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COEFFICIENT REGIONS FOR UNIVALENT TRINOMIALS

Q. I. RAHMAN AND J. WANIURSKI

0. Introduction. The problem of determining necessary and sufficient conditions bearing upon the numbers a_2 and a_3 in order that the polynomial $z + a_2z^2 + a_3z^3$ be univalent in the unit disk $|z| < 1$ was solved by Brannan ([**3**], [**4**]) and by Cowling and Royster [**6**], at about the same time. For his investigation Brannan used the following result due to Dieudonné [**7**] and the well-known Cohn rule [**9**].

THEOREM A (Dieudonné criterion). *The polynomial*

(1) $\quad z + a_2z^2 + \ldots + a_nz^n$

is univalent in $|z| < 1$ *if and only if for every* θ *in* $[0, \pi/2]$ *the associated polynomial*

(2) $\quad 1 + \dfrac{\sin 2\theta}{\sin \theta} a_2 z + \ldots + \dfrac{\sin n\theta}{\sin \theta} a_n z^{n-1}$

does not vanish in $|z| < 1$. *For* $\theta = 0$, (2) *is to be interpreted as the derivative of* (1).

The procedure of Cowling and Royster was based on the observation that $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ is univalent in $|z| < 1$ if and only if for all α such that $0 \leqq |\alpha| \leqq 1, \alpha \neq 1$ the function

$$\frac{z}{f(z) - f(\alpha z)} = \sum_{n=0}^{\infty} b_n(\alpha)z^n$$

is regular in the unit disk. In conclusion they mentioned that although it would be of interest to obtain the precise region of variability of (a_2, a_k) for the univalent trinomial $z + a_2z^2 + a_kz^k$, for $k > 3$, their method yielded only incomplete results in that direction.

In the present paper we take up this problem proposed by Cowling and Royster. We also consider arbitrary trinomials

(3) $\quad z + a_pz^p + a_qz^q, \quad (p < q).$

Trinomials of the form $z + a_pz^p + a_{2p-1}z^{2p-1}$ were studied by Ruscheweyh and Wirths [**12**] and by Rahman and Szynal [**11**].

1. Basic theorem. Let us denote by $V(p, q)$ the region of variability of

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(a_p, a_q), $a_p \in \mathbf{C}$, $a_q \in \mathbf{C}$ for trinomials of the form (3) to be univalent in $|z| < 1$. It is clear that $(a_p, a_q) \in V(p, q)$ if and only if

$$(a_p e^{i(p-1)\alpha}, a_q e^{i(q-1)\alpha}) \in V(p, q) \text{ for every real } \alpha.$$

So in order to determine the region $V(p, q)$ we may restrict ourselves to the case $a_q > 0$. Besides, for sake of simplicity we will write t for a_q. Of course t is to vary in $(0, 1/q]$. Finally, we find it more convenient to write the trinomial in the form $z - a_p z^p + tz^q$ which involves no loss of generality.

Throughout the paper the value of an expression like $(\sin q\theta/\sin\theta)$ at a zero of $\sin\theta$ will be defined by continuity.

If $0 < t < 1/q$ then the curve

$$(4) \qquad w(\varphi) = e^{-i(p-1)\varphi} + t\frac{\sin q\theta}{\sin\theta}e^{i(q-p)\varphi}, \quad 0 \leqq \varphi \leqq 2\pi$$

does not pass through the origin. The same is true if $t = 1/q$ provided $\theta \not\equiv 0 \pmod{\pi}$. Using standard terminology (see for example [**1**, p. 116]) we then denote *the region determined by* the curve (4) and containing the origin by $G_\theta = G_\theta(p, q, t)$. By $G_0(p, q, 1/q)$ we will mean the interval $[-2, 2]$ if $q = 2p - 1$ and $\{0\}$ otherwise.

THEOREM 1. *The trinomial*

$$(5) \qquad f_t(z) = z - a_p z^p + tz^q, \quad (p < q, 0 < t \leqq 1/q)$$

is univalent in $|z| < 1$ *if and only if*

$$(6) \qquad a_p \in \bigcap_{0 \leqq \theta \leqq \pi/2} \frac{\sin\theta}{\sin p\theta}\bar{G}_\theta.$$

For $\theta = \pi/p, 2\pi/p, \ldots, \left[\frac{p}{2}\right]\pi/p$

$$\frac{\sin\theta}{\sin p\theta}\bar{G}_\theta = \mathbf{C}.$$

Proof. By the Dieudonné criterion $f_t(z)$ is univalent in $|z| < 1$ if and only if for every $\theta \in [0, \pi/2]$ the associated polynomial

$$(7) \qquad 1 - \frac{\sin p\theta}{\sin\theta}a_p z^{p-1} + t\frac{\sin q\theta}{\sin\theta}z^{q-1}$$

does not vanish in $|z| < 1$. Since (7) is different from zero at the origin we conclude that a necessary and sufficient condition for $f_t(z)$ to be univalent in $|z| < 1$ is that for every θ in $[0, \pi/2]$ the function

$$(8) \qquad \frac{1}{z^{p-1}} - \frac{\sin p\theta}{\sin\theta}a_p + t\frac{\sin q\theta}{\sin\theta}z^{q-p}$$

does not vanish in $0 < |z| < 1$.

The function

$$\omega(z) := \frac{1}{z^{p-1}} + t\,\frac{\sin q\theta}{\sin \theta}\, z^{q-p}$$

maps $0 < |z| < 1$ onto the complement of $\bar{G}_\theta$ with respect to the complex plane $\mathbf{C}$. For $0 < t < 1/q$ as well as for $t = 1/q$ but $\theta \neq 0 \pmod{\pi}$ this is seen by applying the argument principle to the function

$$\Omega(z) := \begin{cases} 1/\omega(z) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

which is analytic in $|z| < 1$. If $t = 1/q$ and $\theta = 0$ then clearly $\omega(z)$ maps $0 < |z| < 1$ onto $\mathbf{C}\backslash[-2, 2]$ in case $q = 2p - 1$. If $t = 1/q$ and $\theta = 0$ but $q \neq 2p - 1$ then we have to show that $\omega(z)$ maps $0 < |z| < 1$ onto $\mathbf{C}\backslash\{0\}$, i.e.

$$\frac{1}{z^{p-1}} + z^{q-p}$$

assumes every complex value $c \neq 0$ in $0 < |z| < 1$. This is true if and only if the trinomial

$$g(z) := z^{q-1} - cz^{p-1} + 1, \quad (q \neq 2p - 1)$$

has at least one zero in $|z| < 1$. Suppose, if possible, that $g(z)$ does not have any zero in $|z| < 1$. Since the coefficient of z^{q-1} is equal to the constant term, it must then have all its zeros on $|z| = 1$. Consequently

$$z^{q-1}\overline{g(1/\bar{z})} \equiv g(z),$$

and so

$$cz^{p-1} \equiv \bar{c}z^{q-p},$$

which is impossible since $q \neq 2p - 1$.

It follows that (8) does not vanish in $0 < |z| < 1$ if and only if $(\sin p\theta/\sin \theta)\, a_p \in \bar{G}_\theta$, i.e. $a_p \in (\sin \theta/\sin p\theta)\, \bar{G}_\theta$. Thus $f_t(z)$ is univalent in $|z| < 1$ if and only if

$$a_p \in \frac{\sin \theta}{\sin p\theta}\, \bar{G}_\theta \quad \text{for all } \theta \in [0, \pi/2]$$

and hence the result.

Although the equation of the curve which determines the region G_θ is rather simple it is not easy to identify the intersection

$$(9) \qquad D(t, p, q) = \bigcap_{0 \leqq \theta \leqq \pi/2} \frac{\sin \theta}{\sin p\theta}\, \bar{G}_\theta$$

in general. The following lemma helps us to get a simpler description of the region $D(t, p, q)$.

LEMMA 1. *Let $F(z, x)$ be a complex valued function of z (complex) and x (real) having the following properties:*

(i) *there exists an absolute constant $\alpha > 0$ such that for each x belonging to the interval $I: = \{x: a < x \leqq b\}$, $F(z, x)$ is analytic in the annulus*

$$A_\alpha: = \{z: 1 - \alpha < |z| < 1 + \alpha\},$$

and is univalent on the arc

$$\gamma_x: = \{z = e^{i\varphi}: \varphi_1(x) \leqq \varphi \leqq \varphi_2(x)\},$$

where $\varphi_1(x)$, $\varphi_2(x)$ are continuous functions of x satisfying $0 < \varphi_2(x) - \varphi_1(x) < 2\pi$.

(ii) *for each z_0 lying on γ_{x_0} where x_0 is an arbitrary point of I there exists a left-hand neighbourhood*

$$N(x_0; \delta(z_0)): = \{x: x_0 - \delta(z_0) < x \leqq x_0\}$$

of x_0 in which $(\partial F/\partial x)$, $(\partial^2 F/\partial x^2)$, $(\partial^2 F/\partial x \partial z)$ exist and are bounded,

(iii) *there exists an absolute constant M such that for all $x \in I$ and $z \in \bar{A}_{\alpha/2}$,*

$$|F(z, x)| < M.$$

For each $x \in I$, let C_x be the arc

$$w = F(e^{i\varphi}, x), \varphi_1(x) \leqq \varphi \leqq \varphi_2(x).$$

Now, if

$$\text{(10)} \qquad \mathrm{Re}\left[\frac{\partial}{\partial x} F(z, x) \Big/ \left\{z \frac{\partial}{\partial z} F(z, x)\right\}\right] > 0$$

for all $x \in I$, $z \in \gamma_x$, then two arcs C_{x_1}, C_{x_2} where $x_1 \in I$, $x_2 \in I$ do not intersect each other if $|x_1 - x_2|$ is sufficiently small.

The above lemma is similar to a lemma of Bielecki and Lewandowski [**2**]. In this connection also see [**10**, p. 159].

Proof. For a given x_0 in $(a, b]$ we investigate the position of the curve C_x relative to C_{x_0} as x decreases away from x_0. Let $F(z_0, x_0)$ be an arbitrary point on the arc C_{x_0}. For all z in the disk $\{z: |z - z_0| < \alpha/4\}$ and for all $x \in I$, we have [**5**, pp. 72-73]

$$F(z, x) = F(z_0, x) + (z - z_0)\frac{\partial F}{\partial z}(z_0, x) + (z - z_0)^2 \frac{1}{2\pi i}\int_{|w - z_0| = \alpha/2} \frac{F(w, x)}{(w - z_0)^2(w - z)}\, dw.$$

Thus, in particular, if $x \in N(x_0; \delta(z_0))$ then there exist t_x, τ_x in $N(x_0; \delta(z_0))$ such that

$$F(z, x) = F(z_0, x_0) + (x - x_0) \frac{\partial F}{\partial x}(z_0, x_0) + \frac{1}{2!}(x - x_0)^2 \frac{\partial^2 F}{\partial x^2}(z_0, t_x)$$

$$+ (z - z_0) \frac{\partial F}{\partial z}(z_0, x_0) + (z - z_0)(x - x_0) \frac{\partial^2 F}{\partial x \partial z}(z_0, \tau_x)$$

$$+ (z - z_0)^2 \frac{1}{2\pi i} \int_{|w - z_0| = \alpha/2} \frac{F(w, x)}{(w - z_0)^2 (w - z)} dw.$$

Consequently

$$F(z, x) = F(z_0, x_0) + (x - x_0) \frac{\partial F}{\partial x}(z_0, x_0) + (z - z_0) \frac{\partial F}{\partial z}(z_0, x_0)$$

$$+ O(|x - x_0|^2) + O(|z - z_0|^2).$$

From this it readily follows that if (10) holds then, provided $x < x_0$ and $|x - x_0|$, $|z - z_0|$ (where $z_0 \in \gamma_{x_0}$, $z \in \gamma_x$) are sufficiently small, we have

$$\operatorname{Re} \frac{F(z, x) - F(z_0, x_0)}{z_0 \dfrac{\partial F}{\partial z}(z_0, x_0)} < 0.$$

This means that the angle between the vector $F(z, x) - F(z_0, x_0)$ and the vector $\mathbf{N}$ normal (where the normal vector is supposed to be $90°$ behind the tangent vector) to C_{x_0} at the point $F(z_0, x_0)$ is greater than $\pi/2$. Therefore, the arcs C_x being all simple, C_x cannot intersect C_{x_0} if $x < x_0$ and $|x - x_0|$ is sufficiently small.

Rider. In our applications the arcs C_x, except for the end points, remain confined to the interior of a fixed angle $\alpha_1 < \psi < \alpha_2$ of opening $< 2\pi$. Each arc has its initial point on $\psi = \alpha_2$ and its terminal point on $\psi = \alpha_1$. In such a situation if the conditions of Lemma 1 are satisfied, then the arcs C_x being locally disjoint, the sectors S_x determined by the angle and the arcs C_x form a monotonic family of domains. As a consequence, we obtain

$$\bigcap_{a < x \leqq b} \bar{S}_x = \bar{S}_b.$$

Obviously, the conclusion remains valid if (10) fails to hold for $x = b$ and $z = e^{i\varphi_1(b)}, e^{i\varphi_2(b)}$ as long as

$$|F(e^{i\varphi_1(b)}, b)| < |F(e^{i\varphi_1(x)}, x)|, \ |F(e^{i\varphi_2(b)}, b)| < |F(e^{i\varphi_2(x)}, x)|$$

$$\text{for } a < x < b.$$

Since trinomials of the form $z - a_p z^p + t z^q$, $(q = 2p - 1)$ have already been studied by Ruscheweyh and Wirths [**12**] and by Rahman and Szynal [**11**] we will only consider the case $q \neq 2p - 1$.

From Theorem 1 it follows that for $t = 1/q$ the trinomial $z - a_p z^p + tz^q$, $(q \neq 2p - 1)$ is univalent in $|z| < 1$ if and only if $a_p = 0$. So, in our study of such trinomials we will restrict ourselves to values of $t \in (0, 1/q)$.

Since the region G_θ is determined by a curve of the form

$$(11) \quad w(\varphi) = w(b, \varphi) = e^{-i(p-1)\varphi} + be^{i(q-p)\varphi}, \; 0 \leqq \varphi \leqq 2\pi$$

where $-b_0 \leqq b < 1$ with $0 < b_0 < 1$, let us state some of the important properties of the curve Γ_b defined by (11).

P_1: Since $w(\varphi + 2\pi/(q-1)) \equiv w(\varphi) \exp\,(i2(q-p)\pi/(q-1))$, a point w lies on Γ_b if and only if $w \exp\,(i2(q-p)\pi/(q-1))$ does.

P_2: Since $\overline{w(\varphi)} \equiv w(2\pi - \varphi)$, a point w lies on Γ_b if and only if its conjugate does.

P_3: If $b > 0$, then

$$\max_{\varphi} \; |w(b, \varphi)| = \left| w\left(b, \frac{2s}{q-1}\,\pi\right)\right| = 1 + b, \quad s = 0, 1, 2, \ldots, q-2,$$

$$\min_{\varphi} \; |w(b, \varphi)| = \left| w\left(b, \frac{2s-1}{q-1}\,\pi\right)\right| = 1 - b, \quad s = 1, 2, 3, \ldots, q-1.$$

2. Description of the intersection $D(t, p, q)$. In order that our method may not get obscured due to complicated expressions entering into our calculations we first consider trinomials of degree 4.

2.1. *Polynomials of the form* $z - a_2 z^2 + tz^4$, $0 < t < 1/4$. In this particular case the curve Γ_b defined by (11) has the following additional properties.

P_4: Since $\overline{e^{-i\pi/3}\,w(\varphi)} = e^{-i\pi/3}\,w((4\pi/3) - \varphi)$ the curve Γ_b is symmetrical about the line $\arg w = \pi/3$.

P_5: Since $w(-b, \varphi) = -\,w(b, \varphi + \pi)$ the curve Γ_{-b} is the reflection of Γ_b in the imaginary axis.

P_6: If $|b| < 1/2$ then $\varphi_1 \neq \varphi_2$ implies $w(b, \varphi_1) \neq w(b, \varphi_2)$.

P_7: For $|b| < 1/4$, the tangent to the curve Γ_b turns monotonically in the clockwise direction as φ increases from 0 to 2π, and hence the bounded region Δ_b determined by Γ_b is convex (the region Δ_b is convex also for $b = 1/4$). The region is not convex if $|b| > 1/4$ but it is star-shaped with respect to the origin provided $|b| \leqq 1/2$. If $b > 1/2$ then the tangent to the curve Γ_b turns monotonically in the counterclockwise direction as φ increases from 0 to 2π. The region Δ_b determined by Γ_b and containing the origin is star-shaped with respect to the origin. The boundary of Δ_b consists of three congruent regular arcs. It has three corners which lie at the points $b^{-1} - b$, $(b^{-1} - b)e^{2\pi i/3}$ and $(b^{-1} - b)e^{4\pi i/3}$. Depending on the range in which b lies the curve Γ_b looks roughly as sketched in Figures 1, 2, and 3.

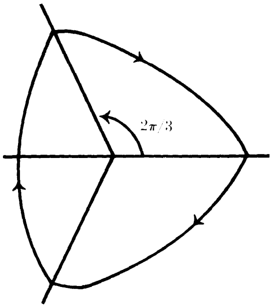

FIG. 1 ($0 < b < 1/4$)

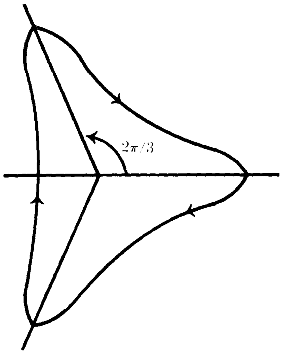

FIG. 2. ($1/4 < b < 1/2$)

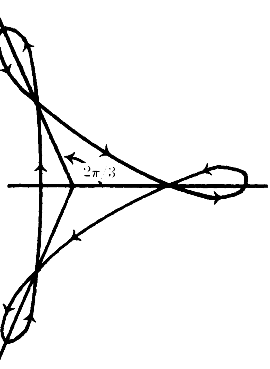

FIG. 3 ($b > 1/2$)

We are now in a position to study the region

$$D(t, 2, 4) = \bigcap_{0 \leqq \theta < \pi/2} \frac{1}{2 \cos \theta} \bar{G}_\theta.$$

Taking into account the properties of the curve Γ_b mentioned above we see that the boundary of the region $(1/2 \cos \theta)\, G_\theta$ is described by the moving point

$$W(\varphi) = \frac{1}{2 \cos \theta} \left(e^{-i\varphi} + t \frac{\sin 4\theta}{\sin \theta} e^{2i\varphi} \right),$$

as φ varies from 0 to 2π provided $t|\sin 4\theta/\sin \theta| \leqq 1/2$, whereas if $1/2 < t \sin 4\theta/\sin \theta < 1$ then φ is to vary over the union of the intervals

$$I_k = [(k - 1)2\pi/3 + \varphi_0, k2\pi/3 - \varphi_0], \quad k = 1, 2, 3$$

where φ_0 is the unique root of the equation

(12) $\quad \operatorname{Im} W(\varphi) = 0$

in $(0, \pi/3)$. Note that

$$\min_{0 \leqq \theta < \pi/2} t \sin 4\theta/\sin \theta \geqq -\sqrt{6}/9 > -1/2$$

which excludes the possibility

$$-1 < t \sin 4\theta/\sin \theta < -1/2.$$

Now let us determine the intersection of $D(t, 2, 4)$ with the real axis. It is an interval $[x^-(t), x^+(t)]$ where $x^-(t) < 0 < x^+(t)$. If $\{x \colon x_\theta^- < x < x_\theta^+\}$ is the intersection of $(1/2 \cos \theta)\, G_\theta$ with the real axis then for all $\theta \in [0, \pi/2)$

$$x_\theta^- = -\frac{1}{2 \cos \theta} \left(1 - t \frac{\sin 4\theta}{\sin \theta} \right).$$

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Since $\max_{0\leqq\theta<\pi/2} x_\theta^- = x_0^-$ we obtain

$$x^-(t) = -(1-4t)/2, \quad 0 < t < 1/4.$$

On the other hand

$$x_\theta^+ = \begin{cases} \dfrac{1}{2\cos\theta}\left(1 + t\dfrac{\sin 4\theta}{\sin\theta}\right) & \text{if } t\left|\dfrac{\sin 4\theta}{\sin\theta}\right| \leqq \frac{1}{2} \\ \dfrac{1}{2\cos\theta}\left(\dfrac{\sin\theta}{t\sin 4\theta} - \dfrac{t\sin 4\theta}{\sin\theta}\right) & \text{if } \frac{1}{2} < t\dfrac{\sin 4\theta}{\sin\theta} < 1. \end{cases}$$

The inequality $t|\sin 4\theta/\sin\theta| \leqq 1/2$ necessarily holds if $t \in (0, 1/8]$ and a simple calculation gives us

$$(13) \qquad \min_{0\leqq\theta<\pi/2} x_\theta^+ = \begin{cases} \frac{1}{2}(1+4t) & \text{if } 0 < t \leqq 1/16 \\ \frac{1}{2}\{3(2t)^{1/3} - 4t\} & \text{if } 1/16 \leqq t \leqq 1/8. \end{cases}$$

If $t \in (1/8, 1/4)$ and we denote by $\theta = \alpha_t$ the only root of the equation

$$t\sin 4\theta/\sin\theta = 1/2$$

in $(0, \pi/2)$, then it is easily seen that

$$x_\theta^+ \geqq \begin{cases} \frac{1}{2}\{3(2t)^{1/3} - 4t\} & \text{if } \theta \in [\alpha_t, \pi/2) \\ \dfrac{1}{2}\left(\dfrac{1}{4t} - 4t\right) & \text{if } \theta \in [0, \alpha_t). \end{cases}$$

Consequently

$$(14) \qquad \min_{0\leqq\theta<\pi/2} x_\theta^+ = \begin{cases} \frac{1}{2}\{3(2t)^{1/3} - 4t\} & \text{if } \frac{1}{8} < t \leqq \frac{1}{2}\left(\frac{1}{6}\right)^{3/4} \\ \dfrac{1}{2}\left(\dfrac{1}{4t} - 4t\right) & \text{if } \frac{1}{2}\left(\frac{1}{6}\right)^{3/4} \leqq t < \frac{1}{4}. \end{cases}$$

The following theorem sums up the above conclusions.

THEOREM 2. *The trinomial*

$$(15) \qquad f_t(z) = z - a_2 z^2 + tz^4, \quad a_2 \in \mathbf{R}, \quad 0 < t \leqq 1/4$$

is univalent in $|z| < 1$ *if and only if*

$$-\tfrac{1}{2}(1-4t) \leqq a_2 \leqq \begin{cases} \frac{1}{2}(1+4t) & \text{if } 0 < t \leqq 1/16 \\ \frac{1}{2}\{3(2t)^{1/3} - 4t\} & \text{if } 1/16 \leqq t \leqq \frac{1}{2}(1/6)^{3/4} \\ \frac{1}{2}((4t)^{-1} - 4t) & \text{if } \frac{1}{2}(1/6)^{3/4} \leqq t \leqq 1/4. \end{cases}$$

COROLLARY 1. *If the trinomial* (15) *is univalent in* $|z| < 1$ *then*

$$|a_2| \leqq \tfrac{1}{4}6^{3/4} - 6^{-3/4}.$$

Now we will use Lemma 1 to obtain a simpler definition of the region $D(t, 2, 4)$.

Let $t \in (0, 1/4)$ be fixed. According as t belongs to $(0, 1/8]$ or to $(1/8, 1/4)$

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we choose ϵ arbitrarily in $(0, 1)$ and in $(0, (16t)^{-1/3})$ respectively. We then apply Lemma 1 (also see the rider) to the function

$$F(z, x) := (2x)^{-1}\{z^{-1} + 4tx(2x^2 - 1)z^2\}$$

where x varies over $(\epsilon, 1]$ or $(\epsilon, (16t)^{-1/3}]$ according as $t \in (0, 1/8]$ or $t \in (1/8, 1/4)$ respectively, and the arc $\{z = e^{i\varphi}: 0 \leqq \varphi \leqq 2\pi/3\}$ is taken for γ_x. Computing $(\partial F/\partial x)$, $(\partial F/\partial z)$ we see that (10) is equivalent to

$$(16) \quad 1 + 128t^2x^4(2x^2 - 1) - 8tx(4x^2 - 1)\cos 3\varphi > 0, \quad 0 \leqq \varphi \leqq 2\pi/3$$

which is easily seen to hold for $0 < x \leqq 1$ if $0 < t < 1/16$. It also holds if $t = 1/16$ and $0 < x < 1$. If $t = 1/16$ and $x = 1$ then it holds only in the open interval $0 < \varphi < 2\pi/3$ but $|F(1, 1)| < |F(1, x)|$, $|F(e^{2\pi i/3}, 1)| < |F(e^{2\pi i/3}, x)|$ for $\epsilon < x < 1$. If $1/16 < t < 1/4$ then (16) holds for $0 < x < (16t)^{-1/3}$ whereas for $x = (16t)^{-1/3}$ it holds only if $0 < \varphi < 2\pi/3$. But, again $|F(1, 1)| < |F(1, x)|$, $|F(e^{2\pi i/3}, 1)| < |F(e^{2\pi i/3}, x)|$ for $\epsilon < x < (16t)^{-1/3}$. Hence

$$\bigcap_{\epsilon<\cos\theta\leqq 1} \frac{1}{2\cos\theta}\bar{G}_\theta = \begin{cases} \frac{1}{2}\bar{G}_0 & \text{if } 0 < t \leqq 1/16 \\ \displaystyle\bigcap_{(16t)^{-1/3}\leqq\cos\theta\leqq 1} \frac{1}{2\cos\theta}\bar{G}_\theta & \text{if } 1/16 < t < 1/4. \end{cases}$$

This, in fact, implies that

$$D(t, 2, 4) = \begin{cases} \frac{1}{2}\bar{G}_0 & \text{if } 0 < t \leqq 1/16 \\ \displaystyle\bigcap_{0\leqq\theta\leqq\theta_0} \frac{1}{2\cos\theta}\bar{G}_0 & \text{if } 1/16 < t < 1/4, \end{cases}$$

where θ_0 is the unique root of the equation

$$(17) \quad \cos\theta = (16t)^{-1/3}$$

in $(0, \pi/2)$. We can do better when $t > 1/8$. In that case, if x satisfies

$$X := 8tx(2x^2 - 1) = 2t\sin 4\theta/\sin\theta > 1$$

then for γ_x we have to take the arc

$$\{z = e^{i\varphi}: \varphi_0 \leqq \varphi \leqq 2\pi/3 - \varphi_0\}$$

where φ_0 is the only root of (12) in $(0, \pi/3)$. Condition (10) is equivalent to

$$(16') \quad 1 + 128t^2x^4(2x^2 - 1) - 8tx(4x^2 - 1)\cos 3\varphi > 0, \varphi_0 \leqq \varphi \leqq 2\pi/3 - \varphi_0.$$

It holds since the left-hand side is larger than $(16tx^3 - 1)(x - 1)$ and hence positive. Thus for $1/8 < t < 1/4$

$$D(t, 2, 4) = \left\{\bigcap_{\theta_1\leqq\theta\leqq\theta_0} \frac{1}{2\cos\theta}\bar{G}_\theta\right\} \cap \tfrac{1}{2}\bar{G}_0$$

where θ_1 is the unique root of the equation

$$(18) \quad 8t(\cos\theta)(2\cos^2\theta - 1) = 1$$

in $(0, \pi/2)$.

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Finally, we observe that if $(1/4)(2-\sqrt{2})^{1/2} \leqq t < 1/4$ then for $\theta_1 \leqq \theta \leqq \theta_0$,

$$\tfrac{1}{2}\bar{G}_0 \subseteq \frac{1}{2\cos\theta}\bar{G}_\theta,$$

so that for such values of t,

$$D(t, 2, 4) = \tfrac{1}{2}\bar{G}_0.$$

Since

$$\tfrac{1}{2}\bar{G}_0 \subseteq \{w: |w| \leqq \tfrac{1}{2}((4t)^{-1} - 4t)\} \text{ and}$$

$$\left\{w : |w| \leqq \frac{1}{2\cos\theta}(1 - 4t(\cos\theta)|2\cos^2\theta - 1|)\right\} \subseteq \frac{1}{2\cos\theta}\bar{G}_\theta$$

we simply check that

$$(19) \qquad (4t)^{-1} - 4t \leqq x^{-1}(1 - 4tx|2x^2 - 1|) \quad \text{if}$$

$$(20) \qquad 1 \leqq 16tx^3 \leqq 1 + 8tx.$$

Case (i). If $x \leqq 1/\sqrt{2}$ then $|2x^2 - 1| = 1 - 2x^2$ and (19) becomes

$$(4t)^{-1} \leqq x^{-1} + 8tx^2$$

which is easily seen to be true.

Case (ii). If $x > 1/\sqrt{2}$ then $|2x^2 - 1| = 2x^2 - 1$ and (19) becomes

$$(4t)^{-1}x - 8tx + 8tx^3 \leqq 1.$$

In view of the second inequality in (20), this certainly holds if

$$x((4t)^{-1} - 4t) \leqq \tfrac{1}{2}.$$

Hence if $x^*(t)$ denotes the unique root of the equation

$$(18') \qquad 16tx^3 - (1 + 8tx) = 0$$

in $(0, 1)$ then it is enough to verify that

$$x^*(t) \leqq 2t/(1 - 16t^2).$$

Replacing x by $2t/(1-16t^2)$ in (18′) we see that the left-hand side is non-negative if $(1/4)(2-\sqrt{2})^{1/2} \leqq t < 1/4$ and hence (19) holds. With this we have proved the following.

THEOREM 3. *The trinomial* $z - a_2z^2 + tz^4$, $0 < t \leqq 1/4$ *is univalent in* $|z| < 1$ *if and only if*

(i) $a_2 \in \tfrac{1}{2}\bar{G}_0$ *in case* $t \in (0, 1/16] \cup [(1/4)(2-\sqrt{2})^{1/2}, 1/4]$,

(ii) $a_2 \in \{\bigcap_{\theta_1 \leqq \theta \leqq \theta_0}(2\cos(\theta))^{-1}\bar{G}_\theta\} \cap \tfrac{1}{2}\bar{G}_0$ *in case* $t \in (1/8, (1/4)(2-\sqrt{2})^{1/2})$,

(iii) $a_2 \in \bigcap_{0 \leqq \theta \leqq \theta_0}(2\cos(\theta))^{-1}\bar{G}_\theta$ *in case* $t \in (1/16, 1/8)$,

where θ_0, θ_1 *are the unique roots in* $(0, \pi/2)$ *of* (17), (18) *respectively.*

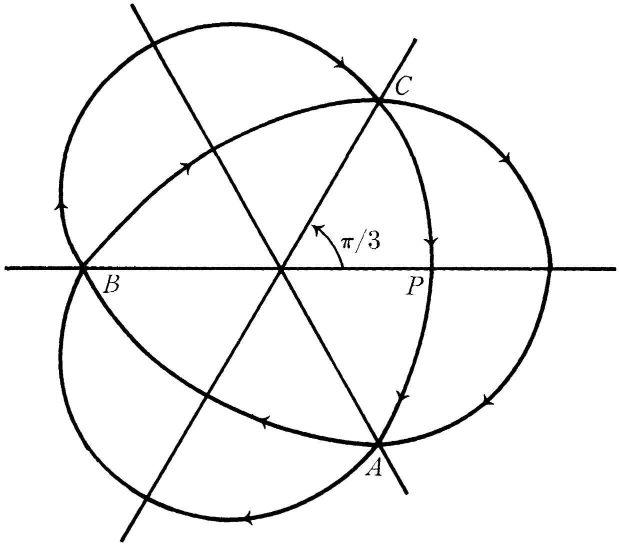

FIGURE 4

2.2. *Polynomials of the form* $z - a_3z^3 + tz^4$, $0 < t < 1/4$. Again, the curve Γ_b defined by (11) has, in addition to the properties P_1, P_2, P_3, the property P_4 since

$$\overline{e^{-i\pi/3}w(\varphi)} = e^{-i\pi/3}w(2\pi/3 - \varphi).$$

However, this time the trace of Γ_{-b} is the same as that of Γ_b. In fact,

$$w(-b, \varphi) = w(b, \varphi + \pi).$$

As φ increases from 0 to 2π the tangent to the curve Γ_b described by the moving point

$$w(\varphi) = e^{-2i\varphi} + be^{i\varphi}$$

turns monotonically in the clockwise direction. The curve cuts itself in the points $(1 - b^2)e^{-i\pi/3}$, $-1 + b^2$, $(1 - b^2)e^{i\pi/3}$ denoted by A, B, C respectively, in Fig. 4. The region Δ_b determined by Γ_b and containing the origin is convex and its boundary consists of three congruent regular arcs, namely AB, CA, BC. The arcs AB, CA, BC are symmetric about $\arg w = -2\pi/3$, $\arg w = 0$, $\arg w = 2\pi/3$ respectively. Now we have to make a distinction according as b is positive or negative. If $0 < b < 1$ and φ_0 denotes the root of the equation

(21) $\quad \cos \varphi = b/2$

in $(\pi/3, \pi/2)$ then the arc AB is described as φ increases from $2\pi/3 - \varphi_0$ to φ_0 whereas the arcs CA and BC are described as φ increases from $4\pi/3 - \varphi_0$ to

$2\pi/3 + \varphi_0$ and from $2\pi - \varphi_0$ to $4\pi/3 + \varphi_0$, respectively. If $b < 0$ the initial point of the curve Γ_b is P and the arc PA is described as φ increases from 0 to $2\pi/3 - \varphi_1$ where φ_1 is the root of the equation (21) in $(\pi/2, 2\pi/3)$.

Now we are in a position to determine the intersection

$$D(t, 3, 4) = \bigcap_{\substack{0 \leqq \theta \leqq \pi/2 \\ \theta \neq \pi/3}} \frac{\sin \theta}{\sin 3\theta} \bar{G}_\theta.$$

First, we use Lemma 1 to show that

$$\bigcap_{0 \leqq \theta \leqq \pi/4} \frac{\sin \theta}{\sin 3\theta} \bar{G}_\theta = \tfrac{1}{3}\bar{G}_0.$$

Note that G_θ is determined by the curve

$$w(\varphi) = e^{-2i\varphi} + 4tx(2x^2 - 1)e^{i\varphi}, \quad (x = \cos \theta)$$

where $0 \leqq 4tx(2x^2 - 1) < 1$. Besides, $(\sin \theta/\sin 3\theta) = (4x^2 - 1)^{-1} > 0$. Hence, setting

$$F(z, x) = (4x^2 - 1)^{-1}\{z^{-2} + 4tx(2x^2 - 1)z\}$$

it is enough (in view of (21) and the various symmetries of the region $(\sin \theta/\sin 3\theta)G_\theta$) to show that (10) holds for $1/\sqrt{2} \leqq x \leqq 1$ and

$$z = e^{i\varphi}, \quad 2\pi/3 - \varphi_0 \leqq \varphi \leqq \varphi_0$$

where φ_0 is the root of

$$\cos \varphi = 2tx(2x^2 - 1)$$

in $(\pi/3, \pi/2)$. Thus we have to verify that

$$2x\{1 + t^2(2x^2 - 1)(8x^4 - 2x^2 + 1)\} - t(16x^4 - 6x^2 + 1) \cos 3\varphi > 0$$

for $0 < t < 1/4$, $1/\sqrt{2} \leqq x \leqq 1$ and $2\pi/3 - \varphi_0 \leqq \varphi \leqq \varphi_0$. But this is clearly true since $\cos 3\varphi < 0$ for $\varphi \in [2\pi/3 - \varphi_0, \varphi_0]$ and $2x\{1 + t^2(2x^2 - 1)(8x^4 - 2x^2 + 1)\}$, $t(16x^4 - 6x^2 + 1)$ are both positive for $0 < t < 1/4$, $1/\sqrt{2} \leqq x \leqq 1$. Next, we note that $\frac{1}{3}\bar{G}_0$ is contained in $(\sin \theta/\sin 3\theta)\, \bar{G}_\theta$ for all $\theta \in (\pi/4, \pi/2]$. In fact, for $\theta \in (\pi/4, \pi/2]$ we have

$$\tfrac{1}{3}\bar{G}_0 \subset \{w : |w| \leqq \tfrac{1}{3}(1 - 16t^2)\} \subset \frac{\sin \theta}{\sin 3\theta} \bar{G}_\theta.$$

Hence

$$\bigcap_{0 \leqq \theta \leqq \pi/2} \frac{\sin \theta}{\sin 3\theta} \bar{G}_\theta = \tfrac{1}{3}\bar{G}_0.$$

Thus we have proved the following.

THEOREM 4. *The trinomial* $z - a_3z^3 + tz^4$, $0 < t \leqq 1/4$ *is univalent in* $|z| < 1$ *if and only if*

$$a_3 \in \tfrac{1}{3}\bar{G}_0$$

where G_0 is the region determined by the curve

$$w(\varphi) = e^{-2i\varphi} + 4te^{i\varphi}$$

and containing the origin.

COROLLARY 2. *If the trinomial $z - a_3z^3 + tz^4$, $0 < t \leqq 1/4$ is univalent in $|z| < 1$ then*

$$|a_3| \leqq \tfrac{1}{3}(1 - 16t^2), \quad -\tfrac{1}{3}(1 - 16t^2) \leqq \operatorname{Re} a_3 \leqq \tfrac{1}{3}(1 - 4t).$$

Now we wish to discuss univalent trinomials of degree 5. Since trinomials of the form $z + a_3z^3 + tz^5$ have already been studied in [**12**] and [**11**] we will consider those of the forms $z + a_2z^2 + tz^5$ and $z + a_4z^4 + tz^5$.

2.3. *Polynomials of the form $z - a_2z^2 + tz^5$, $0 < t < 1/5$.* In this case the curve Γ_b defined by (11) is symmetrical about the lines $\arg w = 0$, $\arg w = \pi/4$, $\arg w = \pi/2$. The situation is in some sense analogous to that in 2.1. It turns out that

$$D(t, 2, 5) = \bigcap_{0 \leqq \theta < \pi/2} (2 \cos \theta)^{-1} \bar{G}_\theta$$

where the boundary of the region G_θ is described by the moving point

$$W(\varphi) = e^{-i\varphi} + t\,(\sin 5\theta/\sin \theta)e^{3i\varphi}$$

as φ varies from 0 to 2π in case $t|\sin 5\theta/\sin \theta| \leqq 1/3$ and over the union of the intervals

$$I_k: = [(k - 1)\pi/2 + \varphi_0, k\pi/2 - \varphi_0], \quad k = 1, 2, 3, 4$$

if $1/3 < t\,(\sin 5\theta/\sin \theta) < 1$, φ_0 being the unique root of the equation

$$\operatorname{Im} W(\varphi) = 0$$

in $[0, \pi/4]$. Note that $\min_{0 \leqq \theta < \pi/2} t\,(\sin 5\theta/\sin \theta) \geqq -1/4 > -1/3$ which excludes the possibility $-1 < t\,(\sin 5\theta/\sin \theta) < -1/3$.

Let us determine the intersection of $D(t, 2, 5)$ with the real axis. It is an interval $[x^-(t), x^+(t)]$ where $x^-(t) = -x^+(t)$. If (x_θ^-, x_θ^+) denotes the intersection of $(2 \cos \theta)^{-1}G_\theta$ with the real axis then

$$-x_\theta^- = x_\theta^+ = \begin{cases} \dfrac{1}{2 \cos \theta}\left(1 + t\dfrac{\sin 5\theta}{\sin \theta}\right) & \text{if } t\left|\dfrac{\sin 5\theta}{\sin \theta}\right| \leqq \tfrac{1}{3} \\[2ex] \dfrac{\left(1 + \dfrac{1}{t}\dfrac{\sin \theta}{\sin 5\theta}\right)^{1/2}}{2 \cos \theta}\left(1 - t\dfrac{\sin 5\theta}{\sin \theta}\right) & \text{if } \tfrac{1}{3} < t\dfrac{\sin 5\theta}{\sin \theta} < 1. \end{cases}$$

Setting $\cos \theta = x$ and

$$g(x) = (2x)^{-1}(1 + t(16x^4 - 12x^2 + 1)]$$

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we see that in case $t|\sin 5\theta/\sin\theta| \leqq 1/3$,

$$\min x_\theta{}^+ = \begin{cases} g(1) & \text{if } 0 < t \leqq \dfrac{1}{35} \\ g(x^*) & \text{if } \dfrac{1}{35} < t < \dfrac{1}{5} \end{cases}$$

where

$$x^* = \left\{\frac{1}{8} + \frac{1}{24}\left(21 + \frac{12}{t}\right)^{1/2}\right\}^{1/2}.$$

On the other hand, if $1/3 < t\,(\sin 5\theta/\sin\theta) < 1$, then

$$\min x_\theta{}^+ = x_0{}^+ = \tfrac{1}{2}(1 - 5t)(1 + (5t)^{-1})^{1/2}.$$

Thus

$$x^+(t) = \begin{cases} g(1) & \text{if } 0 \leqq t \leqq 1/35 \\ \min\{g(x^*), x_0{}^+\} & \text{if } 1/35 < t < 1/5, \end{cases}$$

and we have the following.

THEOREM 5. *A necessary and sufficient condition for the trinomial*

$$f_t(z) = z + a_2 z^2 + tz^5, \quad a_2 \in \mathbf{R}, \quad 0 < t \leqq 1/5$$

to be univalent $|z| < 1$ *is that*

$$|a_2| \leqq \begin{cases} \tfrac{1}{2}(1 + 5t) & \textit{if } 0 < t \leqq 1/35 \\ \tfrac{1}{3}\dfrac{4 + t\left(1 - \sqrt{21 + \dfrac{12}{t}}\right)}{\left(\tfrac{1}{2} + \tfrac{1}{6}\sqrt{21 + \dfrac{12}{t}}\right)^{1/2}} & \textit{if } 1/35 < t \leqq t_0 \\ \tfrac{1}{2}(1 - 5t)\left(1 + \dfrac{1}{5t}\right)^{1/2} & \textit{if } t_0 \leqq t \leqq 1/5 \end{cases}$$

where $t_0 = 0.0726\ldots$ *is the unique root of the equation*

$$\tfrac{1}{3}\frac{4 + t\left(1 - \sqrt{21 + \dfrac{12}{t}}\right)}{\left(\tfrac{1}{2} + \tfrac{1}{6}\sqrt{21 + \dfrac{12}{t}}\right)^{1/2}} = \tfrac{1}{2}(1 - 5t)\left(1 + \frac{1}{5t}\right)^{1/2}$$

in $(1/15, 1/5)$.

Next, we prove

THEOREM 6. *For* $0 < t \leqq 1/35$ *we have*

$$D(t, 2, 5) = \tfrac{1}{2}\bar{G}_0.$$

Proof. We apply Lemma 1 (along with the rider) to the function

$$F(z, x) = (2x)^{-1}\{z^{-1} + t(16x^4 - 12x^2 + 1)z^3\},$$

where x varies over $((\sqrt{5} + 1/4), 1]$ and the arc $\{z = e^{i\varphi}: 0 \leqq \varphi \leqq \pi/2\}$ is taken for γ_x. It turns out that (10) is equivalent to

$$(22)\qquad 1 + 3t^2(48x^4 - 12x^2 - 1)(16x^4 - 12x^2 + 1) \\ - 2t(48x^4 - 24x^2 + 1)\cos 4\varphi > 0, \quad 0 \leqq \varphi \leqq \pi/2$$

which is easily seen to hold for $(\sqrt{5} + 1)/4 < x \leqq 1$ if $0 < t < 1/35$. It also holds if $t = 1/35$ and $(\sqrt{5} + 1)/4 < x < 1$. But for $t = 1/35$ and $x = 1$ it holds only in the open interval $0 < \varphi < \pi/2$. However, $|F(1, 1)| < |F(1, x)|$, $|F(e^{i\pi/2}, 1)| < |F(e^{i\pi/2}, x)|$ for $(\sqrt{5} + 1)/4 < x < 1$. Therefore, for $0 < t \leqq 1/35$ we have

$$D(t, 2, 5) = \{\textstyle\bigcap_{\pi/5 \leqq \theta < \pi/2} (2\cos\theta)^{-1}\bar{G}_\theta\} \cap \frac{1}{2}\bar{G}_0.$$

Further, it can be easily verified that if $\pi/5 \leqq \theta < \pi/2$, $0 < t \leqq 1/35$, then

$$\tfrac{1}{2}\bar{G}_0 \subset \{w: |w| \leqq \tfrac{1}{2}(1 + 5t)\} \subseteq \{w: |w| \leqq (2x)^{-1}(1 - t|16x^4 \\ - 12x^2 + 1|)\} \subseteq (2\cos\theta)^{-1}\bar{G}_\theta,$$

so that (22) holds.

For $t > 1/35$ the definition of the region $D(t, 2, 5)$ can be simplified as in 2.1 and an interested reader may carry out the calculations himself.

2.4. *Polynomials of the form* $z - a_4z^4 + tz^5$, $0 < t < 1/5$. The curve Γ_b defined by (11) is symmetrical about $\arg w = 0$, $\arg w = \pi/4$ and $\arg w = \pi/2$. As φ increases from 0 to 2π the tangent to the curve turns monotonically in the clockwise direction. The curve Γ_b cuts itself in eight different points and for positive b it looks roughly as sketched in Fig. 5. The curve Γ_{-b} is obtained on rotating the curve Γ_b by 45°. The same is true of the region Δ_{-b} determined by Γ_{-b} and containing the origin. For positive b the region Δ_b is bounded by four congruent arcs and has four corners which lie on the real and imaginary axes. One of the four bounding arcs is described as φ increases in $(0, \pi/2)$ from $\arccos \frac{1}{2}\sqrt{3 - b}$ to $\arccos \frac{1}{2}\sqrt{1 + b}$. Further

$$\min_{w \in \partial\Delta_b}|w| = 1 - b, \quad \max_{w \in \Delta_b}|w| = (1 - b)(1 + b)^{1/2}.$$

Now we are in a position to show that in the present case the right-hand side of (9) is $\frac{1}{4}G_0$. It is clearly enough to show that

$$(23)\qquad \textstyle\bigcap_{0 \leqq \theta < \pi/2} \bar{G}_\theta = \bar{G}_0.$$

First we prove that

$$\textstyle\bigcap_{0 \leqq \theta \leqq \pi/5} \bar{G}_\theta = \bar{G}_0.$$

For this we apply Lemma 1 to the function

$$F(z, x) = z^{-3} + t(16x^4 - 12x^2 + 1)z$$

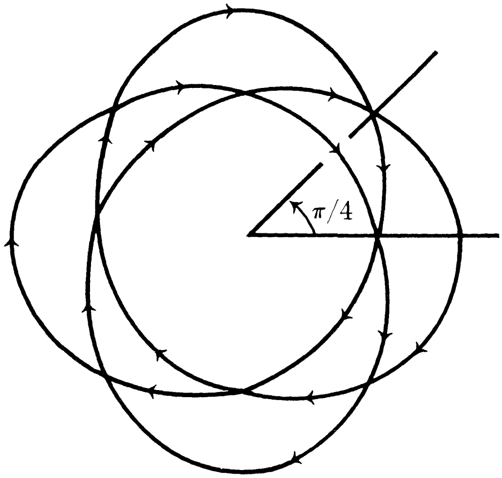

FIGURE 5

where $(\sqrt{5}+1)/4 < x \leqq 1$ and the arc

$$\{z = e^{i\varphi}\colon \pi/6 < \text{arc cos } \tfrac{1}{2}\sqrt{3 - t(16x^4 - 12x^2 + 1)} \leqq \varphi \leqq \text{arc cos } \tfrac{1}{2}\sqrt{1 + t(16x^4 - 12x^2 + 1)} < \pi/3\}$$

is taken for γ_x. Condition (10) turns out to be equivalent to

$$(8x^2 - 3)\{t(16x^4 - 12x^2 + 1) - 3 \cos 4\varphi\} > 0$$

which is easily verified.

Next we observe that

$$\bar{G}_0 \subseteq \{w\colon |w| \leqq (1 - 5t)\sqrt{1 + 5t}\} \subseteq \{w\colon |w| \leqq 1 - t|16x^4 - 12x^2 + 1|\} \subseteq \bar{G}_\theta$$

if $0 < \cos \theta = x \leqq (\sqrt{5}+1)/4$, which completes the proof of (23).

Thus we have the following.

THEOREM 7. *The trinomial $z - a_4z^4 + tz^5$, $0 < t \leqq 1/5$ is univalent in $|z| < 1$ if and only if*

$$a_4 \in \tfrac{1}{4}\bar{G}_0$$

where G_0 is the region determined by the curve

$$w(\varphi) = e^{-3i\varphi} + 5te^{i\varphi},\ \varphi \in [0, 2\pi]$$

and containing the origin.

COROLLARY 3. *If the trinomial* $z + a_4z^4 + tz^5$, $0 < t \leqq 1/5$ *is univalent in* $|z| < 1$ *then*

$$|a_4| \leqq \tfrac{1}{4}(1 - 5t)\sqrt{1 + 5t}.$$

3. Remarks and applications.

3.1. So far we have restricted ourselves to the trinomials of small degree. In general it is very difficult to say anything more than what is given in Theorem 1. Just as the special case $q = 2p - 1$ was handled by Ruscheweyh and Wirths [**12**] and by Rahman and Szynal [**11**] we can extend our earlier reasoning to the cases $q = 3p - 2$; $q = 4p - 3$. In fact, if $q - p = l(p - 1)$, where $l = 2, 3, 4$, etc., then the region $\bar{G}_\theta$ appearing in (6) is determined by the curve

$$w(\psi) = e^{-i\psi} + t\frac{\sin q\theta}{\sin \theta}e^{il\psi}, \quad \psi \in [0, 2\pi]$$

and for $l = 2$ and 3 the determination of the coefficient region for a_p can be carried out in much the same way as in 2.1 and 2.3.

3.2. In situations where the coefficient region $D(t, p, q)$ has been satisfactorily determined we can find the Koebe constant of the family of univalent polynomials (5) by calculating the distance between the boundary of $D(t, p, q)$ and the curve

$$w(\varphi) = e^{-(p-1)i\varphi} + te^{(q-p)i\varphi}, \quad \varphi \in [0, 2\pi].$$

3.3. Our reasoning can be used to determine the coefficient region for meromorphic univalent trinomials of the form

$$z^{-1} + a_pz^p + a_qz^q.$$

Instead of Theorem A of the Introduction we will have to use Lemma B of [**4**] where the corresponding criterion for $\mu_n(z) = z^{-1} + a_1z + \ldots + a_nz^n$ to be univalent in $0 < |z| < 1$ is given.

3.4. It is an open question if for every function $f(z) = z + a_2z^2 + \ldots$ univalent in $|z| < 1$ the integral

$$(24) \qquad \frac{2}{z}\int_0^z f(\zeta)d\zeta$$

is also univalent in $|z| < 1$. In [**8**] the answer was shown to be affirmative for polynomials of degree at most 5. Our study of the coefficient region of (a_p, a_q) for univalent trinomials of the form

$$f(z) = z + a_pz^p + a_qz^q, \quad p < q$$

helps us to answer the above question for such functions.

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THEOREM 8. *If $f(z)$ is a normalized univalent trinomial in the unit disk then so is* (24).

Proof. In view of Theorem 3 in [**8**] we may assume $q \geqq 6$. Clearly we may suppose $0 < t = a_q \leqq 1/q$. Then we have to prove

$$\frac{2}{p+1} D(t, p, q) \subseteq D\left(\frac{2t}{q+1}, p, q\right).$$

It is certainly enough to prove that

$$(25) \quad \frac{2}{p+1} \max_{w \in \partial D(t,p,q)} |w| \leqq \min_{w \in \partial D(2t/(q+1),p,q)} |w| = \frac{1}{p}\left(1 - \frac{2tq}{q+1}\right).$$

Using the crude upper estimate $2/p$ for

$$(26) \quad \max_{w \in \partial D(t,p,q)} |w|$$

we see that (25) holds for $p \geqq 5$ and $p = 4$, $q \geqq 9$.

In case $p = 3$, $q \geqq 7$ we may use

$$\sin (\pi/q)/\sin (3\pi/q) \quad (< \tfrac{1}{2})$$

as an upper bound for (26) to get through.

If $p = 2$, $q \geqq 6$ we consider two cases:

Case (i). $0 < t \leqq 1/q(q - 3)$. The inequality (25) holds if we again use $2/p$ as an upper bound for (26).

Case (ii). $1/q(q - 3) < t \leqq 1/q$. In this case the curve

$$W(\varphi) = e^{-i\varphi} + tqe^{(q-2)i\varphi}, \quad \varphi \in [0, 2\pi]$$

cuts itself since $1/(q - 2) < tq$, and as an upper estimate of (26) we may take

$$\tfrac{1}{2} [\cos \varphi_0 + tq \cos (q - 2)\varphi_0],$$

where $\varphi_0 \in (\pi/2(q - 2), \pi/(q - 2))$ is the unique root of the equation

$$\operatorname{Im} W(\varphi) = 0.$$

Since $\cos (q - 2)\varphi_0 < 0$ we have $\frac{1}{2} [\cos \varphi_0 + tq \cos (q - 2)\varphi_0] < \frac{1}{2}$ and (25) is easily seen to hold if we take $\frac{1}{2}$ as an upper bound for (26).

By a relatively careful study of the coefficient region $D(t, p, q)$ in cases ($p = 3, q = 6$), ($p = 4, q = 6$), ($p = 4, q = 8$) we see that as upper estimates for (26) we may take $\sqrt{2}/3$, $\sqrt{3}/4$, $\sqrt{37}/16$ respectively and (25) holds.

The last case $p = 4$, $q = 7$ can be settled by essentially the same reasoning as used in [**8**] for the proof of Theorem 3.

In fact if $f(z) = z + a_4z^4 + tz^7$ is univalent in $|z| < 1$ then

$$F(\zeta, \theta) = 1 + 2\frac{\sin 4\theta}{\sin \theta}\frac{a_4}{2}\zeta + \frac{\sin 7\theta}{\sin \theta} t\zeta^2 \neq 0 \quad \text{in } |\zeta| < 1$$
$$\text{for } 0 \leqq \theta \leqq \pi/2.$$

Now taking $g(\zeta) = 1 + 2 \cdot \frac{2}{5}\zeta + \frac{1}{4}\zeta^2$ which does not vanish in $|\zeta| < 1$ we see by a well known result (see [**9**], Corollary (16, 1a) on p. 66) that the polynomial

$$F(\zeta, \theta)*g(\zeta) = 1 + 2\frac{1}{5}\frac{\sin 4\theta}{\sin \theta} a_4\zeta + \frac{1}{4}\frac{\sin 7\theta}{\sin \theta} t\zeta^2$$

does not vanish in $|\zeta| < 1$ for $0 \leqq \theta \leqq \pi/2$. Hence also

$$1 + \frac{2}{5}\frac{\sin 4\theta}{\sin \theta} a_4 z^3 + \frac{t}{4}\frac{\sin 7\theta}{\sin \theta} z^6$$

does not vanish in $|z| < 1$ for $0 \leqq \theta \leqq \pi/2$. By the Dieudonné criterion the polynomial

$$z + \frac{2}{5} a_4 z^4 + \frac{t}{4} z^7 = \frac{2}{z}\int_0^z f(\zeta) d\zeta$$

is univalent in $|z| < 1$.

From

$$\frac{1}{p} D(t, p, q) \subset \frac{2}{p+1} D(t, p, q)$$

in conjunction with (25) and

$$\min_{w \in \partial D(2t/(q+1), p, q)} |w| < \min_{w \in \partial D(t/q, p, q)} |w|$$

it follows that

$$\frac{1}{p} D(t, p, q) \subset D\left(\frac{t}{q}, p, q\right)$$

and hence we also have the following.

THEOREM 9. *If the trinomial* $f(z) = z + a_p z^p + a_q z^q$, $p < q$ *is univalent in* $|z| < 1$ *then so is*

$$\int_0^z \frac{f(\zeta)}{\zeta} d\zeta.$$

REFERENCES

1. L. V. Ahlfors, *Complex analysis* (McGraw-Hill, New York, 1966).
2. A. Bielecki et Z. Lewandowski, *Sur certaines familles de fonctions α-étoilées*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A, *15* (1961), 45–55.
3. D. A. Brannan, *On univalent polynomials and related classes of functions*, Thesis, University of London, 1967.
4. ——— *Coefficient regions for univalent polynomials of small degree*, Mathematika *14* (1967), 165–169.
5. E. T. Copson, *An introduction to the theory of functions of a complex variable* (Oxford University Press, 1935).
6. V. F. Cowling and W. C. Royster, *Domains of variability for univalent polynomials*, Proc. Amer. Math. Soc. *19* (1968), 767–772.
7. J. Dieudonné, *Recherches sur quelques problèmes relatifs aux polynomes et aux fonctions bornées d'une variable complexe*, Ann. Ecole Norm. Sup. (3) *48* (1931), 247–358.

8. J. Krzyż and Q. I. Rahman, *Univalent polynomials of small degree*, Ann. Univ. Mariae Curie-Sklodowska, Sectio A, *21* (1967), 79–90.

9. M. Marden, *Geometry of polynomials*, Amer. Math. Soc. Math. Surveys, *3* (1966).

10. Chr. Pommerenke, *Univalent functions* (Vandenhoeck and Ruprecht, Göttingen, 1975).

11. Q. I. Rahman and J. Szynal, *On some classes of polynomials*, Can. J. Math. *30* (1978), 332–349.

12. St. Ruscheweyh and K. J. Wirths, *Über die Koeffizienten spezieller schlichter Polynome*, Ann. Polon. Math. *28* (1973), 341–355.

Université de Montréal,
Montréal, Québec;
Maria Curie-Skłodowska University,
Lublin, Poland