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CLONING IN C^*-ALGEBRAS

KRZYSZTOF KANIOWSKI, KATARZYNA LUBNAUER AND ANDRZEJ ŁUCZAK

Faculty of Mathematics and Computer Science, Łódź University, ulica Stefana Banacha 22, 90-238 Łódź, Poland
(kanio@math.uni.lodz.pl; lubnauer@math.uni.lodz.pl; anluczak@math.uni.lodz.pl)

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Abstract Cloneable sets of states in C^*-algebras are characterized in terms of strong orthogonality of states. Moreover, the relation between strong cloning and distinguishability of states is investigated together with some additional properties of strong cloning in abelian C^*-algebras.

Keywords: cloning states; C^*-algebras; von Neumann algebras; the universal representation

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Introduction

The subject of the cloning and broadcasting of quantum states already has a 30-year history. Its first appearance dates back to [**12**] and [**6**], where a no-cloning theorem was formulated. No-broadcasting variants were also subsequently discovered. Among numerous papers devoted to cloning and broadcasting, [**1**] and [**9**]—in which the problem is analysed in the Hilbert space set-up—deserve special attention, as do [**2**] and [**3**], where the problem is considered in generic probabilistic models. A common feature of these approaches is that they restrict their attention to finite-dimensional models; moreover, in the Hilbert space set-up the map defining cloning or broadcasting is assumed to be completely positive.

The present work is a continuation of our investigations in [**8**] and [**10**] concerning cloning and broadcasting in the general operator algebra framework. This framework consists of considering an arbitrary von Neumann or C^*-algebra of operators on a Hilbert space of arbitrary dimension instead of considering the algebra of all linear operators on a finite-dimensional Hilbert space. In [**8**] and [**10**] we were dealing with von Neumann algebras, and now we focus our attention on C^*-algebras. It is probably worth mentioning that although the origin of broadcasting and cloning lies in quantum physics, these operations nevertheless have a purely mathematical character, being simply some natural maps between states of C^*-algebras.

The main results of the paper are as follows. We characterize cloneable sets of states of an arbitrary C^*-algebra in terms of their strong orthogonality—the notion being a natural generalization of well-known orthogonality, and reducing for normal states on von Neumann algebras to the orthogonality of their supports. Furthermore, we investigate a stronger notion of strong cloning and its relations with almost distinguishability of states in arbitrary C^*-algebras in a manner similar to that in [**2**, **3**]. Some special features of strong cloning in abelian C^*-algebras are also considered.

It may come as a surprise that many rather diverse notions concerning cloning states give a fairly consistent picture describing mutual relations between the various forms of cloneability. We discuss this picture in § 4, and it turns out to be almost complete, at least insofar as it relates to the questions that we consider in this paper.

1. Preliminaries and notation

Let $\mathcal{A}$ be an arbitrary C^*-algebra with identity $\mathbb{1}$. A *state* on $\mathcal{A}$ is a bounded positive linear functional on $\mathcal{A}$ of norm one.

Let $\mathcal{M}$ be a von Neumann algebra. The functionals in the predual $\mathcal{M}_*$ of $\mathcal{M}$ are said to be *normal*. Let $\mathrm{s}(\rho)$ denote the *support* of a positive element $\rho \in \mathcal{M}_*$, i.e. the smallest projection in $\mathcal{M}$ such that $\rho(\mathrm{s}(\rho)) = \rho(\mathbb{1})$.

The crucial role in our analysis will be played by the following notions: (1) tensor products of C^*- and W^*- (or von Neumann) algebras; (2) the universal representation of a C^*-algebra. For an exhaustive treatment of these topics the reader is advised to consult [**7**, **11**]. We will use results from those publications freely, referring only occasionally to the literature.

The main objects of interest to us are the following two operations of *cloning* and *strong cloning* of states.

Let $\mathcal{A}$ be a C^*-algebra and consider the injective tensor product $\mathcal{A}\otimes_{\min}\mathcal{A}$. A linear map $K^*\colon \mathcal{A}^* \to (\mathcal{A}\otimes_{\min}\mathcal{A})^*$ sending states to states is said to *clone* a state ρ if $K^*\rho = \rho\otimes\rho$. It is said to *strongly clone* a state ρ if, in addition, it is the adjoint map to a unital completely positive map $K\colon \mathcal{A}\otimes_{\min}\mathcal{A} \to \mathcal{A}$.

A family of states is said to be *cloneable* (respectively, *strongly cloneable*) if there is a map K^* that clones (respectively, strongly clones) each member of this family. To clarify this definition, let us observe that for an arbitrary state ρ we can always find a map that clones it (even strongly), namely, it is enough to define K^* as

$$K^*\varphi = \varphi(\mathbb{1})\rho\otimes\rho, \quad \varphi \in \mathcal{A}^*.$$

However, problems arise when we want to clone a whole family of states. In particular, the famous no-cloning theorem says that this is impossible for the states of the algebra $\mathbb{B}(\mathcal{H})$ of all linear operators on a finite-dimensional Hilbert space (see [**6**, **12**]). In our earlier works [**8**, **10**] we considered this problem in the general von Neumann algebra set-up. Now we are going to investigate it in the setting of C^*-algebras. Quite remarkably, it turns out that the results for von Neumann and C^*-algebras are strongly interrelated.

The notion of orthogonality of states on $\mathcal{A}$ is well known and consists in the following. Let ρ and φ be states on $\mathcal{A}$. They are said to be *orthogonal* if, for each positive linear functional ω on $\mathcal{A}$ such that $\omega \leqslant \rho$ and $\omega \leqslant \varphi$, we have $\omega = 0$ (see, for example, [**4**]). We shall employ the notion of *strong orthogonality* of states: namely, ρ and φ are said to be *strongly orthogonal* if $\|\rho - \varphi\| = 2$. From the uniqueness of the Jordan decomposition of a hermitian functional on $\mathcal{A}$ it follows that strong orthogonality implies orthogonality.

2. Cloning in arbitrary algebras

In the rest of our paper we will employ the universal representation of a C^*-algebra in various contexts. Let us agree to denote this representation by π_{u}, while for tensor product C^*-algebras we add a tilde on top of π_{u}. Thus, if $\mathcal{A}$ is a C^*-algebra, we have a $*$-isomorphism between $\mathcal{A}$ and $\pi_{\mathrm{u}}(\mathcal{A})$, and for each state ρ on $\mathcal{A}$, its image $\rho \circ \pi_{\mathrm{u}}^{-1}$ on $\pi_{\mathrm{u}}(\mathcal{A})$ is ultraweakly continuous and has a unique extension to a normal state on the universal enveloping von Neumann algebra $\pi_{\mathrm{u}}(\mathcal{A})''$. Our overall strategy consists of the following two steps.

(1) Considering the states ρ on $\mathcal{A}$ as states on $\pi_{\mathrm{u}}(\mathcal{A})$, i.e. precisely speaking, the states $\rho \circ \pi_{\mathrm{u}}^{-1}$, and then considering these states as normal states on the von Neumann algebra $\pi_{\mathrm{u}}(\mathcal{A})''$ (after the unique extension).

(2) Transferring the cloning operation $K^* \colon \mathcal{A}^* \to (\mathcal{A} \otimes_{\min} \mathcal{A})^*$ to a suitable cloning operation $\tilde{K}_* \colon (\pi_{\mathrm{u}}(\mathcal{A})'')_* \to (\pi_{\mathrm{u}}(\mathcal{A})'' \mathbin{\overline{\otimes}} \pi_{\mathrm{u}}(\mathcal{A})'')_*$.

While the first step is pretty obvious, the second requires some care because of the variety of tensor products occurring in the considerations.

In accordance with our idea, for a state ρ on $\mathcal{A}$, by a slight abuse of notation we let $\mathrm{s}(\rho)$ denote the '*support of ρ in* $\pi_{\mathrm{u}}(\mathcal{A})''$', i.e. the support of $\rho \circ \pi_{\mathrm{u}}^{-1}$ after its extension to $\pi_{\mathrm{u}}(\mathcal{A})''$.

We have the following characterization of strong orthogonality.

Lemma 2.1. *Let ρ and φ be states on $\mathcal{A}$. The following conditions are equivalent:*

(i) *ρ and φ are strongly orthogonal;*

(ii) $\mathrm{s}(\rho)\,\mathrm{s}(\varphi) = 0$.

Proof. The strong orthogonality of ρ and φ is equivalent to the equality

$$2 = \|\rho - \varphi\| = \|\rho \circ \pi_{\mathrm{u}}^{-1} - \varphi \circ \pi_{\mathrm{u}}^{-1}\|,$$

meaning that

$$\|\rho \circ \pi_{\mathrm{u}}^{-1} - \varphi \circ \pi_{\mathrm{u}}^{-1}\| = \|\rho \circ \pi_{\mathrm{u}}^{-1}\| + \|\varphi \circ \pi_{\mathrm{u}}^{-1}\|,$$

which in turn is equivalent to the orthogonality of the supports of $\rho \circ \pi_{\mathrm{u}}^{-1}$ and $\varphi \circ \pi_{\mathrm{u}}^{-1}$ (see [**11**, Theorem III.4.2]). □

Let ρ and φ be states on $\mathcal{A}$. Considering $\rho \circ \pi_{\mathrm{u}}^{-1}$ and $\varphi \circ \pi_{\mathrm{u}}^{-1}$ as normal states on $\pi_{\mathrm{u}}(\mathcal{A})''$, we infer that the product state $\rho \circ \pi_{\mathrm{u}}^{-1} \otimes \varphi \circ \pi_{\mathrm{u}}^{-1}$ is a normal state on the tensor product von Neumann algebra $\pi_{\mathrm{u}}(\mathcal{A})'' \mathbin{\overline{\otimes}} \pi_{\mathrm{u}}(\mathcal{A})''$. From [**11**, Proposition IV.4.13] we have the equality

$$\pi_{\mathrm{u}}(\mathcal{A})'' \mathbin{\overline{\otimes}} \pi_{\mathrm{u}}(\mathcal{A})'' = (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}}(\mathcal{A} \otimes_{\min} \mathcal{A}))''.$$

Observe that the following equality also holds true:

$$\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}}(\mathcal{A} \otimes_{\min} \mathcal{A}) = \pi_{\mathrm{u}}(\mathcal{A}) \otimes_{\mathrm{norm}} \pi_{\mathrm{u}}(\mathcal{A}),$$

where 'norm' stands for the norm-closure of the algebraic tensor product of C^*-algebras of bounded operators on a Hilbert space (i.e. on the right-hand side we have the so-called represented tensor product of $\mathcal{A}$ with itself). Since this closure coincides with the injective tensor product $\pi_{\mathrm{u}}(\mathcal{A}) \otimes_{\min} \pi_{\mathrm{u}}(\mathcal{A})$, we shall use the latter notation throughout.

Now the state $(\rho \otimes \varphi) \circ (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^{-1}$ is a state on the C^*-algebra $\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}}(\mathcal{A} \otimes_{\min} \mathcal{A})$, and for each $a, b \in \mathcal{A}$ we have

$$(\rho \otimes \varphi) \circ (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^{-1}(\pi_{\mathrm{u}}(a) \otimes \pi_{\mathrm{u}}(b)) = \rho(a)\varphi(b) = \rho \circ \pi_{\mathrm{u}}^{-1} \otimes \varphi \circ \pi_{\mathrm{u}}^{-1}(\pi_{\mathrm{u}}(a) \otimes \pi_{\mathrm{u}}(b)),$$

showing that

$$(\rho \otimes \varphi) \circ (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^{-1} = \rho \circ \pi_{\mathrm{u}}^{-1} \otimes \varphi \circ \pi_{\mathrm{u}}^{-1}. \tag{2.1}$$

In particular, the state $(\rho \otimes \varphi) \circ (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^{-1}$ on the C^*-algebra $\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}}(\mathcal{A} \otimes_{\min} \mathcal{A})$ has a unique extension to a normal state on the von Neumann algebra $(\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}}(\mathcal{A} \otimes_{\min} \mathcal{A}))''$.

Let $K^* \colon \mathcal{A}^* \to (\mathcal{A} \otimes_{\min} \mathcal{A})^*$. Define

$$K_{\mathrm{u}}^* \colon \pi_{\mathrm{u}}(\mathcal{A})^* \to (\pi_{\mathrm{u}}(\mathcal{A}) \otimes_{\min} \pi_{\mathrm{u}}(\mathcal{A}))^*$$

as

$$K_{\mathrm{u}}^*(\rho \circ \pi_{\mathrm{u}}^{-1}) = (K^*\rho) \circ (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^{-1}, \quad \rho \in \mathcal{A}^*. \tag{2.2}$$

If K^* clones ρ, then on account of (2.1) we have

$$\begin{aligned} K_{\mathrm{u}}^*(\rho \circ \pi_{\mathrm{u}}^{-1}) &= (K^*\rho) \circ (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^{-1} = (\rho \otimes \rho) \circ (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^{-1} \\ &= \rho \circ \pi_{\mathrm{u}}^{-1} \otimes \rho \circ \pi_{\mathrm{u}}^{-1}, \end{aligned}$$

showing that K_{u}^* clones $\rho \circ \pi_{\mathrm{u}}^{-1}$. Consequently, we have transferred the cloning operation from the algebra $\mathcal{A}$ to the algebra $\pi_{\mathrm{u}}(\mathcal{A})$.

The crucial point of our further analysis will be the observation that cloning a state ρ (hence the state $\rho \circ \pi_{\mathrm{u}}^{-1}$) results in obtaining the product state $\rho \circ \pi_{\mathrm{u}}^{-1} \otimes \rho \circ \pi_{\mathrm{u}}^{-1}$, which is an element of the algebraic tensor product $\pi_{\mathrm{u}}(\mathcal{A})^* \otimes \pi_{\mathrm{u}}(\mathcal{A})^*$, and thus belongs to a *suitable tensor product* $\pi_{\mathrm{u}}(\mathcal{A})^* \otimes_\beta \pi_{\mathrm{u}}(\mathcal{A})^*$ (where 'β' is a cross-norm on the algebraic tensor product and, as usual, '$\otimes_\beta$' denotes completion of this tensor product with respect to β). To simplify the notation, set

$$\mathcal{B} = \pi_{\mathrm{u}}(\mathcal{A}).$$

Since $\mathcal{B}$ is a C^*-algebra of bounded operators in a Hilbert space, we have $\mathcal{B} \otimes_{\min} \mathcal{B} = \mathcal{B} \otimes_{\mathrm{norm}} \mathcal{B}$. Let β be the adjoint cross-norm of the injective cross-norm 'min'. Then, according to [**11**, Proposition IV.4.10], $\mathcal{B}^* \otimes_\beta \mathcal{B}^*$ is an invariant subspace of $(\mathcal{B} \otimes_{\min} \mathcal{B})^*$.

Let $\tilde{\pi}_{\mathrm{u}}$ be the universal representation of $\mathcal{B} \otimes_{\min} \mathcal{B}$. Then $(\tilde{\pi}_{\mathrm{u}}^*)^{-1}$ is a linear isometry of $(\mathcal{B} \otimes_{\min} \mathcal{B})^*$ onto $(\tilde{\pi}_{\mathrm{u}}(\mathcal{B} \otimes_{\min} \mathcal{B}))^*$. Since $\tilde{\pi}_{\mathrm{u}}$ is the universal representation, all the elements in $(\tilde{\pi}_{\mathrm{u}}(\mathcal{B} \otimes_{\min} \mathcal{B}))^*$ are ultraweakly continuous, thus they have unique extensions to normal linear functionals on the von Neumann algebra $(\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))''$. Denote this extension by

$$\iota\colon (\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))^* \to ((\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))'')_*,$$

so in particular for each $\tilde{\psi} \in (\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))^*$ we have $\iota(\tilde{\psi}) \mid \tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}) = \tilde{\psi}$.

Set

$$\Theta = \iota \circ (\tilde{\pi}_{\rm u}^*)^{-1}.$$

Then Θ is a linear isometry of $(\mathcal{B} \otimes_{\min} \mathcal{B})^*$ onto $((\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))'')_*$.

Lemma 2.2. *The space $\Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*)$ is a closed invariant subspace of $((\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))'')_*$.*

Proof. Take arbitrary $\tilde{x}, \tilde{y} \in \mathcal{B} \otimes_{\min} \mathcal{B}$, $\tilde{\psi} \in \mathcal{B}^* \otimes_\beta \mathcal{B}^*$. We have

$$\begin{aligned}(\tilde{\pi}_{\rm u}(\tilde{x})\Theta(\tilde{\psi}))(\tilde{\pi}_{\rm u}(\tilde{y})) &= \Theta(\tilde{\psi})(\tilde{\pi}_{\rm u}(\tilde{y}\tilde{x})) = (\iota \circ (\tilde{\pi}_{\rm u}^*)^{-1})(\tilde{\psi})(\tilde{\pi}_{\rm u}(\tilde{y}\tilde{x})) \\ &= ((\tilde{\pi}_{\rm u}^*)^{-1})(\tilde{\psi})(\tilde{\pi}_{\rm u}(\tilde{y}\tilde{x})) = \tilde{\psi}(\tilde{y}\tilde{x}) = (\tilde{x}\tilde{\psi})(\tilde{y}) \\ &= ((\tilde{x}\tilde{\psi}) \circ \tilde{\pi}_{\rm u}^{-1})(\tilde{\pi}_{\rm u}(\tilde{y})) = ((\tilde{\pi}_{\rm u}^*)^{-1}(\tilde{x}\tilde{\psi}))(\tilde{\pi}_{\rm u}(\tilde{y})),\end{aligned}$$

showing that

$$\tilde{\pi}_{\rm u}(\tilde{x})\Theta(\tilde{\psi}) = (\tilde{\pi}_{\rm u}^*)^{-1}(\tilde{x}\tilde{\psi})$$

on $\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B})$. Hence

$$\tilde{\pi}_{\rm u}(\tilde{x})\Theta(\tilde{\psi}) = (\iota \circ (\tilde{\pi}_{\rm u}^*)^{-1})(\tilde{x}\tilde{\psi}), \tag{2.3}$$

since, by definition, $\tilde{\pi}_{\rm u}(\tilde{x})\Theta(\tilde{\psi})$ belongs to $((\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))'')_*$. Taking into account the definition of Θ, we obtain from (2.3) the equality

$$\tilde{\pi}_{\rm u}(\tilde{x})\Theta(\tilde{\psi}) = \Theta(\tilde{x}\tilde{\psi}).$$

The invariance of $\mathcal{B}^* \otimes_\beta \mathcal{B}^*$ yields $\tilde{x}\tilde{\psi} \in \mathcal{B}^* \otimes_\beta \mathcal{B}^*$, showing that

$$\tilde{\pi}_{\rm u}(\tilde{x})\Theta(\tilde{\psi}) = \Theta(\tilde{x}\tilde{\psi}) \in \Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*),$$

which gives the left invariance of $\Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*)$ with respect to multiplication by the elements of $\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B})$. We obtain the right invariance in the same way. Now, since the algebra $\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B})$ is σ-weakly dense in the von Neumann algebra $(\tilde{\pi}_{\rm u}(\mathcal{B} \otimes_{\min} \mathcal{B}))''$, the claim follows from [**11**, Theorem III.2.7]. □

One of the main results of the paper is the following characterization of cloneability.

Theorem 2.3. *Let Γ be an arbitrary set of states of a C^*-algebra $\mathcal{A}$. The following conditions are equivalent:*

(i) *Γ is cloneable;*

(ii) *the states in Γ are pairwise strongly orthogonal.*

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Proof. (i) $\Longrightarrow$ (ii). Let $K^*\colon \mathcal{A}^* \to (\mathcal{A} \otimes_{\min} \mathcal{A})^*$ be an operation cloning all states in Γ, and define

$$K_{\mathrm{u}}^*\colon \mathcal{B}^* \to (\mathcal{B} \otimes_{\min} \mathcal{B})^*$$

by (2.2). Then, as we saw earlier in the proof of Lemma 2.2, K_{u}^* clones the states in $\Gamma_{\mathrm{u}} = \{\rho \circ \pi_{\mathrm{u}}^{-1}\colon \rho \in \Gamma\}$. The invariance of the space $\Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*)$ shown in Lemma 2.2 yields, by virtue of [**11**, Theorem III.2.7], the existence of a central projection $\tilde{p}$ in $(\tilde{\pi}_{\mathrm{u}}(\mathcal{B} \otimes_{\min} \mathcal{B}))''$ such that

$$\Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*) = \tilde{p}((\tilde{\pi}_{\mathrm{u}}(\mathcal{B} \otimes_{\min} \mathcal{B}))'')_*.$$

Define a map $\Pi\colon ((\tilde{\pi}_{\mathrm{u}}(\mathcal{B} \otimes_{\min} \mathcal{B}))'')_* \to \Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*)$ by the formula

$$\Pi\tilde{\psi} = \tilde{p}\tilde{\psi}, \qquad \tilde{\psi} \in ((\tilde{\pi}_{\mathrm{u}}(\mathcal{B} \otimes_{\min} \mathcal{B}))'')_*.$$

Then Π is a projection onto $\Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*)$; in particular, $\Pi\tilde{\psi} = \tilde{\psi}$ for $\tilde{\psi} \in \Theta(\mathcal{B}^* \otimes_\beta \mathcal{B}^*)$, so $\Pi\Theta|\mathcal{B}^* \otimes_\beta \mathcal{B}^* = \Theta|\mathcal{B}^* \otimes_\beta \mathcal{B}^*$.

Since $\mathcal{B} = \pi_{\mathrm{u}}(\mathcal{A})$ is the universal representation of $\mathcal{A}$, we have $\mathcal{B}^* = (\mathcal{B}'')_*$ (after identifying the elements in $\mathcal{B}^*$ with their extensions to $\mathcal{B}''$). Consider the map $\tilde{K}_* = \Theta^{-1}\Pi\Theta K_{\mathrm{u}}^*$. It acts from $(\mathcal{B}'')_*$ into $\mathcal{B}^* \otimes_\beta \mathcal{B}^* = (\mathcal{B}'')_* \otimes_\beta (\mathcal{B}'')_*$; moreover, for all states in Γ_{u} we have

$$\tilde{K}_*(\rho \circ \pi_{\mathrm{u}}^{-1}) = \Theta^{-1}\Pi\Theta(\rho \circ \pi_{\mathrm{u}}^{-1} \otimes \rho \circ \pi_{\mathrm{u}}^{-1}) = \rho \circ \pi_{\mathrm{u}}^{-1} \otimes \rho \circ \pi_{\mathrm{u}}^{-1},$$

since $\Theta^{-1}\Pi\Theta$ is the identity on $\mathcal{B}^* \otimes_\beta \mathcal{B}^*$.

Now

$$\mathcal{B}^* \otimes_\beta \mathcal{B}^* = (\mathcal{B}'')_* \otimes_\beta (\mathcal{B}'')_* = (\mathcal{B}'' \mathbin{\overline{\otimes}} \mathcal{B}'')_*,$$

and thus we have constructed a map

$$\tilde{K}_* = \Theta^{-1}\Pi\Theta K_{\mathrm{u}}^*\colon (\mathcal{B}'')_* \to (\mathcal{B}'' \mathbin{\overline{\otimes}} \mathcal{B}'')_*$$

from the predual of a von Neumann algebra into the predual of its tensor power that clones all states in Γ_{u}. By virtue of [**10**, Theorem 8], the states in Γ_{u} have mutually orthogonal supports, which on account of Lemma 2.1 shows that they are pairwise strongly orthogonal.

(ii) $\Longrightarrow$ (i). This has been proven in [**10**, Theorem 8]. For the sake of completeness let us repeat the main points of the proof here. Denote the states in Γ by ρ_i and put $e_i = \mathrm{s}(\rho_i)$. Then the e_i are pairwise orthogonal. Define a map $K_{\mathrm{u}}^*\colon (\mathcal{B}'')_* \to (\mathcal{B}'' \mathbin{\overline{\otimes}} \mathcal{B}'')_*$ as

$$K_{\mathrm{u}}^*\tilde{\psi} = \sum_i \tilde{\psi}(e_i)(\rho_i \circ \pi_{\mathrm{u}}^{-1} \otimes \rho_i \circ \pi_{\mathrm{u}}^{-1}), \quad \tilde{\psi} \in (\mathcal{B}'')_*.$$

It follows that K_{u}^* clones the states $\rho_i \circ \pi_{\mathrm{u}}^{-1}$, and it is easily seen that a map $K^*\colon \mathcal{A}^* \to (\mathcal{A} \otimes_{\min} \mathcal{A})^*$ defined as

$$K^* = (\pi_{\mathrm{u}} \otimes \pi_{\mathrm{u}})^* K_{\mathrm{u}}^* (\pi_{\mathrm{u}}^*)^{-1}$$

clones the states ρ_i (see (2.2)). □

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Remark 2.4. It is worth observing that we have the following reformulation of the theorem above in terms of the universal enveloping von Neumann algebra: let Γ be a family of states on a C^*-algebra $\mathcal{A}$; Γ is then cloneable if and only if it is cloneable in the universal enveloping von Neumann algebra of $\mathcal{A}$.

Remark 2.5. Let us note that the proof of Theorem 3 in [**5**] essentially shows that if *two pure* states on $\mathcal{A}$ are cloneable, then they are strongly orthogonal.

Now we are going to give yet another characterization of cloneability for a *finite* set of states. To this end let us introduce some necessary notions.

By a *measurement* (or *observable*) we shall mean a finite collection of positive elements $\{e_1, \ldots, e_m\}$ of $\mathcal{A}$ such that

$$\sum_{k=1}^{m} e_k = \mathbb{1}.$$

The notion of distinguishability of states as defined below has been used in [**2**, **3**] for a characterization of joint cloneability in finite dimension. It turns out that a weaker notion of almost distinguishability is also useful for a similar purpose.

States $\rho_1, \ldots, \rho_r$ of $\mathcal{A}$ are said to be *distinguishable* if there exists a measurement $\{e_1, \ldots, e_r\}$ such that $\rho_i(e_j) = \delta_{ij}$ for every $i, j = 1, \ldots, r$. They are said to be *almost distinguishable* if for any $\varepsilon > 0$ there exists a measurement $\{e_1, \ldots, e_r\}$ such that $\rho_k(e_k) > 1 - \varepsilon$ for each $k = 1, \ldots, r$.

Theorem 2.6. *Let $\rho_1, \ldots, \rho_r$ be states of $\mathcal{A}$. The following conditions are equivalent:*

(i) *$\rho_1, \ldots, \rho_r$ are almost distinguishable;*

(ii) *$\rho_1, \ldots, \rho_r$ are strongly orthogonal.*

Proof. (i) $\Longrightarrow$ (ii). Take an arbitrary positive integer n, and let $\{e_1^{(n)}, \ldots, e_r^{(n)}\}$ be a measurement such that

$$\rho_k(e_k^{(n)}) > 1 - \frac{1}{n}, \quad k = 1, \ldots, r.$$

Let π_{u} be the universal representation of $\mathcal{A}$. Then $\{\pi_{\mathrm{u}}(e_1^{(n)}), \ldots, \pi_{\mathrm{u}}(e_r^{(n)})\}$ is a measurement in $\pi_{\mathrm{u}}(\mathcal{A})$ such that

$$(\rho_k \circ \pi_{\mathrm{u}}^{-1})(\pi_{\mathrm{u}}(e_k^{(n)})) > 1 - \frac{1}{n}, \quad k = 1, \ldots, r. \tag{2.4}$$

Since all $\pi_{\mathrm{u}}(e_k^{(n)})$ are bounded in norm, we can find a subnet $\{n'\}$ of $\{n\}$ such that

$$\pi_{\mathrm{u}}(e_k^{(n')}) \to a_k \ \sigma\text{-weakly}, \quad k = 1, \ldots, r,$$

for some a_k in the universal enveloping von Neumann algebra $\pi_{\mathrm{u}}(\mathcal{A})''$ of $\mathcal{A}$. We clearly have $a_k \geqslant 0$ and

$$\sum_{k=1}^{r} a_k = \mathbb{1};$$

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moreover, from the inequalities (2.4) we obtain

$$(\rho_k \circ \pi_{\mathrm{u}}^{-1})(a_k) = 1, \quad k = 1, \dots, r.$$

This in turn shows that for the supports of $\rho_k \circ \pi_{\mathrm{u}}^{-1}$ we have

$$\mathrm{s}(\rho_k \circ \pi_{\mathrm{u}}^{-1}) \leqslant a_k,$$

and the relation

$$\sum_{k=1}^{r} \mathrm{s}(\rho_k \circ \pi_{\mathrm{u}}^{-1}) \leqslant \sum_{k=1}^{r} a_k = \mathbb{1}$$

yields the orthogonality of the supports, which, on account of Lemma 2.1, finishes the proof.

(ii) $\Longrightarrow$ (i). Define

$$e_k = \mathrm{s}(\rho_k), \quad k = 1, \dots, r.$$

On account of the Kaplansky density theorem, there exist nets $\{a_i^{(1)}\}_{i \in I}, \dots, \{a_i^{(r)}\}_{i \in I}$ in the positive part of the unit ball of $\pi_{\mathrm{u}}(\mathcal{A})$ such that

$$a_i^{(k)} \to e_k \text{ strongly}, \quad k = 1, \dots, r.$$

(Passing to subnets if necessary, we may and do assume that the nets have the same index set.) Define inductively the nets $\{b_i^{(r)}\}_{i \in I}, \dots, \{b_i^{(r)}\}_{i \in I}$ as follows:

$$\begin{aligned} b_i^{(1)} &= a_i^{(1)}, \\ b_i^{(n)} &= [\mathbb{1} - (b_i^{(1)} + \cdots + b_i^{(n-1)})] a_i^{(n)} [\mathbb{1} - (b_i^{(1)} + \cdots + b_i^{(n-1)})]. \end{aligned}$$

Taking into account the inequality $0 \leqslant a_i^{(k)} \leqslant \mathbb{1}$ for each $k = 1, \dots, r$, it is easy to show that

$$b_i^{(k)} \geqslant 0$$

for each $k = 1, \dots, r$,

$$b_i^{(1)} + \cdots + b_i^{(r)} \leqslant \mathbb{1},$$

and

$$b_i^{(k)} \to e_k \text{ strongly}$$

for each $k = 1, \dots, r$, because $e_1, \dots, e_r$ are pairwise orthogonal projections. It follows that $b_i^{(k)} \to e_k$ weakly, and since the nets are bounded, the convergence is also σ-weak. In particular,

$$(\rho_k \circ \pi_{\mathrm{u}}^{-1})(b_i^{(k)}) \to (\rho_k \circ \pi_{\mathrm{u}}^{-1})(e_k) = 1$$

for each $k = 1, \dots, r$. Take an arbitrary $\varepsilon > 0$. We can find an index i_0 such that

$$(\rho_k \circ \pi_{\mathrm{u}}^{-1})(b_{i_0}^{(k)}) > 1 - \varepsilon$$

for each $k = 1, \dots, r$. Now we have

$$b_{i_0}^{(1)} + \cdots + b_{i_0}^{(r)} \leqslant \mathbb{1},$$

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so substituting $\mathbb{1} - \sum_{k=2}^{r} b_{i_0}^{(k)} \geqslant b_{i_0}^{(1)}$ for $b_{i_0}^{(1)}$ and leaving the remaining $b_{i_0}^{(k)}$ untouched, we obtain the measurement $\{b_{i_0}^{(1)}, \dots, b_{i_0}^{(r)}\}$ in $\pi_{\mathrm{u}}(\mathcal{A})$ for which the inequality

$$(\rho_k \circ \pi_{\mathrm{u}}^{-1})(b_{i_0}^{(k)}) > 1 - \varepsilon, \quad k = 1, \dots, r,$$

holds. It is obvious that the measurement $\{\pi_{\mathrm{u}}^{-1}(b_{i_0}^{(1)}), \dots, \pi_{\mathrm{u}}^{-1}(b_{i_0}^{(r)})\}$ in $\mathcal{A}$ distinguishes the states $\rho_1, \dots, \rho_r$ up to $\varepsilon > 0$, which completes the proof. □

Remark 2.7. Theorems 2.3 and 2.6 give, for a finite set of states, equivalence of the following three conditions: (i) cloning, (ii) strong orthogonality and (iii) almost distinguishability. Two more natural conditions would be *pairwise almost distinguishability* (*pairwise distinguishability*) and *pairwise cloneability*, their definitions being (obviously) almost distinguishability and cloneability of every pair of the states. (Note that the definition of strong orthogonality has this structure.) It follows that pairwise almost distinguishability is equivalent to each of the conditions (i)–(iii), since by Theorem 2.6 pairwise almost distinguishability implies pairwise strong orthogonality, i.e. just strong orthogonality, and by virtue of Theorem 2.3 the same equivalence holds true for pairwise cloneability. Clearly, pairwise almost distinguishability and pairwise cloneability are easier to check than almost distinguishability and cloneability. However, as noted in [**2**], distinguishability and pairwise distinguishability are *not* equivalent.

The following simple theorem shows that distinguishability is a rather strong condition implying strong cloning even by a completely positive map.

Theorem 2.8. *If states $\rho_1, \dots, \rho_r$ are distinguishable, then they are strongly cloneable by a completely positive map.*

Proof. Indeed, let $\{e_1, \dots, e_r\}$ be a measurement that distinguishes the states $\rho_1, \dots, \rho_r$, and define a map $K\colon \mathcal{A} \otimes_{\min} \mathcal{A} \to \mathcal{A}$ by the formula

$$K(\tilde{a}) = \sum_{k=1}^{r} \rho_k \otimes \rho_k(\tilde{a}) e_k, \quad \tilde{a} \in \mathcal{A} \otimes_{\min} \mathcal{A}.$$

Then K is unital completely positive, and its adjoint has the form

$$K^* \varphi = \sum_{k=1}^{r} \varphi(e_k) \rho_k \otimes \rho_k, \quad \varphi \in \mathcal{A}^*.$$

Since $\rho_i(e_j) = \delta_{ij}$, it follows that K^* clones the ρ_i. □

3. The commutative case

In this section we take a closer look at the case of a commutative C^*-algebra. It turns out that additional structure allows one to consider additional notions of orthogonality of states. As for cloning, the example at the end of the section shows a subtle difference between strong cloneability and distinguishability.

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Let us therefore assume that $\mathcal{A}$ is an abelian C^*-algebra with identity, and identify it with the algebra $C(\Omega)$ of continuous functions on a compact Hausdorff space Ω. The tensor product $\mathcal{A} \otimes_{\min} \mathcal{A}$ is then identified with the algebra $C(\Omega \times \Omega)$. The states on $C(\Omega)$ are Radon probability measures. Let us recall that these measures are defined on the Baire σ-field $\mathcal{B}$ that is generated by closed $\mathbb{G}_\delta$-subsets of Ω. Is is known that Radon measures have unique extensions to Borel measures defined on the Borel σ-field $\mathbb{B}(\Omega)$, thus we can treat the states on $C(\Omega)$ as Borel probability measures.

For $f \in C(\Omega)$ and a state μ, we shall sometimes write $\mu(f)$ to denote $\int_\Omega f \, \mathrm{d}\mu$.

We recall that for a probability measure μ, its *support* is defined (non-uniquely) as any measurable set E that satisfies $\mu(E) = 1$. Two such measures are said to be *singular* if they have disjoint supports. Another classical notion for a probability measure μ, which we will call the *essential support*, is the (uniquely defined) smallest closed set F such that $\mu(F) = 1$ (note that the essential support is often called simply the 'support', especially when topological considerations play an important role). Two probability measures are said to be *strongly singular* if their essential supports are disjoint.

We shall consider mutual relations between cloneability, strong cloneability, distinguishability and orthogonality of states. Besides, we will also give an explicit description of a strong cloning operation.

Lemma 3.1. *Let μ and ν be states on $C(\Omega)$. The following conditions are equivalent:*

(i) *μ and ν are singular;*

(ii) *μ and ν are almost distinguishable;*

(iii) *μ and ν are strongly orthogonal.*

Proof. (i) $\Longrightarrow$ (ii). There exists a measurable set E such that $\mu(E) = 1$ and $\nu(E') = 1$ ($E' = \Omega \setminus E$). Let $\varepsilon > 0$ be arbitrary. From the regularity of μ and ν, it follows that there are compact sets $A \subset E$, $B \subset E'$ such that $\mu(A) > 1 - \varepsilon$ and $\nu(B) > 1 - \varepsilon$. Since A and B are disjoint, there exists a continuous function f, $0 \leqslant f \leqslant 1$, such that $f(\omega) = 1$ for $\omega \in A$ and $f(\omega) = 0$ for $\omega \in B$. Then $(1 - f)(\omega) = 1$ for $\omega \in B$. For a measurement $\{f, 1 - f\}$ we have

$$\mu(f) = \int_\Omega f \, \mathrm{d}\mu \geqslant \int_A f \, \mathrm{d}\mu = \mu(A) > 1 - \varepsilon,$$
$$\nu(1 - f) = \int_\Omega (1 - f) \, \mathrm{d}\nu \geqslant \int_B (1 - f) \, \mathrm{d}\nu = \nu(B) > 1 - \varepsilon,$$

showing that μ and ν are almost distinguishable.

(ii) $\Longleftrightarrow$ (iii). This is Theorem 2.6.

(iii) $\Longrightarrow$ (i). This follows essentially from the uniqueness of the Jordan decomposition of a real signed measure. Namely, define $\varphi = \mu - \nu$, and let $\varphi = \varphi^+ - \varphi^-$ be the Jordan decomposition. Then there are disjoint sets Ω_+ and Ω_- such that $\Omega = \Omega_+ \cup \Omega_-$, $\varphi^+(\Omega_-) = 0$, $\varphi^-(\Omega_+) = 0$, and

$$2 = \|\varphi\| = \|\varphi^+\| + \|\varphi^-\| = \varphi^+(\Omega_+) + \varphi^-(\Omega_-) = \varphi^+(\Omega) + \varphi^-(\Omega).$$

On the other hand,

$$\varphi^+(\Omega) - \varphi^-(\Omega) = \varphi(\Omega) = \mu(\Omega) - \nu(\Omega) = 0,$$

and thus

$$\varphi^+(\Omega_+) = \varphi^+(\Omega) = 1 = \varphi^-(\Omega) = \varphi^-(\Omega_-).$$

Moreover, we have

$$\mu(\Omega_+) - \nu(\Omega_+) = \varphi(\Omega_+) = \varphi^+(\Omega_+) = 1,$$

showing that $\mu(\Omega_+) \geqslant 1$ and thus $\mu(\Omega_+) = 1$; consequently, $\nu(\Omega_+) = 0$, i.e. $\nu(\Omega_-) = 1$, which means that μ and ν are singular (in particular, we also get $\varphi^+ = \mu$, $\varphi^- = \nu$). □

As for a relation between distinguishability and strong singularity we have the following.

Lemma 3.2. *Let μ and ν be states on $C(\Omega)$. Then μ and ν are distinguishable if and only if they are strongly singular.*

Proof. Assume first that μ and ν are distinguishable, and let $\{f_1, f_2\}$ be a measurement ($0 \leqslant f_1, f_2 \leqslant 1$, $f_1 + f_2 = 1$) such that

$$\mu(f_1) = \int_\Omega f_1 \, \mathrm{d}\mu = 1, \qquad \nu(f_2) = \int_\Omega f_2 \, \mathrm{d}\nu = 1.$$

It follows that

$$\mu(\{\omega\colon f_1(\omega) = 1\}) = 1, \qquad \nu(\{\omega\colon f_2(\omega) = 1\}) = 1.$$

Since the sets $E_1 = \{\omega\colon f_1(\omega) = 1\}$ and $E_2 = \{\omega\colon f_2(\omega) = 1\}$ are closed, the essential support of μ is contained in E_1 and the essential support of ν is contained in E_2, and since E_1 and E_2 are disjoint, the conclusion follows.

Assume now that μ and ν are strongly singular, and let E and F be their essential supports, respectively. By the Urysohn lemma, there is a function $0 \leqslant f \leqslant 1$ such that $f(\omega) = 1$ for $\omega \in E$ and $f(\omega) = 0$ for $\omega \in F$. For the measurement $\{f, 1 - f\}$, we have $\mu(f) = 1$, $\nu(1 - f) = 1$, showing that μ and ν are distinguishable. □

Remark 3.3. For the sake of simplicity, in our previous considerations we have restricted our attention to only two states. However, the following natural set-up would be possible. Let $\Gamma = \{\mu_1, \dots, \mu_r\}$ be a finite set of states of $C(\Omega)$. The states in Γ are said to be *singular* (respectively, *strongly singular*) if there exist pairwise disjoint sets (respectively, closed sets) $E_1, \dots, E_r$ such that $\mu_i(E_i) = 1$ for $i = 1, \dots, r$. In this setting we would have obvious counterparts of Lemmas 3.1 and 3.2 in which the set of two states $\{\mu, \nu\}$ would be replaced by Γ. It is not difficult to show that in this case the lemmas above would also hold true.

Now we are going to describe in some detail a strongly cloning map. The first step in this direction is a result that as a matter of fact characterizes linear positive unital maps between algebras of continuous functions on compact Hausdorff spaces. It can safely be said that results of this type belong to the folklore of the field, but for the sake of completeness and the lack of a proper reference we present its proof here. We formulate the theorem in the setting best suited to our purposes but it is easily seen that the same proof applies for a map between arbitrary algebras $C(\Omega_1)$ and $C(\Omega_2)$ of continuous functions on compact Hausdorff spaces Ω_1 and Ω_2.

Theorem 3.4. *Each linear positive unital map* $K\colon C(\Omega\times\Omega)\to C(\Omega)$ *has the form*

$$(K\tilde{f})(\omega)=\int_{\Omega\times\Omega}\tilde{f}\,\mathrm{d}N(\omega,\cdot),\quad \tilde{f}\in C(\Omega\times\Omega),\ \omega\in\Omega, \tag{3.1}$$

where $\{N(\omega,\cdot)\colon\omega\in\Omega\}$ *is a family of probability measures on* $(\Omega\times\Omega,\mathcal{B}\otimes\mathcal{B})$ *with the property*

$$\text{for any net } \{\omega_i\} \text{ in } \Omega \text{ with } \omega_i\to\omega, \text{ we have weak convergence } N(\omega_i,\cdot)\Longrightarrow N(\omega,\cdot). \tag{$*$}$$

Moreover, if the family $\{N(\omega,\cdot)\colon\omega\in\Omega\}$ *satisfies condition* $(*)$*, then for each* $\tilde{E}\in\mathcal{B}\otimes\mathcal{B}$ *the function* $N(\cdot,\tilde{E})$ *on* Ω *is* $\mathcal{B}$*-measurable, i.e.* N *is a Markov kernel on* $\Omega\times(\mathcal{B}\otimes\mathcal{B})$*.*

Proof. Assume that $K\colon C(\Omega\times\Omega)\to C(\Omega)$ is a linear positive unital map. For each $\omega\in\Omega$, the mapping

$$C(\Omega\times\Omega)\ni\tilde{f}\mapsto(K\tilde{f})(\omega)$$

is a linear bounded positive functional on $C(\Omega\times\Omega)$. Denote this functional by T_ω. We have $T_\omega(\mathbb{1})=1$, and there is therefore a probability Radon measure $N(\omega,\cdot)$ on $(\Omega\times\Omega,\mathcal{B}\otimes\mathcal{B})$ such that

$$(K\tilde{f})(\omega)=T_\omega(\tilde{f})=\int_{\Omega\times\Omega}\tilde{f}\,\mathrm{d}N(\omega,\cdot),\quad \tilde{f}\in C(\Omega\times\Omega).$$

Now, if $\omega_i\to\omega$, then for each $\tilde{f}\in C(\Omega\times\Omega)$ we have, by virtue of the continuity of $K\tilde{f}$, the convergence $(K\tilde{f})(\omega_i)\to(K\tilde{f})(\omega)$; that is,

$$\int_{\Omega\times\Omega}\tilde{f}\,\mathrm{d}N(\omega_i,\cdot)\to\int_{\Omega\times\Omega}\tilde{f}\,\mathrm{d}N(\omega,\cdot),$$

which is the weak convergence $N(\omega_i,\cdot)\Longrightarrow N(\omega,\cdot)$.

The converse assertion is immediate; namely, if we assume that K is defined by (3.1), then the weak convergence $N(\omega_i,\cdot)\Longrightarrow N(\omega,\cdot)$ entails that K transforms continuous functions on $\Omega\times\Omega$ into continuous functions on Ω.

Set

$$\mathcal{R}=\{\tilde{E}\in\mathcal{B}\otimes\mathcal{B}\colon N(\cdot,\tilde{E}) \text{ is measurable}\}.$$

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Since $N(\omega, \cdot)$ is a probability measure it follows that $\mathcal{R}$ is a so-called λ-system, i.e. it has the following properties:

(a) $\Omega \times \Omega \in \mathcal{R}$;

(b) if $\tilde{E}_1, \tilde{E}_2 \in \mathcal{R}$ and $\tilde{E}_1 \subset \tilde{E}_2$, then $\tilde{E}_2 \setminus \tilde{E}_1 \in \mathcal{R}$;

(c) if $\tilde{E}_1 \subset \tilde{E}_2 \subset \cdots$ and $\tilde{E}_n \in \mathcal{R}$, then $\bigcup_{n=1}^{\infty} \tilde{E}_n \in \mathcal{R}$.

Now let $\tilde{F}$ be a closed $\mathbb{G}_\delta$-subset of $\Omega \times \Omega$. Then

$$\tilde{F} = \bigcap_{n=1}^{\infty} \tilde{G}_n,$$

where the $\tilde{G}_n$ are open and $\tilde{G}_1 \supset \tilde{G}_2 \supset \cdots$. Let $\tilde{f}_n$ be continuous functions such that $0 \leqslant \tilde{f}_n \leqslant 1$, $\tilde{f}_n(\tilde{\omega}) = 1$ for $\tilde{\omega} \in \tilde{F}$ and $\tilde{f}_n(\tilde{\omega}) = 0$ for $\tilde{\omega} \in \tilde{G}_n'$. Then $\tilde{f}_n \to \chi_{\tilde{F}}$, where $\chi_{\tilde{F}}$ is the characteristic function of $\tilde{F}$. For each $\omega \in \Omega$, we have

$$N(\omega, \tilde{F}) = \int_{\Omega \times \Omega} \chi_{\tilde{F}} \, \mathrm{d}N(\omega, \cdot) = \lim_{n \to \infty} \int_{\Omega \times \Omega} \tilde{f}_n \, \mathrm{d}N(\omega, \cdot) = \lim_{n \to \infty} (K\tilde{f}_n)(\omega),$$

showing that $N(\cdot, \tilde{F})$ is Baire-measurable as a limit of continuous functions. This means that $\tilde{F}$ belongs to $\mathcal{R}$. The family of closed $\mathbb{G}_\delta$-sets is a so-called π-system, i.e. the intersection of two sets from this family belongs to the family too. Consequently, the theorem on π–λ-systems asserts that the smallest σ-field containing all closed $\mathbb{G}_\delta$-subsets of $\Omega \times \Omega$, i.e. $\mathcal{B} \otimes \mathcal{B}$, is contained in $\mathcal{R}$, which means that $N(\cdot, \tilde{E})$ is a measurable function for each $\tilde{E} \in \mathcal{B} \otimes \mathcal{B}$. □

Remark 3.5. It is immediately seen that the correspondence between the map K and the Markov kernel N in Theorem 3.4 is one to one.

From Theorem 3.4 we obtain the following characterization of the adjoint map.

Proposition 3.6. *Let $K \colon C(\Omega \times \Omega) \to C(\Omega)$ be given by (3.1). Its adjoint then has the form*

$$(K^*\mu)(\tilde{E}) = \int_{\Omega} N(\omega, \tilde{E}) \mu(\mathrm{d}\omega), \quad \mu \in C(\Omega)^*, \ \tilde{E} \in \mathcal{B} \otimes \mathcal{B}. \tag{3.2}$$

Proof. It is enough to prove formula (3.2) for a probability measure μ. Define $\tilde{\nu} \colon \mathcal{B} \otimes \mathcal{B} \to \mathbb{R}$ by

$$\tilde{\nu}(\tilde{E}) = \int_{\Omega} N(\omega, \tilde{E}) \mu(\mathrm{d}\omega), \quad \tilde{E} \in \mathcal{B} \otimes \mathcal{B}.$$

It is easily seen that $\tilde{\nu}$ is a probability measure on $\mathcal{B} \otimes \mathcal{B}$. For each $\tilde{f} \in C(\Omega \times \Omega)$ we have

$$\begin{aligned} \tilde{\nu}(\tilde{f}) &= \int_{\Omega \times \Omega} \tilde{f} \, \mathrm{d}\tilde{\nu} = \int_{\Omega} \int_{\Omega \times \Omega} \tilde{f} \, \mathrm{d}N(\omega, \cdot) \mu(\mathrm{d}\omega) \\ &= \int_{\Omega} (K\tilde{f})(\omega) \mu(\mathrm{d}\omega) = \mu(K\tilde{f}) = (K^*\mu)(\tilde{f}), \end{aligned}$$

showing that $\tilde{\nu} = K^*\mu$. □

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The proposition above yields the following corollary characterizing strong cloneability in abelian C^*-algebras.

Corollary 3.7. *Let Γ be an arbitrary family of probability measures on a compact Hausdorff space Ω with Baire σ-field $\mathcal{B}$. Γ is strongly cloneable if and only if there is a Markov kernel N on $\Omega \times (\mathcal{B} \otimes \mathcal{B})$ such that for each $\mu \in \Gamma$ and each $\tilde{E} \in \mathcal{B} \otimes \mathcal{B}$ we have*

$$\mu \otimes \mu(\tilde{E}) = \int_\Omega N(\omega, \tilde{E})\mu(\mathrm{d}\omega).$$

The example below shows that distinguishability is a stronger notion than strong cloneability.

Example 3.8. Let m be the Lebesgue measure on $[0,1]$ and let δ_0 be the Dirac measure concentrated at 0. We are going to show that m and δ_0 are jointly strongly cloneable despite not being strongly orthogonal, and thus not distinguishable either.

For $x \in [0,1]$ define sets $L_x \subset [0,1] \times [0,1]$ by the formulae

$$L_0 = \{(0,0)\}, \qquad L_1 = \{(1,1)\},$$
$$L_x = \begin{cases} \{(u,v)\colon u,v \geqslant 0,\ u+v = \sqrt{2x}\}, & 0 < x \leqslant \frac{1}{2}, \\ \{(u,v)\colon u,v \geqslant 0,\ u+v = 2-\sqrt{2(1-x)}\}, & \frac{1}{2} < x < 1, \end{cases}$$

and let Φ_x be defined as

(a) for $0 \leqslant x \leqslant \frac{1}{2}$,

$$\Phi_x\colon [0, \sqrt{2x}] \to L_x, \quad \Phi_x(t) = (t, \sqrt{2x} - t);$$

(b) for $\frac{1}{2} < x \leqslant 1$,

$$\Phi_x\colon [1-\sqrt{2(1-x)}, 1] \to L_x, \quad \Phi_x(t) = (t, 2-\sqrt{2(1-x)} - t).$$

Define a family $\{N(x,\cdot)\colon x \in [0,1]\}$ of probability measures on $[0,1]\times[0,1]$ as

$$N(0,\cdot) = \delta_{(0,0)} = \delta_0 \otimes \delta_0, \qquad N(1,\cdot) = \delta_{(1,1)} = \delta_1 \otimes \delta_1,$$
$$N(x,\cdot) = \begin{cases} \dfrac{\Phi_x \circ m}{\sqrt{2x}}, & 0 < x \leqslant \frac{1}{2}, \\ \dfrac{\Phi_x \circ m}{\sqrt{2(1-x)}}, & \frac{1}{2} < x < 1. \end{cases}$$

(The $N(x,\cdot)$ for $0 < x < 1$ are simply normalized Lebesgue measures on the segments L_x.) It is easily seen that N is a Markov kernel satisfying condition $(*)$ of Theorem 3.4.

For each continuous function f on $[0,1]\times[0,1]$, we have

$$\int_0^1 \int_0^1 f \,\mathrm{d}N(x,\cdot) = \iint_{L_x} f \,\mathrm{d}N(x,\cdot) \tag{3.3}$$

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and

$$\iint_{L_x} f\,\mathrm{d}N(x,\cdot) = \begin{cases} f(0,0) & \text{for } x=0, \\ f(1,1) & \text{for } x=1, \end{cases} \tag{3.4}$$

$$\iint_{L_x} f\,\mathrm{d}N(x,\cdot) = \begin{cases} \dfrac{1}{\sqrt{2x}}\displaystyle\int_0^{\sqrt{2x}} f(t,\sqrt{2x}-t)\,\mathrm{d}t & \text{for } 0<x\leqslant \frac{1}{2}, \\ \dfrac{1}{\sqrt{2(1-x)}}\displaystyle\int_{1-\sqrt{2(1-x)}}^{1} f(t,2-\sqrt{2(1-x)}-t)\,\mathrm{d}t & \text{for } \frac{1}{2}<x<1. \end{cases} \tag{3.5}$$

Let a strongly cloning map K be defined by (3.1). Then $K^*\mu$ is defined by (3.2); in particular, we have (writing simply $\mathrm{d}x$ instead of $m(\mathrm{d}x)$)

$$\begin{aligned} \int_0^1\int_0^1 f\,\mathrm{d}(K^*m) &= \int_0^1 \left(\int_0^1\int_0^1 f\,\mathrm{d}N(x,\cdot)\right)\mathrm{d}x \\ &= \int_0^{1/2} \left(\int_0^1\int_0^1 f\,\mathrm{d}N(x,\cdot)\right)\mathrm{d}x + \int_{1/2}^1 \left(\int_0^1\int_0^1 f\,\mathrm{d}N(x,\cdot)\right)\mathrm{d}x \\ &= I_1 + I_2. \end{aligned}$$

Taking into account (3.3) and (3.5) we obtain

$$\begin{aligned} I_1 &= \int_0^{1/2} \left(\int_0^1\int_0^1 f\,\mathrm{d}N(x,\cdot)\right)\mathrm{d}x = \int_0^{1/2}\left(\frac{1}{\sqrt{2x}}\int_0^{\sqrt{2x}} f(t,\sqrt{2x}-t)\,\mathrm{d}t\right)\mathrm{d}x \\ &= \int_0^1\int_0^u f(t,u-t)\,\mathrm{d}t\,\mathrm{d}u = \iint_{\Delta_1} f(v,w)\,\mathrm{d}v\,\mathrm{d}w, \end{aligned}$$

where $\Delta_1 = \{(v,w)\colon 0\leqslant v\leqslant 1,\ 0\leqslant w\leqslant 1-v\}$, and

$$\begin{aligned} I_2 &= \int_{1/2}^1 \left(\int_0^1\int_0^1 f\,\mathrm{d}N(x,\cdot)\right)\mathrm{d}x \\ &= \int_{1/2}^1 \left(\frac{1}{\sqrt{2(1-x)}}\int_{1-\sqrt{2(1-x)}}^1 f(t,2-\sqrt{2(1-x)}-t)\,\mathrm{d}t\right)\mathrm{d}x \\ &= \int_0^1\int_{1-u}^1 f(t,2-u-t)\,\mathrm{d}t\,\mathrm{d}u = \iint_{\Delta_2} f(v,w)\,\mathrm{d}v\,\mathrm{d}w, \end{aligned}$$

where $\Delta_2 = \{(v,w)\colon 0\leqslant v\leqslant 1,\ 1-v\leqslant w\leqslant 1\}$. Consequently,

$$\begin{aligned} \int_0^1\int_0^1 f\,\mathrm{d}(K^*m) = I_1+I_2 &= \iint_{\Delta_1} f(v,w)\,\mathrm{d}v\,\mathrm{d}w + \iint_{\Delta_2} f(v,w)\,\mathrm{d}v\,\mathrm{d}w \\ &= \int_0^1\int_0^1 f(v,w)\,\mathrm{d}v\,\mathrm{d}w = \int_0^1\int_0^1 f\,\mathrm{d}(m\otimes m), \end{aligned}$$

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showing that

$$K^*m = m \otimes m,$$

i.e. K^* strongly clones m. Furthermore, (3.3) and (3.4) yield

$$\int_0^1 \int_0^1 f \,\mathrm{d}(K^*\delta_0) = \int_0^1 \left(\int_0^1 \int_0^1 f \,\mathrm{d}N(x, \cdot) \right) \delta_0(\mathrm{d}x)$$
$$= \int_0^1 \int_0^1 f \,\mathrm{d}N(0, \cdot) = f(0, 0) = \int_0^1 \int_0^1 f \,\mathrm{d}(\delta_0 \otimes \delta_0),$$

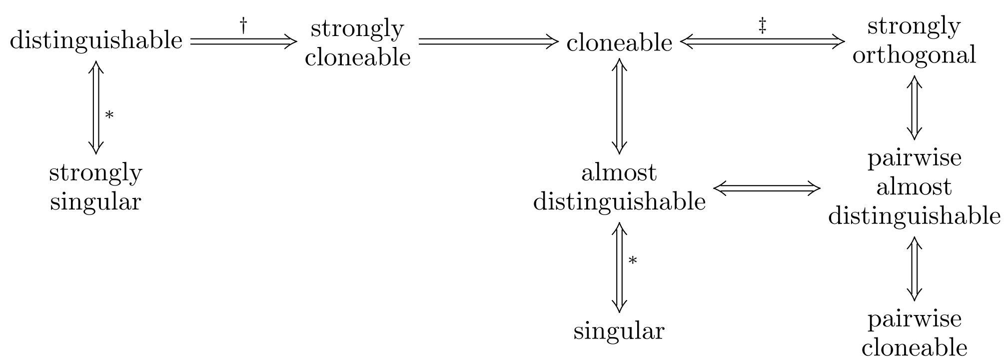
showing that

$$K^*\delta_0 = \delta_0 \otimes \delta_0,$$

i.e. K^* strongly clones δ_0 (the same is of course true of δ_1).

4. Summary

Let us finish with the following picture describing the state of affairs for a finite (with one important exception) number of states in arbitrary C^*-algebras:



'†' means that the reverse implication does not hold even in the abelian case. '‡' means that the relation holds for an arbitrary number of states. '∗' means that the relation holds in the abelian case.

The only missing relation is therefore

$$\text{cloneable} \nRightarrow \text{strongly cloneable}.$$

While this seems highly probable, it does still need to be formally proved.

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