

On the thickness of soap films: an alternative to Frankel's law

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The formation of soap films by vertical withdrawal from a bath is typically described by Frankel's law, which assumes rigid film 'walls' and shear-like dynamics. Since most soap films have interfaces that are not rigid, and as the flow in the withdrawal of thin free films is typically extensional, we reconsider the theory of soap film formation. By assuming extensional flow dominated by surface viscous stresses we find that the film thickness scales as the two-thirds power of the withdrawal speed U . This speed dependence is also predicted by Frankel's law; the difference lies in the origin of the viscous resistance which sets the pre-factor. When bulk viscous stresses are important the speed dependence can vary between $U^{2/3}$ and U^2 .

1. Introduction

It is simple to make a soap film by withdrawing a wire frame from a bath of soapy water as sketched in figure 1(a). Whether demonstrating colourful interference patterns in a soap film or determining properties of an industrial foam, an accurate prediction of the soap film thickness is important. Frankel is credited with showing that if the soap film interfaces are assumed to be rigid and inextensible, then for withdrawal the film thickness h_0 varies as (Mysels, Shinoda & Frankel 1959)

$$\frac{h_0}{\ell_c} = 1.89 \left(\frac{\eta U}{\gamma} \right)^{2/3} = 1.89 C_a^{2/3}, \quad (1.1)$$

where $\ell_c = \sqrt{\gamma/\rho g}$ is the capillary length, η is the viscosity of the liquid, U is the speed at which the film is withdrawn from the bath, γ is the surface tension and C_a is the capillary number. The derivation of (1.1) is identical to that used in the Landau–Levich–Derjaguin (LLD) problem where a flat plate is withdrawn from a bath fluid as sketched in figure 1(b) (see e.g. de Gennes, Brochard-Wyart & Quéré 2004). However, Frankel assumes that both film interfaces act as rigid plates moving up with constant speed U , hence the pre-factor in (1.1) is twice as large as for the LLD result.

It is possible to examine this result using data spanning more than 40 years of work by independent research groups, as shown in figure 2. These experiments have confirmed that $h_0 \propto U^{2/3}$ over a large range of U , with deviations at both low speeds, when the film thickness $h_0 \leq 100$ nm and intermolecular forces play a role, and high speeds, when the film thickness saturates to the diameter of the wire frame. The spread in the pre-factor in the data in figure 2 is larger than can be attributed to the experimental error reported by the investigators and suggests that Frankel's law, with

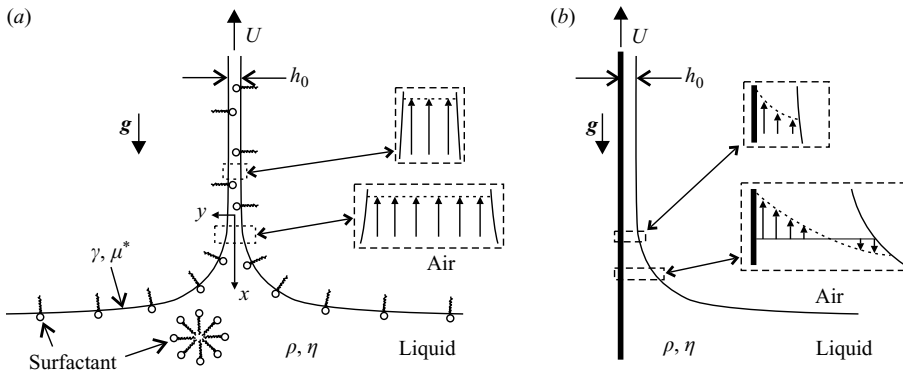


FIGURE 1. (not drawn to scale) (a) Sketch of a soap-film pulling experiment. In all reported experiments, the concentration of the surfactant is close to or above the critical micelle concentration (cmc), so micelles are present in the bulk. The flow is extensional, i.e. the velocity field u is a function of x only as illustrated by the insets. (b) Sketch of the typical LLD film-coating experiment, which can be performed with or without surfactants. The flow is shear-like, as sketched for two different positions in the film.

its sole viscous dependence on the shear viscosity of the liquid, may not be universally valid.

Here, we challenge the underlying assumption in Frankel's analysis that the flow is shear-like (as sketched in figure 1b) and suggest that an extensional flow with extensible film interfaces may be a better description of the physical processes that govern the thickness of a soap film during withdrawal. In particular, we account for the resistance associated with deformation of a surfactant-covered interface, through surface viscosity (Scriven 1960).

2. Alternative to Frankel's law

Assuming that the flow in a soap film is essentially extensional (see figure 1a), and allowing for surface viscous effects, we derive a stress balance for the dynamic meniscus. The dynamic meniscus spans the region of fluid between the static meniscus and the film of constant thickness, and typically extends over a length scale ℓ (to be determined) where $h_0 \ll \ell \ll \ell_c$. In the Appendix, we asymptotically analyse the Stokes equations for small aspect ratios $\epsilon = h_0/\ell$, using normal and tangential stress boundary conditions at the free interfaces which account for Marangoni and surface viscous stresses.

Since all of the experimental withdrawal velocities were slow (between $\mu\text{m s}^{-1}$ and mm s^{-1}), and surfactant concentrations were close to or larger than the critical micelle concentration (cmc), surfactant diffusion across the film and subsequent adsorption to the interface occur more rapidly than surface stretching (see Appendix §A.3. for details). As a result, surface tension gradients are expected to be negligible and Marangoni stresses are not included in what is to follow. Similarly, although gravity sets the length scale and curvature of the static meniscus where fluid and surface motions are negligible, it is not a dominant source of stress in the *dynamic* meniscus owing to the smallness of the film thickness. The resulting one-dimensional stress

balance for the steady dynamic meniscus of thickness $h = h(x)$ is (see the Appendix)

$$\underbrace{4 \eta (hu_x)_x}_{o(\eta h_0 U / \ell^2)} + \underbrace{2 \mu^* (u_x)_x}_{o(\mu^* U / \ell^2)} + \underbrace{\frac{\gamma}{2} h h_{xxx}}_{o(\gamma h_0^2 / \ell^3)} = 0, \tag{2.1}$$

where $u = u(x)$ is the velocity in the x -direction, and $\mu^* = \mu_s + \kappa_s$ is the combination of surface shear (μ_s) and surface dilatational (κ_s) viscosities. The first term in (2.1) accounts for bulk extensional stresses where the factor 4 is the Trouton ratio for viscous stretching of a Newtonian fluid sheet, the second term accounts for surface viscous stresses, and the third term accounts for stresses induced by the curvature of the interface. Both of the viscous terms are described in more detail in the Appendix § A.1. A similar stress balance without surface viscosity but with Marangoni stresses has been used to describe drainage of a foam lamella (Breward & Howell 2002) or, including gravitational effects, a film on a retina (Braun & King-Smith 2007). By including the influence of surface viscosity we assume that the interface is not rigid, and we account for (surface) viscous dissipation associated with shearing or extending an interface with surfactants (Edwards, Brenner & Wasan 1991).

The introduction of surface viscosity μ^* leads to two key dimensionless parameters, namely the Boussinesq number $B_q = \mu^* / (\eta \ell_c)$ and the ‘surface capillary’ number $C_s = \mu^* U / (\gamma \ell_c) = C_a B_q$. The Boussinesq number represents a balance between surface viscous stresses and (extensional) bulk viscous stresses. A balance of the viscous terms in (2.1) shows that surface viscous terms will dominate as long as $\mu^* / (\eta h_0) \equiv B_q \ell_c / h_0 \gg 1$, which is usually the case.

Estimates of μ^* come from measurements of κ_s which is typically orders of magnitude larger than μ_s . Since $\kappa_s \sim 10^{-6} - 10^{-3}$ Pa m s (e.g. Fruhner, Wantke & Lunkenheimer 1999; Liu & Duncan 2006), we expect $B_q \sim 1 - 10^3$, which emphasizes the relative importance of surface properties. Including surface viscous effects in the dynamics of soap films is therefore not uncommon, e.g. it has been used to model film drainage (Schwartz & Roy 1999; Naire, Braun & Snow 2001), and it dominates foam relaxation times (Durand & Stone 2006).

Comparing the typical relative magnitude of the two viscous terms in (2.1), we find that the length scale of the dynamic meniscus will be set by a balance between surface viscous effects and curvature, i.e. $\ell = \gamma h_0^2 / (4 \mu^* U) \equiv h_0^2 / (4 \ell_c C_s)$. Using continuity ($u h = U h_0 = \text{constant}$) to re-write (2.1) as an ODE for h , and non-dimensionalizing with $h = h_0 H$ and $x = \ell X$, we obtain

$$-2 \frac{\eta h_0}{\mu^*} \left(\frac{H_X}{H} \right)_X - \left(\frac{H_X}{H^2} \right)_X + H H_{XXX} = 0. \tag{2.2}$$

There are two sets of boundary conditions for (2.2). On one end, the film evolves to a constant thickness, i.e. $H \rightarrow 1$ and $H_X, H_{XX} \rightarrow 0$ as $X \rightarrow -\infty$. On the other end, the film matches the curvature of a static meniscus: $H_{XX} \rightarrow \alpha$ as $X \rightarrow \infty$. Equation (2.2) is integrated once with respect to X , and the integration constant is set to zero to satisfy the boundary conditions; multiplying the result by $H^{-3/2}$ and integrating again gives

$$\frac{4 h_0 / \ell_c}{3 B_q} H^{-3/2} + \frac{2}{5} H^{-5/2} + H^{-1/2} H_X + c = 0, \tag{2.3}$$

where $c = -\frac{2}{5} - \frac{4}{3} (h_0 / \ell_c) B_q^{-1}$ to satisfy the boundary conditions. For $X \gg 1$, $H = \frac{1}{2} \alpha X^2$ and substitution into (2.3) shows $\alpha = c^2 / 2$. Since the static meniscus has curvature

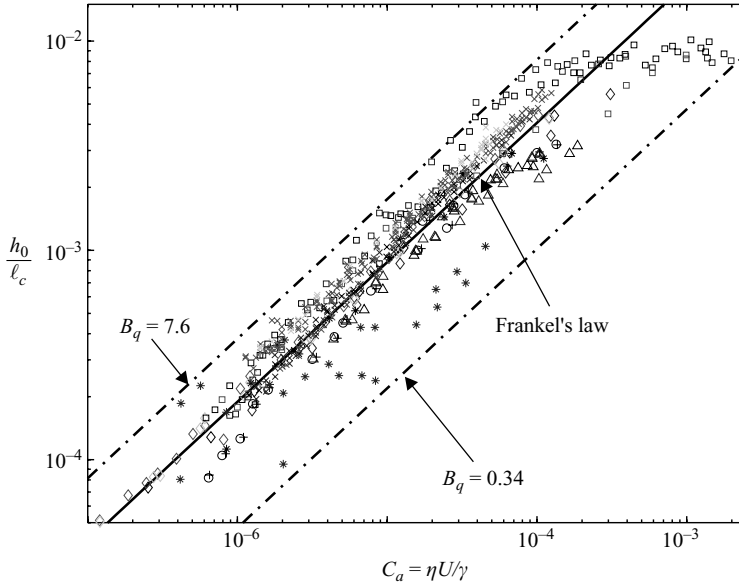


FIGURE 2. All existing data on soap film thicknesses; see table 1 for a full legend. Solid line: Frankel’s law (1.1). Dash-dotted lines: full solutions to (2.4) for values of B_q that correspond to roughly 2 and $\frac{1}{4}$ times Frankel’s law, respectively.

$h_{xx} = \sqrt{2}/\ell_c$, then $\alpha h_0/\ell^2 = \sqrt{2}/\ell_c$, and we obtain

$$\frac{h_0}{\ell_c} = 2^{3/2} \left(\frac{1}{5} + \frac{2 h_0/\ell_c}{3 B_q} \right)^{2/3} C_s^{2/3}, \tag{2.4}$$

which is our main result. Evaluating (2.4) in the limits where surface viscous effects or bulk viscous effects dominate, respectively, gives

$$\frac{h_0}{\ell_c} = \begin{cases} 0.97 C_s^{2/3} & \text{for } C_a \ll C_a^* \\ 10.1 C_a^2 & \text{for } C_a \gg C_a^*, \end{cases} \tag{2.5a}$$

$$\tag{2.5b}$$

where C_a^* is defined as the cross-over between the limits $h_0/\ell_c \propto C_a^{2/3}$ and $h_0/\ell_c \propto C_a^2$, yielding $C_a^* \approx 0.2 B_q^{1/2}$ as shown in figure 3(a).

3. Results and discussion

In the experiments presented in figure 2 the expected values of B_q are such that $C_a \ll C_a^*$. As a result, data that are plotted versus C_a , as in figure 2, are scattered by differences in the ‘pre-factor’, i.e. differences in surface viscosity as measured by B_q . The analytical result (2.4) suggests that it would make more sense to plot film thicknesses versus the surface capillary number C_s rather than C_a . If we use (2.5a) to find the surface viscosity that best fits each data set, we can collapse all existing data as shown in figure 3(b). The values obtained for surface viscosity are reported in table 1; they are all of order 10^{-5} – 10^{-6} Pa m s. which is consistent with existing data albeit on the lower end of the spectrum. The possible relationship of surface viscosity to bulk shear viscosity as observed for two of the solutions used by Mysels & Cox (1962) is unclear, and certainly depends on the specific components present in the film under consideration. For commercial detergents it has been reported that adding

Symbol	Source	η mPa s	γ mN m ⁻¹	μ^* μ Pa m s	Solutions
◇	Mysels & Cox (1962)	0.9	38	5.2	0.25% commercial detergent
		0.9	31	4.5	1% comm.det.
		0.9	32	4.5	5% comm.det.
		0.9	50	5.9	0.29% SDS
		0.9	34	4.7	0.2% SDS + 0.01% dodecanol
		2.7	45	15	0.25% comm. det. +37% glycerol +68% glycerol
+	Bruinsma <i>et al.</i> (1992)	14.5	51	64	+68% glycerol
		1	33	3.3	2% (8 × cmc) SDS (70% pure)
○	Lionti-Addad & di Meglio (1992)	1.05	33	3.8	2% SDS (70% pure)
*	Cohen-Addad & di Meglio (1994)	1.1	17	4.1	0.01% (2.6 × cmc) “fluoro” (zwitter.)
		1.3	30	1.8	0.5% (250 × cmc) C ₁₂ E ₅ (nonionic)
		1.25	23	3.9	4% octanoic acid
△	Lal & di Meglio (1994)	1.24	37	4.9	4% decanoic acid
		1	37	8.4	2% SDS + 4% glycerol
□	Adelizzi & Troian (2004)	1	36	8.8	+8% glycerol
		1	29	5.4	2% (12 × cmc) SDBS
		1	37	7.5	2% SDS
×	Berg <i>et al.</i> (2005)	1	50	7.1	0.08% SDS
		1	45	6.2	0.1% SDS
		1	39	6.0	0.4% SDS
		1	37	7.5	2% SDS
		1	37	7.7	2% SDS + 4% glycerol
		1	35	8.0	+8% glycerol

TABLE 1. Solutions used for the data shown in figure 2. All solutions are aqueous and concentrations of surfactant are reported as weight %. Viscosities and surface tensions are all as reported, except for the data from Mysels & Cox for which γ was calculated. Surface viscosities were calculated using (2.5a) after performing a best fit of the experimental data to the function $\log(h_0/\ell_c) = \frac{2}{3} \log(C_a) + f$, where f is the fitting parameter.

glycerol can increase surface viscosity (Martin & Wu 1995), although the method used in that work only measures surface *shear* viscosities.

In conclusion, taking into account surface viscosity in an extensional flow leads to the same speed dependence for film thickness as Frankel’s law, without assuming unrealistic rigid interfaces. Work is ongoing to probe the regime of low surface viscosity and large capillary number, where film thickness should scale as U^2 . Additionally, we are working to better determine the role of Marangoni stresses at higher withdrawal speeds. If Marangoni stresses are indeed negligible, as assumed in this work, then measuring soap film thicknesses may become a useful method to measure surface viscosities. Other ongoing work will address the effect of adding water-soluble polymers to the film. Such solutions may exhibit unusual scalings since they are generally non-Newtonian and can exhibit large extensional viscosities.

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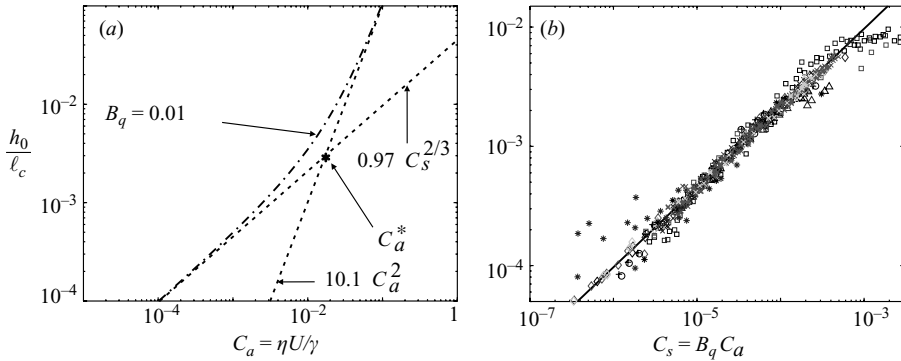


FIGURE 3. (a) Full solution (dash-dotted) and the two limits of (2.4) for $B_q = 0.01$. Note that only one of the two limits depends on B_q , and there is a transition between the limits at C_a^* as described in the text. (b) Data from figure 2 can be collapsed onto a single line corresponding to (2.4) using the fitted values for surface viscosity as reported in table 1.

Appendix. Derivation of the stress balance

The stress balance as shown in (2.1) is derived here along the same lines as the extensional flow derivation given by Breward (1999). We start with the Stokes equations

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \mathbf{0} = -\nabla p + \eta \nabla^2 \mathbf{u} - \rho \mathbf{g}, \tag{A 1}$$

as well as normal and tangential stress conditions at the free surface $y = \tilde{h} = h/2$ (see figure 4), which include the effects of a constant surface viscosity (cf. Naire *et al.* 2001),

$$p_a + \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} = 2\mathcal{H} (\gamma + (\mu^s + \kappa^s) \nabla_s \cdot \mathbf{u}), \tag{A 2a}$$

$$\mathbf{t} \cdot \boldsymbol{\tau} \cdot \mathbf{n} = \mathbf{t} \cdot \nabla_s (\gamma + (\mu^s + \kappa^s) \nabla_s \cdot \mathbf{u}), \tag{A 2b}$$

where p_a is the ambient pressure and $\boldsymbol{\tau}$ is the fluid stress tensor, \mathcal{H} is the mean curvature of the interface, \mathbf{n} and \mathbf{t} are the normal and tangent vectors to the free surface (see figure 4), and $\nabla_s = \nabla - \mathbf{nn} \cdot \nabla$ is the surface gradient operator. We denote the velocity field by $\mathbf{u} = (u, v)$ and use subscripts to indicate derivatives. The origin of the boundary terms in (A 2) is discussed in more detail in §A.1. Finally we note that there is a kinematic boundary condition at the free surface, $v(y = \tilde{h}) = \tilde{h}_t + u \tilde{h}_x$, and there is symmetry at the centreline of the film, i.e. $v(y = 0) = 0$ and $u_y(y = 0) = 0$. We then non-dimensionalize these equations using the following scales:

$$\left. \begin{aligned} x = \ell X, \quad y = h_0 Y, \quad u = u_0 U, \quad v = \epsilon u_0 V, \quad p - p_a = \frac{\eta u_0}{\ell} P, \\ \tilde{h} = h_0 H, \quad t = \frac{\ell}{u_0} T, \quad \gamma = \gamma_0 + \Delta \gamma \sigma, \end{aligned} \right\} \tag{A 3}$$

where σ is the non-dimensional surface tension. Expanding the equations to first order in $\epsilon = h_0/\ell$, the continuity and momentum equations are

$$U_{0X} + V_{0Y} = 0, \quad U_{0YY} = 0, \quad V_{0YY} - P_{0Y} = 0, \tag{A 4}$$

and the boundary conditions on the free interface are

$$-P_0 + 2V_{0Y} - 2U_{0Y} H_X - \frac{\epsilon}{C_a} H_{XX} = 0, \quad U_{0Y} = 0, \quad V_0 = H_T + U_0 H_X. \tag{A 5}$$

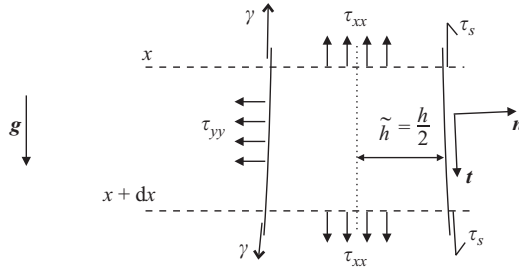


FIGURE 4. Illustration of main sources of stress in an element of a fluid film. For clarity surface tension is only shown on the left and surface viscous stresses are only shown on the right. Likewise, τ_{yy} acts equally and oppositely on the right film interface as well.

Integrating U_{0yy} twice with respect to Y , and applying the boundary and symmetry conditions, we find that $U_0 = U_0(X, T)$, and thus has no Y -dependence. This result is precisely the main feature of an *extensional* flow. At this lowest order, one can find in a straightforward manner that

$$V_0 = -U_{0x}Y, \quad \Rightarrow \quad H_T + (U_0H)_X = 0, \tag{A 6}$$

$$P_0 = -2U_{0x} - \frac{\epsilon}{C_a} H_{XX} \quad \text{at } Y = H. \tag{A 7}$$

To find the stress balance, we must consider higher order in the x -momentum and tangential stress boundary conditions,

$$U_{2yy} = P_{0x} - U_{0xx} + \mathcal{G}, \tag{A 8}$$

$$U_{2y} = \frac{M_a}{\epsilon} \sigma_X + \frac{B_q}{\epsilon} U_{0xx} - V_{0x} - 2(V_{0y} - U_{0x})H_X \quad \text{at } Y = H, \tag{A 9}$$

where $\mathcal{G} = \rho g \ell^2 / (\eta u_0)$ and $M_a = \Delta \gamma / (\eta u_0)$. Integrating (A 8) once and applying (A 9) eventually yields the overall stress balance

$$\frac{M_a}{\epsilon} \sigma_X + \frac{B_q}{\epsilon} U_{0xx} + 4(HU_{0x})_X + \frac{\epsilon}{C_a} HH_{XXX} - H\mathcal{G} = 0, \tag{A 10}$$

or in dimensional terms (replacing \tilde{h} by $h/2$),

$$2\gamma_x + 2\mu^* u_{xx} + 4\eta(hu_x)_x + \frac{\gamma}{2} h h_{xxx} - \rho gh = 0. \tag{A 11}$$

When Marangoni stresses and the gravitational term are neglected, the stress balance as shown in (2.1) is recovered.

A.1. Derivation of the stress balance ‘by inspection’

It can be helpful to ‘derive’ some of the terms in (A 11) by inspection for a unidirectional flow, rather than by rigorous asymptotics. Figure 4 shows a free-body diagram for a small part of the film; surface tension, bulk viscous stresses τ_{xx} , τ_{yy} and surface viscous stresses τ_s (i.e. the deviatoric part of $\boldsymbol{\tau}_s$ below) are indicated. Curvature-induced stresses ($\sim h(\gamma h_{xx})_x$) and Marangoni stresses ($\sim \gamma_x$) are readily constructed from the sketch. For bulk viscous stresses note that by definition

$$\tau_{xx} = -p_0 + 2\eta u_x, \quad \tau_{yy} = -p_0 + 2\eta v_y = -p_0 - 2\eta u_x, \tag{A 12}$$

where continuity has been used in the last expression. In an extensional flow, stress in the y -direction is constant and (therefore) equal to the ambient pressure

p_a . Without loss of generality, we can consider ambient pressure to be 0, so that $\tau_{yy} = 0$ and $p_0 = -2\eta u_x$, and it thus follows from (A 12) that $\tau_{xx} = 4\eta u_x$. Since fluids flow in response to stress gradients, the bulk viscous term in (A 11) is $(h \tau_{xx})_x$. Similarly, surface viscous stress depends on surface velocity gradients, i.e. $\tau_s \propto \mu^* u_x$, and hence the surface viscous term in (A 11) is $\sim \mu^* (u_x)_x$. Note that the factor 2 in the first two terms of (A 11) accounts for the fact that there are two surfaces.

A.2. A note on μ^* and the lack of pure dilatation

In an extensional flow one may be tempted to think that surface motion is dilatational, and therefore that μ^* should be replaced by κ_s in the expressions above. However, this is not so. Since the surface is expanding in one direction only (the direction of film withdrawal), it is not a purely dilatational motion. A comparison of the stress tensor for a three-dimensional Newtonian fluid and a two-dimensional ‘Newtonian surface’ helps reveal why μ^* involves surface shear viscosity. For a Newtonian fluid, the constitutive equation for the stress is

$$\boldsymbol{\tau} = -(p_0 - \kappa (\nabla \cdot \mathbf{u})) \mathbf{I} + 2\eta (\mathbf{E} - \frac{1}{3} \nabla \cdot \mathbf{u} \mathbf{I}), \quad (\text{A } 13)$$

where $\mathbf{E} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the rate-of-strain tensor and κ is the ‘second coefficient of viscosity’ which relates only to dilatation of bulk fluid. In the case of incompressible flows, naturally $\nabla \cdot \mathbf{u} = 0$ and the stress tensor simplifies. Scriven (1960) extended the ideas behind the construction of (A 13) to surfaces. Without doing Scriven’s seminal work justice, (A 13) can be recast into a two-dimensional version for surfaces by ‘replacing’ the internal pressure p_0 by surface tension and using surface operators where appropriate, i.e.

$$\boldsymbol{\tau}_s = (\gamma + \kappa_s (\nabla_s \cdot \mathbf{u})) \mathbf{I}_s + 2\mu_s (\mathbf{E}_s - \frac{1}{2} (\nabla_s \cdot \mathbf{u}) \mathbf{I}_s), \quad (\text{A } 14)$$

where $\mathbf{I}_s = \mathbf{I} - \mathbf{nn}$ is the surface projection operator and \mathbf{E}_s is just the two-dimensional version of \mathbf{E} , evaluated at the interface. For an extensional flow such as the one under consideration $\mathbf{E}_s \propto u_x$ and $\nabla_s \cdot \mathbf{u} = u_x$ so (A 14) reduces to

$$\boldsymbol{\tau}_s = (\gamma + (\kappa_s + \mu_s) u_x) \mathbf{I}_s, \quad (\text{A } 15)$$

which clearly shows why μ^* contains contributions from surface shear, even in this uni-directional extensional flow.

A.3. Time scales of surfactant transport

To determine the potential role of Marangoni stresses, it is useful to consider the time scales that dominate surfactant transport and adsorption, and to compare them to the rate of surface expansion (e.g. Shen *et al.* 2002). If the surfactant moves to new parts of the interface very rapidly, then it is reasonable to expect surface tension gradients to be negligible.

When new interface is generated by stretching on a time scale of $t_s = \ell/U$, surfactant will populate the interface by diffusing there within a typical time $t_d = h_0^2/D$ and then, for surfactants shown in table 1, adsorbing in about $t_a \approx 4 \times 10^{-4}$ s (Shen *et al.* 2002). Comparing t_s and t_d ,

$$\frac{t_d}{t_s} = \frac{h_0^2 U}{D \ell} \approx \frac{\ell_c \gamma}{\eta D} C_a^2 \approx 10^7 C_a^2 B_q, \quad (\text{A } 16)$$

where D is the diffusion coefficient of surfactant in the bulk (of order $10^{-9} \text{ m}^2 \text{ s}^{-1}$), and we have used $\ell \propto h_0^2 / (\ell_c C_s)$. For the typical C_a and B_q values encountered in

film withdrawal experiments, surface stretching is always slower than diffusion and adsorption.

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