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The plasma wake field excitation: Recent developments from thermal to quantum regime

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Abstract. To describe the transverse nonlinear and collective self-consistent interaction of a long relativistic electron or positron beam with an unmagnetized plasma, a pair of coupled nonlinear differential equations were proposed by Fedele and Shukla in 1992 (Fedele, R. and Shukla, P. K. 1992a *Phys. Rev.* A **45**, 4045). They were obtained within the quantum-like description provided by the thermal wave model and the theory of plasma wake field excitation. The pair of equations comprises a 2D Schrödinger-like equation for a complex wave function (whose squared modulus is proportional to beam density) and a Poisson-like equation for the plasma wake potential. The dispersion coefficient of the Schrödinger-like equation is proportional to the beam thermal emittance. More recently, Fedele–Shukla equations have been further applied to magnetized plasmas, and solutions were found in the form of nonlinear vortex states and ring solitons. They have been also applied to plasma focusing problems and extended from thermal to quantum regimes. We present here a review of the original approach, and subsequent developments.

1. Introduction

A self-consistent theory for the interaction between the plasma wake field (PWF) and the driving long relativistic electron (positron) beam in an unmagnetized, overdense, collisionless plasma was proposed 21 years ago by Fedele and Shukla (1992a,b). In the construction of this theory a quantum-like approach was employed for the relativistically charged particle beam propagation, the so-called *thermal wave model* (TWM) (Fedele and Miele 1991, 1992), which was a fresh approach capable of describing the beam transport in terms of a complex function, the so-called *beam wave function* (BWF), whose squared modulus is proportional to the beam density. The BWF is a solution to a Schrödinger-like equation in which the dispersion coefficient is replaced by the (thermal) emittance of the beam. The TWM has been successfully applied to a number of problems in beam physics and particle acceleration. To provide a self-consistent description of our problem, one needs to couple the Schrödinger-like equation with the fluid equations that describe the PWF excitation in the presence of the transverse profile of beam density. In this way one obtains a consistent coupling between the driving beam and the PWFs. Such treatment brought an improvement since the earlier approaches (Chen et al. 1985; Katsouleas 1986; Chen 1987; Rosenzweig and Chen 1989) did not take into account both the self-consistent reaction of the wake field on the driver and the spatial evolution of the beam. In the long beam limit, i.e. when the wavelength of the wake field is much smaller than the longitudinal beam length, the longitudinal beam dynamics can be almost disregarded, and the set of governing equations reduces to a pair of partial differential equations comprising a Schrödinger-like equation for the TWM and a Poisson-like equation for the wake potential.

2. Fedele–Shukla equations (FSEs)

In the framework of TWM (Fedele and Miele 1991), the spatiotemporal evolution of a long relativistic charged particle beam with the transverse emittance ϵ, traveling along the z-axis with the velocity βc (with $\beta \approx 1$), through an overdense unmagnetized plasma, is described by the following Schrödinger-like equation for BWF $\Psi(\mathbf{r}_\perp, \xi)$ (Fedele and Shukla 1992a,b):

$$i\epsilon \frac{\partial \Psi}{\partial \xi} = -\frac{\epsilon^2}{2}\nabla_\perp^2 \Psi + U_w(\mathbf{r}_\perp, \xi)\Psi, \qquad (2.1)$$

where $\nabla_\perp$ is the transverse component of the gradient, ξ is the distance in the moving reference frame, $\xi = z - \beta c t$, and $U_w(\mathbf{r}_\perp, \xi)$ is the dimensionless plasma wake potential energy normalized to $m_0\gamma_0\beta^2c^2$ (m_0

and $\gamma_0 = 1/\sqrt{1-\beta^2}$) are the electron rest mass and the relativistic gamma factor, respectively). Note that the transverse component of the plasma wake electric field induced by the beam is equal to $\mathbf{W}_\perp = -(m_0\gamma\beta^2c^2/q)\nabla_\perp U_w$, where q is the charge of the single particle of the beam ($q = -e$ for electrons, and $q = e$ for positrons). In an analogy to the non-relativistic quantum mechanics, ξ and ϵ play the role of time and Planck's constant, respectively, while in an analogy to the electromagnetic beam optics in a paraxial approximation, ϵ plays the role of the inverse of the wavenumber $k^{-1} = \lambda/2\pi$, and U_w plays the role of the refractive index of the nonlinear medium so that (2.1) corresponds to the well-known Fock–Leontovich equation (Shen 1984). The BWF has the following meaning: If N is the total number of particles of the beam, then the volumetric number density of the beam is given by

$$\rho_b(\mathbf{r}_\perp, \xi) = \frac{N}{\sigma_z}\ |\Psi(\mathbf{r}_\perp, \xi)|^2, \tag{2.2}$$

where σ_z is the beam length. Then the long beam assumption implies $k_p\sigma_z \gg 1$ (where k_p is the plasma wave number). According to (2.2), $|\Psi(\mathbf{r}_\perp, \xi)|^2$ is proportional to the transverse beam density profile. Using the fluid theory, it can be shown that, within the linear approximation, and in an overdense regime (i.e. $n_0 \gg \rho_b$), the plasma density perturbation, say n_1, obeys the following adiabatic shielding condition (Chen et al. 1985): $n_1(\mathbf{r}_\perp, \xi) \approx (q/e)\,\rho_b(\mathbf{r}_\perp, \xi)$, where n_0 is the unperturbed plasma density. Therefore, the Poisson-like equation for U_w can be cast in the form (Fedele and Shukla 1992a),

$$\left(\nabla_\perp{}^2 - k_p^2\right) U_w = \frac{N}{n_0\gamma_0\sigma_z}\ |\Psi(\mathbf{r}_\perp, \xi)|^2, \tag{2.3}$$

where we have used (2.2). Equations (2.1) and (2.3) describe the self-interaction of a long relativistic electron (positron) beam traveling in a collisionless unmagnetized plasma. Hereafter, we refer these equations as FSEs. If we formally solve (2.3) for U_w, we find that it is a functional of $|\Psi|^2$, i.e. $U_w = U_w[|\Psi|^2]$. Consequently, by using the explicit form in (2.1) we obtain a nonlinear Schrödinger equation, which in the general case may be non-local.

3. Relevance of Fedele–Shukla equations for the electron wave optics of non-laminar beams

As shown above, the self-consistent interaction between a beam and a medium (i.e. plasma) involves, through the FSE, a branch of electron optics that describes the situations in which the behavior of a beam containing an extremely large number of charged particles is affected by the electromagnetic interactions that are established within such a system. On the other hand, due to a large number of particles, the effects of a finite temperature cannot be ignored. Therefore, in general, the behavior of such a system is expected to be both collective and affected by the thermal spreading among the particles (thermal regimes). We refer to this branch as *electron optics in thermal regime* (EOTR) (Fedele et al. 2013).

The conventional approach to EOTR is well established and it has been applied in most of the scientific and technological applications in accelerator physics (Lawson 1976, 1988; Chao and Tigner 1998). The thermal spreading introduces, at an arbitrary longitudinal position, an uncertainty in the electron ray positions in the transverse plane. The configuration of the envelope resulting from the electron ray mixing resembles the pattern generated by the paraxial ray diffraction in the beam of electromagnetic radiation. From this conformity one may conclude that the statistical behavior of charged particles in a paraxial beam, which is a fully classical process, *simulates* the paraxial diffraction in the beams of electromagnetic radiation. The experimental evidence for this analogy is found in all processes that are relevant to EOTR (Lawson 1976, 1988) and is also supported by the theoretical kinetic descriptions by the Boltzmann/Vlasov equation (Lawson 1988).

According to the previous sections, an alternative theoretical description has been proposed by extending EOTR in the paraxial approximation to the wave context. Utilizing the analogy between optics and mechanics, a quantization procedure has been performed with TWM (Fedele and Miele 1991) to transit from geometrical to wave description of EOTR, which was the first time that EOTR was formulated in terms of a wave description. With the use of TWM, a number of linear and nonlinear problems in both conventional and plasma-based particle acceleration were successfully described (Fedele and Shukla 1992a,b; Fedele et al. 1993, 1995a,b, 2013). Furthermore, the TWM predictions have been compared with tracking-code simulations and a fair agreement has been demonstrated [the analysis has been carried in both configuration- and phase spaces (Fedele et al. 1995a; Jang et al. 2007, 2010)].

Remarkably, the self-consistent theory of the PWF interaction provided by a pair of FSEs opened up a novel approach to the wave formulation of EOTR, which for the first time took into account the reaction of the medium to the motion of the driving beam (Fedele and Shukla 1992a,b; Fedele et al. 1995b). Then in the local regime, i.e. for $|\nabla_\perp| \ll k_p$, the FSEs have been reduced to a 2D cubic nonlinear Schrödinger equation (Fedele and Shukla 1992a,b; Fedele et al. 1995b) capable of describing the self-focusing/defocusing of the driving relativistic electron (positron) beam in a cold unmagnetized plasma, or to establish the condition for the existence of a 2D stationary transverse profile (2D solitons) and the Weibel instability threshold. It was demonstrated that these physical circumstances were determined by the competition between the dispersive effects (i.e. related to the thermal energy provided by the thermal emittance spreading) and the eigenenergy (i.e. the nonlinear effect produced by the beam, related to the reaction of the medium on the beam). In a strongly non-local regime, i.e. $|\nabla_\perp| \gg k_p$, a pair of FSEs was reduced to an integro-differential nonlinear non-local Schrödinger equation capable of describing, for instance, the Bennett self-pinching equilibrium and the self-broadening of the driving beam. The above successes also led to the self-consistent description of the longitudinal beam dynamics for a beam with a finite length, so that the longitudinal dynamics was non-negligible. For instance, it was very useful for the prediction of soliton-like states of the charged particle beams or to provide a *wave key of reading* for the nonlinear and collective effects of coherent instabilities in high energy accelerating devices (both conventional and plasma-based). It provided the formulation of modulational instability in both deterministic and statistical approaches, with the Landau-type damping playing a fundamental role (Fedele et al. 1993; Johannisson 2004).

Recently, the TWM description, with an appropriate pair of FSEs, of the self-consistent beam–plasma interaction has been also developed for plasmas in a strongly axial magnetic field (Fedele et al. 2011; Tanjia et al. 2011). The collective vortex beam states (orbital angular momentum states) have been predicted for an arbitrary value of the integer effective vortex charge.

More recently, the paraxial electron optics has been extended to the quantum wave context with non-negligible collective interactions among particles. Therefore, the pair of FSEs has been extended to a spinorial form (Fedele et al. 2012a,b; Jovanovic et al. 2012). In this way, a quantum approach to relativistically charged particle beams, named the Quantum Wave Model (QWM) (Fedele et al. 2013), has been proposed and applied to plasma lens for quantum beams (Tanjia et al. 2013). In this approach, similar to TWM, the space charge effects (both capacitive and inductive) are taken into account within the Hartree's mean field approximation. However, the above QWM accounts only for the quantum nature of a single particle, including the single-particle uncertainty principle and the spin of a single particle, while the collective quantum nature of the system related to the overlapping of the single-particle wave functions is disregarded. As a consequence, for typical densities of charged particle beams employed in the present generation of both conventional accelerators and plasma-based acceleration schemes, QWM is appropriate when the temperature of the beam is sufficiently low to preserve the observability of the individual quantum nature of the particles, but sufficiently high to make the overlapping of the single-particle wave functions negligible. Actually, if the quantum uncertainty of the single particle is not concealed by the thermal spreading, within a paraxial picture, one can attribute, to each electron ray, the uncertainty in both the position on the transverse plane and the slope relative to the direction of propagation. It results that now the electron ray mixing is strongly affected by the individual quantum nature of the particles, and this picture corresponds to the analog of paraxial diffraction of the light rays in radiation beams. Since here the diffraction that is exhibited is of the quantum nature, we call *quantum paraxial diffraction* the picture associated with such a kind of mixing of electron rays (Fedele et al. 2012a,b; Jovanovic et al. 2012; Jovanović et al. 2013). It turns out that the spinorial FSEs may be reduced to a 2D spinorial Schrödinger equation. QWM has been recently applied to describe the self-interaction of an electron or positron beam propagating in a strongly magnetized plasma. Quantum ring solitons as the 2D quantum vortex states associated with the angular momentum (orbital plus spin) states have been found, as well as the self-focusing conditions predicted (Fedele et al. 2012a,b; Jovanovic et al. 2012; Jovanović et al. 2013).

4. Conclusions

We summarized the main results obtained with the pair of FSEs, including the subsequent developments with applications to magnetized plasmas and plasma focusing and recent extensions from thermal to quantum regime.

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