{s} \geqslant \boldsymbol{s}_0,\ \mathcal{H}(\omega) = \mathcal{H}(\omega_0)\} \tag{4.26}$$

is positively invariant.

The invariance of this region with respect to all positive semi-flows immediately follows from the non-negativeness of the Nth last components of the vector field entering the definition of (4.21). As a consequence, possible heteroclinic orbits connecting ω_{L} in the past must lie entirely in

$$\Omega(\omega_{\mathrm{L}}) = \{\omega \in \Omega;\ \boldsymbol{s} \geqslant \boldsymbol{s}_{\mathrm{L}},\ \mathcal{H}(\omega) = 0\}. \tag{4.27}$$

The region (4.27) will play a central role in the derivation of positively invariant compact sets.

Here, we exhibit some important features of the linearization $LX(\omega_{\mathrm{c}})$ of the vector field X at equilibrium points ω_{c}, i.e. at states satisfying $\mathcal{F}(\omega_{\mathrm{c}}) = 0$. We check, in particular, that such states are always non-hyperbolic points for which the space $\mathbb{R}^{N+1}$ denotes a direct sum of the eigenspaces associated with $LX(\omega_{\mathrm{c}})$ under the following non-degeneracy condition:

$$\partial_\tau\mathcal{F}(\omega_{\mathrm{c}}) = m^2 + \partial_\tau p(\tau_{\mathrm{c}}, \boldsymbol{s}_{\mathrm{c}}) \neq 0. \tag{4.28}$$

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Towards that aim, let us state some basic facts concerning the linearization $LX(\omega_c)$. The requirement $\mathcal{F}(\omega_c) = 0$ is easily seen to force all the partial derivatives of the Nth last components of X to be identically zero (since these components are all proportional to $\mathcal{F}^2$). Under the non-degeneracy condition (4.28), there consequently exists only one non-trivial eigenvalue, namely $\partial_\tau \mathcal{F}(\omega_c)/\mu(\omega_c)$, while $\lambda = 0$ is a semisimple eigenvalue of $LX(\omega_c)$ of multiplicity N. Furthermore, the corresponding eigenspaces $T(\omega_c)$ and $T^c(\omega_c)$ are respectively the spans of e_1 and $e_2, \dots, e_N$, where $\{e_i\}_{1 \leqslant i \leqslant N+1}$ stands for the canonical orthonormal basis of $\mathbb{R}^{N+1}$.

Equipped with these results, the centre manifold theorem ensures the existence of two locally invariant manifolds $\mathcal{W}(\omega_c)$ and $\mathcal{W}^c(\omega_c)$ (the so-called centre manifold) of class at least C^1 and C^0, respectively, which go through ω_c and are tangent to $T(\omega_c)$ and $T^c(\omega_c)$, respectively, at this point. The regularity properties above are indeed inherited from the continuous differentiability of the vector field, according to this theorem.

Assuming $\partial_\tau \mathcal{F}(\omega_c)$ to be positive (respectively negative), i.e. assuming the corresponding sign for the unique non-trivial eigenvalue of $LX(\omega_c)$, $\mathcal{W}(\omega_c)$ is classically referred as to the unstable (respectively, stable) manifold with superscript 'u' (respectively, 's'). Recall that the unstable manifold of a point is the manifold composed of the totality of the orbits which tend exponentially fast to the point in negative time, the stable manifold being defined conversely. Then, by well-known topological considerations, two rest points, namely ω_L and ω_R, are connected by a heteroclinic orbit γ precisely if $\gamma \subset \mathcal{W}^u(\omega_L) \cap \mathcal{W}^s(\omega_R)$.

An obvious requirement for the existence of a heteroclinic orbit connecting ω_L in the past is then

$$\partial_\tau \mathcal{F}(\omega_L) = m^2 + \partial_\tau p(\omega_L) < 0,$$

but the validity of such an inequality is precisely the argument of the Lax condition (4.18). Conversely, a possible connecting point ω_R in the future is necessarily subject to the condition $\partial_\tau \mathcal{F}(\omega_R) > 0$.

Now, since the unstable manifold $\mathcal{W}^u(\omega_L)$ is one dimensional, there exist locally exactly two solutions of (4.21) which approach ω_L as $t \to -\infty$. With respect to the property of $\mathcal{W}^u(\omega_L)$ is tangential to e_1, the associated almost horizontal orbits approach ω_L from the two opposite directions $\tau \geqslant \tau_L$ and $\tau \leqslant \tau_L$. With clear notation, $\gamma_>(\omega_L)$ (respectively, $\gamma_<(\omega_L)$) will denote the first (respectively, the second) orbit.

The following assertion discards the solution converging to ω_L for negative times from the region $\tau \geqslant \tau_L$. Note that such a result precisely precluded expansion shocks in order to admit viscous profiles.

PROPOSITION 4.5. *There is no heteroclinic orbit of the dynamical system (4.21) in the domain* $\mathcal{N} := \{\omega \in \Omega \colon \tau \geqslant \tau_L,\ \boldsymbol{s} \geqslant \boldsymbol{s}_L\}$.

Consequently, only the second solution can give rise to a heteroclinic orbit. Since the vector field $X \colon \Omega \to \mathbb{R}^{N+1}$ is Lipschitz continuous, the uniqueness part of the celebrated Picard–Lindelöf theorem readily gives the following.

COROLLARY 4.6. *There exists at most one heteroclinic orbit of the dynamical system (4.21) which connects* ω_L *in the past.*

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The proof of proposition 4.5 will follow from the following statement.

LEMMA 4.7. *Any given state ω distinct from ω_{L} in the set $\{\tau \geqslant \tau_{\mathrm{L}},\ \boldsymbol{s} \geqslant \boldsymbol{s}_{\mathrm{L}}\}$ obeys $\mathcal{F}(\omega) > 0$.*

Proof. Observe that the positiveness assumption (4.5) on the Grüneisen numbers implies that, for all ω in the region under interest ($\boldsymbol{s} \geqslant \boldsymbol{s}_{\mathrm{L}}$),

$$f(\tau) := \mathcal{F}(\tau, \boldsymbol{s}_{\mathrm{L}}) \leqslant \mathcal{F}(\omega) \tag{4.29}$$

with equality if and only if $\boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}}$. In particular, for $\tau = \tau_{\mathrm{L}}$ we have $\mathcal{F}(\tau_{\mathrm{L}}, \boldsymbol{s}) > 0$ as soon as $\boldsymbol{s} > \boldsymbol{s}_{\mathrm{L}}$. Next, considering $\tau > \tau_{\mathrm{L}}$, the identity

$$f'(\tau) = m^2 + \partial_\tau p(\tau, \boldsymbol{s}_{\mathrm{L}})$$

clearly yields, under the assumption (4.4) of positive fundamental derivatives, that $\partial^2_{\tau\tau} p(\tau, \boldsymbol{s}) > 0$ for all $\omega \in \Omega$, and therefore

$$f'(\tau) \geqslant f'(\tau_{\mathrm{L}}) = m^2 - (\rho_{\mathrm{L}} c_{\mathrm{L}})^2 > 0,$$

due to the Lax condition (4.18). It immediately follows that $f(\tau) \geqslant f(\tau_{\mathrm{L}}) = \mathcal{F}(\omega_{\mathrm{L}}) = 0$ as soon as $\tau \geqslant \tau_{\mathrm{L}}$ with equality to zero if and only if $\tau = \tau_{\mathrm{L}}$. The inequality (4.29) then gives the required conclusion. □

Proof of proposition 4.5. Let $\boldsymbol{n}_\omega$ be the unit inward normal at the following hypersurfaces: $\{\tau = \tau_{\mathrm{L}},\ \boldsymbol{s} \geqslant \boldsymbol{s}_{\mathrm{L}}\}$ and $\{\tau \geqslant \tau_{\mathrm{L}},\ \boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}}\}$ for all states in these sets. Note that these sets are the lower part of the boundary of $\mathcal{N}$. In view of the definition of the vector field, lemma 4.7 implies that, for such states, $X(\omega) \cdot \boldsymbol{n}_\omega \geqslant 0$. As a (well-known) consequence, $\mathcal{N}$ stays invariant for all positive semi-flows. The required conclusion follows again from lemma 4.7, which says that no critical point exists in $\mathcal{N}$. □

We now stress another important consequence of the local properties of the phase portrait at the rest point ω_{L}. By opposition to the states in the orbit $\gamma_>(\omega_{\mathrm{L}})$; the second orbit $\gamma_<(\omega_{\mathrm{L}})$ emanating from the region $\tau \leqslant \tau_{\mathrm{L}}$ is by definition made of states ω that, at least when close enough to but distinct from ω_{L}, give rise to a compression: locally $\mathcal{F}(\omega) < 0$ in view of the governing equation for τ. In turn, this simple observation implies that the viscous profile under consideration must remain uniformly compressive. This claim is a consequence of the following statement.

LEMMA 4.8. *The set*

$$\mathcal{I} = \{\omega \in \Omega \colon \tau < \tau_{\mathrm{L}},\ \boldsymbol{s} \geqslant \boldsymbol{s}_{\mathrm{L}},\ \mathcal{F}(\omega) \leqslant 0\} \tag{4.30}$$

is positively invariant under the action of the dynamical system (4.21).

Proof. The above assertion is trivial for states $\omega_0 \in \mathcal{I}$ which satisfy $\mathcal{F}(\omega_0) = 0$. Considering states ω_0 with the property $\mathcal{F}(\omega_0) < 0$, we observe that the positive semi-flow $\omega_0 \cdot t$ necessarily satisfies $\mathcal{F}(\omega_0 \cdot t) < 0$ for all time in $[0, t^+(\omega_0))$. Indeed, assuming the existence of a finite time t_{c} in this interval with the property $\mathcal{F}(\omega_0 \cdot t_{\mathrm{c}}) = 0$ would result in a critical point $\omega_0 \cdot t_{\mathrm{c}}$ for the dynamical system (4.21). But, by the Lipschitz-continuity property of the vector field in Ω, it is well known that such a point cannot be reached in finite time. □

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The orbit $\gamma_<(\omega_{\rm L})$ is therefore trapped in the region $\mathcal{I}$. We now establish that, in addition, this orbit must remain within a compact subset K of $\mathcal{I}$. This will imply that $\gamma_<(\omega_{\rm L})$ is relatively compact. Well-known considerations imply that the associated ϖ-limit set is non-empty, compact and connected. The existence of K primary stems from the following result.

LEMMA 4.9. *Let ω_0 be a given state in Ω. Then the positive semi-orbit $\gamma^+(\omega_0)$ has no limit point in the set $\{\tau = 0\}$.*

This assertion immediately gives that the orbit $\gamma_<(\omega_{\rm L})$ has the same property.

Proof. For all time t in $[0, t^+(\omega_0))$, the positive semi-flow $\omega_0 \cdot t$ is known to obey $\mathcal{H}(\omega) = \mathcal{H}(\omega_0) < \infty$ and $\boldsymbol{s} \geqslant \boldsymbol{s}_0$. With respect to the positivity of all the temperatures $T_i = \partial_{s_i}\mathcal{H}$, we immediately get

$$h(\tau) := \mathcal{H}(\tau, \boldsymbol{s}_0) \leqslant \mathcal{H}(\tau, \boldsymbol{s}),$$

with the property that $h(\tau)$ goes to infinity as τ goes to zero (see the asymptotic condition (4.6)).

Assume that $\overline{\gamma^+(\omega_0)} \cap \{\tau = 0\}$ is non-empty. As a consequence, for all $\varepsilon > 0$ there exists $t_\varepsilon \in (0, t^+(\omega_0))$ such that $0 < \tau|_{\omega_0 \cdot t_\varepsilon} < \varepsilon$. Necessarily there exists $\varepsilon_0 > 0$ so that $h(\tau|_{\omega_0 \cdot t_{\varepsilon_0}}) > \mathcal{H}(\omega_0)$ and this gives rise to the contradiction with the preservation of $\mathcal{H}(\omega_0)$ along the orbit. □

We have proven that any given positive semi-flow of the dynamical system (4.21) with initial data ω_0 in Ω satisfies $t^+(\omega_0) = \infty$ (since $\boldsymbol{s} > \boldsymbol{s}_0$). We now conclude with the existence (and therefore uniqueness) of the required viscous profile.

PROPOSITION 4.10. *There exists a state $\omega_{\rm R}$ in $\mathcal{H}^{-1}(0) \cap \mathcal{F}^{-1}(0)$ which is connected by $\gamma_<(\omega_{\rm L})$ in the future.*

Proof. We first establish that the specific entropy vector $\boldsymbol{s}$ stays upper bounded along all positive semi-flows with initial data in $\mathcal{I}$. For fixed τ in $(0, \tau_{\rm L})$, the conditions (4.3) and (4.5) show that $\mathcal{H}(\tau, \boldsymbol{s})$ increases arbitrarily with $\boldsymbol{s}$. The same steps as those involved in the previous proof apply to give the required result. As a consequence, $\gamma_<(\omega_{\rm L})$ must be included in a compact subset, namely K, of the positively invariant region $\mathcal{I}$. This orbit is, therefore, relatively compact and its ϖ-limit set is non-empty. This limit set must be included in $\mathcal{H}^{-1}(0)$. To conclude, observe that τ, when understood as a mapping of the variable ω, trivially yields a Lyapunov function on $\mathcal{I}$ where, by construction, $\mathcal{F}(\omega) \leqslant 0$. The LaSalle invariance principle applied in connection with this Lyapunov function then ensures that the non-empty ϖ-limit set is included in $\{\omega \in \mathcal{I} \colon \mathcal{F}(\omega) = 0\}$. This establishes the existence of $\omega_{\rm R}$. □

5. End states for viscous layers with varying viscosity

The existence (and uniqueness up to translation) of travelling-wave solutions to the multi-pressure NS equations was established in the previous section for N viscosity laws satisfying the non-degeneracy condition (4.9). Given $\omega_{\rm L}$ a fixed state in Ω and a velocity σ according to the condition (4.18), we aim here at characterizing

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the subset of Ω made of all the states ω_{R} that can be reached in the future by a travelling wave with speed σ and connecting ω_{L} in the past. We naturally expect the exit state ω_{R} to depend on the specific form of the N-tuple of viscosity laws. The dynamical system (4.21) shows that such a dependence lies in the ratios of the viscosity laws. As a consequence, possible states ω_{R} to be reached in the future from a fixed ω_{L} (at some speed σ) generically depend on $N-1$ degrees of freedom. The set of exit states we are seeking is thus expected to have $(N-1)$-dimensions.

It will be convenient to study the projection of this set onto the following positive cone of $\mathbb{R}^N$ (understood as the space of the specific entropies $\boldsymbol{s} = (s_1, \ldots, s_N)$) with origin $\boldsymbol{s}_{\mathrm{L}}$:

$$S^+(\boldsymbol{s}) = \{\boldsymbol{s} \in \mathbb{R}^N \mid \boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a},\ \boldsymbol{a} \in \mathcal{S}_+^N,\ \lambda \geqslant 0\}. \tag{5.1}$$

Here, $\mathcal{S}_+^N$ denotes the (positive) part of the unit sphere in $\mathbb{R}^N$ defined by

$$\mathcal{S}_+^N = \{\boldsymbol{a} \in \mathbb{R}_+^N;\ \|\boldsymbol{a}\| = 1\}.$$

For all possible entropies $\boldsymbol{s}_{\mathrm{R}}$ in the cone (5.1), the existence simply comes from the property that the heterocline solutions of theorem 4.2 obey $\boldsymbol{s}_{\mathrm{R}} \geqslant \boldsymbol{s}_{\mathrm{L}}$ and a strict inequality holds for (at least) one specific entropy s_i $(1 \leqslant i \leqslant N)$.

We show hereafter that the projection in the half cone (5.1) of the states ω_{R} that can be reached when varying the definition of the N viscosity laws is a smooth manifold with co-dimension 1:

$$\mathcal{C} = \{\boldsymbol{s} \in S^+(\boldsymbol{s}) \mid \boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}} + \Lambda_0(\boldsymbol{a})\boldsymbol{a},\ \boldsymbol{a} \in \mathcal{S}_+^N\}, \tag{5.2}$$

for some suitable mapping $\Lambda_0(\boldsymbol{a})\colon \boldsymbol{a} \in \mathcal{S}_+^N \mapsto \Lambda_0(\boldsymbol{a}) \in \mathbb{R}$ for which a precise definition will be given latter on. The derivation of the proposed manifold is performed in two steps. In the first step, we analyse closely all the critical points $(\tau_{\mathrm{c}}, \boldsymbol{s}_{\mathrm{c}})$ of the dynamical system (4.21), i.e. the solutions of

$$\mathcal{F}(\tau_{\mathrm{c}}, \boldsymbol{s}_{\mathrm{c}}) = 0, \tag{5.3}$$

without reference to a precise N-tuple of viscosity laws.

We emphasize that eligible critical points that can be reached from the state ω_{L} in the past must preserve the total energy as stated in (4.25). Such states must therefore solve in addition

$$\mathcal{H}(\tau_{\mathrm{c}}, \boldsymbol{s}_{\mathrm{c}}) = 0 \quad \text{with } \boldsymbol{s}_{\mathrm{R}} \geqslant \boldsymbol{s}_{\mathrm{L}}. \tag{5.4}$$

Analysing the solution of (5.3), (5.4) will yield the manifold (5.2).

In a second step, we shall establish that any given value $\boldsymbol{s}$ in the proposed manifold can actually be achieved for at least one suitable N-tuple of viscosity laws. As a consequence, the manifold (5.2) is made entirely of all the specific entropy $\boldsymbol{s}_{\mathrm{R}}$ that can be reached in the future by a travelling-wave solution with speed σ and issued from ω_{L}, when varying the definition of the N viscosity laws.

Let us outline the content of this section. We first analyse the mappings $\boldsymbol{s} \in S^+(\boldsymbol{s}) \mapsto \tau_{\mathcal{F}}(\boldsymbol{s}) \in \mathbb{R}_+$ that solve

$$\mathcal{F}(\tau_{\mathcal{F}}(\boldsymbol{s}), \boldsymbol{s}) = 0. \tag{5.5}$$

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We then characterize the mapping $\boldsymbol{s} \in S^+(\boldsymbol{s}) \mapsto \tau_{\mathcal{H}}(\boldsymbol{s}) \in \mathbb{R}_+$ which solves

$$\mathcal{H}(\tau_{\mathcal{H}}(\boldsymbol{s}), \boldsymbol{s}) = 0. \tag{5.6}$$

Equipped with these two families of functions, we shall consider the existence values of the specific entropy $\boldsymbol{s}_{\mathrm{c}}$ in $S^+(\boldsymbol{s})$ that satisfy $\tau_{\mathcal{F}}(\boldsymbol{s}_{\mathrm{c}}) = \tau_{\mathcal{H}}(\boldsymbol{s}_{\mathrm{c}})$, namely values of $\boldsymbol{s}$ that simultaneously solve (5.5) and (5.6). We now state the main result of this section.

THEOREM 5.1. *Assume that the thermodynamical conditions (4.2)–(4.8) are satisfied. Then there exists a unique map $\mathcal{T} \in \mathcal{C}^0(\bar{\mathcal{K}}, \mathbb{R}_+^*) \cap \mathcal{C}^1(\mathcal{K}, \mathbb{R}_+^*)$, where $\mathcal{K} \subset S^+(\boldsymbol{s})$ is given by*

$$\mathcal{K} = \{\boldsymbol{s} \in S^+(\boldsymbol{s}) \mid \boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a},\ \boldsymbol{a} \in \mathcal{S}_+^N,\ \lambda \in]0, \Lambda_0(\boldsymbol{a})[\}, \tag{5.7}$$

for some smooth application $\Lambda_0 \in \mathcal{C}^1(\mathcal{S}_+^N, \mathbb{R}_+^*)$ with the following properties:

(i) $\mathcal{H}(\mathcal{T}(\boldsymbol{s}), \boldsymbol{s}) = 0$ *for all* $\boldsymbol{s} \in \bar{\mathcal{K}}$,

(ii) $\mathcal{F}(\mathcal{T}(\boldsymbol{s}), \boldsymbol{s}) = 0$ *for all* $\boldsymbol{s} \in \mathcal{C}$, *where*

$$\mathcal{C} = \{\boldsymbol{s} \in S^+(\boldsymbol{s}) \mid \boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}} + \Lambda_0(\boldsymbol{a})\boldsymbol{a},\ \boldsymbol{a} \in \mathcal{S}_+^N\}. \tag{5.8}$$

In addition, $\mathcal{T}$ obeys

(iii) $\mathcal{F}(\mathcal{T}(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}}) = 0$,

(iv) $\mathcal{F}(\mathcal{T}(\boldsymbol{s}), \boldsymbol{s}) < 0$ *for all* $\boldsymbol{s} \in \mathcal{K}$.

The mapping $\Lambda_0 \colon \mathcal{S}_+^N \to \mathbb{R}_+^*$ will be constructed in the course of the proof. Rephrasing the above result, the function $\mathcal{T}(\boldsymbol{s})$ with $\boldsymbol{s} \in \mathcal{C}$ makes $\mathcal{F}$ and $\mathcal{H}$ vanish simultaneously, so that all the values $\boldsymbol{s}$ in the smooth manifold $\mathcal{C}$ are candidates for being reached in the future from the state ω_{L} via a travelling wave with speed σ for a suitable choice of the N viscosity laws.

We now show that all the values $\boldsymbol{s}$ in the manifold $\mathcal{C}$, defined by (5.8), are actually eligible candidates for entering the definition of the specific entropy in exit states ω_{R}.

LEMMA 5.2. *With a state ω_{L} being given in Ω and a velocity σ being prescribed according to (4.18), for any given $\boldsymbol{s} \in \mathcal{C}$ there exists at least one relevant definition of the N-tuple of viscosity laws which yields a travelling-wave solution with speed σ issued from ω_{L} and connecting a state ω_{R} in the future with $\boldsymbol{s}_{\mathrm{R}} = \boldsymbol{s}$.*

Proof. The proof of this result makes use of particular viscosity laws in the form

$$\mu_i(\tau, s_i) = \mu_i^0 T_i(\tau, \boldsymbol{s}_i), \quad \mu_i^0 \geqslant 0, \quad 1 \leqslant i \leqslant N. \tag{5.9}$$

The non-degeneracy condition (4.9) is satisfied as soon as

$$\sum_{i=1}^{N} \mu_i^0 > 0, \tag{5.10}$$

since each of the temperature laws $T_i(\tau, \boldsymbol{s}_i)$ is assumed to be positive. Without loss of generality, we assume $\mu_N^0 > 0$.

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We stress that viscosity laws which depend linearly on the temperature naturally arise in the kinetic theory for dilute gases. We refer the reader to [30].

Observe that viscosity laws in the special form (5.9) let each specific entropy evolve according to

$$\dot{s}_i = \frac{\mu_i^0}{\mu^2(\tau, \boldsymbol{s})} \mathcal{F}^2(\tau, \boldsymbol{s}), \quad 1 \leqslant i \leqslant N.$$

We thus infer the following $(N-1)$ balance equations linking the evolution of the first $(N-1)$ specific entropies s_i to the last one:

$$\dot{s}_i = \frac{\mu_i^0}{\mu_N^0} \dot{s}_N, \quad 1 \leqslant i \leqslant N-1.$$

Since the ratios μ_i^0/μ_N^0 are constant real numbers, we deduce that

$$\left(s_i - \frac{\mu_i^0}{\mu_N^0} s_N \right)(\xi) = s_i^{\mathrm{L}} - \frac{\mu_i^0}{\mu_N^0} s_N^{\mathrm{L}} \quad \text{for all } \xi \in \mathbb{R}.$$

We therefore end up with $(N-1)$ jump relations

$$s_i^{\mathrm{R}} - s_i^{\mathrm{L}} = \frac{\mu_i^0}{\mu_N^0} (s_N^{\mathrm{R}} - s_N^{\mathrm{L}}), \quad 1 \leqslant i \leqslant N. \tag{5.11}$$

We emphasize at this stage that $s_N^{\mathrm{R}} - s_N^{\mathrm{L}} > 0$ in view of our assumption $\mu_N^0 > 0$. From the jump relation (5.11), we therefore get

$$\boldsymbol{s}_{\mathrm{R}} - \boldsymbol{s}_{\mathrm{L}} = \frac{s_N^{\mathrm{R}} - s_N^{\mathrm{L}}}{\mu_N^0} \begin{pmatrix} \mu_1^0 \\ \vdots \\ \mu_N^0 \end{pmatrix}, \tag{5.12}$$

which is obviously in the form $\Lambda_0(\boldsymbol{a})\boldsymbol{a}$, for $\boldsymbol{a}$ in $\mathcal{S}_+^N$, given by

$$\boldsymbol{a} = \frac{1}{\sqrt{\sum_{j=1}^N (\mu_j^0)^2}} (\mu_j^0)_{1 \leqslant j \leqslant N}.$$

Now, up to some relabelling in the viscosity in order to allow μ_N^0 to vanish, any given $\boldsymbol{a} \in \mathcal{S}_+^N$ gives rise to an admissible N-tuple of viscosity coefficients. This concludes the proof. □

Remark 5.3. The identity (5.12) shows in addition that the mapping $\boldsymbol{a} \mapsto \Lambda_0(\boldsymbol{a})$ can be built as soon as the jump in the last specific entropy $s_N^{\mathrm{R}} - s_N^{\mathrm{L}}$ is known. This evaluation can be performed numerically (see, for example, [8]).

We now give a proof of the main result of this section, namely theorem 5.1. Towards this aim, and as already claimed, we propose to first study for existence in $\mathcal{S}_+^N$ the roots $\tau(\boldsymbol{s})$ of $\mathcal{F}(\tau, \boldsymbol{s}) = 0$. Then we shall study their distinctive properties by investigating the values of $\mathcal{H}(\tau(\boldsymbol{s}), \boldsymbol{s})$.

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PROPOSITION 5.4. *There exist two maps, which we denote by* $\tau^{\pm}$, *belonging to* $\mathcal{C}^1(\mathcal{D}\cup\{\boldsymbol{s}_{\rm L}\},\mathbb{R}^*_+)\cap\mathcal{C}^0(\bar{\mathcal{D}},\mathbb{R}^*_+)$, *where* $\mathcal{D}$ *is the subset of* $S^+(\boldsymbol{s}_{\rm L})$ *defined by*

$$\mathcal{D}=\{\boldsymbol{s}\in S^+(\boldsymbol{s}_L)\mid \boldsymbol{s}=\boldsymbol{s}_L+\lambda\boldsymbol{a},\ \boldsymbol{a}\in\mathcal{S}^N_+,\ \lambda\in(0,\bar{\Lambda}(\boldsymbol{a}))\} \tag{5.13}$$

for some $\bar{\Lambda}\in\mathcal{C}^1(\mathcal{S}^N_+,\mathbb{R}^*_+)$ *with* $\bar{\Lambda}(\boldsymbol{a})>\Lambda_0(\boldsymbol{a})$ *for all* $\boldsymbol{a}\in\mathcal{S}^N_+$, *so that*

$$\mathcal{F}(\tau^{\pm}(\boldsymbol{s}),\boldsymbol{s})=0,\quad \boldsymbol{s}\in\bar{\mathcal{D}}.$$

In addition, these two families of roots are interlaced according to

(i) $\tau^-(\boldsymbol{s})<\tau^+(\boldsymbol{s})<\tau_{\rm L}$ *for all* $\boldsymbol{s}\in\mathcal{D}\setminus\{\boldsymbol{s}_{\rm L}\}$,

(ii) $\tau^+(\boldsymbol{s})=\tau_{\rm L}$ *in* $\bar{\mathcal{D}}$ *if and only if* $\boldsymbol{s}=\boldsymbol{s}_{\rm L}$ *with* $\tau^-(\boldsymbol{s}_{\rm L})<\tau^+(\boldsymbol{s}_{\rm L})=\tau_{\rm L}$,

(iii) $\tau^-(\boldsymbol{s})=\tau^+(\boldsymbol{s})$ *in* $\bar{\mathcal{D}}$ *if and only if* $\boldsymbol{s}=\boldsymbol{s}_{\rm L}+\bar{\Lambda}(\boldsymbol{a})\boldsymbol{a}$, $\boldsymbol{a}\in\mathcal{S}^N_+$.

Again, the map $\bar{\Lambda}\colon\mathcal{S}^N_+\to\mathbb{R}^*_+$ will be built in the course of the proof. But, from now on notice that $\bar{\mathcal{K}}\subset\mathcal{D}$. We shall show that, for fixed $\boldsymbol{s}\in\bar{\mathcal{D}}$, $\mathcal{F}(\tau,\boldsymbol{s})=0$ only admits $\tau^{\pm}(\boldsymbol{s})$ as roots and cannot be solved in τ for values of $\boldsymbol{s}$ in $S^+(\boldsymbol{s}_{\rm L})\setminus\bar{\mathcal{D}}$. As a consequence, all the critical points $(\tau(\boldsymbol{s}),\boldsymbol{s})$ of (4.21) are necessarily achieved for $\boldsymbol{s}\in\bar{\mathcal{D}}$ so that $\tau(\boldsymbol{s})$ must coincide with either $\tau^-(\boldsymbol{s})$ or $\tau^+(\boldsymbol{s})$ for suitable values of $\boldsymbol{s}\in\bar{\mathcal{D}}$: i.e. such that $\mathcal{H}(\tau^-(\boldsymbol{s}),\boldsymbol{s})=0$ or $\mathcal{H}(\tau^+(\boldsymbol{s}),\boldsymbol{s})=0$. In this way, let us state some properties of $\mathcal{H}$ with respect to the above two families of roots.

PROPOSITION 5.5. *Using the notation in propositions 5.1 and 5.4, we have*

(i) $\mathcal{H}(\tau^+(\boldsymbol{s}_{\rm L}),\boldsymbol{s}_{\rm L})=0$,

(ii) $\mathcal{H}(\tau^+(\boldsymbol{s}),\boldsymbol{s})>0$ *for all* $\boldsymbol{s}\in\bar{\mathcal{D}}\setminus\{\boldsymbol{s}_{\rm L}\}$,

while

(iii) $\mathcal{H}(\tau^-(\boldsymbol{s}),\boldsymbol{s})<0$ *for all* $\boldsymbol{s}\in\mathcal{K}\cup\{\boldsymbol{s}_{\rm L}\}$,

(iv) $\mathcal{H}(\tau^-(\boldsymbol{s}),\boldsymbol{s})=0$ *for all* $\boldsymbol{s}\in\mathcal{C}=\{\boldsymbol{s}\in\mathcal{D}\mid\boldsymbol{s}=\boldsymbol{s}_{\rm L}+\Lambda_0(\boldsymbol{a})\boldsymbol{a},\ \boldsymbol{a}\in\mathcal{S}^N_+\}$,

(v) $\mathcal{H}(\tau^-(\boldsymbol{s}),\boldsymbol{s})>0$ *for all* $\boldsymbol{s}\in\bar{\mathcal{D}}\setminus\bar{\mathcal{K}}$.

Put in other words, the critical points of the differential system (4.21) necessarily coincide with the set $\{\tau^+(\boldsymbol{s}_{\rm L}),\boldsymbol{s}_{\rm L}\}$ and $\{(\tau^-(\boldsymbol{s}),\boldsymbol{s})\mid\boldsymbol{s}\in\mathcal{C}\}$. Keeping this in mind, we next analyse the roots $\tau(\boldsymbol{s})$ of $\mathcal{H}(\tau(\boldsymbol{s}),\boldsymbol{s})$. The following claim states that $\mathcal{H}$ admits three distinct branches of roots in $\bar{\mathcal{K}}$. Particular attention is paid to singling out a branch $\mathcal{T}$ obeying the requirements

$$\mathcal{T}(\boldsymbol{s}_{\rm L})=\tau^+(\boldsymbol{s}_{\rm L})\quad\text{together with }\mathcal{T}(\boldsymbol{s})=\tau^-(\boldsymbol{s})\text{ for all }\boldsymbol{s}\in\mathcal{C}, \tag{5.14}$$

as put forward in proposition 5.5.

PROPOSITION 5.6. *There exist three maps in* $\mathcal{C}^0(\bar{K},\mathbb{R}^*_+)\cap\mathcal{C}^1(\mathcal{K},\mathbb{R}^*_+)$ *respectively denoted by* $\check{\mathcal{T}},\mathcal{T},\ \hat{\mathcal{T}}\colon\bar{\mathcal{K}}\to\mathbb{R}^*_+$, *so that*

(i) $\mathcal{H}(\mathcal{T}(\boldsymbol{s}),\boldsymbol{s})=\mathcal{H}(\check{\mathcal{T}}(\boldsymbol{s}),\boldsymbol{s})=\mathcal{H}(\hat{\mathcal{T}}(\boldsymbol{s}),\boldsymbol{s})=0$ *for all* $\boldsymbol{s}\in\bar{\mathcal{K}}$.

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These are interlaced with the roots $\tau^{\pm}(\boldsymbol{s})$ of $\mathcal{F}$ as follows:

(ii) $\check{\mathcal{T}}(\boldsymbol{s}) < \tau^{-}(\boldsymbol{s}) < \mathcal{T}(\boldsymbol{s}) < \tau^{+}(\boldsymbol{s}) < \hat{\mathcal{T}}(\boldsymbol{s})$ *for all* $\boldsymbol{s} \in \mathcal{K}$,

(iii) $\check{\mathcal{T}}(\boldsymbol{s}_{\mathrm{L}}) < \mathcal{T}(\boldsymbol{s}_{\mathrm{L}}) = \tau^{+}(\boldsymbol{s}_{\mathrm{L}}) = \hat{\mathcal{T}}(\boldsymbol{s}_{\mathrm{L}})$,

(iv) $\check{\mathcal{T}}(\boldsymbol{s}) = \tau^{-}(\boldsymbol{s}) = \mathcal{T}(\boldsymbol{s}) < \hat{\mathcal{T}}(\boldsymbol{s})$ *for all* $\boldsymbol{s} \in \mathcal{C}$.

Observe that the intermediate mapping $\mathcal{T}$ fulfils the requirements (5.14) so that theorem 5.1 is established.

We now give the proofs of propositions 5.4–5.6. Proposition 5.4 relies on the following two technical lemmas.

LEMMA 5.7. *For all fixed $\boldsymbol{s} \in S^{+}(\boldsymbol{s}_{\mathrm{L}})$, $\mathcal{F}(\cdot, \boldsymbol{s})$ admits a unique minimum in τ. We define $\bar{\tau}(\boldsymbol{s}) < \tau_{\mathrm{L}}$ for all $\boldsymbol{s} \in \bar{\mathcal{D}}$, where $\bar{\tau} \in \mathcal{C}^1(S^{+}(\boldsymbol{s}_{\mathrm{L}}), \mathbb{R}_{+}^{*})$. This minimum obeys the following conditions:*

(i) $\mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) < 0$ *for all* $\boldsymbol{s} \in \mathcal{D} \cup \{\boldsymbol{s}_{\mathrm{L}}\}$,

(ii) $\mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) = 0$ *for all* $\boldsymbol{s} \in \Gamma := \{\boldsymbol{s} \in S^{+}(\boldsymbol{s}_{\mathrm{L}}) \mid \boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}} + \bar{\Lambda}(\boldsymbol{a})\boldsymbol{a},\ \boldsymbol{a} \in \mathcal{S}_{+}^{N}\}$,

(iii) $\mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) > 0$ *for all* $\boldsymbol{s} \in S^{+}(\boldsymbol{s}_{\mathrm{L}}) \setminus \bar{\mathcal{D}}$,

where the set $\mathcal{D}$ has been defined in proposition 5.4.

LEMMA 5.8. *For all fixed $\boldsymbol{s} \in S^{+}(\boldsymbol{s}_{\mathrm{L}})$, $\mathcal{F}(\cdot, \boldsymbol{s})$ is strictly decreasing (respectively, strictly increasing) for all $\tau \in (0, \bar{\tau}(\boldsymbol{s}))$ (respectively, for all $\tau > \bar{\tau}(\boldsymbol{s})$) and achieves the following limits:*

$$\lim_{\tau \to 0^{+}} \mathcal{F}(\tau, \boldsymbol{s}) = +\infty, \qquad \lim_{\tau \to +\infty} \mathcal{F}(\tau, \boldsymbol{s}) = +\infty.$$

As a consequence, $\mathcal{F}(\tau, \boldsymbol{s}) = 0$ can be solved in τ only when $\boldsymbol{s} \in \bar{\mathcal{D}}$, with exactly one solution when $\boldsymbol{s} \in \Gamma$ and exactly two solutions $\tau^{\pm}(\boldsymbol{s})$ for $\boldsymbol{s} \in \bar{\mathcal{D}} \setminus \Gamma$. These solutions define two maps

$$\tau^{\pm} \in \mathcal{C}^0(\bar{\mathcal{D}}, \mathbb{R}_{+}^{*}) \cap \mathcal{C}^1(\mathcal{D} \cup \{\boldsymbol{s}_{\mathrm{L}}\}, \mathbb{R}_{+}^{*})$$

with the following properties:

1. $\tau^{-}(\boldsymbol{s}) < \bar{\tau}(\boldsymbol{s}) < \tau^{+}(\boldsymbol{s}) < \tau_{\mathrm{L}}$ *for all* $\boldsymbol{s} \in \mathcal{D}$,
2. $\tau^{-}(\boldsymbol{s}) = \bar{\tau}(\boldsymbol{s}) = \tau^{+}(\boldsymbol{s}) < \tau_{\mathrm{L}}$ *for all* $\boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}} + \bar{\Lambda}(\boldsymbol{a})\boldsymbol{a},\ \boldsymbol{a} \in \mathcal{S}_{+}^{N}$,
3. $\tau^{-}(\boldsymbol{s}_{\mathrm{L}}) < \bar{\tau}(\boldsymbol{s}_{\mathrm{L}}) < \tau^{+}(\boldsymbol{s}_{\mathrm{L}}) = \tau_{\mathrm{L}}$.

We now establish lemma 5.7, noting that the set $\mathcal{D}$ entering the proposition 5.5 will be explicitly derived in the course of the proof.

Proof of lemma 5.7. Let $\boldsymbol{s}$ be fixed in $S^{+}(\boldsymbol{s})$. With respect to the smoothness of the internal energies, the map $\tau \mapsto \mathcal{F}(\cdot, \boldsymbol{s})$ is at least of class $\mathcal{C}^2(\mathbb{R}_{+}^{*})$. Easy calculations then yield for all $\tau > 0$:

$$\frac{\partial \mathcal{F}}{\partial \tau}(\tau, \boldsymbol{s}) = \frac{\partial p}{\partial \tau}(\tau, \boldsymbol{s}) + m^2, \qquad \frac{\partial^2 \mathcal{F}}{\partial \tau^2}(\tau, \boldsymbol{s}) = \frac{\partial^2 p}{\partial \tau^2}(\tau, \boldsymbol{s}).$$

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On the one hand, the map

$$\tau \mapsto \frac{\partial \mathcal{F}}{\partial \tau}(\tau, \boldsymbol{s})$$

is strictly increasing in view of the genuine nonlinearity assumption (4.4) for the total pressure. On the other hand, assumptions (4.7) on the asymptotic behaviour of $\partial p/\partial \tau$ imply that

$$\lim_{\tau \to 0^+} \frac{\partial \mathcal{F}}{\partial \tau}(\tau, \boldsymbol{s}) = -\infty \quad \text{and} \quad \lim_{\tau \to +\infty} \frac{\partial \mathcal{F}}{\partial \tau}(\tau, \boldsymbol{s}) = m^2 > 0.$$

As a consequence, for all $\boldsymbol{s} \in S^+(\boldsymbol{s}_{\mathrm{L}})$, there exists a unique $\bar{\tau}(\boldsymbol{s}) > 0$ such that

$$\frac{\partial \mathcal{F}}{\partial \tau}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) = 0.$$

This defines a map $\bar{\tau}$ in $\mathcal{C}^1(S^+(\boldsymbol{s}_{\mathrm{L}}), \mathbb{R}_+^*)$ due to the implicit function theorem. Note that the assumption (4.18) on the relative Mach number implies that $\partial_\tau \mathcal{F}(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) > 0$, while $\partial_\tau \mathcal{F}(\bar{\tau}(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}}) = 0$. Therefore, hypothesis (4.5) ensures that

$$\bar{\tau}(\boldsymbol{s}_{\mathrm{L}}) < \tau_{\mathrm{L}}. \tag{5.15}$$

Next, we construct the set $\mathcal{D} \subset S^+(\boldsymbol{s}_{\mathrm{L}})$ introduced in proposition 5.5 when studying for existence the zeros of $\boldsymbol{s} \in S^+(\boldsymbol{s}_{\mathrm{L}}) \mapsto \mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s})$. In this way, we first notice that by definition of $\bar{\tau}(\boldsymbol{s})$ for all $\boldsymbol{s} \in S^+(\boldsymbol{s}_{\mathrm{L}})$ we have

$$\mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) = p(\bar{\tau}(\boldsymbol{s}), s) - p(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) - \frac{\partial p}{\partial \tau}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s})(\bar{\tau}(\boldsymbol{s}) - \tau_{\mathrm{L}}).$$

Introducing the auxiliary function $\phi \colon \mathbb{R}_+ \times \mathcal{S}_+^N \to S^+(\boldsymbol{s}_{\mathrm{L}})$ defined by

$$\phi(\lambda, \boldsymbol{a}) = \mathcal{F}(\bar{\tau}(\boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a}), \boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a}),$$

straightforward calculations give

$$\frac{\partial \phi}{\partial \lambda}(\lambda, \boldsymbol{a}) = \sum_{1 \leqslant i \leqslant N} \left(\frac{\partial p_i}{\partial s_i} a_i + \left(\sum_{1 \leqslant j \leqslant N} \frac{\partial^2 p_j}{\partial \tau^2} \frac{\partial \bar{\tau}}{\partial s_i} + \frac{\partial^2 p_i}{\partial \tau \partial s_i} \right) a_i \right). \tag{5.16}$$

But differentiating the identity $\partial_\tau p(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) = -m^2$, valid for all $\boldsymbol{s} \in S^+(\boldsymbol{s}_{\mathrm{L}})$, easily implies that (5.16) reduces to

$$\frac{\partial \phi}{\partial \lambda}(\lambda, \boldsymbol{a}) = \sum_{1 \leqslant i \leqslant N} \frac{\partial p_i}{\partial s_i}(\bar{\tau}(\boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a}), \boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a}) a_i. \tag{5.17}$$

This derivative is therefore strictly positive for all $\boldsymbol{a} \in \mathcal{S}_+^N$, as follows from (4.5). Next, with respect to the strict convexity in τ of the total pressure and the property (5.15) expressing that $\bar{\tau}(\boldsymbol{s}_{\mathrm{L}}) < \tau_{\mathrm{L}}$, we get

$$\phi(0, \boldsymbol{a}) = p(\bar{\tau}(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}}) - p(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) - \frac{\partial p}{\partial \tau}(\bar{\tau}(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}})(\bar{\tau}(\boldsymbol{s}_{\mathrm{L}}) - \tau_{\mathrm{L}}) < 0.$$

To conclude, we need to check that for all $\boldsymbol{a} \in \mathcal{S}_+^N$ the map $\lambda \mapsto \phi(\lambda, \boldsymbol{a})$ achieves positive values for finite values of λ. Indeed, the implicit function theorem will thus

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ensure the existence of a map $\bar{\Lambda} \in \mathcal{C}^1(\mathcal{S}_+^N, \mathbb{R}_+^*)$ which is well defined in $\mathcal{D}$ with the following property:

$$\phi(\bar{\Lambda}(\boldsymbol{a}), \boldsymbol{a}) = 0 \quad \text{for all } \boldsymbol{a} \in \mathcal{S}_+^N \tag{5.18}$$

with $\phi(\lambda, \boldsymbol{a}) < 0$ (respectively, $\phi(\lambda, \boldsymbol{a}) > 0$) for all $\lambda < \bar{\Lambda}(\boldsymbol{a})$ (respectively, $\lambda > \bar{\Lambda}(\boldsymbol{a})$). This is plainly the required result. To establish the validity of (5.18), we show that for all $\boldsymbol{a} \in \mathcal{S}_+^N$ there exists $\lambda_\star(\boldsymbol{a}) > 0$ such that

$$\bar{\tau}(\boldsymbol{s}_{\mathrm{L}} + \lambda_\star(\boldsymbol{a})) = \tau_{\mathrm{L}}. \tag{5.19}$$

Indeed, for such values of λ, ϕ reduces to

$$\phi(\lambda_\star(\boldsymbol{a}), \boldsymbol{a}) = p(\tau_{\mathrm{L}}, \boldsymbol{s}) - p(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) > 0,$$

as follows from (4.5). To derive (5.19), we introduce the auxiliary smooth function $\theta \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ defined for all $\boldsymbol{a} \in \mathcal{S}_+^N$ by

$$\theta(\lambda) = \frac{\partial p}{\partial \tau}(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a}) + m^2.$$

For any given $\boldsymbol{a}$ in $\mathcal{S}_+^N$, we establish the existence of solutions to $\theta(\lambda) = 0$, $\lambda_\star(a)$ being chosen to be, for instance, the smallest smallest possible solution. The existence of such solutions readily follows from the assumption (4.18) on the relative Mach number, ensuring that $\theta(0) > 0$, while the asymptotic condition (4.7) ensures $\theta(\lambda) < 0$ for large enough λ. Note that the solutions under consideration are strictly positive. In addition, since $\bar{\Lambda}(\boldsymbol{a}) < \lambda_\star(\boldsymbol{a})$ for all $\boldsymbol{a} \in \mathcal{S}_+^N$, we obtain

$$\bar{\tau}(\boldsymbol{s}) < \tau_{\mathrm{L}}, \quad \boldsymbol{s} \in \bar{\mathcal{D}}, \tag{5.20}$$

which concludes the proof of lemma 5.7. □

Proof of lemma 5.8. Let $\boldsymbol{s}$ in $S^+(\boldsymbol{s}_{\mathrm{L}})$ be given. By the definition of $\bar{\tau}(\boldsymbol{s})$, $\mathcal{F}(\cdot, \boldsymbol{s})$ achieves the monotonicity properties stated in lemma 5.8; the required limits immediately follow from the asymptotic conditions (4.6). Consideration of the sign of $\mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s})$ (as we proposed earlier) obviously implies that for fixed $\boldsymbol{s} \in S^+(\boldsymbol{s}_{\mathrm{L}})$ the equation $\mathcal{F}(\tau, \boldsymbol{s}) = 0$ has exactly two solutions τ^-, τ^+ in $\bar{\mathcal{D}} \setminus \Gamma$ so that $\tau^- < \bar{\tau}(\boldsymbol{s}) < \tau^+$; this equation has exactly one solution, namely $\bar{\tau}(\boldsymbol{s})$, when $\boldsymbol{s} \in \Gamma$, and has no solution whenever $\boldsymbol{s} \in S^+(\boldsymbol{s}_{\mathrm{L}}) \setminus \bar{\mathcal{D}}$. In addition, using the notation introduced in the proof of lemma 5.7, it can easily be seen that the following limits hold true:

$$\lim_{\lambda \to \bar{\Lambda}(\boldsymbol{a})^-} \tau^{\pm}(\boldsymbol{s}_{\mathrm{L}} + \lambda \boldsymbol{a}) = \bar{\tau}(\boldsymbol{s}_{\mathrm{L}} + \bar{\Lambda}(\boldsymbol{a})\boldsymbol{a}) \quad \text{for all } \boldsymbol{a} \in \mathcal{S}_+^N. \tag{5.21}$$

These observations allow for the definition of two maps $\tau^{\pm} \colon \bar{\mathcal{D}} \to \mathbb{R}_+^*$ satisfying

$$\mathcal{F}(\tau^{\pm}(\boldsymbol{s}), \boldsymbol{s}) = 0 \quad \text{for all } \boldsymbol{s} \in \bar{\mathcal{D}},$$

and so that

$$\tau^-(\boldsymbol{s}) < \bar{\tau}(\boldsymbol{s}) < \tau^+(\boldsymbol{s}), \quad \boldsymbol{s} \in \bar{\mathcal{D}} \setminus \Gamma; \qquad \tau^-(\boldsymbol{s}) = \bar{\tau}(\boldsymbol{s}) = \tau^+(\boldsymbol{s}), \quad \boldsymbol{s} \in \Gamma. \tag{5.22}$$

Then the above inequalities yield $\partial_\tau \mathcal{F}(\tau^{\pm}(\boldsymbol{s}), \boldsymbol{s}) \neq 0$ for all $\boldsymbol{s} \in \mathcal{D} \cup \{\boldsymbol{s}_{\mathrm{L}}\}$ in view of (5.22), so that the implicit function theorem ensures that $\tau^{\pm} \in \mathcal{C}^1(\mathcal{D} \cup \{\boldsymbol{s}_{\mathrm{L}}\}, \mathbb{R}_+^*)$, while (5.21) gives that $\tau^{\pm} \in \mathcal{C}^0(\bar{\mathcal{D}}, \mathbb{R}_+^*)$.

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Next, focusing on some given $\boldsymbol{s} \in \bar{\mathcal{D}} \setminus \{\boldsymbol{s}_{\rm L}\}$, we observe that

$$\mathcal{F}(\tau_{\rm L}, \boldsymbol{s}) = p(\tau_{\rm L}, \boldsymbol{s}) - p(\tau_{\rm L}, \boldsymbol{s}_{\rm L}) > 0$$

so that, necessarily, either $\tau_{\rm L} < \tau^-(\boldsymbol{s})$ or $\tau^+(\boldsymbol{s}) < \tau_{\rm L}$. In addition, the identity $\mathcal{F}(\tau_{\rm L}, \boldsymbol{s}_{\rm L}) = 0$ expresses that either $\tau^-(\boldsymbol{s}_{\rm L}) = \tau_{\rm L}$ or $\tau^+(\boldsymbol{s}_{\rm L}) = \tau_{\rm L}$. But lemma 5.7 ensures that $\bar{\tau}(\boldsymbol{s}) < \tau_{\rm L}$ for all $\boldsymbol{s} \in \bar{\mathcal{D}}$. This concludes the proof. □

Equipped with these two lemmas, the proof of proposition 5.4 is essentially completed: the required inequality $\bar{\Lambda}(\boldsymbol{a}) > \Lambda_0(\boldsymbol{a})$ for all $\boldsymbol{a} \in \mathcal{S}_+^N$ will be deduced from the derivation of the set $\mathcal{K}$ we propose hereafter. We shall need the following technical result.

LEMMA 5.9. *For all* $\boldsymbol{s} \in \Gamma$, $\mathcal{H}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) > 0$.

This statement actually indicates that there is no critical point on Γ.

Proof. To shorten the notation, let us introduce

$$\epsilon(\tau, \boldsymbol{s}) = \sum_{i=1}^{N} \epsilon_i(\tau, s_i),$$

and consider the auxiliary function $\psi \in \mathcal{C}^1(\Gamma, \mathbb{R})$ defined for all $\boldsymbol{s} \in \Gamma$ by

$$\psi(\boldsymbol{s}) = \epsilon(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) - \epsilon(\tau_{\rm L}, \boldsymbol{s}) + H_{\rm L}(\bar{\tau}(\boldsymbol{s}) - \tau_{\rm L}) - \tfrac{1}{2}m^2(\bar{\tau}(\boldsymbol{s})^2 - \tau_{\rm L}^2).$$

Here we set

$$H_{\rm L} = m^2\tau_{\rm L} + p(\tau_{\rm L}, \boldsymbol{s}_{\rm L}),$$

so that $\mathcal{H}(\tau, \boldsymbol{s})$ may be recast as

$$\mathcal{H}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) = \psi(\boldsymbol{s}) + \epsilon(\tau_{\rm L}, \boldsymbol{s}) - \epsilon(\tau_{\rm L}, \boldsymbol{s}_{\rm L}).$$

With respect to the identity $\mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) = 0$ valid for all $\boldsymbol{s} \in \Gamma$ (see lemma 5.7(ii)), we have

$$\left.\begin{aligned} H_{\rm L} &= p(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) + m^2\bar{\tau}(\boldsymbol{s}), \\ m^2(\bar{\tau}(\boldsymbol{s}) - \tau_{\rm L}) &= p(\tau_{\rm L}, \boldsymbol{s}_{\rm L}) - p(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}), \end{aligned}\right\} \tag{5.23}$$

which give successively, for all $\boldsymbol{s} \in \Gamma$,

$$\begin{aligned} \psi(\boldsymbol{s}) &= \epsilon(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) - \epsilon(\tau_{\rm L}, \boldsymbol{s}) + (\bar{\tau}(\boldsymbol{s}) - \tau_{\rm L})(p(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) + \tfrac{1}{2}m^2(\bar{\tau}(\boldsymbol{s}) - \tau_{\rm L})) \\ &= \epsilon(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) - \epsilon(\tau_{\rm L}, \boldsymbol{s}) + (\bar{\tau}(\boldsymbol{s}) - \tau_{\rm L})(p(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) - \tfrac{1}{2}(p(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) - p(\tau_{\rm L}, s_{\rm L}))). \end{aligned}$$

Moreover, the two identities

$$\mathcal{F}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) = 0 \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial \tau}(\bar{\tau}, \boldsymbol{s}) = 0,$$

valid for all $\boldsymbol{s} \in \Gamma$, are easily seen to give, for the $\boldsymbol{s}$ under consideration,

$$p(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}) - p(\tau_{\rm L}, \boldsymbol{s}_{\rm L}) = (\bar{\tau}(\boldsymbol{s}) - \tau_{\rm L})\frac{\partial p}{\partial \tau}(\bar{\tau}(\boldsymbol{s}), \boldsymbol{s}).$$

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Consequently, for all $\boldsymbol{s} \in \Gamma$,

$$\psi(\boldsymbol{s}) = \epsilon(\bar\tau(\boldsymbol{s}), \boldsymbol{s}) - \epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}) + (\bar\tau(\boldsymbol{s}) - \tau_{\mathrm{L}}) p(\bar\tau(\boldsymbol{s}), \boldsymbol{s}) - \tfrac{1}{2}(\bar\tau(\boldsymbol{s}) - \tau_{\mathrm{L}})^2 \frac{\partial p}{\partial \tau}(\bar\tau(\boldsymbol{s}), \boldsymbol{s}).$$

To conclude, we show that

$$\mathcal{H}(\bar\tau(\boldsymbol{s}), \boldsymbol{s}) = \theta(\bar\tau(\boldsymbol{s}), \boldsymbol{s}) + \epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}) - \epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) > 0. \tag{5.24}$$

Since

$$\frac{\partial \epsilon_i}{\partial s_i}(\tau_{\mathrm{L}}, s_i) = T_i(\tau_{\mathrm{L}}, s_i) > 0,$$

we have $\epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}) - \epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) > 0$ for all $s \in \Gamma$ because $\boldsymbol{s}_{\mathrm{L}} \notin \Gamma$. Indeed, observe that lemma 5.8 implies that equality to zero holds if and only if $\boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}}$ but $\boldsymbol{s}_{\mathrm{L}} \notin \Gamma$.

To show (5.24), we study the following auxiliary function: $\Psi \in \mathcal{C}^1(\mathbb{R}_+^*, \mathbb{R})$, setting, for fixed $s \in \Gamma$,

$$\Psi(\tau) = \epsilon(\tau, \boldsymbol{s}) - \epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}) + (\tau - \tau_{\mathrm{L}}) p(\tau, \boldsymbol{s}) - \tfrac{1}{2}(\tau - \tau_{\mathrm{L}})^2 \frac{\partial p}{\partial \tau}(\tau, \boldsymbol{s}).$$

Easy calculations give

$$\frac{\partial \Psi}{\partial \tau}(\tau) = -\tfrac{1}{2}(\tau - \tau_{\mathrm{L}})^2 \frac{\partial^2 p}{\partial \tau^2}(\tau, \boldsymbol{s}) \leqslant 0,$$

with $\Psi(\tau_{\mathrm{L}}) = 0$. Consequently, $\Psi(\tau) > 0$ for all $\tau < \tau_{\mathrm{L}}$. Since $\bar\tau(\boldsymbol{s}) < \tau_{\mathrm{L}}$ (see lemma 5.7), $\Psi(\bar\tau(\boldsymbol{s})) > 0$ for all $\boldsymbol{s} \in \Gamma$, and we thus obtain the required inequality: $\mathcal{H}(\bar\tau(\boldsymbol{s}), \boldsymbol{s}) > 0$. □

Proof of proposition 5.5. We first establish the required properties of $\mathcal{H}$ related to the branch of solutions τ^+. With respect to the identity $\tau^+(\boldsymbol{s}) = \bar\tau(\boldsymbol{s})$ for all $\boldsymbol{s} \in \Gamma$, the technical lemma 5.9 allows us to restrict ourselves to $\boldsymbol{s} \in \bar{\mathcal{D}} \setminus \Gamma$ where τ^+ is continuously differentiable. For such $\boldsymbol{s}$, the identity $\mathcal{F}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) = 0$ can be equivalently re-expressed as

$$m^2(\tau^+(\boldsymbol{s}) - \tau_{\mathrm{L}}) = (p(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) - p(\tau^+(\boldsymbol{s}), \boldsymbol{s})). \tag{5.25}$$

Let us evaluate $\mathcal{H}(\tau^+(\boldsymbol{s}), \boldsymbol{s})$ as follows:

$$\mathcal{H}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) = \epsilon(\tau^+(\boldsymbol{s}), \boldsymbol{s}) - \epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) + (\tau^+(\boldsymbol{s}) - \tau_{\mathrm{L}})(p(\tau^+(\boldsymbol{s}), s) + \tfrac{1}{2} m^2 (\tau^+(\boldsymbol{s}) - \tau_{\mathrm{L}})),$$

where $\epsilon(\tau, \boldsymbol{s}) = \sum_{i=1}^N \epsilon_i(\tau, s_i)$. Using (5.25), we then obtain

$$\mathcal{H}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) = \epsilon(\tau^+(\boldsymbol{s}), \boldsymbol{s}) - \epsilon(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}) - \frac{1}{2m^2}(p^2(\tau^+(\boldsymbol{s}), \boldsymbol{s}) - p^2(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}})).$$

Let us introduce the auxiliary function $\Theta\colon \mathbb{R}_+^* \times \bar{\mathcal{D}} \to \mathbb{R}$ by setting

$$\Theta(\tau, \boldsymbol{s}) = \epsilon(\tau, \boldsymbol{s}) - \frac{1}{2m^2} p^2(\tau, \boldsymbol{s}),$$

so that, for all $\boldsymbol{s} \in \bar{\mathcal{D}}$,

$$\mathcal{H}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) = \Theta(\tau^+(\boldsymbol{s}), \boldsymbol{s}) - \Theta(\tau_{\mathrm{L}}, \boldsymbol{s}_{\mathrm{L}}), \tag{5.26}$$

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with $\boldsymbol{s} \mapsto \Theta(\tau^+(\boldsymbol{s}), \boldsymbol{s}) \in \mathcal{C}^1(\bar{\mathcal{D}} \setminus \Gamma, \mathbb{R})$. Since $\tau^+(\boldsymbol{s}_{\mathrm{L}}) = \tau_{\mathrm{L}}$ by lemma 5.8, (5.26) can equivalently be written as

$$\mathcal{H}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) = \Theta(\tau^+(\boldsymbol{s}), \boldsymbol{s}) - \Theta(\tau^+(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}}).$$

Moreover, for all $\boldsymbol{s} \in \bar{\mathcal{D}} \setminus \Gamma$ we have

$$\begin{aligned}
\frac{\partial}{\partial s_i}\theta(\tau^+(\boldsymbol{s}), \boldsymbol{s}) &= \frac{\partial \tau^+}{\partial s_i}(\boldsymbol{s})\frac{\partial \mathcal{H}}{\partial \tau}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) + \frac{\partial \mathcal{H}}{\partial s_i}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) \\
&= -\frac{\partial \tau^+}{\partial s_i}(\boldsymbol{s})\mathcal{F}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) + \frac{\partial \epsilon_i}{\partial s_i}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) \\
&= \frac{\partial \epsilon_i}{\partial s_i}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) \\
&= T_i(\tau^+(\boldsymbol{s}), \boldsymbol{s}) > 0,
\end{aligned} \tag{5.27}$$

where we have used the identity $\mathcal{F}(\tau^+(\boldsymbol{s}), \boldsymbol{s}) = 0$. Consequently, we deduce that

$$\theta(\tau^+(\boldsymbol{s}), \boldsymbol{s}) - \theta(\tau^+(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}}) \geqslant 0 \quad \text{for all } \boldsymbol{s} \in \bar{\mathcal{D}} \setminus \Gamma$$

with equality to zero if and only if $\boldsymbol{s} = \boldsymbol{s}_{\mathrm{L}}$ (see lemma 5.8). Combining the previous steps with lemma 5.9 gives the required properties (i) and (ii).

We now derive the remaining properties of $\mathcal{H}$ related to τ^-. Observe that the technical lemma 5.9 immediately gives

$$\mathcal{H}(\tau^-(\boldsymbol{s}), \boldsymbol{s}) > 0, \quad \boldsymbol{s} \in \Gamma, \tag{5.28}$$

since $\tau^-(\boldsymbol{s}) = \bar{\tau}(\boldsymbol{s})$ for the $\boldsymbol{s}$ under consideration. We can now obtain the following estimate:

$$\mathcal{H}(\tau^-(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}}) < 0. \tag{5.29}$$

To prove lemma 5.9, let us introduce the following auxiliary function $\psi\colon \mathbb{R}_+^* \to \mathbb{R}$, setting

$$\psi(\tau) = \mathcal{H}(\tau, \boldsymbol{s}_{\mathrm{L}}).$$

Since $\psi'(\tau) = -\mathcal{F}(\tau, \boldsymbol{s}_{\mathrm{L}})$ for all $\tau > 0$, lemma 5.8 is easily seen to imply that ψ strictly increases in $(\tau^-(\boldsymbol{s}_{\mathrm{L}}), \tau^+(\boldsymbol{s}_{\mathrm{L}}))$ with $\mathcal{H}(\tau^+(\boldsymbol{s}_{\mathrm{L}}), \boldsymbol{s}_{\mathrm{L}}) = 0$, as just established. This yields inequality (5.29).

To conclude the proof, we follow exactly the same steps as those developed in the proof of lemma 5.7 devoted to the derivation of the subset $\mathcal{K} \in S^+(\boldsymbol{s}_{\mathrm{L}})$.

We introduce the following auxiliary function defined by

$$\Phi(\lambda, \boldsymbol{a}) = \mathcal{H}(\tau^-(\boldsymbol{s}_{\mathrm{L}} + \lambda\boldsymbol{a}), \boldsymbol{s}_{\mathrm{L}} + \lambda\boldsymbol{a}), \quad \boldsymbol{a} \in \mathcal{S}_+^N, \quad \lambda \in [0, \bar{\Lambda}(\boldsymbol{a})[\,.$$

Note that this function is continuously differentiable on its domain of definition since, in view of lemma 5.8, the function $(\tau, \lambda) \mapsto \tau^-(\boldsymbol{s}_{\mathrm{L}} + \lambda\boldsymbol{a})$ is differentiable. Straightforward calculations then give

$$\begin{aligned}
\frac{\partial \Phi}{\partial \lambda}(\lambda, \boldsymbol{a}) = -\mathcal{F}(\tau^-(\boldsymbol{s}_{\mathrm{L}} + \lambda\boldsymbol{a}), \boldsymbol{s}_{\mathrm{L}} + \lambda\boldsymbol{a})\Bigg(\sum_{1 \leqslant i \leqslant N} \frac{\partial \tau^-}{\partial s_i} a_i\Bigg) \\
+ \sum_{1 \leqslant i \leqslant N} T_i(\tau^-(\boldsymbol{s}_{\mathrm{L}} + \lambda\boldsymbol{a}), \boldsymbol{s}_{\mathrm{L}} + \lambda\boldsymbol{a}) a_i.
\end{aligned} \tag{5.30}$$

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But the identity $\mathcal{F}(\tau^-(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a}),\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a})=0$ holds true by definition for all $\boldsymbol{a}\in\mathcal{S}_+^N$ and $\lambda\in[0,\bar{\Lambda}(\boldsymbol{a})]$, so that (5.30) reduces to

$$\frac{\partial\Phi}{\partial\lambda}(\lambda,\boldsymbol{a})=\sum_{1\leqslant i\leqslant N}T_i(\tau^-(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a}),\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a})a_i>0.$$

With respect to the inequalities (5.28) and (5.29), the implicit function theorem implies the existence of a map $\Lambda_0\in\mathcal{C}^1(\mathcal{S}_+^N,\mathbb{R}_+^*)$ with the following properties:

$$\Phi(\Lambda_0(\boldsymbol{a}),\boldsymbol{a})=0,\quad \boldsymbol{a}\in\mathcal{S}_+^N,$$

together with $\Phi(\lambda,\boldsymbol{a})<0$ for all $\lambda\in[0,\Lambda_0(\boldsymbol{a})[$ and $\Phi(\lambda,\boldsymbol{a})>0$ for all $\lambda\in]\Lambda_0(\boldsymbol{a}),\bar{\Lambda}(\boldsymbol{a})]$. This concludes the proof of proposition 5.5. □

Proof of proposition 5.6. With respect to the identity $\partial_\tau\mathcal{H}(\tau,\boldsymbol{s})=-\mathcal{F}(\tau,\boldsymbol{s})$ valid for all $(\tau,\boldsymbol{s})\in\mathbb{R}_+^*\times\mathcal{K}$, lemma 5.8 immediately implies that the smooth map $\tau\mapsto\mathcal{H}(\tau,\boldsymbol{s})$ ($\boldsymbol{s}$ being fixed in $\mathcal{K}$) strictly decreases in $]0,\tau^-(s)[$ and $]\tau^+(s),+\infty[$, while it strictly increases in $]\tau^-(\boldsymbol{s}),\tau^+(\boldsymbol{s})[$ with the following limits:

$$\lim_{\tau\to0^+}\mathcal{H}(\tau,\boldsymbol{s})=+\infty\quad\text{and}\quad\lim_{\tau\to\infty}\mathcal{H}(\tau,\boldsymbol{s})=-\infty$$

in view of the asymptotic conditions (4.2). In addition, for all $\boldsymbol{s}\in\mathcal{K}$, we infer from proposition 5.5 that $\mathcal{H}(\tau^-(\boldsymbol{s}),\boldsymbol{s})<0$ and $\mathcal{H}(\tau^+(\boldsymbol{s}),\boldsymbol{s})>0$. These observations allow the definition of three maps, namely $\check{\mathcal{T}}$, $\mathcal{T}$, $\hat{\mathcal{T}}\colon\mathcal{K}\to\mathbb{R}_+^*$, with the following properties:

$$\mathcal{H}(\check{\mathcal{T}}(\boldsymbol{s}),\boldsymbol{s})=\mathcal{H}(\mathcal{T}(\boldsymbol{s}),\boldsymbol{s})=\mathcal{H}(\hat{\mathcal{T}}(\boldsymbol{s}),\boldsymbol{s})=0,\quad\boldsymbol{s}\in\mathcal{K},$$

together with

$$0<\check{\mathcal{T}}(\boldsymbol{s})<\tau^-(\boldsymbol{s})<\mathcal{T}(\boldsymbol{s})<\tau^+(\boldsymbol{s})<\hat{\mathcal{T}}(\boldsymbol{s}),\quad\boldsymbol{s}\in\mathcal{K}.$$

Next, using the notation introduced in the proof of lemma 5.7, we first compute, for all $\boldsymbol{a}\in\mathcal{S}_+^N$,

$$\lim_{\lambda\to0^+}\check{\mathcal{T}}(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a})<\lim_{\lambda\to0^+}\mathcal{T}(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a})=\lim_{\lambda\to0^+}\hat{\mathcal{T}}(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a})=\tau^+(\boldsymbol{s}_{\rm L}),$$

since $\mathcal{H}(\tau^-(\boldsymbol{s}_{\rm L}),\boldsymbol{s}_{\rm L})<\mathcal{H}(\tau^+(\boldsymbol{s}_{\rm L}),\boldsymbol{s}_{\rm L})=0$ in view of (iii) and (i) in proposition 5.5. In the same way, we get

$$\begin{aligned}\lim_{\lambda\to\Lambda_0(\boldsymbol{a})}\check{\mathcal{T}}(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a})&=\lim_{\lambda\to\Lambda_0(\boldsymbol{a})}\mathcal{T}(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a})\\&=\tau^-(\boldsymbol{s}_{\rm L}+\Lambda_0(\boldsymbol{a})\boldsymbol{a})\\&<\lim_{\lambda\to\Lambda_0(\boldsymbol{a})}\hat{\mathcal{T}}(\boldsymbol{s}_{\rm L}+\lambda\boldsymbol{a}),\end{aligned}$$

since $\mathcal{H}(\tau^-(\boldsymbol{s}),\boldsymbol{s})=0<\mathcal{H}(\tau^+(\boldsymbol{s}),\boldsymbol{s})$ for all $\boldsymbol{s}\in\mathcal{C}$, in view of proposition 5.5. To conclude, we have to establish the smoothness properties put forward in proposition 5.6. In view of the monotonicity properties of $\tau\mapsto\mathcal{H}(\tau,\boldsymbol{s})$ we have just established for all $\boldsymbol{s}\in\mathcal{K}$ that all three maps are obviously in $\mathcal{C}^1(\mathcal{K},\mathbb{R}_+^*)\cap\mathcal{C}^0(\bar{\mathcal{K}},\mathbb{R}_+^*)$ due to the implicit function theorem. This concludes the proof of proposition 5.6. □

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