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FINITELY PRESENTED ORDERED GROUPS

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Theorem. *There exist non-Abelian finitely presented lattice-ordered groups which are totally ordered. This disproves a previous conjecture of the author* [**5**].

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A group G with a partial order on it that satisfies ($a \leqq b \rightarrow fag \leqq fbg$) for all $a, b, f, g \in G$ is called a *p.o. group*. If the partial order is a lattice (for every $a, b \in G$, there is a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$) the p.o. group is said to be a *lattice-ordered group*, or *l-group* for short. A p.o. group in which the partial order is total (for all a, b, $a \leqq b$ or $b \leqq a$) is called an *o-group*.

The class of l-groups is an equationally defined class of algebras under the operations $\cdot$, $^{-1}$, $\vee$ and $\wedge$. Hence free l-groups on arbitrary sets X exist [**2**, Chapter IV].

If G is an l-group and H is a subgroup of G that is closed under the lattice operations $\vee$ and $\wedge$, we call H an l-subgroup of G. A homomorphism (embedding, isomorphism) between l-groups that preserves the lattice and group operations is said to be an *l-homomorphism* (*l-embedding*, *l-isomorphism*). The kernels of l-homomorphisms are precisely the convex normal l-subgroups (C is said to be *convex* in G if $x \in G$, $c_1, c_2 \in C$ and $c_1 \leqq x \leqq c_2$ imply $x \in C$). If K is a convex normal l-subgroup of G, then G/K is an l-group under the naturally induced order ($Kf \leqq Kg$ iff $hf \leqq g$ for some $h \in K$); see [**1**,Section 2.3] where, as usual, *iff* is shorthand for if and only if. If an l-group G contains an Abelian convex normal l-subgroup A such that G/A is Abelian, then G is said to be *l-metabelian*.

An l-group G is said to be *finitely presented* (as an l-group) if there is a finite set $x_1, \ldots, x_m$ and a finite set $w_1, \ldots, w_n$ of elements of the free l-group F on $\{x_1, \ldots, x_m\}$ such that G is l-isomorphic to F/N where N is the convex normal l-subgroup of F generated by $w_1, \ldots, w_n$. In this case we simply write $\langle x_1, \ldots, x_m; w_1 = e, \ldots, w_n = e \rangle$ for F/N, where throughout e denotes the identity element of a group. The set $\{w_1 = e, \ldots, w_n = e\}$ is called the *set of* (*defining*) *relations* for F/N.

In any l-group G, let $|g| = g \vee g^{-1}$ for $g \in G$. It is easy to see [**1**, 1.3.10 and 1.3.11] that $|g| \geqq e$, and $|g| = e$ iff $g = e$. Hence ($w_1 = e$ & $\ldots$ & $w_n = e$) iff $|w_1| \vee \cdots \vee |w_n| = e$; thus any

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finitely presented l-group can be given by a single defining relation and so is an m generator one relator l-group for some finite m.

In [5] we conjectured that the only finitely presented l-groups that are o-groups are $\mathbb{Z}$, the additive group of integers under the usual ordering, and $\{e\}$; also, that the only finitely presented l-groups that are subdirect products of o-groups are Abelian. This was shown to be the case if the defining $w_1, \ldots, w_n$ were all *group* words, see [3]. However, in this note we prove both conjectures are false with an easy example.

Theorem. *There is a countably infinite set of pairwise non-l-isomorphic two generator one relator l-metabelian non-Abelian o-groups.*

Clearly there are only countably many finitely presented l-groups. Moreover, one generator l-groups are Abelian and free l-groups on at least two generators are not subdirect products of o-groups. Hence the theorem is the best (or worst?) possible.

Throughout we use $\mathbb{Q}$ for the additive group of rationals with the usual order; $A \overleftarrow{\times} B$ for a semidirect product of A by an o-group B where $a_1 b_1 \leqq a_2 b_2$ iff $b_1 < b_2$ in B or both $b_1 = b_2$ and $a_1 \leqq a_2$ in A; and $a \ll b$ for $a^n \leqq b$ for all $n \in \mathbb{Z}$.

For any further background, see [**1, 4, 5**] if necessary.

We first give a permutation proof in outline and then provide a more formal proof in detail.

Permutation Proof. Let $m > 1$ be a positive integer and g be the order-preserving permutation of the real line given by: $\alpha \mapsto \alpha + 1$. Then there are order-preserving permutations f of the real line conjugating g_0 to g_0^m but for any such f, there are real numbers α and β such that $\alpha f < \alpha$ and $\beta f > \beta$ (see [**4**, Lemma 2.2.1]). Hence if f and g are any order-preserving permutations of the real line that move no point down and $f^{-1} g f = g^m$, then g has infinitely many intervals of support and f moves each interval of support of g to one strictly to the right. Consequently, $g \ll f$. If $L(m)$ is the l-subgroup generated by f and g, then the normal subgroup C_m of $L(m)$ generated by g is convex and Abelian. Moreover, it is an o-group whence $L(m)$ is an l-metabelian o-group. Since every countable l-group can be l-embedded in the l-group of all order-preserving permutations of the real line [**4**, Corollary 2L], $L(m) \cong \langle x, y;\ x^{-1} y x = y^m,\ x \wedge y = y,\ y \wedge e = e \rangle$. Clearly $L(m_1) \cong L(m_2)$ iff $m_1 = m_2$. The theorem follows. □

Proof of Theorem. Let m be a positive integer exceeding 1 and

$$L_m = \langle x, y;\ x^{-1} y x = y^m,\ x \wedge y = y,\ y \wedge e = e \rangle.$$

So L_m is a finitely presented l-group for each m. We will prove that L_m is actually an l-metabelian o-group.

By definition, $y \leqq x$. If $y^n \leqq x$ then $y^{m+n} \leqq x y^m = y x$; hence $y^{n+1} \leqq y^{m+n-1} \leqq x$ since $m \geqq 2$ and $y \geqq e$. Thus $y^n \leqq x$ for all integers n by induction; so $y \ll x$. Consequently, $x^{-j} y x^j \ll x$ for all integers j. So if C_m is the normal l-subgroup of L_m generated by y, then C_m is convex; clearly it is Abelian.

We now examine C_m. We first note that

$$x^jy^ix^{-j} \leqq x^sy^rx^{-s} \text{ iff } i/m^j \leqq r/m^s.$$

For if $j \leqq s$, then $x^jy^ix^{-j} \leqq x^sy^rx^{-s}$ iff $x^{-(s-j)}y^ix^{s-j} \leqq y^r$ iff $y^{im^{s-j}} \leqq y^r$ iff $y^{im^s} \leqq y^{rm^j}$ iff $im^s \leqq rm^j$; similarly if $s \leqq j$. Moreover,

$$x^jy^ix^{-j} \cdot x^sy^rx^{-s} = \begin{cases} x^sy^{im^{s-j}+r}x^{-s} & \text{if } j \leqq s \\ x^jy^{i+rm^{j-s}}x^{-j} & \text{if } s \leqq j. \end{cases}$$

Therefore if $\phi: C_m \to \mathbb{Q}$ is given by: $(x^jy^ix^{-j})\phi = i/m^j$, then ϕ is an embedding and $z \leqq t$ iff $z\phi \leqq t\phi$ for all $z, t \in C_m$. Consequently C_m is an Abelian o-group.

Each element of L_m has the form wx^k for some $w \in C_m$ and unique integer k. Furthermore $w_1x^j \leqq w_2x^k$ iff $j < k$ or both $j = k$ and $w_1 \leqq w_2$ $(w_1, w_2 \in C_m;\ j, k \in \mathbb{Z})$. Therefore L_m is an l-metabelian o-group. Indeed if $\mathbb{Q}(m) = \{r/m^s : r, s \in \mathbb{Z}\}$, an l-subgroup of the o-group $\mathbb{Q}$, and $\psi \in \mathrm{Aut}(\mathbb{Q}(m), +, 0, \leqq)$ is multiplication by m, then we have shown that L_m is l-isomorphic to $\mathbb{Q}(m) \,\overleftarrow{\times}\, \langle \psi \rangle$. It follows that $L(m_1)$ and $L(m_2)$ are not l-isomorphic if $m_1 \neq m_2$ and the theorem is proved. □

I know of no other examples of finitely presented l-groups that are o-groups. Therefore, the following questions remain:

(I) Is every finitely presented l-group that is an o-group in fact l-soluble?

(II) Is every finitely presented l-group that is an l-soluble o-group actually l-metabelian?

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