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ATIYAH CLASS AND CHERN CHARACTER FOR GLOBAL MATRIX FACTORISATIONS

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Abstract We define the Atiyah class for global matrix factorisations and use it to give a formula for the categorical Chern character and the boundary-bulk map for matrix factorisations, generalising the formula in the local case obtained in [12]. Our approach is based on developing the Lie algebra analogies observed by Kapranov [7] and Markarian [9].

Keywords and phrases: matrix factorisation; Atiyah class; categorical Chern character; boundary-bulk map; Hochschild-Kostand-Rosenberg isomorphism

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Introduction

Recall that for a vector bundle E over a smooth algebraic variety X, one has a natural class $\mathrm{at}(E) \in \mathrm{Ext}^1(E, \Omega^1_X \otimes E)$, called the *Atiyah class* of E, such that the Chern character of E is obtained as $\mathrm{tr}(\exp(\mathrm{at}(E)))$ when X is a projective smooth variety over $\mathbb{C}$ (see [1]). This construction also generalises to bounded complexes of vector bundles (see [6]). Furthermore, it shows up in the formula for the categorical Chern character and, more generally, the boundary–bulk map (see [3], [15]).

The goal of this article is to generalise the construction of the Atiyah class to the case of (global) matrix factorisations and to give a formula for the categorical boundary–bulk map for matrix factorisations.

Thus, we start with a smooth scheme X over $\mathbb{C}$ equipped with a function w. We refer to [5], [8], [14] for the background on categories of matrix factorisations.

We want to construct an analogue of Atiyah class for a matrix factorisation (E,δ) of w. In the case when X is affine, the construction of such an Atiyah class is known (see [4], [18]).

It turns out that in the case when $w \neq 0$ one can still construct a certain class $\hat{\mathrm{at}}(E)$ (see below) that reduces to $1 + \mathrm{at}(E)$ when $w = 0$.

Below we denote by $\mathrm{Hom}^*_{\mathrm{MF}(w)}(\cdot,\cdot)$ the cohomology of the ($\mathbb{Z}/2$-graded) morphisms spaces in the category of matrix factorisations of w. When considering the Hochschild homology/cohomology of $\mathrm{MF}(w)$, we work over $k[u,u^{-1}]$, where $\deg(u)=2$ (using the $\mathbb{Z}/2$-grading on the category $\mathrm{MF}(w)$).

Main construction (see Section 1.1). *For every matrix factorisation E of w we construct a natural class*

$$\hat{\mathrm{at}}(E)\in \mathrm{Hom}^0_{\mathrm{MF}(w)}(E,[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]\otimes E),$$

such that the image of $\hat{\mathrm{at}}(E)$ under the natural projection $\mathrm{Hom}^0_{\mathrm{MF}(w)}(E,[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]\otimes E)\to \mathrm{Hom}^0_{\mathrm{MF}(w)}(E,E)$ is the identity element id_E. The formation of $\hat{\mathrm{at}}(E)$ is functorial and compatible with pull-backs.

Next, we define the class

$$\exp(\mathrm{at}(E))\in \mathrm{Hom}^0_{\mathrm{MF}(w)}(E,(\Omega^\bullet_X,\wedge dw)\otimes E)$$

as the composition of the iterated map

$$\hat{\mathrm{at}}(E)^{(n)}:E\to[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]^{\otimes n}\otimes E\to S^n[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]\otimes E,$$

where $n=\dim X$, with the isomorphism

$$S^n[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]\otimes E\xrightarrow{\sim}(\Omega^\bullet_X,\wedge dw)\otimes E$$

induced by the isomorphim

$$S^n[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]=[\mathcal{O}_X\xrightarrow{ndw}\Omega^1\xrightarrow{(n-1)dw}\to\dots]\to(\Omega^\bullet_X,\wedge dw) \tag{0.1}$$

given by $\alpha_0\mapsto\alpha_0$, $\alpha_i\mapsto\frac{\alpha_i}{n(n-1)\dots(n-i+1)}$, where $\alpha_i\in\Omega^i_X$.

The above definition may look a bit strange; however, it is easily explained by the fact that when one tries to recover $\exp(x)$ from $(1+x)^n$ in the ring $\mathbb{Q}[x]/(x^{n+1})$, one has to rescale x^i by the factor $\frac{1}{n(n-1)\dots(n-i+1)}$.

Below we view $\exp(\mathrm{at}(E))$ as an element of $\mathbb{H}^0$ of the $\mathbb{Z}/2$-graded complex $\underline{\mathrm{Hom}}(E,E)\otimes(\Omega^\bullet_X,\wedge dw)$ and denote by

$$\mathrm{str}:\underline{\mathrm{Hom}}(E,E)\otimes(\Omega^\bullet_X,\wedge dw)\to(\Omega^\bullet_X,\wedge dw)$$

the supertrace morphism. Here is our main result.

Theorem A. *Assume now that $w=0$ on the critical locus of w (set-theoretically). Under the natural identification*

$$HH_*(\mathrm{MF}(w))\simeq H^*(X,(\Omega^\bullet_X,\wedge dw)), \tag{0.2}$$

the categorical boundary–bulk map

$$\mathrm{Hom}^*_{\mathrm{MF}(w)}(E,E)\to HH_*(\mathrm{MF}(w))$$

for a matrix factorisation E of w is equal to the map induced on hypercohomology by the map

$$\underline{\mathrm{Hom}}(E,E)\to(\Omega^\bullet_X,\wedge dw):x\mapsto\mathrm{str}(\exp(\mathrm{at}(E))\cdot x)$$

in the $\mathbb{Z}/2$-graded derived category of X.

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Note that we give two constructions of isomorphism (0.2): an abstract one coming from analogies with Lie theory described below and the one given by an explicit chain map (this construction is due to [8]).

In the case of matrix factorisations over a regular local ring there is a simpler formula for the categorical Chern character obtained in [12]. It can be derived from the above theorem using a connection on the underlying vector bundle of E (see Remark 1.2). In the case of global matrix factorisations, a formula similar to the one in Theorem A was obtained by Platt [10]. However, his definition of $\exp(\mathrm{at}(E))$ is much more complicated (based on some explicit resolutions of the relevant objects in the derived category).

Note that for $w = 0$ – that is, in the classical case of vector bundles – we get a new proof of the compatibility of the categorical Chern character with the classical one, originally proved by Caldararu [3]. Our proof is more conceptual than that in [3]: We check the key techical statement (see Lemma 3.1 below) without using Čech representatives.[1]

The proof of Theorem A uses analogues of some constructions of Markarian [9] for matrix factorisations. Recall that he used the Atiyah classes to equip the shifted tangent bundle $T_X[-1]$ with a structure of a Lie algebra in the derived category of X (this construction goes back to Kapranov [7]) and showed that in an appropriate sense the universal enveloping of this algebra can be identified with the sheafified Hochschild cohomology $\underline{HH}^*(X)$. Furthermore, the Hochschild-Kostant-Rosenberg isomorphism can be viewed as an analogue of the Poincaré-Birkhoff-Witt theorem in this case.

It turns out that there is a similar Lie context for the sheafified Hochschild cohomology of the category of matrix factorisations, $\underline{HH}^* \mathrm{MF}(X, w)$. The general principle is that the picture for $w \neq 0$ should be a deformation of the picture for $w = 0$. In Lie theory there is a well-known way of deforming the universal enveloping algebras $U(\mathfrak{g})$ of a Lie algebra starting from a central extension of Lie algebras

$$0 \to k \cdot \mathbf{1} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0. \tag{0.3}$$

Namely, one can view $\mathbf{1} \in \widetilde{\mathfrak{g}}$ as a central element of $U(\widetilde{\mathfrak{g}})$ and consider the quotient $U(\widetilde{\mathfrak{g}})/(\mathbf{1} - 1)$, which is a deformation of $U(\widetilde{\mathfrak{g}})/(\mathbf{1}) \simeq U(\mathfrak{g})$.

Given a function w, we will equip the $\mathbb{Z}/2$-graded complex

$$L_w := [T_X \xrightarrow{i_{dw}} \mathcal{O}_X]$$

(where $\mathcal{O}_X$ is placed in degree 0) with a structure of a Lie algebra, so that the exact triangle

$$\mathcal{O}_X \to [T_X \xrightarrow{i_{dw}} \mathcal{O}_X] \to T_X[1]$$

can be viewed as a central extension of Lie algebras in $D(X)$, the $\mathbb{Z}/2$-graded derived category of X (see Subsection 2.3). Note that such a construction would not work in the usual $\mathbb{Z}$-graded derived category because it is $T_X[-1]$ that has the Lie algebra structure, not $T_X[1]$.

Extending the picture of Markarian [9], we will show that $\underline{HH}^* \mathrm{MF}(X, w)$ can be viewed as the corresponding quotient of the universal enveloping algebra, $U(L_w)/(\mathbf{1} - 1)$.

[1] We were not able to understand Markarian's proof of a similar statement [9, Proposition 5].

In Section 4 we consider a version of the above picture for a $\mathbb{Z}$-graded category of matrix factorisations defined in the presence of a $\mathbb{G}_m$-action. More precisely, we assume that X is equipped with a $\mathbb{G}_m$-action and we have a function W on X satisfying $W(\lambda x) = \lambda W(x)$. In this context one has a natural $\mathbb{Z}$-graded dg-category $\mathrm{MF}_{\mathbb{G}_m}(X, W)$ of $\mathbb{G}_m$-*equivariant matrix factorisations of* W. We prove that the analogue of Theorem A holds in this context. In the particular case when $W = 0$ and the action of $\mathbb{G}_m$ is trivial, the category $\mathrm{MF}_{\mathbb{G}_m}(X, 0)$ is equivalent to the usual $\mathbb{Z}$-graded derived category of X, so in this case we recover the classical picture for the latter category as described in [3], [15].

Note that a more natural context for $\mathbb{Z}$-graded categories of matrix factorisations involves equivariant matrix factorisations with respect to an algebraic group Γ equipped with a surjective homomorphism $\chi : \Gamma \to \mathbb{G}_m$ such that $\ker(\chi)$ is finite. However, the corresponding picture already has a stacky flavour. We intend to consider it elsewhere, along with a stacky version of Theorem A.

Conventions and notation. We work with matrix factorisations of a regular function w on a smooth scheme X of dimension n over a field of characteristic 0. By $\mathrm{MF}(w) = \mathrm{MF}(X, w)$ we denote the corresponding derived category of matrix factorisations. Whenever we need to use the Hochschild-Kostant-Rosenberg (HKR) isomorphism for sheafified Hochschild homology of the category of matrix factorisations of w, we assume that 0 is the only critical value of w. We denote by $D(X)$ the $\mathbb{Z}/2$-graded derived category of X and by $D(X)_{\mathbb{Z}}$ the usual $\mathbb{Z}$-graded derived category.

1. Definition of the Atiyah class

1.1. Global definition

Recall that the Atiyah class for vector bundles (or bounded complexes thereof) is defined using the canonical exact sequence

$$0 \to \Delta_*\Omega^1_X \to \mathcal{O}_{\Delta^{(2)}} \to \mathcal{O}_\Delta \to 0. \tag{1.1}$$

Here $\mathcal{O}_{\Delta^{(2)}} := \mathcal{O}_{X^2}/I_\Delta^2$, where $I_\Delta \subset \mathcal{O}_{X^2}$ is the ideal sheaf of the diagonal $\Delta \subset X^2$. For a vector bundle E (or any object in $D(X)$), we tensor the above sequence with p_2^*E and apply p_{1*} to get the extension sequence

$$0 \to \Omega^1_X \otimes E \to J(E) \to E \to 0$$

representing $\mathrm{at}(E)$.

Equivalently, we can view $[\Delta_*\Omega^1_X \to \mathcal{O}_{\Delta^{(2)}}]$ (in degrees -1 and 0) as a complex quasi-isomorphic to $\mathcal{O}_\Delta$. Then the natural projection to $(\Omega^1_X)_\Delta[1] := \Delta_*\Omega^1_X[1]$ gives a morphism in $D(X^2)$,

$$\mathrm{at}^{\mathrm{univ}} : \mathcal{O}_\Delta \to (\Omega^1_X)_\Delta[1], \tag{1.2}$$

called the *universal Atiyah class*. From this morphism of kernels we get a morphism of functors, whose value on E is $\mathrm{at}(E)$.

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Lemma 1.1. *The universal Atiyah class,* $\mathrm{at}^{\mathrm{univ}}$*, is equal to the composition*

$$\mathcal{O}_\Delta \xrightarrow{\mathrm{at}(\mathcal{O}_\Delta)} \Omega^1_{X^2} \otimes \mathcal{O}_\Delta[1] \simeq \Delta_*\Delta^*\Omega^1_{X^2}[1] \to \Delta_*\Omega^1_X[1],$$

where the last arrow is induced by the canonical map $\Delta^*\Omega^1_{X^2} \to \Omega^1_X$*. Equivalently, if we use the decomposition* $\Omega^1_{X^2} = p_1^*\Omega^1_X \oplus p_2^*\Omega^1_X$*, then* $\mathrm{at}^{\mathrm{univ}}$ *is equal to the component*

$$\mathrm{at}^1(\mathcal{O}_\Delta) \in \mathrm{Hom}(\mathcal{O}_\Delta, p_1^*\Omega^1_X \otimes \mathcal{O}_\Delta[1]) \simeq \mathrm{Hom}(\mathcal{O}_\Delta, \Delta_*\Omega^1_X[1])$$

of $\mathrm{at}(\mathcal{O}_\Delta)$.

Proof. (See [16, Section 5.5].) The class $\mathrm{at}(\mathcal{O}_\Delta)$ corresponds to the canonical extension

$$0 \to \Omega^1_{X^2} \otimes \mathcal{O}_\Delta \to J(\mathcal{O}_\Delta) \to \mathcal{O}_\Delta \to 0,$$

where $J(?)$ denotes the sheaf of first-order jets. By definition, $J(\mathcal{O}_\Delta)$ is obtained as the push-forward of $\mathcal{O}_{X^4}/(J_{\Delta_{13}} + J_{\Delta_{24}})^2 \otimes \mathcal{O}_{\Delta_{34}}$ under $p_{12} : X^4 \to X^2$, where $\Delta_{ij} \subset X^4$ are partial diagonals. Using the identification of Δ_{34} with X^3 we get

$$J(\mathcal{O}_\Delta) \simeq p_{12*}\mathcal{O}_{X^3}/(J_{\Delta_{13}} + J_{\Delta_{23}})^2.$$

Hence, we have a natural map

$$J(\mathcal{O}_\Delta) \simeq p_{12*}\mathcal{O}_{X^3}/(J_{\Delta_{13}} + J_{\Delta_{23}})^2 \to p_{12*}\mathcal{O}_{X^3}/(J^2_{\Delta_{13}} + J_{\Delta_{23}}) \simeq \mathcal{O}_{X^2}/J^2_\Delta.$$

One can check that it is compatible with the projection $\Omega^1_{X^2} \otimes \mathcal{O}_\Delta \to p_1^*\Omega^1_X \otimes \mathcal{O}_\Delta$. Thus, we get a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_{X^2} \otimes \mathcal{O}_\Delta & \longrightarrow & J(\mathcal{O}_\Delta) & \longrightarrow & \mathcal{O}_\Delta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & p_1^*\Omega^1_X \otimes \mathcal{O}_\Delta & \longrightarrow & \mathcal{O}_{\Delta^{(2)}} & \longrightarrow & \mathcal{O}_\Delta & \longrightarrow & 0
\end{array}$$

and our assertion follows. □

We observe that the sequence (1.1) has the following analogue in the category of (coherent) matrix factorisations $\mathrm{MF}(\widetilde{w})$, where $\widetilde{w} = w \otimes 1 - 1 \otimes w \in H^0(X^2, \mathcal{O})$. Note that we have a natural functor

$$\Delta_* : \mathrm{MF}(X, 0) \to \mathrm{MF}(X^2, \widetilde{w}),$$

because $\widetilde{w}|_\Delta = 0$. We denote by $\mathcal{O}^{\widetilde{w}}_\Delta \in \mathrm{MF}(X^2, \widetilde{w})$ the image of $\mathcal{O}_X$ under this functor. Let us define the matrix factorisaton $\mathcal{O}^{(2)}_{\Delta,\widetilde{w}}$ of $\widetilde{w}$ as follows:

$$\mathcal{O}^{(2)}_{\Delta,\widetilde{w}} = \mathcal{O}_{X^2}/I^2_\Delta \oplus \mathcal{O}_\Delta[1]$$
$$\delta_0 = -\widetilde{w} \bmod I^2_\Delta = -dw : \mathcal{O}_\Delta \to I_\Delta/I^2_\Delta \subset \mathcal{O}_{\Delta^{(2)}}, \quad \delta_1 = -1 : \mathcal{O}_{\Delta^{(2)}} \to \mathcal{O}_\Delta.$$

Then we have an exact sequence of matrix factorisations

$$0 \to \Delta_*[\mathcal{O}_X \xrightarrow{dw} \Omega^1_X][1] \to \mathcal{O}^{(2)}_{\Delta,\widetilde{w}} \to \mathcal{O}^{\widetilde{w}}_\Delta \to 0 \tag{1.3}$$

that can be viewed as a morphism in the derived category of $\mathrm{MF}(X^2,\widetilde{w})$,

$$\hat{\mathrm{at}}^{\mathrm{univ}}:\mathcal{O}_\Delta^{\widetilde{w}}\to\Delta_*[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X].$$

Note that by the definition of the shift functor on complexes, the complex $\Delta_*[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X][1]$ has $\mathcal{O}_X$ in odd degree, Ω^1_X in even degree and the differential is given by $-dw$. This is compatible with the sign in the definition of the differential on $\mathcal{O}^{(2)}_{\Delta,\widetilde{w}}$.

Now given a matrix factorisation $E\in\mathrm{MF}(X,w)$, we can tensor the exact sequence (1.3) with p_2^*E and then apply Rp_{1*}. This will produce a class

$$\hat{\mathrm{at}}(E)\in\mathrm{Hom}^0(E,[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]\otimes E).$$

Equivalently, we can obtain $\hat{\mathrm{at}}(E)$ from the morphism of kernels

$$\mathcal{O}_\Delta^{\widetilde{w}}\to\Delta_*[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]$$

in $\mathrm{MF}(\widetilde{w})$ corresponding to the sequence (1.3). By definition, this morphism in $\mathrm{MF}(\widetilde{w})$ uses the quasi-isomorphism of the cone of the first arrow in (1.3) with $\mathcal{O}_\Delta^{\widetilde{w}}$. It will be convenient to use a slightly more compact resolution of $\mathcal{O}_\Delta^{\widetilde{w}}$, which should be viewed as a curved analogue of the resolution $[(\Omega^1_X)_\Delta\to\mathcal{O}_{\Delta^{(2)}}]$ of $\mathcal{O}_\Delta$. Namely, let us equip $\mathcal{O}_{\Delta^{(2)}}\oplus(\Omega^1_X)_\Delta[1]$ with the structure of a matrix factorisation of $\widetilde{w}$ using the maps

$$\delta_0:(\Omega^1_X)_\Delta\simeq I_\Delta/I_\Delta^2\hookrightarrow\mathcal{O}_{\Delta^{(2)}},\quad \delta_1:\mathcal{O}_{\Delta^{(2)}}\xrightarrow{1}\mathcal{O}_\Delta\xrightarrow{dw}(\Omega^1_X)_\Delta.$$

Lemma 1.2.

(i) *The natural map in* $\mathrm{MF}(\widetilde{w})$,

$$q:[\mathcal{O}_{\Delta^{(2)}}\oplus\Delta_*\Omega^1_X[1],\delta]\to\mathcal{O}_\Delta^{\widetilde{w}},$$

induced by the projection $\mathcal{O}_{\Delta^{(2)}}\to\mathcal{O}_\Delta$, *is an isomorphism in the derived category and* $\hat{\mathrm{at}}^{\mathrm{univ}}$ *is equal to the composition of* q^{-1} *with the natural morphism in* $\mathrm{MF}(\widetilde{w})$,

$$[\mathcal{O}_{\Delta^{(2)}}\oplus\Delta_*\Omega^1_X[1],\delta]\to\Delta_*[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X],\tag{1.4}$$

which is identity on $\Delta_*\Omega^1_X[1]$ *and the natural projection to* $\mathcal{O}_\Delta$ *on* $\mathcal{O}_{\Delta^{(2)}}$.

(ii) *The composition*

$$\mathcal{O}_\Delta^{\widetilde{w}}\to\Delta_*[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X]\to\Delta_*\mathcal{O}_X=\mathcal{O}_\Delta^{\widetilde{w}}$$

is the identity map. Hence, the image of $\hat{\mathrm{at}}(E)$ *under the natural projection* $\mathrm{Hom}^0(E,E\otimes[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X])\to\mathrm{Hom}^0(E,E)$ *is the identity element* id_E.

(iii) *The universal Atiyah class* $\hat{\mathrm{at}}^{\mathrm{univ}}$ *is obtained as the composition of the Atiyah class*

$$\hat{\mathrm{at}}(\mathcal{O}_\Delta^{\widetilde{w}}):\mathcal{O}_\Delta^{\widetilde{w}}\to[\mathcal{O}_{X^2}\xrightarrow{d\widetilde{w}}\Omega^1_{X^2}]\otimes\mathcal{O}_\Delta^{\widetilde{w}}$$

with the projection

$$[\mathcal{O}_{X^2}\xrightarrow{d\widetilde{w}}\Omega^1_{X^2}]\otimes\mathcal{O}_\Delta^{\widetilde{w}}\to[\mathcal{O}_{X^2}\xrightarrow{d(w\otimes 1)}p_1^*\Omega^1_X]\otimes\mathcal{O}_\Delta^{\widetilde{w}}\simeq\Delta_*[\mathcal{O}_X\xrightarrow{dw}\Omega^1_X].$$

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Proof.

(i) Let $f : \Delta_*[\mathcal{O}_X \xrightarrow{dw} \Omega^1_X][1] \to \mathcal{O}^{(2)}_{\Delta,\widetilde{w}}$ be the map from the sequence (1.3) (f is the identity on $\mathcal{O}_\Delta$ and is the natural embedding into $\mathcal{O}_{\Delta^{(2)}}$ on $\Delta_*\Omega^1_X$). We have a natural quasi-isomorphism

$$\mathrm{Cone}(f) \to [\mathcal{O}_{\Delta^{(2)}} \oplus (\Omega^1_X)_\Delta[1], \delta],$$

which is identical on $\Delta_*\Omega^1_X[1]$ and $\mathcal{O}_{\Delta^{(2)}}$, zero on $\mathcal{O}_\Delta[1]$ and equal to $-dw : \mathcal{O}_\Delta \to \Delta_*\Omega^1_X$ on $\mathcal{O}_\Delta$. One can check that its composition with (1.4) is homotopic to the canonical projection $\mathrm{Cone}(f) \to \Delta_*[\mathcal{O}_X \xrightarrow{dw} \Omega^1_X]$ (one uses $\mathrm{id} : \mathcal{O}_\Delta[1] \to \mathcal{O}_\Delta[1]$ as a homotopy). This implies our assertion.

(ii) This follows easily from (i): The composition of (1.4) with the natural projection $\Delta_*[\mathcal{O}_X \xrightarrow{dw} \Omega^1_X] \to \Delta_*\mathcal{O}_X = \mathcal{O}^{\widetilde{w}}_\Delta$ is exactly the map q.

(iii) This is proved similarly to Lemma 1.1. □

1.2. Čech representative

Let (U_i) be an affine open covering of X. Over every U_i we can choose an algebraic connection

$$\nabla_i : E|_{U_i} \to \Omega_{U_i} \otimes E_{U_i},$$

which is even; that is, compatible with the $\mathbb{Z}/2$-grading on $E|_{U_i}$. Over each intersection $U_{ij} = U_i \cap U_j$ we have a 1-form with values in $\underline{\mathrm{End}}^0(E)$, $\alpha_{ij} \in \Omega^1 \otimes \underline{\mathrm{End}}^0(E)|_{U_{ij}}$, such that

$$\nabla_j - \nabla_i = \alpha_{ij}.$$

Assume for simplicity that X is separated, so all intersections $U_{i_1\ldots i_k}$ are still affine. Then for any matrix factorisations of w, E and F, we can calculate the space $\mathrm{Hom}^0(E,F)$ as the zeroth cohomology of the $\mathbb{Z}/2$-graded complex

$$(C^\bullet(\underline{\mathrm{Hom}}(E,F)), [\delta, ?] + d_C),$$

where $C^\bullet(?)$ denotes the Čech complex and d_C is the Čech differential. More precisely, the differential on $\alpha \in C^p(\underline{\mathrm{Hom}}(E,F))$ is $(-1)^p[\delta,\alpha] + d_C(\alpha)$.

Proposition 1.3. *For a matrix factorisation $E \in \mathrm{MF}(X,w)$, the Atiyah class $\hat{\mathrm{at}}(E)$ is represented by the cocycle*

$$(\mathrm{id}_E, -[\nabla_i,\delta], \alpha_{ij}) \in C^\bullet([\mathcal{O} \xrightarrow{dw} \Omega^1_X] \otimes \underline{\mathrm{End}}(E)).$$

Proof. We use the following general fact: If

$$0 \to E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3 \to 0 \tag{1.5}$$

is an exact sequence of matrix factorisations, then the corresponding class in $\mathrm{Hom}^1(E_3, E_1)$ is represented by the following Čech cocycle. First, we find local retractions $r_i : E_2 \to E_1$ of the embedding $E_1 \to E_2$, which are morphisms of $\mathbb{Z}/2$-graded $\mathcal{O}$-modules over U_i. Then we consider $\alpha_{ij} : E_3 \to E_1$ over U_{ij} such that $\alpha_{ij}g = r_j|_{U_{ij}} - r_i|_{U_{ij}}$. On the other hand, we

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consider $\beta_i : E_3 \to E_1[1]$ over U_i such that $\beta_i g = [\delta, r_i] = \delta_{E_1} r_i - r_i \delta_{E_2}$. Now we claim that the Čech cocycle $c = (\beta_i, \alpha_{ij})$ represents the class corresponding to our extension. Indeed, by definition, this class corresponds to the obvious projection $\mathrm{Cone}(f) \to E_1[1]$ under the isomorphism

$$\mathrm{Hom}^1(E_3, E_1) \xrightarrow{\sim} \mathrm{Hom}^1(\mathrm{Cone}(f), E_1)$$

induced by g. Now the image of c under the morphism

$$C^1(\underline{\mathrm{Hom}}(E_3, E_1)) \to C^1(\underline{\mathrm{Hom}}(\mathrm{Cone}(f), E_1))$$

is given by the cocycle

$$(\beta_i g, \alpha_{ij} g) = ([\delta, r_i], r_j - r_i).$$

Subtracting the coboundary of the element $(r_i) \in C^0(\underline{\mathrm{Hom}}(\mathrm{Cone}(f), E_1))$, we get the cocycle given by $(r_i f)$ – that is, by id_{E_1} – as claimed.

We apply the above general fact to the sequence

$$0 \to [\mathcal{O}_X \xrightarrow{dw} \Omega^1_X] \otimes E[1] \to \hat{J}(E) \to E \to 0 \tag{1.6}$$

obtained from (1.3) by tensoring with $p_2^* E$ and taking the push-forward p_{1*}. Note that

$$\hat{J}(E) = J(E) \oplus E[1]$$

as a $\mathbb{Z}/2$-graded vector bundle. A connection ∇_i on $E|_{U_i}$ can be viewed as a retraction

$$\nabla_i : J(E) \to \Omega^1_X \otimes E : 1 \otimes s \mapsto \nabla_i(s).$$

This leads to the claimed formula. □

Remark. In the case when the underlying vector bundle E is trivialised, we can take ∇ to be the corresponding connection (defined globally). Then $[\nabla, \delta]$ is the matrix of 1-forms obtained by taking differentials of the entries of δ (viewed as a matrix of functions). Using this one can check that in this case the formula of Theorem A for the Chern character is equivalent to the one obtained in [12] in the case of matrix factorisations over a regular local ring.

Using Čech representatives one can easily derive the following compatibility of the Atiyah class construction with the tensor product of a matrix factorisation by a complex of vector bundles.

Lemma 1.4. *For $E \in \mathrm{MF}(X, w)$ and $F \in D(X)$, let us consider $E \otimes F \in \mathrm{MF}(X, w)$. Then one has*

$$\hat{\mathrm{at}}(E \otimes F) = \hat{\mathrm{at}}(E) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{at}(F).$$

1.3. Dolbeault representative

In the case when X is a complex manifold and w is a holomorphic function, the space $\mathrm{Hom}^0(E, F)$ can be calculated using Dolbeault complex

$$(\Omega^{0,*}(\underline{\mathrm{Hom}}(E, F)), [\delta, ?] + \overline{\partial}).$$

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In the case $w = 0$ it is well known that the Atiyah class is represented by the $(1,1)$-part of the curvature of any C^∞-connection on E, compatible with the holomorphic structure.

We have the following analogue for matrix factorisations.

Proposition 1.5. *Let (E,δ) be a holomorphic matrix factorisation of w. Let ∇ be an even C^∞-connection on E, compatible with the holomorphic structure, and let $F^{1,1}$ be the $(1,1)$-part of the curvature of ∇. Then $\hat{\mathrm{at}}(E)$ is represented by the cocycle*

$$(\mathrm{id}_E, -[\nabla,\delta], F^{1,1}) \in \Omega^{\le 1,*}(\underline{\mathrm{End}}(E,E)),$$

where the differential on the latter complex is given by $[\delta, ?] + \wedge dw + \overline{\partial}$.

Proof. The proof is similar to that of Proposition 1.3, with the Čech resolution replaced by the Dolbeault resolution. First, one checks that for an exact sequence of matrix factorisations (1.5), a choice of C^∞-retractions $r : E_2 \to E_1$ gives a Dolbeault representative for the corresponding class in $\mathrm{Hom}^1(E_3,E_1)$. Namely, one should consider

$$(\beta,\alpha) \in \Omega^{0,0}(\underline{\mathrm{Hom}}(E_3,E_1)_1) \oplus \Omega^{(0,1)}(\underline{\mathrm{Hom}}(E_3,E_1)_0),$$

where

$$\alpha g = \overline{\partial}(r), \quad \beta g = [\delta, r]$$

(the proof uses the Dolbeault complex of $\mathrm{Cone}(f)$, similar to Proposition 1.3).

Now we apply this for the exact sequence (1.6). We use $\nabla^{1,0}$, the $(1,0)$-part of the connection ∇, to get a C^∞-retraction of the embedding $\Omega^{1,0} \otimes E \to J(E)$. This leads to the claimed formula, where the $(1,1)$-curvature $F^{1,1}$ appears as $[\overline{\partial}, \nabla^{1,0}]$. □

2. Lie algebra analogies

2.1. Lie bracket in the case $w = 0$

Recall that the Atiyah class $\mathrm{at}(\Omega^1) : \Omega^1_X \to \Omega^1_X \otimes \Omega^1_X[1]$ factors through a map $\Omega^1_X \to S^2(\Omega^1_X)[1]$ whose dual can be viewed as a Lie bracket on $T_X[-1]$ (see [7], [9]).

In addition, there is a morphism

$$\iota : T_X[-1] \to \underline{HH}^*(X) = Rp_{1*}\underline{\mathrm{End}}(\mathcal{O}_\Delta) \tag{2.1}$$

obtained by adjunction from the component

$$\mathrm{at}^1(\mathcal{O}_\Delta) \in \mathrm{Hom}(\mathcal{O}_\Delta, p_1^*\Omega^1_X \otimes \mathcal{O}_\Delta[1]) \simeq \mathrm{Hom}(p_1^* T_X[-1], \underline{\mathrm{End}}(\mathcal{O}_\Delta))$$

of the Atiyah class $\mathrm{at}(\mathcal{O}_\Delta) \in \mathrm{Hom}(\mathcal{O}_\Delta, \Omega^1_{X^2} \otimes \mathcal{O}_\Delta[1])$.

2.2. HKR-isomorphisms in the case $w = 0$

Recall that in [9, Theorem 1] Markarian showed that $\underline{HH}^*(X)$ can be identifed with the universal enveloping algebra $U(T_X[-1])$. By definition, this means that the above map ι satisfies the identity

$$\iota([x,y]) = \iota(x)\iota(y) - \iota(y)\iota(x) \tag{2.2}$$

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understood as the equality of morphisms $T_X[-1]\otimes T_X[-1]\to \underline{HH}^*(X)$, and the morphism

$$I^{abs}: S^*(T_X[-1]) \to T^*(T_X[-1]) \to \underline{HH}^*(X), \tag{2.3}$$

induced by ι and by the multiplication on $\underline{HH}^*(X)$, is an isomorphism (see [9, Definition 4]).

One of the key tools in the arguments of [9] is the natural duality between $\underline{HH}_*(X)$ and $\underline{HH}^*(X)$ induced by the canonical functional

$$\varepsilon : \underline{HH}_*(X) = \Delta^*(\mathcal{O}_\Delta) \to \mathcal{O}_X$$

and by the canonical action

$$\mathcal{D}_0 : \underline{HH}^*(X)\otimes \underline{HH}_*(X) \to \underline{HH}_*(X).$$

Lemma 2.1. *The composition*

$$\varepsilon\circ\mathcal{D}_0 : \underline{HH}^*(X)\otimes \underline{HH}_*(X) \to \mathcal{O}_X$$

is a perfect pairing and corresponds to the natural isomorphism

$$\begin{aligned}\underline{\mathrm{Hom}}(\underline{HH}_*(X),\mathcal{O}_X) &= \underline{\mathrm{Hom}}(\Delta^*\mathcal{O}_\Delta,\mathcal{O}_X) \simeq Rp_{1*}\Delta_*\underline{\mathrm{Hom}}(\Delta^*\mathcal{O}_\Delta,\mathcal{O}_X)\\ &\simeq Rp_{1*}\underline{\mathrm{Hom}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta).\end{aligned}$$

Proof. Note that for any morphism $f: X\to Y$ and sheaves F on Y and G on X, the composition of the natural maps

$$\underline{\mathrm{Hom}}(F,f_*G) \xrightarrow{f^*} Rf_*\underline{\mathrm{Hom}}(f^*F,f^*f_*G) \xrightarrow{\mathrm{can}_G} Rf_*\underline{\mathrm{Hom}}(f^*F,G)$$

is precisely the (sheafified) adjunction isomorphism. Applying this to $f=\Delta$, $F=\mathcal{O}_\Delta$, $G=\mathcal{O}_X$, we obtain that the composition

$$\underline{\mathrm{Hom}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta) \xrightarrow{\Delta^*} \Delta_*\underline{\mathrm{Hom}}(\Delta^*\mathcal{O}_\Delta,\Delta^*\mathcal{O}_\Delta) \xrightarrow{\epsilon} \Delta_*\underline{\mathrm{Hom}}(\Delta^*\mathcal{O}_\Delta,\mathcal{O}_X)$$

is the adjunction isomorphism. Now the assertion follows from the fact that applying Rp_{1*} to the first arrow we get the map $\underline{HH}^*(X)\to \underline{\mathrm{Hom}}(\Delta^*\mathcal{O}_\Delta,\Delta^*\mathcal{O}_\Delta)$ defining $\mathcal{D}_0$. □

Dualising the composition

$$S^*(T_X[-1])\otimes \underline{HH}_*(X) \xrightarrow{I^{abs}\otimes\mathrm{id}} \underline{HH}^*(X)\otimes \underline{HH}_*(X) \xrightarrow{\varepsilon\circ\mathcal{D}_0} \mathcal{O}_X$$

we get a map

$$I_{abs}: \underline{HH}_*(X) \to S^*(T_X[-1])^\vee \simeq S^*(\Omega_X[1]), \tag{2.4}$$

which is an isomorphism (see [9, Proposition 3]).

Note that essentially by definition, our isomorphisms I^{abs} and I_{abs} are compatible with the duality of Lemma 2.1 and the natural duality between $S^*(T_X[-1])$ and $S^*(\Omega_X[1])$.

Theorem 2.2. *The isomorphisms I^{abs} and I_{abs} in $D(X)$ coincide with the HKR isomorphisms I^{HKR} and I_{HKR}, given by the explicit chain maps in [3].*

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Proof. Recall that the action of $T_X[-1]$ on $\underline{HH}_*(X)$ is obtained by applying Δ^* to the map $p_1^* T_X[-1] \otimes \mathcal{O}_\Delta \to \mathcal{O}_\Delta$ given by $\mathrm{at}^{\mathrm{univ}} = \mathrm{at}^1(\mathcal{O}_\Delta)$. We are going to realise the latter map by an explicit chain map of complexes on X^2, replacing $\mathcal{O}_\Delta$ by its completed bar resolution $\mathcal{B}_\bullet$ (see [17], [3, Section 4]). Recall that $\mathcal{B}_i$ is the push-forward to X^2 of the formal completion of X^{i+2} along the small diagonal. We denote local sections of $\mathcal{B}_i$ as $[a_0 \otimes a_1 \otimes \ldots \otimes a_{i+1}]$, where a_j are local functions on X.

We claim that the map $\mathrm{at}^{\mathrm{univ}}$ is represented by the chain map

$$p_1^* T_X \otimes \mathcal{B}_\bullet \to \mathcal{B}_\bullet[1] : v \otimes [a_0 \otimes a_1 \otimes \ldots \otimes a_{i+1}] \mapsto (-1)^{i+1}[a_0 v(a_1) \otimes a_2 \otimes \ldots \otimes a_{i+1}].$$

Indeed, it is enough to look at the induced map of complexes $p_1^* T_X \otimes \mathcal{B}_\bullet \to \mathcal{O}_\Delta[1]$ or, equivalently, $\mathcal{B}_\bullet \to \Delta_* \Omega_X^1[1]$ induced by the map $f_1 : \mathcal{B}_1 \to \Delta_* \Omega^1 : [a_0 \otimes a_1 \otimes a_2] \mapsto a_2 a_0 da_1$. Now we observe that there is a quasi-isomorphism of complexes,

$$\begin{array}{ccccc}
\mathcal{B}_2 & \longrightarrow & \mathcal{B}_1 & \longrightarrow & \mathcal{B}_0 \\
\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle 1} \\
0 & \longrightarrow & \Delta_* \Omega_X^1 & \longrightarrow & \mathcal{O}_{\Delta^{(2)}} .
\end{array}$$

Because $\mathrm{at}^{\mathrm{univ}}$ is induced by the exact sequence (1.1), this implies our claim.

It follows that the map

$$(\mathrm{at}^{\mathrm{univ}})^{(i)} : p_1^* \bigwedge^i T_X \otimes \mathcal{O}_\Delta \to \mathcal{O}_\Delta[i]$$

obtained by iterating $\mathrm{at}^{\mathrm{univ}}$ and restricting to skew-symmetric tensors is represented by the chain map

$$\begin{aligned} p_1^* \bigwedge^i T_X \otimes \mathcal{B}_\bullet \to \mathcal{B}_\bullet[i] &: (v_1 \wedge \ldots \wedge v_i) \otimes [a_0 \otimes a_1 \otimes \ldots \otimes a_m] \\ &\mapsto (-1)^{im}[a_0 \langle v_1 \wedge \ldots \wedge v_i, da_1 \wedge \ldots \wedge da_i \rangle \otimes a_{i+1} \otimes \ldots \otimes a_m]. \end{aligned}$$

Composing with the projection $\mathcal{B}_\bullet[i] \to \mathcal{O}_\Delta[i]$, we see that $(\mathrm{at}^{\mathrm{univ}})^{(i)}$ is represented by the chain map $p_1^* \bigwedge^i T_X \otimes \mathcal{B}_\bullet \to \mathcal{O}_\Delta[i]$, which corresponds by duality to the map

$$\mathcal{B}^i \to p_1^* \Omega_X^i \otimes \mathcal{O}_\Delta \simeq \Delta_* \Omega_X^i : [a_0 \otimes a_1 \otimes \ldots \otimes a_{i+1}] \mapsto a_{i+1} a_0 da_1 \wedge \ldots \wedge da_i,$$

which is exactly the ith component of I_{HKR}. This proves the equality $I_{abs} = I_{HKR}$.

Recall (see [3, Section 4]) that the other HKR map

$$I^{HKR} : \bigoplus \bigwedge^i T_X[-i] \to Rp_{1*}\underline{\mathrm{End}}(\mathcal{O}_\Delta)$$

is essentially the dual of I_{HKR}: it is given by the composition

$$\bigoplus \bigwedge^i T_X[-i] = \underline{\mathrm{Hom}}(\bigoplus \Omega_X^i[i], \mathcal{O}_X) \to \underline{\mathrm{Hom}}(\Delta^* \mathcal{O}_\Delta, \mathcal{O}_X) \simeq Rp_{1*}(\underline{\mathrm{Hom}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

where the last isomorphism follows from the adjunction of (Δ^*, Δ_*).

On the other hand, it is easy to see that the composition

$$\underline{HH}^* \otimes \underline{HH}_* \xrightarrow{\mathcal{D}_0} \underline{HH}_* \xrightarrow{\varepsilon} \mathcal{O}_X$$

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corresponds to the same duality, so the equality $I^{abs} = I^{HKR}$ follows from the equality $I_{abs} = I_{HKR}$. □

Remark. The above theorem can be easily deduced from the arguments in the proof of [3, Proposition 4.4]. Note that I^{HKR} (and hence I^{abs}) is not an algebra homomorphism with respect to the natural algebra structures on $S^*(T_X[-1])$ and $\underline{HH}^*(X)$; to become one it has to be twisted by the square root of the Todd class (see [3], [2]).

2.3. The general case

The (usual) Atiyah class of the complex $L_w^\vee = [\mathcal{O}_X \xrightarrow{dw} \Omega_X^1]$ is an element

$$\mathrm{at}(L_w^\vee) : L_w^\vee \to \Omega_X^1[1] \otimes L_w^\vee.$$

In the $\mathbb{Z}/2$-graded derived category we have a natural morphism $\Omega_X^1[1] \to L_w^\vee$. Thus, composing the map $\mathrm{at}(L_w^\vee)$ with this morphism, we get a map

$$L_w^\vee \to L_w^\vee \otimes L_w^\vee,$$

or dualising, a map

$$[\cdot,\cdot] : L_w \otimes L_w \to L_w,$$

which factors through $T_X[1] \otimes L_w$.

Lemma 2.3. *The dual of the bracket $[\cdot,\cdot]$ factors in the $\mathbb{Z}/2$-graded derived category as a composition*

$$L_w^\vee \to S^2\Omega_X^1 \to \Omega_X^1 \otimes \Omega_X^1 \to L_w^\vee \otimes L_w^\vee.$$

Hence, the bracket $[\cdot,\cdot]$ is skew-symmetric and $\mathcal{O}_X \subset L_w$ is central with respect to it.

Proof. The proof is a slight variation of the proof of [9, Proposition 1.1]. By definition, the map $\mathrm{at}(L_w^\vee)$ corresponds to an exact sequence of complexes

$$0 \to p_{1*}[I_\Delta/I_\Delta^2 \otimes p_2^* L_w^\vee] \to p_{1*}[\mathcal{O}_{X^2}/I_\Delta^2 \otimes p_2^* L_w^\vee] \to p_{1*}[\mathcal{O}_\Delta \otimes p_2^* L_w^\vee] \to 0.$$

Now we use the natural morphism induced by the canonical differential $d_{X^2} : \mathcal{O}_{X^2} \to \Omega_{X^2}^1$:

$$I_\Delta/I_\Delta^3 \to \mathcal{O}_{X^2}/I_\Delta^3 \xrightarrow{-d_{X^2}} \mathcal{O}_{X^2}/I_\Delta^2 \otimes \Omega_{X^2}^1 \to \Omega_{X^2}/I_\Delta^2 \otimes p_2^*\Omega_X^1.$$

Note that under this morphism, $f \otimes 1 - 1 \otimes f \bmod I_\Delta^3$ is sent to $1 \otimes p_2^*(df)$. This morphism extends to a chain map of complexes

$$[\mathcal{O}_X \xrightarrow{\delta} p_{1*}(I_\Delta/I_\Delta^3)] \to p_{1*}[\mathcal{O}_{X^2}/I_\Delta^2 \otimes p_2^* L_w^\vee],$$

where $\delta(f) = f \cdot (1 \otimes w - w \otimes 1)$, and the map $\mathcal{O}_X \to p_{1*}(\mathcal{O}_{X^2}/I_\Delta^2)$ is given by $f \mapsto f \otimes 1 \bmod I_\Delta^2$. Furthermore, this chain map extends naturally to a morphism of exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_{1*}(I_\Delta^2/I_\Delta^3)[-1] & \longrightarrow & [\mathcal{O}_X \to p_{1*}(I_\Delta/I_\Delta^3)] & \longrightarrow & [\mathcal{O}_X \to p_{1*}(I_\Delta/I_\Delta^2)] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & p_{1*}[I_\Delta/I_\Delta^2 \otimes p_2^* L_w^\vee] & \longrightarrow & p_{1*}[\mathcal{O}_{X^2}/I_\Delta^2 \otimes p_2^* L_w^\vee] & \longrightarrow & p_{1*}[\mathcal{O}_\Delta \otimes p_2^* L_w^\vee] & \longrightarrow & 0 \end{array}$$

in which the leftmost vertical arrow can be identified with the natural map $S^2\Omega_X^1[-1] \to \Omega_X^1 \otimes L_w^\vee$. This implies our assertion. □

By analogy with morphism (2.1) we want to define a morphism

$$\iota_w : L_w \to \underline{HH}^*(\mathrm{MF}(X,w)) = Rp_{1*}\underline{\mathrm{End}}(\mathcal{O}_\Delta^{\widetilde{w}}) \tag{2.5}$$

in $D(X)$. For this we consider the universal Atiyah class

$$\hat{\mathrm{at}}^{\mathrm{univ}} = \hat{\mathrm{at}}^1(\mathcal{O}_\Delta^{\widetilde{w}}) \in \mathrm{Hom}(\mathcal{O}_\Delta^{\widetilde{w}}, p_1^* L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}}) \tag{2.6}$$

(see Lemma 1.2(iii)). Dualising it we get a morphism

$$p_1^* L_w \to \underline{\mathrm{End}}(\mathcal{O}_\Delta^{\widetilde{w}})$$

from which ι_w is obtained by adjunction.

Note that to identify an associative algebra U with the algebra of the form $U(\widetilde{\mathfrak{g}})/(\mathbf{1}-1)$ for a central extension of Lie algebras (0.3), one has to provide a linear map $\iota : \widetilde{\mathfrak{g}} \to U$ satisfying the universal enveloping algebra identity (2.2), such that $\iota(\mathbf{1}) = 1$ and the natural map

$$S(\widetilde{\mathfrak{g}})/(\mathbf{1}-1) \to U(\widetilde{\mathfrak{g}})/(\mathbf{1}-1)$$

induced by ι is an isomorphism. Note that the source of this map can be identified with $\varinjlim_i S^i(\widetilde{\mathfrak{g}})$, which is a better expression for us because it makes sense also in nonabelian categories.

We want to check analogues of these properties for the map (2.5) in the derived category. Similar to the proof of [9, Theorem 1], the universal enveloping algebra identity (2.2) for ι_w is equivalent to the condition that the skew-symmetrisation of the composition

$$\mathcal{O}_\Delta^{\widetilde{w}} \xrightarrow{\hat{\mathrm{at}}^1(\mathcal{O}_\Delta^{\widetilde{w}})} p_1^* L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}} \xrightarrow{\mathrm{id}\otimes\hat{\mathrm{at}}^1(\mathcal{O}_\Delta^{\widetilde{w}})} p_1^* L_w^\vee \otimes p_1^* L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}}$$

is equal to the composition

$$\mathcal{O}_\Delta^{\widetilde{w}} \xrightarrow{\hat{\mathrm{at}}^1(\mathcal{O}_\Delta^{\widetilde{w}})} p_1^* L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}} \xrightarrow{\mathrm{at}^1(p_1^* L_w^\vee)\otimes\mathrm{id}} p_1^* L_w^\vee \otimes p_1^* L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}}.$$

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But this follows immediately from the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_\Delta^{\widetilde{w}} & \xrightarrow{\hat{\mathrm{at}}^1(\mathcal{O}_\Delta^{\widetilde{w}})} & p_1^*L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}} \\ {\scriptstyle \hat{\mathrm{at}}^1(\mathcal{O}_\Delta^{\widetilde{w}})}\downarrow & & \downarrow{\scriptstyle \hat{\mathrm{at}}^1(p_1^*L_w^\vee\otimes\mathcal{O}_\Delta^{\widetilde{w}})} \\ p_1^*L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}} & \xrightarrow{\mathrm{id}\otimes\hat{\mathrm{at}}^1(\mathcal{O}_\Delta^{\widetilde{w}})} & p_1^*L_w^\vee \otimes p_1^*L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}} \end{array}$$

by applying Lemma 1.4 to the expression $\hat{\mathrm{at}}^1(p_1^*L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}})$.

The fact that $\hat{\mathrm{at}}(E)$ projects to the identity id_E implies (by taking $E = \mathcal{O}_\Delta^{\widetilde{w}}$) that the composition

$$\mathcal{O}_X \xrightarrow{\mathbf{1}} L_w \xrightarrow{\iota_w} \underline{HH}^*(\mathrm{MF}(X,w))$$

is the natural embedding of a unit.

Lastly, we need to check that the map

$$I^{abs,w} : \varinjlim_i S^i(L_w) \to \underline{HH}^*(\mathrm{MF}(X,w)), \tag{2.7}$$

induced by ι_w, is an isomorphism. It is easy to see that the limit here stabilises and we have

$$\varinjlim_i S^i(L_w) = S^n(L_w) \simeq (\textstyle\bigwedge^\bullet(T_X), i_{dw}),$$

where $n = \dim X$. Here the second isomorphism is dual to (0.1). The fact that the map (2.7) is an isomorphism can be checked formally locally using the Koszul resolution of the diagonal matrix factorisation (it also follows from Theorem 2.5 below and from the results of [8]).

Similar to the case of Lie algebras, where the Poincaré-Birkhoff-Witt theorem can be used to derive the Jacobi identity (see, e.g., [11, Chapter 5]), one can show that the properties proved above imply that the bracket $[\cdot,\cdot]$ on L_w satisfies the Jacobi identity (we will not use this fact).

Similar to the case $w = 0$, we have a canonical functional

$$\varepsilon : \underline{HH}_*(\mathrm{MF}(X,w)) = \Delta^*(\mathcal{O}_\Delta^{\widetilde{w}}) \to \mathcal{O}_X$$

coming from the adjoint pair of functors (Δ^*, Δ_*) between $\mathrm{MF}(X,0)$ and $\mathrm{MF}(X^2, \widetilde{w})$. On the other hand, we have the natural action of $\underline{HH}^*(\mathrm{MF}(X,w)) = Rp_{1*}\underline{\mathrm{End}}(\mathcal{O}_\Delta^{\widetilde{w}})$ on $\underline{HH}_*(\mathrm{MF}(X,w))$,

$$\mathcal{D} : \underline{HH}^*(\mathrm{MF}(X,w)) \otimes \underline{HH}_*(\mathrm{MF}(X,w)) \to \underline{HH}_*(\mathrm{MF}(X,w)).$$

Lemma 2.4. *The composition*

$$\varepsilon \circ \mathcal{D} : \underline{HH}^*(\mathrm{MF}(X,w)) \otimes \underline{HH}_*(\mathrm{MF}(X,w)) \to \mathcal{O}_X$$

is a perfect pairing, which corresponds to the natural isomorphism

$$\underline{\mathrm{Hom}}(\underline{HH}_*(\mathrm{MF}(X,w)),\mathcal{O}_X) = \underline{\mathrm{Hom}}(\Delta^*\mathcal{O}_\Delta^{\widetilde{w}},\mathcal{O}_X) \simeq Rp_{1*}\Delta_*\underline{\mathrm{Hom}}(\Delta^*\mathcal{O}_\Delta^{\widetilde{w}},\mathcal{O}_X)$$
$$\simeq Rp_{1*}\underline{\mathrm{Hom}}(\mathcal{O}_\Delta^{\widetilde{w}},\mathcal{O}_\Delta^{\widetilde{w}}),$$

induced by the adjoint pair of functors (Δ^*,Δ_*) *between* $\mathrm{MF}(X,0)$ *and* $\mathrm{MF}(X^2,\widetilde{w})$.

Proof. The proof is very similar to that of Lemma 2.1 and is left to the reader. □

Let us consider the composition

$$S^n(L_w)\otimes \underline{HH}_*(\mathrm{MF}(X,w)) \xrightarrow{I^{abs,w}\otimes\mathrm{id}} \underline{HH}^*(\mathrm{MF}(X,w))\otimes \underline{HH}_*(\mathrm{MF}(X,w)) \xrightarrow{\varepsilon\circ\mathcal{D}} \mathcal{O}_X.$$

Dually, we get a morphism

$$I_{abs,w} : \underline{HH}_*(\mathrm{MF}(X,w)) \to S^n(L_w)^\vee \simeq [\Omega_X^\bullet, \wedge\, dw], \tag{2.8}$$

where the last isomorphism is (0.1).

Theorem 2.5. *The maps* $I^{abs,w}$ *and* $I_{abs,w}$ *in* $D(X)$ *coincide with the maps* $I_{HKR,w}$ *and* $I^{HKR,w}$ *defined using the completed bar resolution in* [8] *and* [10].

Proof. As in the proof of Theorem 2.2, to check the equality $I_{abs,w} = I_{HKR,w}$, we first realise

$$\hat{\mathrm{at}}^{\mathrm{univ}} : p_1^*L_w\otimes\mathcal{O}_\Delta^{\widetilde{w}} \to \mathcal{O}_\Delta^{\widetilde{w}}$$

by a closed map of matrix factorisations.

Recall that the completed bar resolution $(\mathcal{B}_\bullet, b)$ is equipped with the second differential

$$B_w[a_0\otimes a_1\otimes\ldots\otimes a_{m+1}] = \sum_{i=0}^{m}(-1)^i a_0\otimes\ldots\otimes a_i\otimes w\otimes a_{i+1}\otimes\ldots\otimes a_{m+1},$$

so that $\mathcal{B}^{\widetilde{w}} := (\mathcal{B}_\bullet, b+B_w)$ is a (quasicoherent) matrix factorisation of $\widetilde{w} = w\otimes 1 - 1\otimes w$.

Similar to the case $w=0$, we define a closed morphism of matrix factorisations

$$p_1^*L_w\otimes\mathcal{B}^{\widetilde{w}} \to \mathcal{B}^{\widetilde{w}} : (v+f)\otimes[a_0\otimes a_1\otimes\ldots\otimes a_m]$$
$$= (-1)^m[a_0 v(a_1)\otimes a_2\otimes\ldots\otimes a_m] + [fa_0\otimes a_1\otimes\ldots\otimes a_m].$$

We claim that it represents $\hat{\mathrm{at}}^{\mathrm{univ}}$. Indeed, it is enough to consider the composed map

$$p_1^*L_w\otimes\mathcal{B}^{\widetilde{w}} \to \mathcal{O}_\Delta^{\widetilde{w}}$$

and compare its dualisation

$$\mathcal{B}^{\widetilde{w}} \to p_1^*L_w^\vee\otimes\mathcal{O}_\Delta^{\widetilde{w}} \simeq \Delta_*[\mathcal{O}_X \xrightarrow{dw} \Omega_X^1] \tag{2.9}$$

with $\hat{\mathrm{at}}^{\mathrm{univ}}$. It remains to observe that (2.9) factors as the composition

$$\mathcal{B}^{\widetilde{w}} \to (\mathcal{O}_{\Delta^{(2)}}\oplus(\Omega_X^1)_\Delta[1],\delta) \to (\mathcal{O}_\Delta\oplus(\Omega_X^1)_\Delta[1], dw),$$

where the first map is an isomorphism of resolutions of $\mathcal{O}_\Delta^{\widetilde{w}}$, and the second map induces $\hat{\mathrm{at}}^{\mathrm{univ}}$ by Lemma 1.2(i).

Considering the induced map $p_1^* S^n L_w \otimes \mathcal{B}^{\widetilde{w}} \to \mathcal{B}^{\widetilde{w}} \to \mathcal{O}_\Delta^{\widetilde{w}}$, we deduce that I_{abs} coincides with the HKR isomorphism $I_{HRK,w}$ given by the map

$$\Delta^* \mathcal{B}^{\widetilde{w}} \to (\Omega_X^\bullet, \wedge dw) : [a_0 \otimes \ldots \otimes a_{i+1}] \mapsto a_{i+1} a_0 da_1 \wedge \ldots \wedge da_i.$$

The equality $I^{abs,w} = I^{HKR,w}$ follows by duality as in the case $w = 0$. □

3. Boundary–bulk map

3.1. Generalities

Recall that the diagonal matrix factorisation $\mathcal{O}_\Delta^{\widetilde{w}} \in \mathrm{MF}(X^2, \widetilde{w})$ corresponds to the identity functor on $\mathrm{MF}(X, w)$. The categorical trace functor can be identified with the composition

$$\mathrm{Tr} : \mathrm{MF}(X^2, \widetilde{w}) \xrightarrow{\Delta^*} D(X) \xrightarrow{R\Gamma} D(k).$$

Thus, the Hochschild homology of the category $\mathrm{MF}(X, w)$ can be computed as

$$HH_*(\mathrm{MF}(X, w)) = R\Gamma(X, \underline{HH}_*(\mathrm{MF}(X, w)),$$

where

$$\underline{HH}_*(\mathrm{MF}(X, w)) = \Delta^* \mathcal{O}_\Delta^{\widetilde{w}}.$$

Furthermore, we have an isomorphism

$$\underline{HH}_*(\mathrm{MF}(X, w)) \simeq [\Omega^\bullet, \wedge dw].$$

The sheafified boundary–bulk map

$$\underline{\mathrm{End}}(E) \to \underline{HH}_*(\mathrm{MF}(X, w)), \tag{3.1}$$

which is a map in $D(X)$, is obtained by applying Δ^* to the evaluation morphism in $\mathrm{MF}(X^2, \widetilde{w})$,

$$\mathrm{ev}_E : E \boxtimes E^\vee \to \mathcal{O}_\Delta^{\widetilde{w}}.$$

The latter morphism is obtained by dualisation from the morphism

$$\eta_E : p_1^* E \to E_\Delta \simeq p_2^* E \otimes \mathcal{O}_\Delta^{\widetilde{w}}$$

in $\mathrm{MF}(X^2, w \otimes 1)$, which corresponds by adjunction to the isomorphism $E \to Rp_{1*}(p_2^* E \otimes \mathcal{O}_\Delta^{\widetilde{w}})$.

Because the boundary–bulk map is obtained from ev_E by applying the categorical trace functor Tr, it is obtained from the sheafified boundary–bulk map (3.1) by passing to derived global sections.

3.2. Exponentials

Our exponentials are analogues of the following Lie-theoretic construction. Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. Then for every $i \geq 0$, we have a morphism given by the iterated action

$$u_i : \mathfrak{g}^{\otimes i} \otimes M \to M : x_1 \otimes \ldots \otimes x_i \otimes m \mapsto x_1 \cdot (\ldots \cdot (x_{i-1} \cdot (x_i \cdot m)) \ldots).$$

We denote by $s_i : S^i(\mathfrak{g}) \otimes M \to M$ the restriction of u_i to symmetric tensors. We can think of s_i as an element of $S^i(\mathfrak{g})^* \otimes \mathrm{End}(M)$. Combining these elements together we get the element

$$\exp_M \in S^\bullet(\mathfrak{g})^* \otimes \mathrm{End}(M).$$

Now given an object E in $D(X)$, the Atiyah class of E defines a map

$$T_X[-1] \otimes E \to E,$$

which is an action of $T_X[-1]$ (viewed as a Lie algebra) on E. Thus, we get the corresponding element

$$\exp_E \in \mathrm{Hom}(E, S^\bullet(\Omega_X[1]) \otimes E) = \mathrm{Hom}(E, \bigoplus_i \Omega_X^i[i] \otimes E).$$

Unraveling the definitions, we see that

$$\exp_E = \exp(\mathrm{at}(E)),$$

where the right-hand side is defined in the standard way (see, e.g., [3, Section 4]).

Similarly, we can consider $\mathcal{O}_\Delta$ as a module over $p_1^* T_X[-1]$ using the universal Atiyah class, $\mathrm{at}^{\mathrm{univ}} : p_1^* T_X[-1] \otimes \mathcal{O}_\Delta \to \mathcal{O}_\Delta$. This gives rise to the element

$$\exp(\mathrm{at}^{\mathrm{univ}}) \in \mathrm{Hom}(\mathcal{O}_\Delta, \bigoplus_i p_1^* \Omega_X^i[i] \otimes \mathcal{O}_\Delta) = \mathrm{Hom}(\mathcal{O}_\Delta, \Delta_* \bigoplus_i \Omega_X^i[i]). \tag{3.2}$$

It is easy to check that if we view $\exp(\mathrm{at}^{\mathrm{univ}})$ as a morphism of Fourier-Mukai kernels then the induced morphism of functors $D(X) \to D(X)$,

$$E \mapsto \bigoplus_i \Omega_X^i[i] \otimes E,$$

is precisely $\exp(\mathrm{at}(E))$.

In the same way for a matrix factorisation $E \in \mathrm{MF}(X, w)$ we define

$$\exp(\mathrm{at}(E)) \in \mathrm{Hom}^0(E, (\Omega_X^\bullet, \wedge dw) \otimes E)$$

using the action of L_w on E given by $\hat{\mathrm{at}}(E)$, passing to the induced map

$$S^n(L_w) \otimes E \to E$$

(where $n = \dim X$), dualising, and using the isomorphism of $S^n(L_w^\vee)$ with $(\Omega_X^\bullet, \wedge dw)$ (see (0.1)). This is equivalent to the definition given in Theorem A.

Similarly, the universal Atiyah class

$$\hat{\mathrm{at}}^{\mathrm{univ}} \in \mathrm{Hom}(\mathcal{O}_\Delta^{\widetilde{w}}, p_1^* L_w^\vee \otimes \mathcal{O}_\Delta^{\widetilde{w}})$$

gives an action of $p_1^* L_w$ on $\mathcal{O}_\Delta^{\widetilde{w}}$ that induces an element

$$\exp(\mathrm{at}^{\mathrm{univ}}) \in \mathrm{Hom}(\mathcal{O}_\Delta^{\widetilde{w}}, p_1^* S^n(L_w^\vee) \otimes \mathcal{O}_\Delta^{\widetilde{w}}) \simeq p_1^*(\Omega_X^\bullet, \wedge dw) \otimes \mathcal{O}_\Delta^{\widetilde{w}}.$$

Again, if we view this as a morphism of Fourier-Mukai kernels, then the induced morphism of functors $\mathrm{MF}(X, w) \to \mathrm{MF}(X, w)$ is given by $\exp(\mathrm{at}(E))$ on $E \in \mathrm{MF}(X, w)$.

3.3. Key lemma

First, let us formulate the key assertion about the exponential of the universal Atiyah class (3.2) in the case $w = 0$.

Lemma 3.1. *One has a commutative triangle*

$$\begin{array}{ccc} \mathcal{O}_\Delta & \xrightarrow{\text{can}} & \Delta_*\Delta^*\mathcal{O}_\Delta \\ & \searrow^{\exp(\text{at}^{\text{univ}})} & \downarrow{\scriptstyle \Delta_* I_{abs}} \\ & & \Delta_* \bigoplus_i \Omega_X^i[i] \end{array}$$

Proof. Because the action of $T_X[-1]$ on $\underline{HH}_*(X) = \Delta^*\mathcal{O}_\Delta$ is obtained by applying Δ^* to at^{univ}, by naturality of can, we get the following commutative diagram

$$\begin{array}{ccc} p_1^* T_X[-1] \otimes \mathcal{O}_\Delta & \xrightarrow{\text{id}\otimes\text{can}} & p_1^* T_X[-1] \otimes \Delta_*\Delta^*\mathcal{O}_\Delta \\ \downarrow{\scriptstyle \text{at}^{\text{univ}}} & & \downarrow \\ \mathcal{O}_\Delta & \xrightarrow{\text{can}} & \Delta_*\Delta^*\mathcal{O}_\Delta \end{array}$$

where the right vertical arrow corresponds to the action of $T_X[-1]$ on $\underline{HH}_*(X)$. In other words, the map $\text{can} : \mathcal{O}_\Delta \to \Delta_*\Delta^*\mathcal{O}_\Delta$ is compatible with the action of $p_1^* T_X[-1]$. Hence, it is also compatible with the iteration of this action and its restriction to (skew)-symmetric tensors:

$$\begin{array}{ccc} p_1^* S^\bullet(T_X[-1]) \otimes \mathcal{O}_\Delta & \xrightarrow{\text{id}\otimes\text{can}} & p_1^* S^\bullet(T_X[-1]) \otimes \Delta_*\Delta^*\mathcal{O}_\Delta \\ \downarrow{\scriptstyle \exp(\text{at}^{\text{univ}})} & & \downarrow \\ \mathcal{O}_\Delta & \xrightarrow{\text{can}} & \Delta_*\Delta^*\mathcal{O}_\Delta \end{array}$$

where the left vertical arrow corresponds to $\exp(\text{at}^{\text{univ}})$ by duality. Composing with the map $\Delta_*\varepsilon : \Delta_*\Delta^*\mathcal{O}_\Delta \to \mathcal{O}_\Delta$ (whose composition with can is the identity), we get the commutative triangle

$$\begin{array}{ccc} p_1^* S^\bullet(T_X[-1]) \otimes \mathcal{O}_\Delta & \xrightarrow{\text{id}\otimes\text{can}} & p_1^* S^\bullet(T_X[-1]) \otimes \Delta_*\Delta^*\mathcal{O}_\Delta \\ & \searrow^{\exp(\text{at}^{\text{univ}})} & \downarrow \\ & & \mathcal{O}_\Delta \end{array}$$

where the vertical arrow is the push-forward by Δ of the composition

$$\varepsilon \circ \mathcal{D}_0 \circ (I^{abs} \otimes \text{id}) : S^\bullet(T_X[-1]) \otimes \Delta^*\mathcal{O}_\Delta \to \mathcal{O}_X.$$

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Now the assertion follows from the fact that I_{abs} is obtained from the latter map by dualisation. □

Remark. Note that modulo Theorem 2.2 the assertion of Lemma 3.1 is equivalent to that of Proposition 4.4 in [3] (which refers to the standard HKR isomorphism defined using the completed bar resolution). Thus, we get a more conceptual proof of that proposition.

Now let us consider the case of matrix factorisations.

Lemma 3.2. *One has a commutative triangle*

$$\begin{array}{ccc} \mathcal{O}_\Delta^{\widetilde{w}} & \xrightarrow{\text{can}} & \Delta_*\Delta^*\mathcal{O}_\Delta^{\widetilde{w}} \\ & \searrow^{\exp(\mathrm{at}_{\mathrm{univ}})} & \downarrow{\scriptstyle \Delta_* I_{abs,w}} \\ & & \Delta_*(\Omega_X^\bullet, \wedge\, dw) \end{array}$$

Proof. The proof is similar to that of Lemma 3.1. We use the fact that the dualisation of the universal Atiyah class induces an action of $p_1^*L_w$ on $\mathcal{O}_\Delta^{\widetilde{w}}$, so we get a commutative diagram

$$\begin{array}{ccc} p_1^*L_w \otimes \mathcal{O}_\Delta^{\widetilde{w}} & \xrightarrow{\mathrm{id}\otimes\mathrm{can}} & p_1^*L_w \otimes \Delta_*\Delta^*\mathcal{O}_\Delta^{\widetilde{w}} \\ {\scriptstyle \hat{\mathrm{at}}^{\mathrm{univ}}}\downarrow & & \downarrow \\ \mathcal{O}_\Delta^{\widetilde{w}} & \xrightarrow{\text{can}} & \Delta_*\Delta^*\mathcal{O}_\Delta^{\widetilde{w}} \end{array}$$

and then use the iteration of this action to get a commutative diagram

$$\begin{array}{ccc} p_1^*S^n(L_w) \otimes \mathcal{O}_\Delta^{\widetilde{w}} & \xrightarrow{\mathrm{id}\otimes\mathrm{can}} & p_1^*S^n(L_w) \otimes \Delta_*\Delta^*\mathcal{O}_\Delta^{\widetilde{w}} \\ {\scriptstyle \exp(\mathrm{at}^{\mathrm{univ}})}\downarrow & & \downarrow \\ \mathcal{O}_\Delta^{\widetilde{w}} & \xrightarrow{\text{can}} & \Delta_*\Delta^*\mathcal{O}_\Delta^{\widetilde{w}}. \end{array}$$

Finally, composing with $\Delta_*\varepsilon : \Delta_*\Delta^*\mathcal{O}_\Delta^{\widetilde{w}} \to \mathcal{O}_\Delta^{\widetilde{w}}$ and dualising, we get the result. □

3.4. Proof of Theorem A

Let us first consider the case $w = 0$. We can view the commutative triangle of Lemma 3.1 as the triangle of Fourier-Mukai functors from $D(X)$ to $D(X)$ (where to a kernel K in $D(X \times X)$ we associate the functor $Rp_{1*}(K \otimes p_2^*(\cdot))$). Applying these functors to an

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object $E \in D(X)$, we get a commutative triangle of the form

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{can}_E} & \Delta^*\mathcal{O}_\Delta \otimes E \\ & \searrow^{\exp(\mathrm{at}(E))} & \downarrow{\scriptstyle I_{abs}\otimes \mathrm{id}} \\ & & \Omega^\bullet_X \otimes E. \end{array} \tag{3.3}$$

We claim that the morphism can_E is obtained by applying Δ^* to the canonical morphism

$$\eta_E : p_1^*E \to \Delta_*E \simeq \mathcal{O}_\Delta \otimes p_2^*E.$$

Indeed, first let us observe that for any $F, G \in D(X \times X)$ we have a commutative triangle

$$\begin{array}{ccc} F \otimes G & \xrightarrow{a_F \otimes \mathrm{id}_G} & \Delta_*\Delta^*F \otimes G \\ & \searrow^{a_{F\otimes G}} & \downarrow{\scriptstyle \sim} \\ & & \Delta_*\Delta^*(F \otimes G) \end{array}$$

where $a_F : F \to \Delta_*\Delta^*F$ is the adjunction map, and the vertical arrow is the composition of the natural isomorphisms

$$\Delta_*\Delta^*F \otimes G \simeq \Delta_*(\Delta^*F \otimes \Delta^*G) \simeq \Delta_*\Delta^*(F \otimes G).$$

Applying this to $F = \mathcal{O}_\Delta$ and $G = p_2^*E$ we get the commutativity of the triangle in the diagram

$$\begin{array}{ccccc} p_1^*E & \xrightarrow{\eta_E} & \mathcal{O}_\Delta \otimes p_2^*E & \xrightarrow{\mathrm{can}\otimes\mathrm{id}} & \Delta_*\Delta^*\mathcal{O}_\Delta \otimes p_2^*E \\ \downarrow & & \downarrow & \swarrow_{\sim} & \\ \Delta_*\Delta^*p_1^*E & \xrightarrow{\Delta_*\Delta^*\eta_E} & \Delta_*\Delta^*(\mathcal{O}_\Delta \otimes p_2^*E). & & \end{array}$$

Note that here the vertical arrows are the adjunction maps, so the square in the above diagram is also commutative. Using the adjointness of (p_1^*, p_1^*), we get a commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{\sim} & Rp_{1*}(\mathcal{O}_\Delta \otimes p_2^*E) & \xrightarrow{Rp_{1*}(\mathrm{can}\otimes\mathrm{id})} & \Delta^*\mathcal{O}_\Delta \otimes p_2^*E \\ \downarrow{\scriptstyle\sim} & & \downarrow & \swarrow_{\sim} & \\ \Delta^*p_1^*E & \xrightarrow{\Delta^*\eta_E} & \Delta^*(\mathcal{O}_\Delta \otimes p_2^*E). & & \end{array}$$

By definition, the composition of arrows in the first row is can_E, and our claim follows.

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This implies that can_E corresponds by dualisation to the sheafified boundary–bulk map

$$E \otimes E^\vee \to \Delta^*\mathcal{O}_\Delta = \underline{HH}_*(X)$$

(obtained as Δ^* of the evaluation map $E \boxtimes E^\vee \to \mathcal{O}_\Delta$). Composing with I_{abs} and using commutativity of (3.3) we get its expression in terms of $\exp(\mathrm{at}(E))$.

Now we can repeat the same argument in the case of matrix factorisations. For $E \in \mathrm{MF}(X, w)$, using Lemma 3.2, we get a commutative triangle

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{can}_E} & \Delta^*\mathcal{O}_\Delta^{\widetilde{w}} \otimes E \\ & \searrow^{\exp(\mathrm{at}(E))} & \downarrow I_{abs,w}\otimes \mathrm{id} \\ & & (\Omega_X^\bullet, \wedge dw) \otimes E, \end{array}$$

where the morphism can_E is obtained by applying Δ^* to η_E. Because ev_E corresponds to η_E by dualisation, this implies that the sheafified boundary–bulk map

$$\Delta^*(\mathrm{ev}_E) : \underline{\mathrm{End}}(E) \to \Delta^*(\mathcal{O}_\Delta^{\widetilde{w}})$$

is given by $x \mapsto \mathrm{str}(\exp(\mathrm{at}(E)) \cdot x)$. □

4. Graded case

4.1. Basics

4.1.1. Category of $\mathbb{G}_m$-equivariant matrix factorisations. Let X be a smooth scheme equipped with a $\mathbb{G}_m$-action, and let W be a regular function on X satisfying

$$W(\lambda x) = \lambda W(x)$$

for $\lambda \in \mathbb{G}_m$.

Let χ denote the identity character of $\mathbb{G}_m$. The category $\mathrm{MF}_{\mathbb{G}_m}(X, W)$ of $\mathbb{G}_m$-equivariant matrix factorisations of W has as objects $\mathbb{G}_m$-equivariant $\mathbb{Z}_2$-graded bundles $E = E_0 \oplus E_1$ on X equipped with $\mathcal{O}_X$-linear maps

$$\delta_1 : E_1 \to E_0, \quad \delta_0 : E_0 \to E_1 \otimes \chi,$$

such that $\delta_0 \circ \delta_1 = \delta_1 \circ \delta_0 = W$. In order to define morphisms it is more convenient to replace (E, δ) with the $\mathbb{Z}$-graded bundle equipped with a degree one endomorphism δ such that $\delta^2 = W$,

$$C(E) = C(E, \delta) : \ldots E_1 \xrightarrow{\delta_1} E_0 \xrightarrow{\delta_0} E_1 \otimes \chi \xrightarrow{\delta_1} E_0 \otimes \chi \to \ldots$$

where the $\mathbb{Z}$-grading on $C(E)$ is determined by $C(E)_0 = E_0$. This complex is equipped with a chain 2-*quasi-periodicity* isomorphism

$$\alpha_E : C(E) \simeq C(E) \otimes \chi[-2].$$

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Now for a pair of matrix factorisations E, F we define the $\mathbb{Z}$-graded complex of $\mathbb{G}_m$-equivariant bundles, $\underline{\mathrm{Hom}}(E, F)$, as a subcomplex in the sheafified internal Hom complex of the corresponding 2-quasi-periodic complexes, $\underline{\mathrm{Hom}}(C(E), C(F))$, consisting of morphisms respecting isomorphism α_E and α_F.

Then the morphisms from E to F are defined as

$$\mathrm{Hom}(E, F) := R\Gamma(X, \underline{\mathrm{Hom}}(E, F))^{\mathbb{G}_m}.$$

More precisely, here $R\Gamma$ should be replaced by some functorial multiplicative resolution. The resulting category $\mathrm{MF}_{\mathbb{G}_m}(X, W)$ is a $\mathbb{Z}$-graded dg-category, unlike the usual category of matrix factorisations, which is only $\mathbb{Z}_2$-graded (see [13, Section 1] for more details).

Note that the complex $\underline{\mathrm{Hom}}(E, F)$ is still 2-quasi-periodic, so in fact we have

$$\underline{\mathrm{Hom}}(E, F) = C(\underline{\mathrm{Hom}}^{mf}(E, F)),$$

where $\underline{\mathrm{Hom}}^{mf}(E, F)$ is a $\mathbb{G}_m$-equivariant matrix factorisation of 0 on X.

Note that in the case $W = 0$ we can associate with every bounded $\mathbb{Z}$-graded complex of $\mathbb{G}_m$-equivariant vector bundles $(V^\bullet, d)$, a $\mathbb{G}_m$-equivariant matrix factorisation $\mathrm{mf}(V^\bullet)$ of 0, given by

$$\mathrm{mf}(V^\bullet)_0 = \bigoplus_{n\in\mathbb{Z}} V^{2n} \otimes \chi^{-n}, \quad \mathrm{mf}(V^\bullet)_1 = \bigoplus_{n\in\mathbb{Z}} V^{2n-1} \otimes \chi^{-n}.$$

Note that the corresponding $\mathbb{Z}$-graded complex $C(\mathrm{mf}(V^\bullet))$ is simply

$$C(\mathrm{mf}(V^\bullet)) = \bigoplus_{n\in\mathbb{Z}} V \otimes \chi^n[-2n].$$

Example. In the case when the action of $\mathbb{G}_m$ on X is trivial, one can easily see that the composed functor from $D(X)_{\mathbb{Z}}$, the usual $\mathbb{Z}$-graded derived category of coherent sheaves on X,

$$D(X)_{\mathbb{Z}} \to D_{\mathbb{G}_m}(X)_{\mathbb{Z}} \xrightarrow{\mathrm{mf}} \mathrm{MF}_{\mathbb{G}_m}(X, 0),$$

where the first functor equips a complex in $D(X)_{\mathbb{Z}}$ with the trivial $\mathbb{G}_m$-action, is an equivalence (see [13, Section 1.2]).

The above definitions also make sense for more general categories of quasicoherent (respectively coherent) matrix factorisations.

4.1.2. Tensor product of graded matrix factorisations. If W and W' are functions satisfying $W(\lambda x) = \lambda W(x)$, $W'(\lambda x) = \lambda W'(x)$, then we have a natural operation of tensor product

$$\otimes : \mathrm{MF}_{\mathbb{G}_m}(X, W) \times \mathrm{MF}_{\mathbb{G}_m}(X, W') \to \mathrm{MF}_{\mathbb{G}_m}(X, W + W'),$$

which is uniquely determined by

$$C(E \otimes F) = \mathrm{equaliser}\Big(C(E) \otimes C(F) \underset{\mathrm{id}\otimes\alpha_F}{\overset{\alpha_E\otimes\mathrm{id}}{\rightrightarrows}} C(E) \otimes C(F) \otimes \chi[-2]\Big).$$

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More explicitly,

$$(E\otimes F)_0 = E_0\otimes F_0\oplus E_1\otimes F_1\otimes\chi,\quad (E\otimes F)_1 = E_0\otimes F_1\oplus E_1\otimes F_0$$

(see [13, Section 1.1]).

In the particular case $W=0$ the tensor product gives a structure of a symmetric monoidal category on $\mathrm{MF}_{\mathbb{G}_m}(X,0)$.

There is also a natural duality functor

$$\mathrm{MF}_{\mathbb{G}_m}(X,W)^{op}\to \mathrm{MF}_{\mathbb{G}_m}(X,-W): E\mapsto E^\vee,$$

such that

$$C(E^\vee)\simeq C(E)^\vee,$$

with $\alpha_{E^\vee}$ induced by α_E. One has an isomorphism of $\mathbb{G}_m$-equivariant matrix factorisations of 0,

$$\underline{\mathrm{Hom}}^{mf}(E,F)\simeq F\otimes E^\vee$$

(see [13, Lemma 1.1.6]).

Note that we have a natural forgetful functor

$$\mathrm{MF}_{\mathbb{G}_m}(X,0)\to \mathrm{MF}(X,0) = D(X)_{\mathbb{Z}_2}, \tag{4.1}$$

where $D(X)_{\mathbb{Z}_2}$ is the $\mathbb{Z}_2$-graded derived category of coherent sheaves on X. It is easy to see that this functor is compatible with tensor products.

4.2. Atiyah classes

Let us consider $\widetilde{W} = W\otimes 1 - 1\otimes W$ as a function on X^2. Note that it still satisfies $\widetilde{W}(\lambda x) = \lambda\widetilde{W}(x)$, where we equip X^2 with the diagonal $\mathbb{G}_m$-action. We can associate with each object K of $\mathrm{MF}_{\mathbb{G}_m}(X^2,\widetilde{W})$, with support proper with respect to p_1, a Fourier-Mukai-type functor $Rp_{1*}(K\otimes p_2^*(\cdot)$ from $\mathrm{MF}_{\mathbb{G}_m}(X,W)$ to itself.

Because $\widetilde{W}|_{\Delta(X)}=0$, we have natural functors

$$\Delta_*: \mathrm{MF}_{\mathbb{G}_m}(X,0)\to \mathrm{MF}_{\mathbb{G}_m}(X^2,\widetilde{W}),\quad \Delta^*: \mathrm{MF}_{\mathbb{G}_m}(X^2,\widetilde{W})\to \mathrm{MF}_{\mathbb{G}_m}(X,0). \tag{4.2}$$

We can view $\mathcal{O}_X$ sitting in degree 0 as a $\mathbb{G}_m$-equivariant matrix factorisation of 0 via the functor mf. We denote by $\mathcal{O}_\Delta^{\widetilde{W}}$ the corresponding object $\Delta_*(\mathcal{O}_X)\in \mathrm{MF}_{\mathbb{G}_m}(X^2,\widetilde{W})$. Equivalently,

$$\mathcal{O}_\Delta^{\widetilde{W}} = \mathrm{mf}(\mathcal{O}_\Delta).$$

Next, let us define a $\mathbb{G}_m$-equivariant matrix factorisation $\mathcal{O}^{(2)}_{\Delta,\widetilde{W}}$ of $\widetilde{W}$ by

$$C(\mathcal{O}^{(2)}_{\Delta,\widetilde{W}}) = [\dots\mathcal{O}_{\Delta^{(2)}}\xrightarrow{-1}\mathcal{O}_\Delta\xrightarrow{-dW}\mathcal{O}_{\Delta^{(2)}}\otimes\chi\to\dots],$$

where $\mathcal{O}_{\Delta^{(2)}}$ sits in degree 0.

Also, let us define a $\mathbb{G}_m$-equivariant matrix factorisation $L_W^\vee$ of 0 by

$$L_W^\vee = \mathrm{mf}([\mathcal{O}_X\xrightarrow{dW}\Omega_X^1\otimes\chi]),$$

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where the 2-complex is placed in degrees [0,1]. Then we have a natural exact sequence of $\mathbb{G}_m$-equivariant matrix factorisations of $\widetilde{W}$,

$$0 \to \Delta_* L_W^\vee[-1] \xrightarrow{\phi} \mathcal{O}^{(2)}_{\Delta,\widetilde{W}} \xrightarrow{\psi} \mathcal{O}_\Delta^{\widetilde{W}} \to 0,$$

where the map ϕ has as components the identity map on $\mathcal{O}_\Delta$ and the natural embedding $\Delta_*\Omega^1_X \to \mathcal{O}_{\Delta^{(2)}}$, and the nontrivial component of ψ is given by the natural projection $\mathcal{O}_{\Delta^{(2)}} \to \mathcal{O}_\Delta$.

From the above exact sequence we get a morphism in $\mathrm{MF}_{\mathbb{G}_m}(X^2,\widetilde{W})$,

$$\hat{\mathrm{at}}^{\mathrm{univ}}: \mathcal{O}_\Delta^{\widetilde{W}} \to \Delta_* L_W^\vee \simeq p_1^* L_W^\vee \otimes \mathcal{O}_\Delta^{\widetilde{W}}.$$

Applying the corresponding Fourier-Mukai functors, we get for every $\mathbb{G}_m$-equivariant matrix factorisation E of W a morphism

$$\hat{\mathrm{at}}(E): E \to L_W^\vee \otimes E.$$

Example. Assume that the action of $\mathbb{G}_m$ on X is trivial and $W=0$. Then $\mathrm{MF}_{\mathbb{G}_m}(X,0)$ is identified with $D(X)_{\mathbb{Z}}$ (see Example 4.1.1). It is easy to see that in this case $\hat{\mathrm{at}}(E) = \mathrm{id}_E + \mathrm{at}(E)$, where $\mathrm{at}(E): E \to \Omega^1_X[1] \otimes E$ is the usual Atiyah class of E.

4.3. Lie algebra structure, HKR, and the boundary–bulk map

We are going to equip $L_W^\vee$ with a Lie algebra structure in the symmetric monoidal category $\mathrm{MF}_{\mathbb{G}_m}(X,0)$, such that the embedding $\mathrm{mf}(\mathcal{O}_X) \to L_W^\vee$ is in the center.

For this we start by applying the construction of the Atiyah class to $L_W^\vee$ viewed as a matrix factorisation of 0:

$$\hat{\mathrm{at}}_{L_W^\vee}: L_W^\vee \to L_0^\vee \otimes L_W^\vee.$$

Next, we observe that there are canonical chain maps of complexes placed in degrees [0,1],

$$[\mathcal{O}_X \xrightarrow{0} \Omega^1_X] \to [0 \to \Omega^1_X \otimes \chi] \to [\mathcal{O}_X \xrightarrow{dW} \Omega^1_X \otimes \chi].$$

Applying the functor mf we get a canonical morphism in $\mathrm{MF}_{\mathbb{G}_m}(X,0)$,

$$\varphi_W: L_0^\vee \to L_W^\vee.$$

Now we define the dual of the Lie bracket as the composition

$$L_W^\vee \xrightarrow{\hat{\mathrm{at}}_{L_W^\vee}} L_0^\vee \otimes L_W^\vee \xrightarrow{\varphi_W \otimes \mathrm{id}} L_W^\vee \otimes L_W^\vee.$$

Next, using adjunction, from $\hat{\mathrm{at}}^{\mathrm{univ}}$, we get a morphism in $\mathrm{MF}_{\mathbb{G}_m}(X,0)$,

$$\iota_W: L_W \to Rp_{1*}\underline{\mathrm{End}}^{mf}(\mathcal{O}_\Delta^{\widetilde{W}}) =: \underline{HH}^*(\mathrm{MF}_{\mathbb{G}_m}(X,W))$$

(recall that $\underline{\mathrm{End}}^{mf}(\mathcal{O}_\Delta^{\widetilde{W}})$ is a $\mathbb{G}_m$-equivariant matrix factorisation of 0 on X^2). Using the product in $\underline{HH}^*(\mathrm{MF}_{\mathbb{G}_m}(X,W))$ we obtain a map of $\mathbb{G}_m$-equivariant matrix

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factorisations of 0

$$I^{abs,\mathbb{G}_m} : S^n(L_W) \to \underline{HH}^*(\mathrm{MF}_{\mathbb{G}_m}(X, W)),$$

where $n = \dim X$.

On the other hand, setting

$$\underline{HH}_*(\mathrm{MF}_{\mathbb{G}_m}(X, W)) := \Delta^* \mathcal{O}_\Delta^{\widetilde{W}} \in \mathrm{MF}_{\mathbb{G}_m}(X, 0),$$

we have a natural action of $\underline{HH}^*(\mathrm{MF}_{\mathbb{G}_m}(X, W))$ on $\underline{HH}^*(\mathrm{MF}_{\mathbb{G}_m}(X, W))$ (understood in terms of the tensor structure on $\mathrm{MF}_{\mathbb{G}_m}(X, 0)$) and a canonical functional

$$\varepsilon : \underline{HH}_*(\mathrm{MF}_{\mathbb{G}_m}(X, W)) \to \mathrm{mf}(\mathcal{O}_X)$$

coming from the adjunction for the functors (4.2). Thus, iterating the action of L_W on $\underline{HH}_*(\mathrm{MF}_{\mathbb{G}_m}(X, W))$ and applying ε, we get similarly to the $\mathbb{Z}_2$-graded case a map

$$I_{abs,\mathbb{G}_m} : \underline{HH}_*(\mathrm{MF}_{\mathbb{G}_m}(X, W)) \to S^n L_W^\vee.$$

Because the maps $I^{abs,\mathbb{G}_m}$ and $I_{abs,\mathbb{G}_m}$ lift the previously defined maps I^{abs} and I_{abs} using the forgetful functor (4.1), they are quasi-isomorphisms.

Note that we have an isomorphism defined in the same way as (0.1),

$$S^n L_W^\vee \simeq \mathrm{mf}(S^n[\mathcal{O}_X \xrightarrow{dW} \Omega_X^1 \otimes \chi]) \simeq \mathrm{mf}(\textstyle\bigwedge^\bullet(\Omega_X^1 \otimes \chi), \wedge dW).$$

Thus, iterating $\hat{\mathrm{at}}(E)$, as in the $\mathbb{Z}_2$-graded case, we get for $E \in \mathrm{MF}_{\mathbb{G}_m}(X, W)$ a map

$$\exp(\mathrm{at}(E)) : E \to \mathrm{mf}(\textstyle\bigwedge^\bullet(\Omega_X^1 \otimes \chi), \wedge dW) \otimes E.$$

Now all of the previous arguments generalise immediately to the graded case and give the following analogue of Theorem A.

Theorem B. *Assume that $W = 0$ on the critical locus of W (set-theoretically). There is a natural identification*

$$HH_*(\mathrm{MF}_{\mathbb{G}_m}(X, W)) \simeq H^*(X, \textstyle\bigwedge^\bullet(\Omega_X^1 \otimes \chi), \wedge dW)^{\mathbb{G}_m}. \tag{4.3}$$

Under this identification, the categorical boundary–bulk map for $E \in \mathrm{MF}_{\mathbb{G}_m}(X, W)$,

$$\mathrm{Hom}^*_{\mathrm{MF}_{\mathbb{G}_m}(X, W)}(E, E) \to HH_*(\mathrm{MF}_{\mathbb{G}_m}(X, W)),$$

is equal to the map induced on the $\mathbb{G}_m$-invariant part of hypercohomology by the map

$$\underline{\mathrm{Hom}}^{mf}(E, E) \to \mathrm{mf}(\textstyle\bigwedge^\bullet(\Omega_X^1 \otimes \chi), \wedge dW) : x \mapsto \mathrm{str}(\exp(\mathrm{at}(E)) \cdot x)$$

in $\mathrm{MF}_{\mathbb{G}_m}(X, 0)$.

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