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EQUIVALENCE OF SEMI-NORMS RELATED TO SUPER WEAKLY COMPACT OPERATORS

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Abstract

We study super weakly compact operators through a quantitative method. We introduce a semi-norm $\sigma(T)$ of an operator $T : X \to Y$, where X, Y are Banach spaces, the so-called measure of super weak noncompactness, which measures how far T is from the family of super weakly compact operators. We study the equivalence of the measure $\sigma(T)$ and the super weak essential norm of T. We prove that Y has the super weakly compact approximation property if and only if these two semi-norms are equivalent. As an application, we construct an example to show that the measures of T and its dual T^* are not always equivalent. In addition we give some sequence spaces as examples of Banach spaces having the super weakly compact approximation property.

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1. Introduction

In view of [7] and [10], a Banach space may be given an equivalent uniformly convex norm if and only if it is super-reflexive. Super weakly compact operators were introduced by Beauzamy [4] in 1976 in terms of ultrapowers of operators (see Definition 3.1). A bounded linear operator $T : X \to Y$, where X, Y are Banach spaces, is super weakly compact if $T_{\mathcal{U}}$ is weakly compact for any free ultrafilter $\mathcal{U}$ (see Section 2 for the definition of ultrafilter). Super weakly compact operators are uniformly convexifiable [4]. Ultrapowers are indispensable in this theory. In this paper we present a different approach.

A bounded linear operator $T : X \to Y$ is weakly compact if $T(B_X)$ is relatively weakly compact, where B_X is the closed unit ball of X. Thus, Beauzamy's definition is different from the classical notion of compactness of operators. It makes many techniques, effective in the theory of weakly compact operators, useless for studying super weakly compact operators. A recent concept of super weakly compact sets seems to be useful to narrow the gap. A subset A of a Banach space X is relatively super

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weakly compact if and only if $A_{\mathcal{U}}$ is relatively weakly compact for any free ultrafilter $\mathcal{U}$. Equivalently, A is relatively super weakly compact if and only if, for every $\varepsilon > 0$, there is a super weakly compact set $S \subset X$ such that $A \subset S + \varepsilon B_X$. For more details on super weakly compact sets, see [5, 6, 16, 17, 20].

The notion of a super weakly compact set inspires us to consider a new way to define super weak compactness for operators (see Definition 2.4). We say that T is super weakly compact if $T(B_X)$ is relatively super weakly compact and prove that our definition is equivalent to Beauzamy's. This allows us to study super weakly compact operators via quantitative methods.

Measures of (weak) noncompactness are widely used in the theory of bounded linear operators (see, for instance, [2, 12, 21]). Recently, the author [19] introduced the measure of super weak noncompactness of a bounded subset A of a Banach space X by

$$\sigma(A) := \inf\{t > 0 \mid A \subset S + tB_X,\ S \text{ is super weakly compact}\}. \tag{1.1}$$

Here, we introduce and study the induced measure of super weak noncompactness of bounded linear operators.

Denote by $L(X, Y)$ the Banach space of bounded linear operators between Banach spaces X and Y. For any $T \in L(X, Y)$, the measure of super weak noncompactness is

$$\sigma(T) := \sigma(TB_X).$$

In fact, σ is a semi-norm on $L(X, Y)$ and $\sigma(T) = 0$ if and only if T is super weakly compact. On $S(X, Y)$, the Banach space of super weakly compact operators between X and Y, the super weak essential norm $\|\cdot\|_S$ of $T \in L(X, Y)$ is given by

$$\|T\|_S := \inf\{\|T - S\| : S \in S(X, Y)\}.$$

Obviously, $\|\cdot\|_S$ is the quotient norm on $L(X, Y)/S(X, Y)$ and $\|T\|_S = 0$ if and only if $T \in S(X, Y)$. It is natural to ask whether the semi-norms σ and $\|\cdot\|_S$ are equivalent, that is, whether there are constants $a, b > 0$ such that for any $T \in L(X, Y)$,

$$a\sigma(T) \le \|T\|_S \le b\sigma(T).$$

In view of the results in [2, Theorem 2.5] and [3, Theorem 1] concerning compact and weakly compact cases, we infer that the problem involves some type of approximation property of the Banach spaces X and Y and we introduce the super weakly compact approximation property. In Section 3, we prove (Theorem 3.6) that the two semi-norms $\|\cdot\|_S$ and σ are equivalent if and only if Y has the super weakly compact approximation property. This also means that $L(X, Y)/S(X, Y)$ is σ-complete if and only if Y has the super weakly compact approximation property.

From the viewpoint of [11, page 489], quantitative theorems replace the respective implications by inequalities. A bounded linear operator $T \in L(X, Y)$ is compact if and only if its dual T^* is compact. Goldenstein and Markus [8, Theorem 3] quantified the result by proving that $\frac{1}{2}\gamma(T) \le \gamma(T^*) \le 2\gamma(T)$ for any $T \in L(X, Y)$, where γ is the Hausdorff measure of noncompactness. For the weakly compact case, Gantmacher's

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theorem says that T and T^* are weakly compact simultaneously. Unlike the compact case, Astala and Tylli [3, Corollary 5] proved that $\omega(T)$ and $\omega(T^*)$ are not equivalent, where ω is the De Blasi measure of weak noncompactness, sometimes called the Hausdorff measure of weak noncompactness, because ω behaves similarly to the Hausdorff measure of noncompactness γ, which can be defined as in (1.1) by replacing a super weakly compact set S by a compact one.

As for the super weakly compact case, it is proved in [9, page 177] that T is super weakly compact if and only if T^* is. To study the quantitative relationship of the super weak compactness of T and T^*, we present in Section 4 some basic properties of Banach spaces having the super weakly compact approximation property and show that some sequence spaces have this property. Then we complete the paper by applying Theorem 3.6 to show that $\sigma(T)$ and $\sigma(T^*)$ are not always equivalent.

2. Preliminaries

Let X, Y be real infinite-dimensional Banach spaces and let B_X, B_Y be the closed unit balls in X and Y, respectively. Assume that Ω is an infinite set.

DEFINITION 2.1. A filter $\mathcal{F}$ is a collection of subsets of a set Ω satisfying:

(1) $\emptyset \notin \mathcal{F}$;
(2) $A \cap B \in \mathcal{F}$ if $A, B \in \mathcal{F}$;
(3) $B \in \mathcal{F}$ if $A \subset B \subset \Omega$ and $A \in \mathcal{F}$.

A filter $\mathcal{F}$ is said to be free if $\cap\{F \in \mathcal{F}\} = \emptyset$. A filter $\mathcal{U}$ is called an ultrafilter if for any $A \subset \Omega$, either $A \in \mathcal{U}$ or $\Omega \backslash A \in \mathcal{U}$.

Suppose that $\mathcal{U}$ is a filter on Ω and $x_\omega, x \in X$ $(\omega \in \Omega)$. We say that x is a limit of x_ω (written $x = \lim_{\mathcal{U}} x_\omega$) if for any neighbourhood U of x, there is $F \in \mathcal{U}$ such that $\{x_\omega : \omega \in F\} \subset U$. If $\{x_\omega : \omega \in \Omega\} \subset X$ is relatively compact, then for any ultrafilter $\mathcal{U}$, there exists $x \in X$ such that $\lim_{\mathcal{U}} x_\omega = x$.

For a nonempty set Ω, let $\{X_\omega : \omega \in \Omega\}$ be a collection of Banach spaces. Assume that $\mathcal{U}$ is an ultrafilter on Ω and consider the Banach space $\ell_\infty(\Omega, X_\omega)$ consisting of all bounded (x_ω), where $x_\omega \in X_\omega$ $(\omega \in \Omega)$, normed by

$$\|(x_\omega)\| = \sup_\omega \|x_\omega\|.$$

Let $N_{\mathcal{U}}$ be the set of all $(x_\omega) \in \ell_\infty(\Omega, X_\omega)$ such that

$$\lim_{\mathcal{U}} \|x_\omega\| = 0.$$

Then $N_{\mathcal{U}}$ is a closed subspace of $\ell_\infty(\Omega, X_\omega)$. The ultraproduct of $(X_\omega)_{\omega\in\Omega}$ with respect to the ultrafilter $\mathcal{U}$ is

$$(X_\omega)_{\mathcal{U}} := \ell_\infty(\Omega, X_\omega)/N_{\mathcal{U}}$$

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with the canonical quotient norm. As any $(x_\omega) \in (X_\omega)_\mathcal{U}$ is actually an equivalence class, we will use $(x_\omega)_\mathcal{U}$ to denote an element in $(X_\omega)_\mathcal{U}$. Then

$$\|(x_\omega)_\mathcal{U}\| = \lim_\mathcal{U} \|x_\omega\|.$$

Notice that $(x_\omega)_\mathcal{U} = (y_\omega)_\mathcal{U}$ if and only if $\lim_\mathcal{U} \|x_\omega - y_\omega\| = 0$. If $X_\omega = X$, we write $X_\mathcal{U}$ and call it the ultrapower of X. For a subset A of X, $A_\mathcal{U}$ denotes the set of all elements $(x_\omega)_\mathcal{U} \in X_\mathcal{U}$ such that $x_\omega \in A$.

Definition 2.2. A subset A of X is said to be relatively super weakly compact if $A_\mathcal{U}$ is relatively weakly compact in $X_\mathcal{U}$ for any free ultrafilter $\mathcal{U}$. Further, A is said to be super weakly compact if it is weakly closed and relatively super weakly compact.

We refer to [5] for the properties of super weakly compact sets. It is easy to see that a subset of a relatively super weakly compact set is also relatively super weakly compact. We recall here that if $A, B \subset X$ are relatively super weakly compact, then the sets $A \cup B$, $A + B$, $A \times B$ and $A \backslash B$ are relatively super weakly compact. In the recent paper [19], the author studied the super weak compactness of a bounded set A and its convex hull $\mathrm{co}(A)$.

Lemma 2.3. *Assume that A is a super weakly compact subset of a Banach space X. Then its closed convex hull $\overline{\mathrm{co}}(A)$ is also super weakly compact.*

Recall that super weak compactness is stable under a bounded linear operator. We use the notion of super weakly compact sets to define super weakly compact operators.

Definition 2.4. A bounded linear operator $T : X \to Y$ is said to be super weakly compact if $T(B_X)$ is relatively super weakly compact.

3. Semi-norms related to super weakly compact operators

In this section, we introduce the measure σ of super weak noncompactness of operators and study the equivalence of σ and the super weak essential norm $\|\cdot\|_S$. The main result in this section is Theorem 3.6.

We denote by $L(X, Y)$ the collection of all bounded linear operators mapping X to Y. Further, $W(X, Y) \subset L(X, Y)$ and $S(X, Y) \subset L(X, Y)$ represent the collections of all weakly compact operators and super weakly compact operators, respectively. When $X = Y$, we abbreviate them to $L(X)$, $W(X)$ and $S(X)$. By an operator, we always mean a bounded linear operator.

Beauzamy [4] introduced the notion of a super weakly compact operator by means of ultrapowers of operators, which is different from our Definition 2.4. Assume that $T : X \to Y$ is a bounded linear operator and $\mathcal{U}$ is a free ultrafilter on an infinite set Ω. Define $T_\mathcal{U} : X_\mathcal{U} \to Y_\mathcal{U}$ as

$$T_\mathcal{U}((x_\omega)_\mathcal{U}) := (Tx_\omega)_\mathcal{U}.$$

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DEFINITION 3.1 (Beauzamy). An operator $T : X \to Y$ is said to be super weakly compact if $T_{\mathcal{U}}$ is weakly compact for any free ultrafilter $\mathcal{U}$.

In fact, T is super weakly compact if and only if $T_{\mathcal{U}}$ is weakly compact for any free ultrafilter on $\mathbf{N}$. We will see that this notion is equivalent to Definition 2.4.

THEOREM 3.2. *If $T : X \to Y$ is an operator and $\mathcal{U}$ is a free ultrafilter on Ω, then $T_{\mathcal{U}} B_{X_{\mathcal{U}}} = (TB_X)_{\mathcal{U}}$. Thus, $T(B_X)$ is super weakly compact if and only if $T_{\mathcal{U}}$ is weakly compact, that is, the two definitions of super weakly compact operators coincide.*

PROOF. For any $x_\omega \in B_X$ with $\omega \in \Omega$, it is easy to see that $(x_\omega)_{\mathcal{U}} \in B_{X_{\mathcal{U}}}$. Hence, $(TB_X)_{\mathcal{U}} \subset T_{\mathcal{U}} B_{X_{\mathcal{U}}}$. For any $(x_\omega)_{\mathcal{U}} \in B_{X_{\mathcal{U}}}$, without loss of generality, assume that $\lim_{\mathcal{U}} \|(x_\omega)_{\mathcal{U}}\| = \alpha > 0$. Then $(x_\omega)_{\mathcal{U}} = (y_\omega)_{\mathcal{U}}$, where $y_\omega = (\alpha/\|x_\omega\|)x_\omega \in B_X$ with $\omega \in \Omega$. Thus, $(x_\omega)_{\mathcal{U}} \in (B_X)_{\mathcal{U}}$, which yields $(B_X)_{\mathcal{U}} = B_{X_{\mathcal{U}}}$ and, therefore, $T_{\mathcal{U}} B_{X_{\mathcal{U}}} = (TB_X)_{\mathcal{U}}$. □

In [19], the author introduced the notion of the measure of super weak noncompactness σ. For any bounded subset $A \subset X$,

$$\sigma(A) := \inf\{t > 0 \mid A \subset S + tB_X \text{ with a relatively super weakly compact set } S\}.$$

Making use of the measure σ, we may study some quantitative properties of super weakly compact sets. From [19], σ has the following properties.

LEMMA 3.3. *If A is a bounded subset of a Banach space X and $\overline{A}^w$ denotes the weakly closed hull of A, then:*

(i) $\sigma(A) = 0$ *if and only if A is relatively super weakly compact;*
(ii) $\sigma(A) \le \sigma(B)$ *if $A \subset B$;*
(iii) $\sigma(A) = \sigma(\overline{A}^w)$*;*
(iv) $\sigma(A) = \sigma(\mathrm{co}A)$*;*
(v) $\sigma(A \cup B) = \max\{\sigma(A), \sigma(B)\}$*;*
(vi) $\sigma(A + B) \le \sigma(A) + \sigma(B)$*;*
(vii) $\sigma(tA) = |t|\sigma(A)$ *for $t \in \boldsymbol{R}$;*
(viii) $\omega(A) \le \sigma(A) \le \gamma(A)$*,*

where ω is the De Blasi measure of weak noncompactness and γ is the Hausdorff measure of noncompactness.

LEMMA 3.4. *Let B_X denote the unit ball of X. If A is a bounded subset of a Banach space X, then $\sigma(A + tB_X) = \sigma(A) + t\sigma(B_X)$ for $t \ge 0$.*

We introduce the measure of super weak noncompactness for operators induced by σ. For any $T \in L(X, Y)$, let

$$\sigma(T) := \sigma(TB_X).$$

It is clear that $\sigma(T) = 0$ if and only if $T \in S(X, Y)$. For any $T, G \in L(X, Y)$ and $t \in \mathbf{R}$,

$$\sigma(tT) = \sigma(tT(B_X)) = |t|\sigma(T), \tag{3.1}$$

$$\sigma(T + G) \leq \sigma(T(B_X)) + G(B_X)) \leq \sigma(T(B_X)) + \sigma(G(B_X)) = \sigma(T) + \sigma(G). \tag{3.2}$$

Moreover, for any $T \in L(X, Y)$ and $A \subset X$,

$$\sigma(T(A)) \leq \sigma(T)\sigma(A).$$

Indeed, assume that $\sigma(T) < \alpha$ and $\sigma(A) < \beta$. There exist super weakly compact sets S_1 and S_2 such that

$$T(B(X)) \subset S_1 + \alpha B(Y), \quad A \subset S_2 + \beta B(X).$$

We get the result by noticing that $T(A) \subset T(S_2) + \beta S_1 + \alpha\beta B(Y)$ and $T(S_2) + \beta S_1$ is super weakly compact. So, it is easy to see that for any $G \in L(X, Y)$, $T \in L(Y, Z)$,

$$\sigma(TG) \leq \sigma(T)\sigma(G). \tag{3.3}$$

Formulas (3.1) and (3.2) imply that the function $\sigma(\cdot)$ is a semi-norm on $L(X, Y)$.

In order to proceed, let us observe that $S(X, Y)$ is a closed ideal in $L(X, Y)$. Indeed, given a sequence $T_n \in S(X, Y)$ converging to $T \in L(X, Y)$ in the operator norm topology, for any $\varepsilon > 0$, there is $n \in \mathbf{N}$ such that $\|T - T_n\| < \varepsilon$. Therefore, $T(B_X) \subset T_n(B_X) + \varepsilon B_Y$. Since $T_n(B_X)$ is relatively super weakly compact, so is $T(B_X)$ by (1.1), that is, $T \in S(X, Y)$. For any $G \in L(X, Y)$, it is easy to observe from (3.3) that $TG, GT \in S(X, Y)$.

The super weak essential norm $\|\cdot\|_S$ of $T \in L(X, Y)$ is the semi-norm induced from the quotient space $L(X, Y)/S(X, Y)$, that is,

$$\|T\|_S := \inf\{\|T - S\| : S \in S(X, Y)\}. \tag{3.4}$$

Observe that if $\|T\|_S < \alpha$, there is $S \in S(X, Y)$ such that $\|T - S\| < \alpha$. Therefore, $T(B_X) \subset S(B_X) + \alpha B_Y$, which actually means that $\sigma(T) \leq \|T\|_S$.

It is natural to ask whether these two semi-norms are equivalent. To solve this problem, we introduce the super weakly compact approximation property.

DEFINITION 3.5. A Banach space X is said to have the super weakly compact approximation property (SWAP for short) if there is a real number $\lambda > 0$ such that for any super weakly compact set $A \subset X$ and any $\varepsilon > 0$, there is a super weakly compact operator $R : X \to X$ with

$$\sup_{x \in A} \|x - Rx\| \leq \varepsilon \quad \text{and} \quad \|R\| \leq \lambda.$$

THEOREM 3.6. *A Banach space Y has the super weakly compact approximation property if and only if the semi-norms σ and $\|\cdot\|_S$ are equivalent in $L(X, Y)$ for any Banach space X.*

PROOF. Assume that Y has the super weakly compact approximation property and take an arbitrary $\alpha > 0$ and $T \in L(X, Y)$ with $\sigma(T) < \alpha$. Then there is a super weakly compact set $S \subset Y$ such that $T(B_X) \subset S + \alpha B_Y$. Using the assumption, there is a

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positive λ such that for any $\varepsilon > 0$, there is a super weakly compact operator $R : Y \to Y$ satisfying $\sup_{x\in S}\|Rx - x\| < \varepsilon$ and $\|R\| \le \lambda$. Let $J := RT$. Clearly, the operator J is super weakly compact. For any $x \in B_X$, there is $s \in S$ such that $\|Tx - s\| < \alpha$; hence,

$$\|Tx - Jx\| = \|Tx - s + s - Rs + Rs - Jx\| \le \|(R + I)(s - Tx)\| + \|s - Rs\| \le (\lambda + 1)\alpha + \varepsilon.$$

Consequently, $\|T\|_S \le (\lambda + 1)\alpha$, that is, $\|T\|_S \le (\lambda + 1)\sigma(T)$.

Conversely, assume that Y does not have the super weakly compact approximation property. Then, for any $n \in \mathbf{N}$, there are a super weakly compact set $S_n \subset Y$ and $\varepsilon_n > 0$ such that R is not super weakly compact whenever $R \in L(Y)$ satisfies

$$\sup_{x\in S_n} \|x - Rx\| \le \varepsilon_n \quad \text{and} \quad \|R\| \le n + 1.$$

Without loss of generality, assume that $\varepsilon_n \to 0$ as $n \to \infty$ and S_n is closed absolutely convex (recall that, in view of Lemma 2.3, the closed absolutely convex hull of a super weakly compact set is super weakly compact). Set $A_n := (\varepsilon_n/n)B_Y + S_n$. Then A_n is absorbing and the Minkowski functional

$$|x|_n = \inf\{t > 0 \mid x \in tA_n\}$$

determined by A_n defines an equivalent norm on Y, that is, there is $a_n > 0$ such that

$$\frac{\varepsilon_n}{n}|x|_n \le \|x\| \le a_n|x|_n.$$

Let $Y_n = (Y, |\cdot|_n)$ and let T_n be the identity from Y_n onto Y. It is immediate to see that $\sigma(T_n) = \varepsilon_n/n$ by Lemma 3.4 and $T_n(B_{Y_n}) = A_n$.

Next we show that $\|T_n\|_S \ge \varepsilon_n$. Suppose that there is $R \in S(Y_n, Y)$ such that $\|T_n - R\| < \varepsilon_n$ and still denote by R the induced operator from Y to Y. Since $S_n \subset B_{Y_n}$ and $(\varepsilon_n/n)B_Y \subset B_{Y_n}$,

$$\|x - Rx\| < \varepsilon_n, \forall x \in S_n \quad \text{and} \quad \|x - Rx\| \le n, \forall x \in B_Y.$$

Thus, R is not super weakly compact by the assumption.

Let $X = c_0(Y_n)$ and P_n ($n \in \mathbf{N}$) be the canonical projection from X onto Y_n, that is, $P_n((y_i)) = y_n$ for any n. Let $J_n = T_nP_n \in L(X, Y)$. Then

$$\sigma(J_n) = \sigma(J_n(B_X)) = \sigma(A_n) = \frac{\varepsilon_n}{n}$$

and

$$\|J_n\|_S = \|T_n\|_S \ge \varepsilon_n$$

since $J_nI_n = T_n$, where $I_n : Y_n \to X$ is the natural inclusion from Y_n into X. This completes the proof. □

4. Quantitative relationship of an operator and its dual

In order to give an application of Theorem 3.6, in this section we study some basic properties of Banach spaces having the super weakly compact approximation property.

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We focus on some classical Banach sequence spaces. Then, by applying Theorem 3.6, we construct an example to show that $\sigma(T)$ is not equivalent to $\sigma(T^*)$.

Astala and Tylli [3] introduced the weakly compact approximation property (WAP for short) and proved that ℓ_1 has WAP while c_0, $C[0,1]$ and $L^1(0,1)$ fail to have WAP. Odell and Tylli [15] and Saksman and Tylli [18] considered more examples involving WAP and showed that, for example, $\ell_1(\ell_p)$ and $\ell_p(\ell_1)$ $(1 < p < \infty)$ do have WAP. They also obtained results about the quasi-reflexive James' space J and its dual J^*, the James' tree space JT, the nuclear operator space $N(\ell_p, \ell_q)$ and others.

It is trivial that super reflexive spaces have SWAP, so ℓ_p $(1 < p < \infty)$ has SWAP by its uniform convexity. Since ℓ_1 is a Schur space, the super weak compactness is equivalent to weak compactness according to Lemma 3.3. So, SWAP is equivalent to WAP in ℓ_1 and, hence, ℓ_1 has SWAP.

THEOREM 4.1. *ℓ_p $(1 \le p < \infty)$ has the super weakly compact approximation property.*

We next prove that c_0 does not have SWAP. Recall that an operator $T \in L(X, Y)$ is completely continuous if the sequence (Tx_n) is norm convergent whenever a sequence (x_n) is weakly convergent. A Banach space X is called an $\mathcal{L}^\infty$ space if there is $\lambda \ge 1$ such that any finite-dimensional subspace E of X is contained in a finite-dimensional subspace F such that

$$d_M(F, \ell_\infty^n) \le \lambda,$$

where $d_M(F, \ell_\infty^n)$ is the Banach–Mazur distance of F and ℓ_∞^n. Note that c_0 and ℓ_∞ are $\mathcal{L}^\infty$ spaces. It is known that every weakly compact operator in $W(X)$ is completely continuous whenever X is an $\mathcal{L}^\infty$ space [14].

THEOREM 4.2. *c_0 and ℓ_∞ do not have the super weakly compact approximation property.*

PROOF. Let $A := \{e_i\}$, where $\{e_i\}$ is the standard unit vector basis of c_0. By the result in [5, Example 2.2], A is super weakly compact. Assume to the contrary that c_0 has SWAP. Then, for $\varepsilon = 1/3$, there is a super weakly compact operator $R : c_0 \to c_0$ such that

$$\sup_{x \in A} \|x - Rx\| \le \varepsilon.$$

Since $\|e_i - e_j\| = 1$ whenever $i \ne j$,

$$\|Re_i - Re_j\| \ge 1/3. \tag{4.1}$$

On the other hand, $R(A)$ is relatively compact, because c_0 is an $\mathcal{L}^\infty$ space and R is a weakly compact operator by Lemma 3.3. The relative compactness of $R(A)$ conflicts with (4.1). So, by the assumption, c_0 does not have SWAP.

Similarly, $A = \{e_i\} \subset \ell_\infty$ is super weakly compact since it is the image of $A \subset c_0$ under the canonical inclusion operator. Following the above procedure, we conclude that ℓ_∞ does not have SWAP. □

THEOREM 4.3. *If X has the super weakly compact approximation property, then so does any complemented subspace Y of X.*

PROOF. Suppose that X has SWAP and let $A \subset Y$ be a super weakly compact set. It is clear that A as a subset of X is super weakly compact, too. By the assumption, there is $\lambda > 0$ such that for any $\varepsilon > 0$, there is a super weakly compact operator $R : X \to X$ satisfying

$$\sup_{x \in A} \|x - Rx\| \le \varepsilon \quad \text{and} \quad \|R\| \le \lambda.$$

Since Y is complemented in X, there is a bounded projection $P : X \to Y$. Clearly, the operator $PR : Y \to Y$ is super weakly compact and

$$\|PR\| \le \lambda \|P\|.$$

Furthermore, for any $x \in A$,

$$\|x - PRx\| = \|Px - PRx\| \le \|P\| \|x - Rx\| \le \|P\| \varepsilon.$$

This completes the proof. □

Assume that $(X_i, \|\cdot\|_i)$, with $i \in \mathbf{N}$, is a sequence of Banach spaces and that $1 \le p < \infty$. Consider the Banach space $\ell_p(X_i)$ of all sequences (x_i) with $x_i \in X_i$ such that $\sum_{i=1}^{\infty} \|x_i\|^p < \infty$. For any $x = (x_i) \in \ell_p(X_i)$, the norm of x is

$$\|x\| = \|(x_i)\| = \Big(\sum_{i=1}^{\infty} \|x_i\|^p \Big)^{1/p}.$$

We write $\ell_p(Y)$ for short when $X_i = Y$ for each $i \in \mathbf{N}$. Let P_i, $i \in \mathbf{N}$, be the canonical projection from $\ell_p(X_i)$ onto X_i. Clearly, $P_i(B_X) = B_{X_i}$, where $X := \ell_p(X_i)$. If A is a subset of $\ell_p(X_i)$, set $A_i = P_i(A)$. We wish to investigate under what conditions the space $\ell_p(X_i)$ has SWAP.

For any $x \in \ell_1(X_i)$, let $d_n(x) := d_n(x_1, x_2, \ldots) = (0, 0, \ldots, 0, x_n, x_{n+1}, \ldots)$. For $A \subset \ell_1(X_i)$, set $d_n(A) := \{d_n(x) : x \in A\}$ and $\|d_n(A)\| := \{\|d_n(x)\| : x \in A\}$.

LEMMA 4.4 [20, Corollary 7]. *A subset A of $\ell_1(X_i)$ is relatively super weakly compact if and only if:*

(a) *A is bounded;*
(b) *A_i is relatively super weakly compact for all $i \in \mathbf{N}$;*
(c) $\lim_n \|d_n(A)\| = 0$.

THEOREM 4.5. *$\ell_1(X_i)$ has the super weakly compact approximation property if and only if every X_i has the super weakly compact approximation property with a common constant for all $i \in \mathbf{N}$.*

PROOF. Let $X = \ell_1(X_i)$ and suppose that X has SWAP with a constant λ. Since X_i is a 1-complemented subspace in X, by the proof of Theorem 4.3, each X_i has SWAP with the constant λ.

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On the other hand, suppose that every X_i has SWAP with a uniform constant λ. Let $\varepsilon > 0$. For any super weakly compact set $A \subset X = \ell_1(X_i)$ and $i \in \mathbf{N}$, A_i is relatively super weakly compact. By the assumption, there is a super weakly compact operator $R_i : X_i \to X_i$ such that

$$\sup_{x \in A_i} \|x - R_i x\| \leq \frac{\varepsilon}{2^{i/p}} \quad \text{and} \quad \|R_i\| \leq \lambda.$$

Since A is super weakly compact, by Lemma 4.4, there is $n \in \mathbf{N}$ such that

$$\sup_{x \in A} \sum_{i=n+1}^{\infty} \|x_i\| \leq \varepsilon.$$

For any $x \in X$, let $Rx = (R_1 x_1, \ldots, R_n x_n, 0, 0, \ldots)$. Since $\|R_i\| \leq \lambda$, we have $R \in L(X)$ and $\|R\| \leq \lambda$. By Lemma 4.4, the set $R(B_X)$ is relatively super weakly compact, because $R_i(B_{X_i})$ $(1 \leq i \leq n)$ is relatively super weakly compact. Hence, R is a super weakly compact operator. For any $x \in A$, it is easy to see that

$$\|x - Rx\| = \sum_{i=1}^{n} \|x_i - R_i x_i\| + \varepsilon \leq 2\varepsilon.$$

Consequently, X has SWAP. □

COROLLARY 4.6. *$\ell_1(\ell_p)$ $(1 < p < \infty)$ has the super weakly compact approximation property.*

REMARK 4.7. Odell and Tylli [15] proved that $\ell_p(\ell_1)$ $(1 < p < \infty)$ enjoys the weakly compact approximation property. But we do not know whether or not such a conclusion is true for the super weakly compact case, that is, whether the Banach space $\ell_p(\ell_1)$ $(1 < p < \infty)$ has SWAP.

It is known that an operator T is super weakly compact if and only if its dual T^* is super weakly compact [9, page 177]. It is natural to study the quantitative relationship of super weak compactness between an operator T and its dual T^*, that is, whether $\sigma(T)$ and $\sigma(T^*)$ are equivalent.

Let $\mathcal{A}$ be an ideal of operators. Astala [1] introduced the measure $\gamma_{\mathcal{A}}$ induced by the ideal $\mathcal{A}$. For any bounded $D \subset Y$,

$$\gamma_{\mathcal{A}}(D) := \inf\{t > 0 \mid D \subset T(B_Z) + tB_Y, T \in \mathcal{A}(Z, Y)\}, \tag{4.2}$$

where the infimum is taken over all Banach spaces Z. For any $T \in L(X, Y)$, set $\gamma_{\mathcal{A}}(T) := \gamma_{\mathcal{A}}(TB_X)$ and $\|T\|_{\mathcal{A}} = \inf\{\|T - J\| \mid J \in \mathcal{A}\}$.

Since the collection of super weakly compact operators $S(X, Y)$ is an ideal of operators, we may take $\mathcal{A} = S(X, Y)$. For any bounded $D \subset Y$, it is easy to see that $\sigma(D) \leq \gamma_{\mathcal{A}}(D)$ since the set $T(B_Z)$ in (4.2) is relatively super weakly compact. On the other hand, for any super weakly compact convex set $K \subset Y$, there are a Banach space X and a super weakly compact operator $T : X \to Y$ such that $K \subset T(B_X)$ (see

[16, Theorem 4.5]). Hence, $\sigma(D) \geq \gamma_{\mathcal{A}}(D)$. Consequently, $\sigma(T) = \gamma_{\mathcal{A}}(T)$ when $\mathcal{A} = S(X, Y)$.

An ideal $\mathcal{A}$ of operators is called symmetric if $T \in \mathcal{A}(X, Y)$ implies that $T^* \in \mathcal{A}(Y^*, X^*)$. It is clear that $S(X, Y)$ is a symmetric ideal of operators. The following lemmas from Astala [1] are crucial in what follows.

LEMMA 4.8 (Astala). *If $\mathcal{A}$ is a symmetric ideal of operators, then $\gamma_{\mathcal{A}}(T^{**}) \leq \gamma_{\mathcal{A}}(T)$ for any $T \in L(X, Y)$.*

A Banach space Y is said to have the extension property if for any closed subspace M of an arbitrary Banach space X and any $T \in L(M, Y)$, there exists an operator $S \in L(X, Y)$ such that $T = SJ_M$ and $\|T\| = \|S\|$, where $J_M : M \to X$ is the canonical inclusion. It is well known that the Banach space ℓ_∞ has the extension property. For more information about the extension property, we refer to [13, Section 2.f].

LEMMA 4.9 (Astala). *Suppose that $\mathcal{A}$ is a symmetric ideal of operators and $T \in (X, Y)$. If Y has the extension property, then $\gamma_{\mathcal{A}}(T^*) = \|T^*\| = \|T\|_{\mathcal{A}}$.*

Let us recall that Theorem 4.2 states that c_0 fails to have the super weakly compact approximation property. We use Theorem 3.6 to construct an example showing that $\gamma(T)$ and $\gamma(T^*)$ are not equivalent.

THEOREM 4.10. *There are a separable space X and a sequence $T_n \in L(X, c_0)$ such that*

$$\sigma(T_n^*) = 1 \quad \textit{and} \quad \sigma(T_n^{**}) \leq \sigma(T_n) \to 0$$

for each $n \in \mathbf{N}$.

PROOF. Note that c_0 does not have the super weakly compact approximation property. By the proof of Theorem 3.6, there is a separable Banach space $X = c_0(X_i)$ such that each X_i is isomorphic to c_0. Replace $T_n : X \to c_0$ in the proof of Theorem 3.6 by $T_n/\|T_n\|_S$ and still denote it by T_n. Obviously, $T_n \in L(X, c_0)$. Moreover, by Theorem 3.6, it is easy to see that $\|T_n\|_S = 1$ and $\sigma(T_n) \leq 1/n$. Lemma 4.8 implies that $\sigma(T_n^{**}) \leq \sigma(T_n) \leq 1/n$. Let $J : c_0 \to \ell_\infty$ be the natural embedding.

Claim 1: $\|T_n\|_S \leq 2\|JT_n\|_S$. For any $\varepsilon > 0$, by (3.4), there is a super weakly compact operator $S \in S(X, \ell_\infty)$ such that $\|JT_n - S\| \leq \|JT_n\|_S + \varepsilon$. Since $S(B_X)$ is relatively super weakly compact (and, hence, weakly compact) and a weakly compact set in ℓ_∞ is separable, the space $Y := \overline{\mathrm{span}}\{Jc_0, S(X)\}$ is separable. By [13, Theorem 2.f], c_0 is separable injective and there is a projection $P : X \to c_0$ with $\|P\| \leq 2$. Thus,

$$\|T_n\|_S \leq \|T_n - PS\| = \|P(JT_n - S)\| \leq 2\|JT_n\|_S + 2\varepsilon.$$

The claim follows immediately since ε is arbitrary.

Claim 2: $\sigma(T_n^*) = \|JT_n\|_S$. Since $J^* : \ell_\infty^* \to \ell_1$ is surjective and $J^*(B_{\ell_\infty^*}) \subset B_{\ell_1}$, we have $\sigma(T_n^*) = \sigma(T_n^* J^*)$. Moreover, because ℓ_∞ has the extension property, by Lemma 4.9, it is clear that

$$\sigma(T_n^* J^*) = \|(JT_n)^*\|_S = \|JT_n\|_S.$$

We complete the proof by replacing T_n by $T_n/\sigma(T_n^*)$. □

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References

[1] K. Astala, 'On Measures of Noncompactness and Ideal Variations in Banach Spaces', *Ann. Acad. Sci. Fenn. Math. Diss.* **29** (1980), 42 pages.
[2] K. Astala and H.-O. Tylli, 'On the bounded compact approximation property and measures of noncompactness', *J. Funct. Anal.* **70**(2) (1987), 388–401.
[3] K. Astala and H.-O. Tylli, 'Seminorms related to weak compactness and to Tauberian operators', *Math. Proc. Cambridge Philos. Soc.* **107**(2) (1990), 367–375.
[4] B. Beauzamy, 'Opérateurs uniformément convexifiants', *Studia Math.* **57**(2) (1976), 103–139.
[5] L. Cheng, Q. Cheng, S. Luo, K. Tu and J. Zhang, 'On super weak compactness of subsets and its equivalences in Banach spaces', *J. Convex Anal.* **25**(3) (2018), 899–926.
[6] L. Cheng, Q. Cheng, B. Wang and W. Zhang, 'On super-weakly compact sets and uniformly convexifiable sets', *Studia Math.* **199**(2) (2010), 145–169.
[7] P. Enflo, 'Banach spaces which can be given an equivalent uniformly convex norm', *Israel J. Math.* **13** (1972), 281–288.
[8] L. S. Goldenstein and A. S. Markus, 'On a measure of noncompactness of bounded sets and linear operators', in: *Studies in Algebra and Mathematical Analysis* (Karta Moldovenjaski, Kishinev, 1965), 45–54.
[9] M. González and A. M. Abejón, *Tauberian Operators,* Operator Theory: Advances and Applications, 194 (Birkhäuser, Basel, 2010).
[10] R. C. James, 'Some self-dual properties of normed linear spaces', in: *Sympos. Infinite-Dimensional Topology (Louisiana State University, Baton Rouge, LA, 1967)*, Annals of Mathematics Studies, 69 (ed. B. D. Anderson) (Princeton University Press, Princeton, NJ, 1972), 159–175.
[11] M. Kacena, O. F. K. Kalenda and J. Spurny, 'Quantitative Dunford–Pettis property', *Adv. Math.* **234** (2013), 488–527.
[12] A. Kryczka, S. Prus and M. Szczepanik, 'Measure of weak noncompactness and real interpolation of operators', *Bull. Aust. Math. Soc.* **62**(3) (2000), 389–401.
[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I. Sequence Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 92 (Springer, Berlin–New York, 1977).
[14] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. II. Function Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 97 (Springer, Berlin–New York, 1979).
[15] E. Odell and H.-O. Tylli, 'Weakly compact approximation in Banach spaces', *Trans. Amer. Math. Soc.* **357**(3) (2005), 1125–1159.
[16] M. Raja, 'Finitely dentable functions, operators and sets', *J. Convex Anal.* **15**(2) (2008), 219–233.
[17] M. Raja, 'Super WCG Banach spaces', *J. Math. Anal. Appl.* **439**(1) (2016), 183–196.
[18] E. Saksman and H.-O. Tylli, 'New examples of weakly compact approximation in Banach spaces', *Ann. Acad. Sci. Fenn. Math.* **33**(2) (2008), 429–438.
[19] K. Tu, 'Convexification of super weakly compact sets and measure of super weak noncompactness', *Proc. Amer. Math. Soc.* **149**(6) (2021), 2531–2538.

[20] K. Tu, ‘Measure of super weak noncompactness in some Banach sequence spaces’, *J. Math. Anal. Appl.* **496**(2) (2021), Article ID 124813, 11 pages.

[21] H.-O. Tylli, ‘The essential norm of an operator is not self-dual’, *Israel J. Math.* **91**(1–3) (1995), 93–110.

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