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Poles of the Standard $\mathcal{L}$-function of G_2 and the Rallis–Schiffmann Lift

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Abstract. We characterize the cuspidal representations of G_2 whose standard $\mathcal{L}$-function admits a pole at $s = 2$ as the image of the Rallis–Schiffmann lift for the commuting pair $(\widetilde{\mathrm{SL}}_2, G_2)$ in $\widetilde{\mathrm{Sp}}_{14}$. The image consists of non-tempered representations. The main tool is the recent construction, by the second author, of a family of Rankin–Selberg integrals representing the standard $\mathcal{L}$-function.

1 Introduction

Let G be a reductive algebraic group defined over a number field F. In the theory of automorphic forms, for any automorphic irreducible representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ and a finite-dimensional complex representation ρ of the Langlands dual complex group ${}^L G$, one can associate a partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\rho)$, where S is a finite set of places of the number field F outside of which π_v is unramified. It is defined by

$$\mathcal{L}^S(s,\pi,\rho) = \prod_{v\notin S}\Big(\det(I-\rho(t_{\pi_v})q_v{}^{-s})\Big)^{-1},$$

where t_{π_v} is a representative of the Satake conjugacy class associated with π_v and q_v is the order of the residue field of F_v. Conjecturally, all such $\mathcal{L}$-functions admit meromorphic continuation. The poles of the $\mathcal{L}$-functions are of special interest, since images of functorial lifts can often be characterized in terms of these poles.

Precisely, given an algebraic reductive group H, a map of dual groups $r\colon {}^L H \to {}^L G$, and an irreducible automorphic representation $\sigma = \otimes_v \sigma_v$ of $H(\mathbb{A})$, we say that an automorphic representation π of $G(\mathbb{A})$ is a *weak lift of σ with respect to r* if for almost all places, $r(t_{\sigma_v})$ is conjugate to t_{π_v}. We can now formulate the main result of our paper.

Let G be the split exceptional group G_2. In particular, its Langlands dual group is $G_2(\mathbb{C})$. Let $\mathfrak{st}$ denote the standard seven-dimensional representation of $G_2(\mathbb{C})$. Consider the map $r\colon \mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C}) \to G_2(\mathbb{C})$, where the map restricted to the first (resp. second) copy of $\mathrm{SL}_2(\mathbb{C})$ corresponds to the unipotent conjugacy class generated by a long (resp. short) root of G_2.

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Theorem 1.1 (Main Theorem) *For a cuspidal irreducible representation π of $G_2(\mathbb{A})$, the following are equivalent.*

(i) *There exists an irreducible square-integrable automorphic representation τ of*

$$\mathrm{SO}(2,1)(\mathbb{A})$$

such that π is a weak lift of the representation $\tau \boxtimes 1$ of $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$ with respect to the map r.

(ii) *The partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\mathfrak{st})$ has a pole at $s=2$.*

The pole of the $\mathcal{L}$-function is simple unless π is a weak lift of $1 \boxtimes 1$, in which case the pole is of order two.

The direction (i) $\Rightarrow$ (ii) is easy. This is the content of the following lemma.

Lemma 1.2 *Let π be an irreducible representation of $G_2(\mathbb{A})$ that is a weak lift of a square-integrable automorphic representation $\tau \boxtimes 1$ of $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$. Then the following hold.*

(i) *$\mathcal{L}^S(s,\pi,\mathfrak{st})$ has a pole at $s=2$.*

(ii) *The pole is simple unless $\tau = 1$, in which case the pole is of order 2.*

Proof We identify the algebraic groups $\mathrm{SO}(2,1)$ and PGL_2. The representation τ is either cuspidal or of the form $\chi \circ \det$ for a quadratic automorphic character χ.

In the first case

$$\mathcal{L}^S(s,\pi,\mathfrak{st}) = \zeta^S(s-1)\,\mathcal{L}^S(s-1/2,\tau)\zeta^S(s)\,\mathcal{L}^S(s+1/2,\tau)\zeta^S(s+1).$$

The factor $\zeta^S(s-1)$ contributes a simple pole at $s=2$, whereas the other $\mathcal{L}$-functions are non-zero at $s=2$.

In the second case

$$\mathcal{L}^S(s,\pi,\mathfrak{st}) = \zeta^S(s-1)\,\mathcal{L}^S(s-1,\chi)\,\mathcal{L}^S(s,\chi)^2\zeta^S(s)\,\mathcal{L}^S(s+1,\chi)\zeta^S(s+1),$$

and hence has a simple pole at $s=2$ for $\chi \neq 1$ and a pole of order 2 for $\chi = 1$. ∎

The proof of (ii) $\Rightarrow$ (i) requires both some information on the poles of the $\mathcal{L}$-function and a proof of the existence of the weak functorial lift. We use our recent result on a Rankin–Selberg integral representation for $\mathcal{L}^S(s,\pi,\mathfrak{st})$ to achieve the information about its poles.

1.1 The Standard $\mathcal{L}$-function of G_2

The meromorphic continuation of $\mathcal{L}^S(s,\pi,\mathfrak{st})$ has been proved [Seg17, Seg16] by constructing a family of new-way Rankin–Selberg integrals for $\mathcal{L}^S(s,\pi,\mathfrak{st})$. The integrals in the family are parameterized by étale cubic algebras E over F. Precisely, for any étale cubic algebra E there is an associated simply-connected quasi-split group H_E of type D_4 with Heisenberg maximal parabolic subgroup P. Let $\mathcal{E}^*_P(f,s,h)$ denote the normalized Eisenstein series associated with the normalized induced representation

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$\mathrm{Ind}_P^{H_E}\,\delta_P^s$. Consider a family of integrals

$$\mathcal{Z}_E(\varphi, f, s) = \int_{G_2(F)\backslash G_2(\mathbb{A})} \varphi(g)\mathcal{E}_P^*(f, s, g)\,dg, \tag{1.1}$$

where φ belongs to the space of a cuspidal representation π of $G_2(\mathbb{A})$.

For each cuspidal representation π, the integral $\mathcal{Z}_E(\cdot,\cdot,s)$ either represents the standard $\mathcal{L}$-function or is identically zero, depending on whether or not π supports the Fourier coefficient corresponding to the étale algebra E along the Heisenberg unipotent subgroup. See Section 2 for details on Fourier coefficients. Wee Teck Gan [Gan05, Theorem 3.1] showed that any cuspidal representation supports such a coefficient for at least one étale cubic algebra E. For such E one has [Seg17, Theorem 3.1]

$$\mathcal{Z}_E(\varphi, f, s) = \mathcal{L}^S(5s + 1/2, \pi, \mathfrak{st}) d_S(f, \varphi, s) \tag{1.2}$$

and for any given point $s_0 \in \mathbb{C}$, the data φ and f can be chosen so that $d_S(f, \varphi, s)$ is holomorphic and non-zero in a neighborhood of s_0. In this case, we say that the integral $\mathcal{Z}_E(\varphi, f, s)$ represents the L-function $\mathcal{L}^S(5s + 1/2, \pi, \mathfrak{st})$.

If $\mathcal{Z}_E(\varphi, f, s)$ represents $\mathcal{L}^S(5s+1/2, \pi, \mathfrak{st})$, then $\mathcal{Z}_E(\varphi, f, s)$ can be used to study the special values of $\mathcal{L}^S(s, \pi, \mathfrak{st})$. In particular, for any s_0, the order of $\mathcal{L}^S(s, \pi, \mathfrak{st})$ at s_0 is bounded by the order of $\mathcal{E}_P^*(f, s, g)$ at $\frac{1}{5}(s_0 - \frac{1}{2})$. In the right half-plane, the poles of $\mathcal{E}_P^*(f, s, g)$ coincide with the poles of the unnormalized Eisenstein series $\mathcal{E}_P(f, s, g)$. The poles of these Eisenstein series for $\mathrm{Re}(s) > 0$ were studied in [Seg18].

The possible poles of $\mathcal{E}_P(f, s, g)$ in the right half-plane can occur at $s = 1/10$, $s = 3/10$ or $s = 1/2$. Thus for any cuspidal representation π, possible poles of the $\mathcal{L}$-function $\mathcal{L}^S(s, \pi, \mathfrak{st})$ are at $s = 1$, $s = 2$ or $s = 3$. The possibility $s = 3$ does not occur, since the residue of $\mathcal{E}_P(f, s, g)$ at $s = 1/2$ is a constant function for any E, and hence the residue of $\mathcal{Z}_E(\varphi, f, s)$ is zero. Hence, $\mathcal{L}^S(s, \pi, \mathfrak{st})$ is always holomorphic at $s = 3$.

In this paper we describe the cuspidal representations π whose $\mathcal{L}$-function $\mathcal{L}^S(s, \pi, \mathfrak{st})$ admits a pole at $s = 2$. The order of $\mathcal{L}^S(s, \pi, \mathfrak{st})$ at $s = 2$ is bounded by the order of $\mathcal{E}_P(f, s, g)$ at $s = 3/10$. The pole of $\mathcal{E}_P(f, s, g)$ at $s = 3/10$ has been studied for $E = F \times F \times F$ [GGJ02] and for general E [Seg18, Seg16].

Theorem 1.3 ([Seg16, GGJ02]) *Let $\mathcal{E}_P(f, s, g)$ be an Eisenstein series associated with the representation $\mathrm{Ind}_P^{H_E}\,\delta_P^s$.*

(i) *Let E be a cubic field extension. Then $\mathcal{E}_P(f, s, g)$ is holomorphic at $s = 3/10$.*

(ii) *Let $E = F \times K$, where K is a quadratic field extension. Then $\mathcal{E}_P(f, s, g)$ has at most a simple pole at $s = 3/10$. This pole is attained by the spherical section (as defined in item (14) of Section 3.1). The residual representation at this point is not square integrable.*

(iii) *Let $E = F \times F \times F$. Then $\mathcal{E}_P(f, s, g)$ has a pole of order at most 2 at $s = 3/10$. This pole is attained by the spherical section. The leading term of the Laurent expansion generates the minimal representation of the group H_E.*

The integrals $\mathcal{Z}_E(\cdot,\cdot,s)$ can also be used to characterize functorial lifts in terms of poles and special values of the standard $\mathcal{L}$-function.

Let us give an example. The dual pair $S_3 \times G_2 \hookrightarrow \mathrm{Spin}_8 \rtimes S_3$ has been studied both locally [HMS98] and globally [GGJ02]. The automorphic minimal representation $\Pi_{\min}$

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of the split group $H = H_{F\times F\times F} = \mathrm{Spin}_8$ is generated by functions of the form

$$[(s-3/10)^2\mathcal{E}_P(f,s,g)]\big|_{s=3/10}, \quad f\in \mathrm{Ind}_P^H(s).$$

and it can be extended to a representation of $\mathrm{Spin}_8 \rtimes S_3$. Let us denote by Θ the theta correspondence for this dual pair. It was shown [GGJ02] that, whenever $\Theta(\pi)\neq 0$, one has, for almost all v,

$$r(\mathrm{diag}(q_v^{1/2},q_v^{-1/2}),\mathrm{diag}(q_v^{1/2},q_v^{-1/2}))\sim t_{\pi_v}.$$

Here $\sim$ indicates that the elements are conjugate in $G_2(\mathbb{C})$.

An immediate corollary of [GS15, Theorem 1.1] and Lemma 1.2 is the following.

Theorem 1.4 *Let π be a cuspidal representation of $G_2(\mathbb{A})$. The following statements are equivalent.*

(i) *The partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\mathfrak{st})$ admits a double pole at $s=2$.*
(ii) *The theta lift $\Theta(\pi)$ is not zero.*
(iii) *π is a weak lift of $1\boxtimes 1$ with respect to r.*

The explicit construction of the weak lift from any $\tau\boxtimes 1$ with respect to r is fully realized using the Rallis–Schiffmann lift that will be described below.

Remark 1.5 The pole of $\mathcal{E}_P^*(f,s,g)$ at $s=1/10$ is simple for all E and the residue of $\mathcal{E}_P^*(f,s,g)$ is described in [Sega]. The description of the cuspidal representations π of G_2 whose $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\mathfrak{st})$ has a pole at $s=1$ is a work in progress.

1.2 The Rallis–Schiffmann Lift

1.2.1 Construction

A remarkable construction of a class of cuspidal non-tempered representations of the exceptional group G_2 was obtained by Rallis and Schiffmann [RS89]. The pair $(\widetilde{\mathrm{SL}}_2,G_2)$ is not a dual pair, but merely a commuting pair inside $\widetilde{\mathrm{Sp}}_{14}$. Indeed, the centralizer of a certain embedding of $\widetilde{\mathrm{SL}}_2$ in $\widetilde{Sp}_{14}$ is the group $\mathrm{SO}(V^7)\times\{\pm1\}$, where V^7 is a seven-dimensional split quadratic space. The group G_2 is naturally embedded into the split special orthogonal group $\mathrm{SO}(V^7)$.

Let $\psi=\bigotimes\psi_v$ be a fixed additive complex character of $F\backslash\mathbb{A}$. For any cuspidal representation σ of the metaplectic cover $\widetilde{\mathrm{SL}}_2$, its theta lift $\theta_\psi(\sigma)$ to $\mathrm{SO}(V^7)$ is a non-zero and non-cuspidal automorphic representation. However, the restriction of functions in $\theta_\psi(\sigma)$ to the subgroup $G_2(\mathbb{A})$ of $\mathrm{SO}(V^7)(\mathbb{A})$ defines a square-integrable automorphic representation of $G_2(\mathbb{A})$. This representation of $G_2(\mathbb{A})$ is cuspidal whenever the theta lift of σ to $\mathrm{SO}(2,1)$ with respect to ψ is zero. This lift is denoted by $RS_\psi(\sigma)$. It is not difficult to extend the definition of the lift from the cuspidal spectrum to all the summands of the discrete spectrum of $\widetilde{\mathrm{SL}}_2$ [GG06, §12.7].

The lift $RS_\psi(\sigma)$ is not necessarily irreducible. The question of reducibility was studied in [GG06] by a complete determination of the local lift. The local lift is used in order to define certain non-tempered A-packets on G_2, and the global lift is used to prove Arthur's multiplicity formula for these packets in most cases.

The construction has proved to be functorial. Although the group $\widetilde{\mathrm{SL}}_2$ is not algebraic, the Satake parameters of an irreducible automorphic representation σ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ are defined via the Waldspurger map Wd_ψ, which associates, with every irreducible square-integrable automorphic representation σ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, an irreducible square integrable representation $\tau = Wd_\psi(\sigma)$ of $\mathrm{SO}(2,1)(\mathbb{A})$. The map is finite-to-one.

Thus, whenever σ_v is spherical, the Satake parameter $t_{\psi_v,\sigma_v} \in \mathrm{SL}_2(\mathbb{C})$ is defined to be the Satake parameter of $Wd_{\psi_v}(\sigma_v)$. Note the dependence on the character ψ.

Proposition 1.6 ([RS89, §5, Theorem 1]) *Let $\sigma = \bigotimes_v \sigma_v$ be a cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. Let $\pi = \bigotimes_v \pi_v$ be an irreducible summand of $RS_\psi(\sigma)$. For all v such that σ_v, π_v are unramified, denote by t_{ψ_v,σ_v} the Satake parameter of σ_v, and by t_{π_v} the Satake parameter of π_v, which is a representative of a semisimple conjugacy class in $G_2(\mathbb{C})$. Then $r(t_{\psi_v,\sigma_v}, \mathrm{diag}(q_v^{1/2}, q_v^{-1/2})) \sim t_{\pi_v}$. Here $\sim$ indicates that the elements are conjugate in $G_2(\mathbb{C})$.*

In particular, π is a *weak functorial lift*, with respect to the map r, of the automorphic representation $Wd_\psi(\sigma) \boxtimes 1$ of $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$.

The lift in the opposite direction is naturally defined. For a cuspidal representation π of $G_2(\mathbb{A})$, its lift to a representation $RS_\psi(\pi)$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ is the span of the functions

$$RS_\psi(\phi, \varphi)(g) = \int_{G_2(F)\backslash G_2(\mathbb{A})} \theta_\psi(\phi)(g,h)\varphi(h)\,dh,$$

where $\varphi \in \pi$, ϕ is a Schwartz function on $V^7(\mathbb{A})$, and $\theta_\psi(\phi)$ is an automorphic theta function on $\widetilde{Sp}_{14}$ restricted to $\widetilde{\mathrm{SL}}_2 \times G_2$. Computing the constant term of $RS_\psi(\pi)$ using the Schrödinger model, it is easy to see that $RS_\psi(\pi)$ is necessarily a cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

1.2.2 Exhaustion

In this subsection, we show how the results of [GG06] imply the following.

Proposition 1.7 *Let π be an irreducible cuspidal representation of $G_2(\mathbb{A})$. The following statements are equivalent.*

(i) $RS_\psi(\pi) \neq 0$.

(ii) *There exists an irreducible square-integrable automorphic representation τ of* $\mathrm{SO}(2,1)(\mathbb{A})$ *such that π is a weak lift of $\tau \boxtimes 1$ with respect to r.*

Proof We first recall the structure of the space $\mathcal{A}_2(\widetilde{\mathrm{SL}}_2)$ that is the sum of all irreducible representations contained in the space of square integrable genuine automorphic forms of $\widetilde{\mathrm{SL}}_2$. Recall [Wal91, Wal80] the decomposition of the space $\mathcal{A}_2(\widetilde{\mathrm{SL}}_2)$:

$$\mathcal{A}_2(\widetilde{\mathrm{SL}}_2) = (\bigoplus_\tau \mathcal{A}_\tau) \oplus (\bigoplus_\chi \mathcal{A}_\chi),$$

where τ runs over the cuspidal representations of $\mathrm{SO}(2,1)$, and χ runs over the set of quadratic Hecke characters of $F^\times\backslash\mathbb{A}^\times$.

For each cuspidal representation τ of $\mathrm{SO}(2,1)$, the space $\mathcal{A}_\tau$ is a sum of nearly equivalent representations σ of $\widetilde{\mathrm{SL}}_2$ such that $Wd_\psi(\sigma) = \tau$. All summands appear in $\mathcal{A}_\tau$ with multiplicity one and form a full near-equivalence class.

For each quadratic character χ, the space $\mathcal{A}_\chi$ is a sum of irreducible summands of the Weil representation ω_χ associated with the one-dimensional orthogonal space, whose discriminant defines the quadratic character χ via class field theory. Again, all the summands appear in $\mathcal{A}_\chi$ with multiplicity one and form a full near-equivalence class. For each irreducible representation σ in $\mathcal{A}_\chi$ one has $Wd_\psi(\sigma) = \chi \circ \det$, considered as an automorphic representation of $\mathrm{SO}(2,1) \simeq PGL_2$.

Denote by V_χ and V_τ the Rallis–Schiffman lift of the spaces $\mathcal{A}_\chi$ and $\mathcal{A}_\tau$, respectively.

Theorem 1.8 ([GG06, Theorem 16.1]) *Let π be an irreducible cuspidal representation of $G_2(\mathbb{A})$.*

(i) *If π is a weak lift of $(\chi \circ \det) \boxtimes 1$ for some quadratic Hecke character χ, then π is contained in V_χ.*

(ii) *If π is a weak lift of $\tau \boxtimes 1$ for some cuspidal representation τ of* $\mathrm{SO}(2,1)$, *then π is not orthogonal to V_τ.*

In both cases $RS_\psi(\pi) \neq 0$.

Conversely, if σ is contained in $RS_\psi(\pi)$, then π is isomorphic to an irreducible summand of $RS_\psi(\sigma)$ and hence is a weak lift of $Wd_\psi(\sigma) \boxtimes 1$ with respect to r. The proposition follows. ∎

Now to prove Theorem 1.1, it remains to show that if $\mathcal{L}^S(s, \pi, \mathfrak{st})$ has a pole at $s = 2$, then $RS_\psi(\pi) \neq 0$. This will be proved in Theorem 5.1.

2 Wave Front

The Fourier coefficients of an automorphic form on G_2 along the Heisenberg unipotent subgroup are parameterized by cubic algebras over F [Seg17, §2]. In this section, we prove that if $\mathcal{L}^S(s, \pi, \mathfrak{st})$ has a pole at $s = 2$, then π admits a Fourier coefficient of type $F \times K$, where K is an étale quadratic algebra over F. This allows us to relate the analytic properties of $\mathcal{L}^S(s, \pi, \mathfrak{st})$ to those of $\mathcal{Z}_{F\times K}(\cdot, \cdot, s)$.

2.1 Fourier Coefficients of $\widetilde{\mathrm{SL}}_2$

Let $B = T \cdot N$ denote the Borel subgroup of SL_2. We denote by α the unique positive root of SL_2 and denote by $x_\alpha : \mathbb{G}_a \to N$ the associated one-parametric subgroup. The torus $T(F)$ acts on the set of non-trivial characters of $N(\mathbb{A})$ that are trivial on $N(F)$, and the orbits are parameterized by quadratic étale algebras. Fix a non-trivial unitary character $\psi \colon F\backslash\mathbb{A} \to \mathbb{C}^\times$. For any square class a and its associated quadratic algebra K, define the character $\Psi_K : N(\mathbb{A}) \to \mathbb{C}$ given by $\Psi_K(x_\alpha(r)) = \psi(ar)$.

Let $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ denote the metaplectic cover of $\mathrm{SL}_2(\mathbb{A})$. The groups $N(\mathbb{A})$ and $T(F)$ split in $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

For any automorphic form φ on $\widetilde{\mathrm{SL}}_2$ define

$$\varphi^{N,\Psi_K}(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\overline{\Psi_K(n)}\,dn.$$

The wave front of an automorphic representation σ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ is defined by

$$\widehat{F}_\psi(\sigma) = \left\{K \mid \exists\varphi \in \sigma : \varphi^{N,\Psi_K} \neq 0\right\}.$$

Proposition 2.1 ([GG06, §12.3]) *Let σ be an irreducible summand of $\mathcal{A}_2(\widetilde{\mathrm{SL}}_2)$. If $\widehat{F}_\psi(\sigma) = \{K\}$, then σ is a summand of $\mathcal{A}_\chi$, where the quadratic character χ is associated with the algebra K by class field theory.*

2.2 Fourier Coefficients of $G = G_2$

We shall start with an overview of $G = G_2$ as a Chevalley group defined over $\mathbb{Z}$. We fix a maximal split torus T_G and a Borel subgroup $B_G = T_G \cdot N_G$. This determines the root datum of the group. Denote by α and β the short and long simple roots, respectively. There are six positive roots: $\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$. With any positive root γ we associate a one-parametric subgroup $x_\gamma : \mathbb{G}_a \to N_G$ [GGJ02, GS15, Jia98, Seg17], and denote its image by U_γ.

We denote by $P_1 = M_1 \cdot U_1$ and $P_2 = M_2 \cdot U_2$, the maximal standard parabolic subgroups such that $U_\alpha \subset U_1$ and $U_\beta \subset U_2$. By [GGS02], the choice of one-parametric subgroups induces an isomorphism between $U_2/[U_2, U_2]$ and the space of binary cubic forms.

The Levi factor $M_2(F)$ acts on the set of unitary characters on $U_2(\mathbb{A})$ that are trivial on $U_2(F)$, and the orbits are indexed by cubic algebras over F. The generic orbits correspond to étale cubic algebras. For any cubic algebra E, choose a representative Ψ_E of the associated orbit.

For any automorphic form φ on G_2 and étale cubic algebra E, denote

$$\varphi^{U_2,\Psi_E}(g) = \int_{U_2(F)\backslash U_2(\mathbb{A})} \varphi(ug)\overline{\Psi_E(u)}\,du.$$

For any automorphic representation π of $G_2(\mathbb{A})$, define the wave front of π with respect to U_2 by

$$\widehat{F}_\psi(\pi) = \left\{E \text{ étale} \mid \exists\varphi \in \pi : \varphi^{U_2,\Psi_E} \neq 0\right\}.$$

We shall write down explicitly a character $\Psi_{F\times K}$ on $U_2(\mathbb{A})$ that is a representative of the generic orbit corresponding to the étale cubic algebra $F\times K$, where K is a quadratic étale algebra. Let a be the square class in $F^\times$ associated with K. Then

$$\Psi_{F\times K}(x_\beta(r_1)x_{\alpha+\beta}(r_2)x_{2\alpha+\beta}(r_3)x_{3\alpha+\beta}(r_4)x_{3\alpha+2\beta}(r_5)) = \psi(-ar_1 + r_3).$$

A family of reductive periods is closely related to the family of Fourier coefficients corresponding to the algebras of type $F \times K$. The group G_2 acts on the seven-dimensional space V^7, preserving the split quadratic form. The stabilizer of a vector in this space, whose norm is a square class in $F^\times$ corresponding to the algebra K, is isomorphic to the special unitary group SU_3^K. In particular, when $K = F \times F$, the stabilizer is isomorphic to SL_3.

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For an irreducible cuspidal representation π of $G_2(\mathbb{A})$ define

$$\widetilde{F}(\pi) = \Big\{ K \;\Big|\; \exists\, \varphi \in \pi : \int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} \varphi(g)\, dg \neq 0 \Big\}.$$

Proposition 2.2 *Let π be an irreducible cuspidal representation of $G_2(\mathbb{A})$. Then $RS_\psi(\pi) \neq 0$ if and only if $\widetilde{F}(\pi) \neq \varnothing$.*

Proof The proof is essentially contained in [RS89, §3, Lemma 2]. We repeat it here for the convenience of the reader. Let $a \in F^\times$ be a representative of a square class associated with the algebra K. We use the Schrödinger model of the Weil representation, realized on the space of Schwartz functions $\mathcal{S}(V^7(\mathbb{A}))$.

Computing the Ψ_K Whittaker coefficient of $RS_\psi(\phi, \varphi)$ for $\phi \in \mathcal{S}(V^7(\mathbb{A}))$ and $\varphi \in \pi$, we obtain

$$\begin{aligned}
&\int_{N(F)\backslash N(\mathbb{A})} RS_\psi(\phi,\varphi)(nh)\overline{\Psi_K(n)}\, dn \\
&\qquad = \int_{G_2(F)\backslash G_2(\mathbb{A})} \Big[\int_{N(F)\backslash N(\mathbb{A})} \Big(\sum_{\xi \in V(F)} \omega_\psi^{14}(nh, g)\phi(\xi) \Big) \overline{\Psi_K(n)}\, dn \Big] \varphi(g)\, dg \\
&\qquad = \int_{G(F)\backslash G(\mathbb{A})} \Big(\sum_{\substack{\xi \in V^7(F),\\ \langle \xi, \xi\rangle = a}} \omega_\psi^{14}(h, g)\phi(\xi) \Big) \varphi(g)\, dg.
\end{aligned}$$

The group $G_2(F)$ acts transitively on the set $\mathcal{O}_a(F) = \{\xi \in V^7(F), \langle \xi, \xi \rangle = a\}$ and the stabilizer of each point is $\mathrm{SU}_3^K(F)$. Let ξ_a be a representative of the orbit $\mathcal{O}_a(F)$. The integral above equals

$$\begin{aligned}
&\int_{G_2(F)\backslash G_2(\mathbb{A})} \Big(\sum_{\gamma \in SU_3^K(F)\backslash G_2(F)} \omega_\psi^{14}(h, \gamma g)\phi(\xi_a) \Big) \varphi(g) dg \\
&\qquad = \int_{SU_3^K(\mathbb{A})\backslash G_2(\mathbb{A})} \omega_\psi^{14}(h, g)\phi(\xi_a) \Big(\int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} \varphi(g_1 g)\, dg_1 \Big) dg
\end{aligned}$$

In particular, if $RS_\psi(\pi) \neq 0$, then there exists $K \in \widetilde{F}(\pi)$.

Conversely, assume that $K \in \widetilde{F}(\pi)$. Thus there exists $\varphi \in \pi$ such that

$$I_K(\varphi) = \int_{SU_3^K(F)\backslash SU_3^K(\mathbb{A})} \varphi(g_1)\, dg_1 \neq 0.$$

For $\xi = g^{-1}\xi_a \in \mathcal{O}_a(\mathbb{A})$, the function $I_K(\varphi)(\xi) = I_K(g\cdot\varphi)$ is a continuous function on $\mathcal{O}_a(\mathbb{A})$. Similarly, $\omega_\psi(g)\phi(\xi_a) = \phi(g^{-1}\xi_a) = \phi(\xi)$ is the restriction of ϕ to $\mathcal{O}_a(\mathbb{A})$. Since $\mathcal{O}_a(\mathbb{A})$ is closed in $V^7(\mathbb{A})$, there exists a Schwartz function ϕ on $V^7(\mathbb{A})$ whose restriction to $\mathcal{O}_a(\mathbb{A})$ is a non-negative Schwartz function and has sufficiently small support to ensure that $\int_{\mathcal{O}_a(\mathbb{A})} \phi(\xi) I_K(\varphi)(\xi)\, d\xi \neq 0$.

Hence, for some functions $\phi \in \mathcal{S}(V^7(\mathbb{A}))$ and $\varphi \in \pi$, the function $RS_\psi(\phi,\varphi)$ has a non-zero Ψ_K-Whittaker coefficient and hence is non-zero. It follows that $RS_\psi(\pi)$ is not zero. ∎

2.3 Fourier–Jacobi Coefficients

Let $P_1 = M_1U_1$ be a non-Heisenberg maximal parabolic subgroup of G_2. The unipotent radical U_1 is the three-step unipotent subgroup. One has $U_1 \supset Z \supset Z_1$, where $Z = [U_1, U_1]$ and Z_1 is the center of U_1. The group U_1/Z_1 is a Heisenberg group whose center is $Z/Z_1 \simeq \mathbb{G}_a$. We regard ψ as a character of Z/Z_1. The Weil representation ω_ψ^2 of $U_1(\mathbb{A})/Z_1(\mathbb{A})$ is realized on the space of Schwartz functions $\mathcal{S}(U_\alpha)$ and is extended to the group $\widetilde{M'}_1(\mathbb{A})\cdot[U_1(\mathbb{A})/Z_1(\mathbb{A})]$, where $M_1' \simeq \mathrm{SL}_2$ is the derived group of M_1.

The representation ω_ψ^2 has an automorphic realization in the space of Jacobi forms by

$$\theta_\psi(\phi)(g) = \sum_{x\in U_\alpha(F)} \omega_\psi(g)\phi(x), \quad g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})\cdot[U_1(\mathbb{A})/Z_1(\mathbb{A})].$$

For an automorphic form φ of G_2, its Fourier–Jacobi coefficient is defined by

$$\varphi^{Z,\psi}(g) = \int_{Z(F)\backslash Z(\mathbb{A})} \varphi(zg)\overline{\psi(z)}\,dz,$$

which is a Jacobi form on $\mathrm{SL}_2(\mathbb{A})\cdot[U_1(\mathbb{A})/Z_1(\mathbb{A})]$. The space generated by all such coefficients is denoted by $\pi_{Z,\psi}$. This is a representation of the Jacobi group.

The automorphic representation $\mathrm{FJ}_\psi(\pi)$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ is defined as the span of all the functions

$$\mathrm{FJ}_\psi(\varphi,\phi)(h) = \int_{U_1(F)\backslash U_1(\mathbb{A})} \varphi^{Z,\psi}(uh)\overline{\theta_\psi(\phi)(uh)}\,du, \quad h \in \widetilde{\mathrm{SL}}_2(\mathbb{A}), \varphi \in \pi, \phi \in \omega_\psi^2.$$

By the result of Ikeda [Ike94], one has the following isomorphism of

$$\mathrm{SL}_2(\mathbb{A})\cdot[U_1(\mathbb{A})/Z_1(\mathbb{A})]\text{-representations:}$$

$\pi_{Z,\psi} \simeq \mathrm{FJ}_\psi(\pi)\widehat{\otimes}\omega_\psi^2$.

An easy computation, contained in the proof of [Gan05][Lemma 3.10], shows the following.

Proposition 2.3 *Let π be an automorphic representation of G_2 and K be a quadratic étale algebra over F. Then $K \in \widehat{F}_{\overline{\psi}}(\mathrm{FJ}_\psi(\pi)) \Leftrightarrow F\times K \in \widehat{F}_\psi(\pi)$.*

2.4 The Pole of the $\mathcal{L}$-function and Fourier Coefficients

The next theorem gives information on the Fourier coefficients supported by an irreducible cuspidal representation π of $G_2(\mathbb{A})$, whose standard $\mathcal{L}$-function admits a pole at $s = 2$.

Theorem 2.4 *Let π be an irreducible cuspidal representation of $G_2(\mathbb{A})$ such that the partial $\mathcal{L}$-function $\mathcal{L}^S(s,\pi,\mathfrak{st})$ admits a pole at $s = 2$. Then the following hold.*

(i) *The representation π is not nearly equivalent to a generic cuspidal representation.*
(ii) *There exists a quadratic algebra K such that $F \times K \in \widehat{F}_\psi(\pi)$.*
(iii) *If $\widehat{F}_\psi(\pi) = \{F \times F \times F\}$, then $RS_\psi(\pi) \neq 0$.*

Proof (i) D. Ginzburg [Gin93] constructed a Rankin–Selberg integral for the standard $\mathcal{L}$-function of *generic* cuspidal representations of G_2. The construction used an Eisenstein series on $\widetilde{\mathrm{SL}}_2$. In particular, it was shown that in the right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ the partial $\mathcal{L}$-function of a generic cuspidal representation can have a pole only at $s = 1$.

(ii) Recall from [Gan05, Theorem 3.1] that any cusp form of $G_2(\mathbb{A})$ supports some generic coefficient along U_2. No field E belongs to $\widehat{F}(\pi)$ by Theorem 1.3. Hence, there exists $F \times K \in \widehat{F}_\psi(\pi)$, where K is an étale quadratic algebra over F. Note that K may be split.

(iii) The proof is essentially contained in the proof of [GG06, Theorem 16.1] We repeat it for the convenience of the reader.

The proof uses a global theta lift θ_{E_6} for the exceptional dual pair (G_2, PGL_3) in the adjoint group of type E_6 [GJ01, GRS97b].

Claim *Let π be an irreducible cuspidal representation of $G_2(\mathbb{A})$.*

(i) *If the Shalika functional β on π defined by*

$$\beta(\varphi) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \int_{Z(F)\backslash Z(\mathbb{A})} \varphi(zg)\overline{\psi(z)}\, dz dg$$

does not vanish, then $\theta_{E_6}(\pi) \neq 0$ [GJ01, Theorem 1.1].

(ii) *If $\theta_{E_6}(\pi)$ is a cuspidal representation of $\mathrm{PGL}_3(\mathbb{A})$, then π is isomorphic to a generic representation* [GJ01, Theorem 3.1].

(iii) *If $\theta_{E_6}(\pi)$ is non-cuspidal, then π admits a non-zero SL_3-period* [GRS97b, Theorem A, Theorem 4.1(5)].

Assume that $\widehat{F}_\psi(\pi) = \{F \times F \times F\}$. Then by Proposition 2.3 $\widehat{F_{\overline{\psi}}}(\mathrm{FJ}_\psi(\pi)) = \{F \times F\}$, so that $\mathrm{FJ}_\psi(\pi) \subset \overline{\mathcal{A}_{\chi_0}}$, where χ_0 is the trivial character.

In particular, $\pi_{Z,\psi} \simeq \mathrm{FJ}_\psi(\pi)\widehat{\otimes}\omega_\psi \subset \overline{\mathcal{A}_{\chi_0}}\widehat{\otimes}\omega_\psi$, and hence the Shalika functional β does not vanish on π. It follows, by part (i) of the claim that $\theta_{E_6}(\pi) \neq 0$. Combining Theorem 2.4 (i) and part (ii) of the claim, we derive that $\theta_{E_6}(\pi)$ is not cuspidal. Hence, by part (iii) of the claim the representation π admits a non-zero SL_3 period. It follows by Proposition 2.2 that $RS_\psi(\pi) \neq 0$. ■

3 The Eisenstein Series and the Siegel–Weil Identity

The proof of Theorem 1.1 involves an identity between the leading terms of two Eisenstein series on the group $H = H_{F\times K}$. More precisely, we consider two degenerate principal series representations: one induced from the maximal parabolic subgroup P and the other induced from another parabolic subgroup Q and also the degenerate Eisenstein series associated with them. We prove a Siegel–Weil type identity relating the leading terms of the two series at certain points.

In fact, in the proof of Theorem 1.1 we shall use the identity only when K is a field. However, in this section we formulate and prove the identity for any étale quadratic algebra K.

3.1 Notations and Preliminaries

Let F be a local or global field, and K be a quadratic étale algebra over F.

We begin with the notations on the algebraic groups involved in this paper.

(1) Let $H = H_{F\times K}$ be a quasi-split simply-connected group over F of type D_4 associated with the cubic algebra $F \times K$.

(2) Let $B_H = N_H \cdot T_H$ be a Borel subgroup of H with the unipotent radical N_H and maximal torus T_H. Let $T_S \subseteq T_H$ be a maximal split torus.

(3) The simple roots in the absolute root system of H are denoted by α_i, $i = 1\dots 4$. The Dynkin diagram has the form

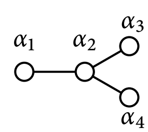

Figure 1: The Dynkin diagram of type D_4

(4) Let $\Phi(G, T_S)$ be a relative root system of H, Φ^+ be the set of positive roots and Δ be the set of simple roots. For any root $\alpha \in \Phi(G, T_S)$, we denote by F_α the field of definition of α. In the split case $K = F \times F$, the positive roots of H with respect to $T_H = T_S$ are denoted by

$$\Phi^+ = \Big\{ [1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1], [1,1,1,1], [1,2,1,1], \\ [1,1,0,0], [0,1,1,0], [0,1,0,1], [1,1,1,0], [1,1,0,1], [0,1,1,1] \Big\}.$$

In the case where K is a field, the relative root system of H with respect to T_S is of type B_3 and the positive roots are denoted by

$$\Phi^+ = \Big\{ [1,0,0], [0,1,0], [0,0,1], [1,1,0], \\ [0,1,1], [0,1,2], [1,1,1], [1,2,1], [1,2,2] \Big\}$$

The roots defined over K are $[0,0,1]$, $[0,1,1]$, and $[1,1,1]$.

(5) For any root $\alpha \in \Phi^+$, we fix a pinning, that is, a collection of maps

$$\varphi_\alpha \colon \mathrm{SL}_2(F_\alpha) \to H(F).$$

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Define the unipotent subgroups

$$X_\alpha(F) = \{\varphi_\alpha\big(\big(\begin{smallmatrix}1 & r\\ 0 & 1\end{smallmatrix}\big)\big), r \in F_\alpha\},$$
$$X_{-\alpha}(F) = \{\varphi_\alpha\big(\big(\begin{smallmatrix}1 & 0\\ r & 1\end{smallmatrix}\big)\big), r \in F_\alpha\},$$

and

$$w_\alpha = \varphi_\alpha\big(\big(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\big)\big).$$

The relative Weyl group W_H of H is generated by the images of the elements w_α for the simple roots α. By $w[i_1 \cdots i_k]$ we denote the Weyl word $w_{\alpha_{i_1}} \cdots w_{\alpha_{i_k}}$.

(6) We parameterize T_H using coroots.

$$T_H \ni t = \begin{cases} (t_1,t_2,t_3,t_4) = \alpha_1^\vee(t_1)\alpha_2^\vee(t_2)\alpha_3^\vee(t_3)\alpha_4^\vee(t_4), \text{ for } t_i \in \mathbb{G}_m, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } K = F \times F, \\ (t_1,t_2,t_3) = \alpha_1^\vee(t_1)\alpha_2^\vee(t_2), \alpha_3^\vee(t_3), \text{ for } t_1,t_2 \in \mathbb{G}_m, t_3 \in \operatorname{Res}_{K/F}\mathbb{G}_m, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } K \neq F \times F. \end{cases}$$

In particular, for a local field F, the modular character is given by

$$\delta_{B_H}(t) = \begin{cases} |t_1t_2t_3t_4|_F^2 & \text{if } K = F \times F, \\ |t_1t_2|_F^2|t_3|_K^2 & \text{if } K \neq F \times F. \end{cases}$$

(7) Let $X^*(T_H)$ denote the the lattice of F-rational characters of T_H and let $\mathfrak{a}_{\mathbb{C}}^* = X^*(T_H) \otimes \mathbb{C}$. Also, let C^+ denote the positive Weyl chamber in $\mathfrak{a}_{\mathbb{C}}^*$.

For a local non-archimedean field F, the space $\mathfrak{a}_{\mathbb{C}}^*$ can be identified with the group of unramified characters of $T_H(F)$ via the map $\overline{s} \mapsto \lambda_{\overline{s}}$ given by

$$\lambda_{\overline{s}}(t) = \begin{cases} |t_1|_F^{s_1}|t_2|_F^{s_2}|t_3|_F^{s_3}|t_4|_F^{s_4} & \text{if } K = F \times F, \\ |t_1|_F^{s_1}|t_2|_F^{s_2}|t_3|_K^{s_3} & \text{if } K \neq F \times F. \end{cases}$$

(8) Let $P = M \cdot U$ be the Heisenberg parabolic group of H. Its Levi subgroup is isomorphic to

$$M \simeq (\mathrm{GL}_2 \times \operatorname{Res}_{K/F} GL_2^0)^{\det} = \{(g_1,g_2) \in \mathrm{GL}_2 \times \operatorname{Res}_{K/F}\mathrm{GL}_2^0 \mid \det(g_1) = \det(g_2)\},$$

where

$$\operatorname{Res}_{K/F} GL_2^0 = \begin{cases} \{(g',g'') \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid \det(g') = \det(g'')\} & \text{if } K = F \times F, \\ \{g \in \operatorname{Res}_{K/F}\mathrm{GL}_2 \mid \det(g) \in \mathbb{G}_m\} & \text{if } K \neq F \times F. \end{cases}$$

Under this isomorphism it holds that

$$\delta_P(g_1,g_2) = |\det(g_1)|^5 \quad \forall g_1 \in \mathrm{GL}_2,\ g_2 \in \operatorname{Res}_{K/F}\mathrm{GL}_2^0.$$

(9) Let $Q = L \cdot V$ be the maximal parabolic subgroup generated by B_H, $X_{-\alpha_2}$, $X_{-\alpha_3}$, and $X_{-\alpha_4}$ if $K = F \times F$ and by B_H, $X_{-\alpha_2}$, and $X_{-\alpha_3}$ if K is a field.

The Levi subgroup L fits into a short exact sequence

$$1 \longrightarrow GL_1 \longrightarrow L \longrightarrow \mathrm{SO}(V_K^6) \longrightarrow 1,$$

where V_K^6 is a quadratic space of dimension 6 and discriminant K. Its modular character δ_Q restricted to the torus of H satisfies $\delta_Q(t) = |t_1|^6$.

(10) We denote by W_M and W_L the Weyl groups of M and L, respectively.

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(11) The group $\overline{H}$ is the group of isometries of the quadratic space V_K^8 of dimension eight and discriminant K. It is a group of type D_4 and there is an isogeny p of algebraic groups $1 \to \mu_2 \to H \xrightarrow{p} \overline{H} \to 1$. The associated map $p\colon H(F) \to \overline{H}(F)$ is not necessarily surjective but for a local field F the image has finite index. There is a bijection between the parabolic groups of H and of $\overline{H}$. We write $V_K^8 = V_K^6 \oplus \mathbb{H}$, where $\mathbb{H} = \operatorname{Span}\{e_0, e_0^*\}$ is a hyperbolic plane. The parabolic subgroup $\overline{Q} = \overline{L} \cdot \overline{V}$ is the subgroup stabilizing $\operatorname{Span}\{e_0\}$. The Levi subgroup $\overline{L}$ is isomorphic to $\mathrm{GL}_1 \times \mathrm{SO}(V_K^6)$, and we have $\delta_{\overline{Q}}(g_1, g_2) = |g_1|^6$ for all $g_1 \in \mathrm{GL}_1$, $g_2 \in \mathrm{SO}(V_K^6)$.

We continue with generalities on induced representations associated with parabolic subgroups B_H, P, Q. Below, F will denote a global field and F_v a local field.

(12) For every place v fix a maximal compact subgroup $\mathcal{K}_v$ of $H(F_v)$ that is special for every finite place v. Let $\mathcal{K} = \Pi_v \mathcal{K}_v$ be a maximal compact subgroup of $H(\mathbb{A})$.

(13) For any place v and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ denote by $I_{B_H v}(\lambda)$ the normalized smooth induction $\operatorname{Ind}_{B_H(F_v)}^{H(F_v)} \lambda$. This is a smooth admissible representation of $H(F_v)$ for finite v and is a smooth admissible Frechet representation for infinite v. Thus, $I_{B_H}(\lambda) = \otimes_v I_{B_H v}(\lambda)$ is an $H(\mathbb{A})$-module.

(14) Consider a section $f_\lambda \in I_{B_H}(\lambda)$.

- f_λ is *Standard* if f_λ is independent of λ when restricted to $\mathcal{K}$.
- f_λ is *Spherical* if it is standard and $\mathcal{K}$ invariant. We further call such a section *normalized* if $f_\lambda(1) = 1$. Since the space of spherical vectors in $I_{B_H}(\lambda)$ is one-dimensional the normalized spherical vector is unique.
- f_λ is *Holomorphic* if $f_\lambda(g)$ is a holomorphic function of λ for any $g \in H(\mathbb{A})$.

(15) We fix a set of representatives $\widetilde{W}_H$ in $H(F)$ of W_H so that $\widetilde{w} \in \mathcal{K}$ for any $\widetilde{w} \in \widetilde{W}_H$. As in [Ste68, §6, §11], for $\alpha \in \Delta$, let $\widetilde{w}_\alpha = x_\alpha(1)x_{-\alpha}(1)x_\alpha(1)$. For a reduced word $w = w_{\alpha_{i_1}} \cdots w_{\alpha_{i_k}}$, let $\widetilde{w} = \widetilde{w}_{\alpha_{i_1}} \cdots \widetilde{w}_{\alpha_{i_k}}$. For any $w \in W_H$, we consider the standard global intertwining operator $M_w(\lambda)\colon I_{B_H}(\lambda) \to I_{B_H}(w^{-1} \cdot \lambda)$ given by

$$M_w(\lambda)(f_\lambda)(g) = \int_{N_H(\mathbb{A}) \cap \widetilde{w}^{-1} N_H(\mathbb{A}) \widetilde{w} \backslash N_H(\mathbb{A})} f_\lambda(\widetilde{w} n g)\, dn,$$

where f_λ is a standard section of $I_B(\lambda)$. Note that this integral is independent of the choice of the representative $\widetilde{w}$. This integral converges absolutely for $\mathrm{Re}(\lambda)$ in some positive cone and extends to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$. The main properties of these operators are the following.

- For any $w, w' \in W$, the intertwining operators satisfy the following cocycle equation $M_{ww'}(\lambda) = M_{w'}(w^{-1} \cdot \lambda) \circ M_w(\lambda)$.
- The global intertwining operator $M_w(\lambda)$ decomposes as a product $\otimes M_{w,v}(\lambda)$ of local intertwining operators $M_{w,v}(\lambda)\colon I_{B_{H,v}}(\lambda) \to I_{B_{H,v}}(w^{-1} \cdot \lambda)$ given for $\mathrm{Re}(\lambda) \gg 0$, by

$$M_{w,v}(\lambda)(f_{\lambda,v})(g) = \int_{N_H(F_v) \cap \widetilde{w}^{-1} N_H(F_v) \widetilde{w} \backslash N_H(F_v)} f_{\lambda,v}(\widetilde{w} n g)\, dn.$$

• (Gindikin–Karpelevich formula). The local intertwining operator $M_{w,v}(\lambda)$ depends on the choice of representative $\widetilde{w}$. However, since $\widetilde{w} \in \mathcal{K}_v$, for the normalized spherical vector $f^0_{\lambda,v} \in I_{B_H,v}(\lambda)$, it holds that

$$M_{w,v}(\lambda)(f^0_{\lambda,v}) = J_v(w,\lambda) f^0_{w^{-1}\cdot\lambda,v}, \tag{3.1}$$

where

$$J_v(w,\lambda) = \prod_{\substack{\alpha>0,\\ w^{-1}\alpha<0}} \frac{\zeta_{F_\alpha,v}(\langle\lambda,\alpha^\vee\rangle)}{\zeta_{F_\alpha,v}(\langle\lambda,\alpha^\vee\rangle+1)}.$$

We denote by $\zeta_{F_\alpha}(s)$ a complete zeta function of F_α normalized such that $\zeta_{F_\alpha}(s) = \zeta_{F_\alpha}(1-s)$. Moreover we denote $R_{F_\alpha} = \lim_{s\to 1}(s-1)\zeta_{F_\alpha}(s)$. The global Gindikin–Karpelevich factor $J(w,\lambda) = \Pi_v J_v(w,\lambda)$ is a ratio of products of complete zeta functions.

• The induced representations $I_{P_v}(s) = \operatorname{Ind}_{P(F_v)}^{H(F_v)} \delta_P^s$ and $I_{Q_v}(s) = \operatorname{Ind}_{Q(F_v)}^{H(F_v)} \delta_Q^s$ (smooth normalized induction) will play a central role in this paper. We define

$$I_P(s) = \bigotimes I_{P_v}(s), \quad I_Q(s) = \bigotimes I_{Q_v}(s).$$

Their normalized spherical sections will be denoted $f^0_s \in I_P(s)$ and $\widetilde{f}^0_s \in I_Q(s)$. By induction in stages, we observe that $I_P(s)$ and $I_Q(s)$ are subrepresentations of the following principal series: $I_P(s) \hookrightarrow I_{B_H}(\chi_{P,s})$ and $I_Q(s) \hookrightarrow I_{B_H}(\chi_{Q,s})$, where

$$\chi_{P,s} = \delta_P^{s+1/2}\delta_B^{-1/2}, \quad \chi_{Q,s} = \delta_Q^{s+1/2}\delta_B^{-1/2}.$$

The representation $I_{\overline{Q}}(s)$ of $\overline{H}(\mathbb{A})$ is defined similarly to $I_Q(s)$.

We shall make use of the following lemma.

Lemma 3.1 *The map $p\colon H(F_v) \to \overline{H}(F_v)$ induces the map $p^*: I_{\overline{Q}_v}(s) \to I_{Q_v}(s)$.*

(i) *$p^*: I_{\overline{Q}_v}(s) \to I_{Q_v}(s)$ is an isomorphism of vector spaces and*

$$h\cdot(p^*(\overline{f})) = p^*(p(h)\cdot\overline{f}) \quad \forall h \in H(F_v), \overline{f} \in I_{\overline{Q}_v}(s).$$

(ii) *If the representation $I_{Q_v}(s)$ is generated by the normalized spherical section f^0, then $I_{\overline{Q}_v}(s)$ is generated by the normalized spherical section $\overline{f}^0$. If $v\mid\infty$, then the converse is also true.*

(iii) *If the representation $I_{Q_v}(s)$ is irreducible, then $I_{\overline{Q}_v}(s)$ is irreducible. If $v\mid\infty$, then the converse is also true.*

Proof Part (i) follows from the fact that $\overline{H}(F_v) = \overline{Q}(F_v)\cdot p(H(F_v))$. If $I_{Q_v}(s)$ is irreducible (or generated by a spherical section), then obviously the same is true for $I_{\overline{Q}_v}(s)$.

Let us show the converse statements for $v|\infty$. They rely on the following decompositions,, that hold for archimedean local fields:

$$\overline{H}(F_v) = p(H(F_v))\overline{\mathcal{K}}_v, \tag{3.2}$$

$$\overline{H}(F_v) = \overline{Q}(F_v)p(\mathcal{K}_v). \tag{3.3}$$

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The first decomposition holds, since the group $p(H(F_v))$ is the topological identity component of $\overline{H}(F_v)$ and $\overline{\mathcal{K}}_v$ meets every connected component of $\overline{H}(F_v)$. Similarly, the second decomposition holds, since the group $p(\mathcal{K}_v)$ is the topological identity component of $\overline{\mathcal{K}_v}$ and $\overline{Q}(F_v)$ meets every component of $\overline{H}(F_v)$.

Assume that $I_{\overline{Q}_v}(s)$ is generated by the spherical vector $\overline{f}^0$, and let Π be the subrepresentation of $I_{Q_v}(s)$ generated by f^0. Then $(p^*)^{-1}(\Pi)$ is a $p(H(F_v))$-subrepresentation of $I_{\overline{Q}_v}(s)$ containing $\overline{f}^0$. It follows from (3.2) that $(p^*)^{-1}(\Pi) = I_{\overline{Q}_v}(s)$ and hence $\Pi = I_{Q_v}(s)$.

Assume that the representation $I_{\overline{Q}_v}(s)$ is irreducible as a representation of $\overline{H}(F_v)$. The restriction of $I_{\overline{Q}_v}(s)$ to $p(H(F_v))$ is a direct sum of irreducible representations. By (3.2) the projection of $\overline{f}^0$ to each summand is non-trivial. Hence, every summand has a non-zero $p(\mathcal{K}_v)$-invariant vector. However, by (3.3) it follows that $\dim(I_{\overline{Q}_v}(s)^{p(\mathcal{K}_v)}) = 1$. Hence the restriction of $I_{\overline{Q}_v}(s)$ to $p(H(F_v))$ is irreducible and so $I_{Q_v}(s)$ is irreducible, as required. ■

3.2 Induced Representations

We begin with the study of the reducibility of the induced representations $I_P(3/10)$ and $I_Q(1/6)$. Consider the following element of the Weyl group W_H:

$$w = \begin{cases} w[2342] & \text{if } K_v = F_v \times F_v, \\ w[232] & \text{if } K_v \neq F_v \times F_v. \end{cases} \tag{3.4}$$

The following properties are checked directly using the Gindikin–Karpelevich formula (3.1).

Lemma 3.2 (i) *For w as in (3.4), it holds that $w^{-1}(\chi_{P,3/10}) = \chi_{Q,1/6}$.*

(ii) *For any place v, the factor $J_v(w, \chi_{P,3/10})$ is finite and non-zero.*

(iii) *The factor $J(w, \chi_{P,s})$ admits a simple pole at $s = 3/10$ and*

$$J_w \overset{\text{def}}{=} \lim_{s\to 3/10}(5s - 3/2)J(w, \chi_{P,s}) = \begin{cases} \frac{R_F\zeta_F(2)}{\zeta_F(3)\zeta_F(4)} & \text{if } K = F\times F, \\ \frac{R_F\zeta_K(2)\zeta_F(3)}{\zeta_F(2)\zeta_K(3)\zeta_F(4)} & \text{if } K \neq F\times F. \end{cases} \tag{3.5}$$

Theorem 3.3 *Let v be any local place of F.*

(i) *The representation $I_{P_v}(3/10)$ has a unique irreducible quotient and that quotient is spherical.*

(ii) *The representations $I_{Q_v}(1/6)$ and $I_{\overline{Q}_v}(1/6)$ have a unique irreducible quotient and that quotient is spherical. Both representations are irreducible when K_v is a field.*

(iii) *The restriction of $M_{w,v}(\chi_{P,3/10})$ to $I_{P_v}(3/10)$ defines a surjective map onto $I_{Q_v}(1/6)$.*

Proof (i) The representation $I_{P_v}(3/10)$ is a quotient of $I_{B_{H_v}}(\delta_P^{-1/5}\delta_B^{1/2})$ which, by induction in stages, can be written as a standard module $\operatorname{Ind}_{R_v}^{H_v}[\delta_{R_v}^{1/3}\operatorname{Ind}_{A_v\cap B_{H_v}}^{A_v}\chi_0]$. Here R_v is a standard parabolic subgroup of H_v whose Levi subgroup A_v is generated by $T_{H,v}$, and $X_{\pm\alpha_2,v}$, χ_0 denotes the trivial character. By Langlands's classification

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theorem, it admits a unique irreducible quotient Π_v^0 that is the image of the operator $M_{\widetilde{w},v}(\delta_P^{-1/5}\delta_B^{1/2})$, where $\widetilde{w}$ is the shortest representative of the longest coset in $W_A\backslash W_H$. Since $M_{\widetilde{w},v}(\delta_P^{-1/5}\delta_B^{1/2})f^0 \neq 0$, it follows that this quotient, Π_v^0, is a spherical representation. Thus $I_{P_v}(3/10)$ also admits Π_v^0 as its unique irreducible quotient.

(ii) The special feature of Q and $\overline{Q}$ is that they have Abelian unipotent radical and are conjugate to their opposite. The degenerate principal series associated with maximal parabolic subgroups with these properties were studied by Sahi for local Archimedean fields using $\mathcal{K}$-types and by Weissman for non-Archimedean fields using the Fourier–Jacobi functor.

First assume that F_v is non-Archimedean local field. M. Weissman proved the statement for $I_{Q_v}(1/6)$ in case $K_v = F_v \times F_v$, [Wei03, §5.1]. The case where K_v is a field will be dealt with in AppendixA, where we adapt his approach to the quasi-split case. By Lemma 3.1, the claims follow for $I_{\overline{Q}_v}(1/6)$.

Now assume that F_v is Archimedean. Sahi [Sah95]studied the reducibility of $I_{\overline{Q}_v}(s)$; the details are given in Appendix B. Lemma 3.1 implies the result for $I_{Q_v}(1/6)$.

(iii) This follows directly from Lemma 3.2 (i), (ii), and the fact that $f^0_{\chi_{P,3/10}}$ (resp. $f^0_{\chi_{Q,1/6}}$) generates $I_{P_v}(3/10)$ (resp. $I_{Q_v}(1/6)$). ∎

3.3 Eisenstein Series

The Eisenstein series $\mathcal{E}_{B_H}(\cdot,\cdot,\lambda)$ associated with an induced representation $I_{B_H}(\lambda)$ is an operator that maps every standard section $f_\lambda \in I_{B_H}(\lambda)$ to an automorphic function

$$\mathcal{E}_{B_H}(f,g,\lambda) = \sum_{\gamma\in B_H(F)\backslash H(F)} f_\lambda(\gamma g).$$

The series converges for λ in a positive cone and admits a meromorphic continuation for all λ. This is a classical result for $\mathcal{K}$-finite sections, and for smooth sections it follows from [Lap08]. The constant term of $\mathcal{E}_{B_H}(f,g,\lambda)$ along N_H is given by

$$\mathcal{E}_{B_H}(f,g,\lambda)_{N_H} = \sum_{w\in W_H} M_w(\lambda)(f_\lambda)(g).$$

The Eisenstein series $\mathcal{E}_P(f,g,s)(g)$, $\mathcal{E}_Q(f,g,s)(g)$, and $\mathcal{E}_{\overline{Q}}(f,g,s)$ associated with $I_P(s)$, $I_Q(s)$, and $I_{\overline{Q}}(s)$, respectively are defined similarly. For example, when $\mathrm{Re}(s)$ is large, one has $\mathcal{E}_P(f,g,s) = \sum_{\gamma\in P(F)\backslash H(F)} f_s(\gamma g), \quad f_s \in I_P(s)$.

By a standard computation [GRS97a], the constant term of $\mathcal{E}_P(f,g,s)$ along N_H is computed as

$$\mathcal{E}_P(f,g,s)_{N_H} = \sum_{w\in W(M,T)} M_w(\chi_{P,s})(f_s)(g),$$

where $W(M,T) = \{w \in W_H \mid w^{-1}\cdot\alpha_i > 0, \text{for all } i \neq 2\}$ is the set of the shortest representatives of cosets in $W_M\backslash W_H$. Similarly

$$\mathcal{E}_Q(f,g,s)_{N_H} = \sum_{w\in W(L,T)} M_w(\chi_{Q,s})(f_s)(g),$$

where $W(L,T) = \{w \in W_H \mid w^{-1}\cdot\alpha_i > 0, \text{for all } i \neq 1\}$ is the set of the shortest representatives of cosets in $W_L\backslash W_H$.

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Note also that

$$\chi_{P,s} = \begin{cases} \lambda_{(-1,5s+\frac{3}{2},-1,-1)} & \text{if } K = F \times F, \\ \lambda_{(-1,5s+\frac{3}{2},-1)} & \text{if } K \neq F \times F, \end{cases}$$

$$\chi_{Q,s} = \begin{cases} \lambda_{(6s+2,-1,-1,-1)} & \text{if } K = F \times F, \\ \lambda_{(6s+2,-1,-1)} & \text{if } K \neq F \times F. \end{cases}$$

Lemma 3.4 *For a general $s \in \mathbb{C}$, it holds that*

$$\mathcal{E}_{B_H}(f^0, g, \chi_{P,s}) = \mathcal{E}_P(f^0, g, s), \quad \mathcal{E}_{B_H}(f^0, g, \chi_{Q,s}) = \mathcal{E}_Q(\widetilde{f}^0, g, s).$$

Proof Consider the constant term along the unipotent radical N_H of the Borel subgroup B_H: $\mathcal{E}_{B_H}(f^0, g, \lambda)_{N_H} = \sum_{w \in W} J_w(\lambda) f^0_{w^{-1}\lambda}(g)$. Let $\lambda = \chi_{P,s}$. We shall show that for $\operatorname{Re}(s) \gg 0$, the term $J_w(\chi_{P,s})$ is holomorphic and vanishes unless $w \in W(M,T)$. Indeed, recall that $J(w, \chi_{P,s}) = \prod_{\alpha>0, w^{-1}\alpha<0} J_\alpha(\chi_{P,s})$, where

$$J_\alpha(\chi_{P,s}) = \frac{\zeta_{F_\alpha}(\langle \chi_{P,s}, \alpha^\vee \rangle)}{\zeta_{F_\alpha}(\langle \chi_{P,s}, \alpha^\vee \rangle + 1)}.$$

For $\alpha \in \Phi$, let $\alpha^\vee = \sum_{\gamma \in \Delta} n_\gamma(\alpha) \gamma^\vee$. For $\operatorname{Re}(s) \gg 0$ it holds that

$$\langle \chi_{P,s}, \alpha^\vee \rangle = \begin{cases} -\sum_{\gamma \in \Delta_M} n_\gamma(\alpha) & \text{if } n_{\alpha_2}(\alpha) = 0, \\ \text{has real part bigger than 1} & \text{if } n_{\alpha_2}(\alpha) \neq 0. \end{cases}$$

In particular, for $\operatorname{Re}(s) \gg 0$, the term $J_\alpha(\chi_{P,s})$ is holomorphic for every $\alpha \in \Phi^+$ and is zero if and only if $\langle \chi_{P,s}, \alpha \rangle = -1$, *i.e.*, for $\alpha \in \Delta \backslash \{\alpha_2\}$. As a result $J_w(\chi_{P,s})$ is holomorphic (for $\operatorname{Re}(s) \gg 0$) and does not vanish if and only if $w^{-1} \cdot \alpha \in \Phi^+$ for any $\alpha \in \Delta \smallsetminus \{\alpha_2\}$. This exactly means that $w \in W(M,T)$. Hence, $\mathcal{E}_{B_H}(f^0, g, \chi_{P,s})_{N_H} = \mathcal{E}_P(f^0, g, s)_{N_H}$ for $\operatorname{Re}(s) \gg 0$. By meromorphic continuation, the equality holds for all $s \in \mathbb{C}$.

In addition, the cuspidal components of both $\mathcal{E}_{B_H}(f^0, g, \chi_{P,s})$ and $\mathcal{E}_P(f^0, g, s)$ are zero along any standard parabolic subgroup strictly containing the Borel subgroup B_H. Hence, by [MW95, Proposition I.3.4], there is an equality of the automorphic forms, *i.e.*, $\mathcal{E}_{B_H}(f^0, g, \chi_{P,s}) = \mathcal{E}_P(f^0, g, s)$. The equality $\mathcal{E}_{B_H}(f^0, g, \chi_{Q,s}) = \mathcal{E}_Q(\widetilde{f}^0, g, s)$ is proven similarly. ∎

A. Segal [Seg18] studied the poles of $\mathcal{E}_P(f, g, s)$ for $\operatorname{Re}(s) > 0$ and in particular at $s = 3/10$. The result is quoted in Theorem 1.3.

Our goal is to study the behavior of $\mathcal{E}_Q(f, g, s)$ at $s = 1/6$ for various K.

Proposition 3.5 (i) *Let K be a field. For any standard section $f_s \in I_Q(s)$, the Eisenstein series $\mathcal{E}_Q(f_s, g, s)$ is holomorphic at $s = 1/6$.*

(ii) *Let $K = F \times F$. For any standard section $f_s \in I_Q(s)$, the Eisenstein series $\mathcal{E}_Q(f_s, g, s)$ has at most a simple pole at $s = 1/6$. The pole is attained by a spherical function.*

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$w \in W(L,T)$	$J(w,\chi_{Q,s})$	Order at 1/6	$w^{-1}\cdot\chi_{Q,\frac{1}{6}}(t)$
1	1	0	$\frac{\lvert t_1\rvert_F^3}{\lvert t_2\rvert_F\lvert t_3\rvert_K}$
$w[1]$	$\frac{\zeta_F(6s+2)}{\zeta_F(6s+3)}$	0	$\frac{\lvert t_2\rvert_F^3}{\lvert t_1\rvert_F^3\lvert t_3\rvert_K}$
$w[12]$	$\frac{\zeta_F(6s+1)}{\zeta_F(6s+3)}$	0	$\frac{\lvert t_3\rvert_K}{\lvert t_1\rvert_F\lvert t_1\rvert_F^2}$
$w[123]$	$\frac{\zeta_F(6s+1)}{\zeta_F(6s+3)}\frac{\zeta_K(6s)}{\zeta_K(6s+1)}$	1	$\frac{1}{\lvert t_1\rvert_F\lvert t_3\rvert_K}$
$w[1232]$	$\frac{\zeta_F(6s+1)}{\zeta_F(6s+3)}\frac{\zeta_F(6s-1)}{\zeta_F(6s)}\frac{\zeta_K(6s)}{\zeta_K(6s+1)}$	1	$\frac{1}{\lvert t_1\rvert_F\lvert t_3\rvert_K}$
$w[12321]$	$\frac{\zeta_F(6s+1)}{\zeta_F(6s+3)}\frac{\zeta_F(6s-2)}{\zeta_F(6s)}\frac{\zeta_K(6s)}{\zeta_K(6s+1)}$	0	$\frac{\lvert t_1\rvert_F}{\lvert t_2\rvert_F\lvert t_3\rvert_K}$

Table 1

Proof The proof is standard. The order of a pole of degenerate Eisenstein series at any point coincides with the order of a pole of its constant term along N_H. Since $I_Q(3/10)$ is generated by a spherical vector for any K, it is enough to check that $\mathcal{E}_Q(f^0,g,s)_{N_H}$ is holomorphic when K is a field, and admits a simple pole when $K = F\times F$.

(i) Let K be a field. Table 1 describes the Gindikin–Karpelevich factor associated with each $w\in W(L,T)$ and the exponents $w^{-1}\cdot\chi_{Q,1/6}$.

Simple poles are attained for $J(w[123],\chi_{Q,s})$ and $J(w[1232],\chi_{Q,s})$ at $s=1/6$.

$$(M_{w[1232]}(\chi_{Q,s}) + M_{w[123]}(\chi_{Q,s}))(f_s^0)$$
$$= J(w[123],\chi_{Q,s})\cdot\Big(\Big(\frac{\zeta_F(6s-1)}{\zeta_F(6s)}\Big) f^0_{w[123]^{-1}\cdot\chi_{Q,s}} + f^0_{w[1232]^{-1}\cdot\chi_{Q,s}}\Big).$$

Since $w[2]$ fixes $w[123]^{-1}\cdot\chi_{Q,1/6}$ and $\lim_{s\to 1/6}\frac{\zeta(6s-1)}{\zeta(6s)} = -1$, the expression above is holomorphic at $s=1/6$.

(ii) Now assume that $K=F\times F$. Table 2 describes the Gindikin–Karpelevich factors associated with each $w\in W(L,T)$ and the relevant exponent $w^{-1}\cdot\chi_{Q,s}$.

A pole of order 2 at $s=1/6$ is attained by $M_{w[1234]}$ and $M_{w[12342]}$. Reasoning as above,

$$(M_{w[12342]}(\chi_{Q,s}) + M_{w[1234]}(\chi_{Q,s}))(f^0_{\chi_{Q,s}})$$
$$= J(w[1234],\chi_{Q,s})\cdot\Big(\Big(\frac{\zeta_F(6s-1)}{\zeta_F(6s)}\Big) f^0_{w[1234]^{-1}\cdot\chi_{Q,s}} + f^0_{w[12342]^{-1}\cdot\chi_{Q,s}}\Big)$$

has at most a simple pole at $s=1/6$. The pole is attained for f_s^0, since $M_{w[123]}(\chi_{Q,s})f_s^0$ contributes a simple pole that cannot be cancelled by other terms. Indeed, the exponent $w[123]^{-1}\cdot\chi_{Q,1/6}$ is not equal to $w^{-1}\cdot\chi_{Q,1/6}$ for any other $w\in W(L,T)$. ■

$w \in W(L,T)$	$J(w, \chi_{Q,s})$	Order of pole at 1/6	$w^{-1} \cdot \chi_{Q,\frac{1}{6}}(t)$
1	1	0	$\frac{\lvert t_1\rvert^3}{\lvert t_2 t_3 t_4\rvert}$
$w[1]$	$\frac{\zeta_F(6s+2)}{\zeta_F(6s+3)}$	0	$\frac{\lvert t_2\rvert^2}{\lvert t_1^3 t_3 t_4\rvert}$
$w[12]$	$\frac{\zeta_F(6s+1)}{\zeta_F(6s+3)}$	0	$\frac{\lvert t_3 t_4\rvert}{\lvert t_1 t_2^2\rvert}$
$w[123]$	$\frac{\zeta_F(6s)}{\zeta_F(6s+3)}$	1	$\frac{\lvert t_4\rvert}{\lvert t_1 t_2 t_3\rvert}$
$w[124]$	$\frac{\zeta_F(6s)}{\zeta_F(6s+3)}$	1	$\frac{\lvert t_3\rvert}{\lvert t_1 t_2 t_4\rvert}$
$w[1234]$	$\frac{\zeta_F(6s)^2}{\zeta_F(6s+3)\,\zeta_F(6s+1)}$	2	$\frac{1}{\lvert t_1 t_3 t_4\rvert}$
$w[12342]$	$\frac{\zeta_F(6s)\,\zeta_F(6s-1)}{\zeta_F(6s+3)\,\zeta_F(6s+1)}$	2	$\frac{1}{\lvert t_1 t_3 t_4\rvert}$
$w[123421]$	$\frac{\zeta_F(6s)\,\zeta_F(6s-2)}{\zeta_F(6s+3)\,\zeta_F(6s+1)}$	1	$\frac{\lvert t_1\rvert}{\lvert t_2 t_3 t_4\rvert}$

Table 2

Corollary 3.6 *For $h \in H(\mathbb{A})$, $\overline{f} \in I_{\overline{Q}}(s)$, and for* $\mathrm{Re}(s) \gg 0$ *one has*

$$\mathcal{E}_Q(p^*(\overline{f}), h, s) = \mathcal{E}_Q(\overline{f}, p(h), s).$$

In particular, the Eisenstein series $\mathcal{E}_{\overline{Q}}(f, g, s)$ on $\overline{H}(\mathbb{A})$ is holomorphic at $s = 1/6$ when K is a field. When $K = F \times F$, it has at most a simple pole that is attained by the spherical section $\overline{f}^0_s$.

3.4 Siegel–Weil Identity

Let $\mathcal{A}(H)$ denote the space of automorphic forms on H. Define

$$\Lambda_Q(1/6)\colon I_Q(1/6) \longrightarrow \mathcal{A}(H), \quad \Lambda_P(3/10)\colon I_P(3/10) \longrightarrow \mathcal{A}(H)$$

by

$$\Lambda_Q(1/6)(f_s) = \begin{cases} \lim_{s\to 1/6}(6s-1)\mathcal{E}_Q(f,g,s) & \text{if } K = F \times F, \\ \mathcal{E}_Q(f,g,1/6) & \text{if } K \neq F \times F, \end{cases}$$

$$\Lambda_P(3/10)(f_s) = \begin{cases} \lim_{s\to 3/10}(5s-3/2)^2\mathcal{E}_P(f,g,s) & \text{if } K = F \times F, \\ \lim_{s\to 3/10}(5s-3/2)\mathcal{E}_P(f,g,s) & \text{if } K \neq F \times F. \end{cases}$$

These operators are $H(\mathbb{A})$-equivariant. Denote by Π_Q and Π_P their images in the space $\mathcal{A}(H)$ of automorphic forms. The map $\Lambda_{\overline{Q}}(1/6)\colon I_Q(1/6) \to \mathcal{A}(\overline{H})$ and the representation $\Pi_{\overline{Q}}$ are defined similarly to $\Lambda_Q(1/6)$ and Π_Q.

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Lemma 3.7 *The restriction of the global intertwining operator $M_w(\chi_{P,s})$ to $I_P(s)$ admits a simple pole at $s = 3/10$ that is attained by the spherical section. The operator*

$$\mathfrak{X}_w = \lim_{s\to 3/10}(5s-3/2)M_w(\chi_{P,s})\colon I_P(3/10) \longrightarrow I_Q(1/6),$$

where w is given as in (3.4), is $H(\mathbb{A})$-equivariant, surjective, and satisfies

$$\mathfrak{X}_w(f^0_{\chi_{P,3/10}}) = J_w \cdot f^0_{\chi_{Q,1/6}},$$

where J_w is defined in (3.5).

Proof For a section $f_s \in I_{B_H}(\chi_{P,s})$, let S denote a finite set of places of F such that $f_{s,v} = f^0_{s,v}$ for all $v \notin S$. By the Gindikin–Karpelevich formula, we have

$$\begin{aligned} M(w,\chi_{P,s})f_s &= \Big(\bigotimes_{v\in S} M_{w,v}(\chi_{P,s})f_{s,v}\Big) \otimes \Big(\bigotimes_{v\notin S} J_v(w,\chi_{P,s})f^0_{s,v}\Big) \\ &= J(w,\chi_{P,s})\Big(\bigotimes_{v\in S} J_v(w,\lambda_{P,s})^{-1}M_{w,v}(\lambda)f_{\lambda,v}\Big) \otimes \Big(\bigotimes_{v\notin S} f^0_{w^{-1}\cdot\chi_{P,s},v}\Big). \end{aligned}$$

The term $J(w,\chi_{P,s})$ has a simple pole at $s = 3/10$. The partially normalized intertwining operators

$$\frac{1}{\Pi_{\alpha>0,w^{-1}\alpha<0}\zeta_{F_\alpha,v}(\langle\chi_{P,s},\alpha^\vee\rangle)} M_{w,v}(\chi_{P,s})(f_s)$$

are entire due to [Win78] for $v \nmid \infty$ and [Sha80] for $v \mid \infty$. The term

$$\Pi_{\alpha>0,w^{-1}\alpha<0}\zeta_{F_\alpha,v}(\langle\chi_{P,s},\alpha^\vee\rangle + 1)$$

is holomorphic for $s = 3/10$. Thus, the terms $J_v(w,\chi_{P,s})^{-1}M_{w,v}(\chi_{P,s})(f_s)$ are holomorphic at $s = 3/10$. In particular, the operator $M_w(\chi_{P,s})$ has at most a simple pole on $I_{B_H}(\chi_{P,s})$. Its restriction to $I_P(s)$ admits a simple pole on the normalized spherical section and hence the residue operator is $H(\mathbb{A})$-equivariant.

The claim now follows from Lemma 3.2 (iii), Section 3.1 (15), and Theorem 3.3 (ii). ∎

Our goal is to prove the following.

Theorem 3.8

(i) (Siegel–Weil identity) *There is an equality of operators*

$$\Lambda_P(3/10) = \Lambda_Q(1/6)\circ\mathfrak{X}_w\colon I_P(3/10) \longrightarrow \mathcal{A}(H).$$

(ii) *There is an equality of automorphic representations $\Pi_P = \Pi_Q$.*

Proof Since the spherical vector f^0 generates the representation $I_P(3/10)$, it is sufficient to show the equality for f^0. Using Lemma 3.7, it remains to show that

$$\Lambda_P(3/10)(f^0_{3/10}) = J_w \cdot \Lambda_Q(1/6)(\widetilde{f}^0_{1/6})$$

holds. The proof is similar to the proof of [Ike92, Proposition 1.8] and relies on the properties of the normalized spherical Eisenstein series

$$\mathcal{E}^\sharp_{B_H}(\lambda,g) = \Big[\prod_{\alpha\in\Phi^+}\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle+1)(\langle\lambda,\alpha^\vee\rangle+1)(\langle\lambda,\alpha^\vee\rangle-1)\Big]\mathcal{E}_{B_H}(\lambda,f^0_\lambda,g), \tag{3.6}$$

where f^0_λ is the normalized spherical section in $I_{B_H}(\lambda)$.

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Proposition 3.9 *The normalized Eisenstein series $\mathcal{E}_B^\sharp(\lambda, g)$ is entire and W_H-invariant in the sense that for any $w \in W_H$ it holds that $\mathcal{E}_B^\sharp(w \cdot \lambda, g) = \mathcal{E}_B^\sharp(\lambda, g)$.*

This result is considered standard and is mentioned without proof in several papers, *e.g.*, [Ike92]. For the sake of completeness of presentation, we prove it in Appendix C.

Since $w^{-1}(\chi_{P,3/10}) = \chi_{Q,1/6}$, for w as in (3.4), it follows that

$$\mathcal{E}_{B_H}^\sharp(\chi_{P,3/10}, g) = \mathcal{E}_{B_H}^\sharp(\chi_{Q,1/6}, g).$$

Let us prove Theorem 3.8 in the case of $K = F \times F$. The case where K is a field follows similarly.

One has $\chi_{P,3/10} = \lambda_{(-1,3,-1,-1)}$. Substituting the formula for the normalizing factor and using Lemma 3.4, one checks that

$$\begin{aligned}\mathcal{E}_{B_H}^\sharp(\chi_{P,3/10}, g) &= \lim_{s_2 \to 3} \lim_{s_1, s_3, s_4 \to -1} \mathcal{E}_{B_H}^\sharp(\lambda_{\bar{s}}, g)\\ &= -2^{12} 3^3 R^4 \zeta_F(2)^3 \zeta_F(3)^3 \zeta_F(4)^2 \cdot \lim_{s \to 3} (s-3)^2 \cdot \mathcal{E}_P\Big(f^0, g, \frac{2s-3}{10}\Big).\end{aligned}$$

Similarly, one has $\chi_{Q,1/6} = \lambda_{(3,-1,-1,-1)}$ and

$$\begin{aligned}\mathcal{E}_{B_H}^\sharp(\chi_{Q,1/6}, g) &= \lim_{s_1 \to 3} \lim_{s_2, s_3, s_4 \to -1} \mathcal{E}_{B_H}^\sharp(\lambda_{\bar{s}}, g)\\ &= -2^{12} 3^3 R^5 \zeta_F(2)^4 \zeta_F(3)^2 \zeta_F(4) \lim_{s \to 3} (s-3) \cdot \mathcal{E}_Q\Big(\widetilde{f}^0, g, \frac{s-2}{6}\Big).\end{aligned}$$

Dividing both sides by $-2^{12} 3^3 R_F^4 \zeta_F(2)^3 \zeta_F(3)^3 \zeta_F(4)^2$ and making a linear change of variables, we obtain

$$\Lambda_P(3/10)(f^0_{3/10}) = \frac{R_F \zeta_F(2)}{\zeta_F(3)\zeta_F(4)} \Lambda_Q(1/6)(\widetilde{f}^0_{1/6}),$$

as required.

The second part follows immediately, since $\mathfrak{X}_w$ is surjective. ∎

Remark 3.10 When $K = F \times F$, the representation $\Pi_P = \Pi_Q$ is the minimal representation and is contained in the space of square integrable automorphic functions. This was proved in [GGJ02], but it also follows from Table 2 by Jacquet's criterion [MW95, p. 74]. When K is a field, the image $\Pi_P = \Pi_Q$ is a special value of an Eisenstein series and is not contained in the space of square-integrable automorphic functions.

4 Global Theta Lift

The goal of this section is to show that $\Pi_{\overline{Q}}$ is a regularized theta lift of a certain non-cuspidal representation for the dual pair $\mathrm{SL}_2(\mathbb{A}) \times \overline{H}(\mathbb{A})$.

4.1 Notations and Setup

We recall the notations and the setup for the theta lift.

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(1) Let $B = T \cdot N$ be the Borel subgroup of SL_2 and let χ_K be a quadratic automorphic character of $T(\mathbb{A})$ associated with the quadratic algebra K by class field theory. Let $\mathcal{K}_2 = \prod_v \mathcal{K}_{2v}$ denote the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{A})$.

(2) Let $I_B(\chi_K, s)$ denote the smooth normalized induction $\mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{SL}_2(\mathbb{A})} \chi_K \delta_B^s$. series $\mathcal{E}_B(\chi_K, \cdot, g, s)$ is holomorphic when K is a field and has at most a simple pole at $s = 1/2$ when $K = F \times F$. In the latter case the pole is attained by the standard spherical section $\underline{f}^0(\cdot, s) \in \mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{SL}_2(\mathbb{A})} \chi_K \delta_B^s$ and the residual representation is the trivial representation.

(3) Denote by $\Lambda_B(\chi_K, 1/2) \colon I_B(\chi_K, 1/2) \to \mathcal{A}(\mathrm{SL}_2)$ the operator that is the leading term of the Laurent expansion of the Eisenstein series $\mathcal{E}_B(\chi_K, \cdot, g, s)$ at $s = 1/2$. Let $\Pi_B(\chi_K, 1/2)$ denote the image of $\Lambda_B(\chi_K, 1/2)$.

(4) The pair $(\mathrm{SL}_2, \overline{H})$ is a dual pair inside Sp_{16}. There is a splitting $\mathrm{SL}_2(\mathbb{A}) \times \overline{H}(\mathbb{A}) \to \widetilde{Sp}_{16}(\mathbb{A})$ that depends on the form q_K on $\overline{H}(\mathbb{A})$. We denote the pullback of the Weil representation ω_ψ to $\mathrm{SL}_2(\mathbb{A}) \times \overline{H}(\mathbb{A})$ by ω_{ψ, q_K}.

(5) The representation ω_{ψ, q_K} acts by way of the Schrödinger model on the space of Schwartz functions $\mathcal{S}(V_K^8(\mathbb{A}))$. The representation ω_{ψ, q_K} is realized automorphically via

$$\theta_{\psi, q_K} \colon \mathcal{S}(V_K^8(\mathbb{A})) \longrightarrow \mathcal{A}(\mathrm{SL}_2 \times \overline{H})$$
$$\theta_{\psi, q_K}(\phi)(g, h) = \sum_{v \in V_K^8(F)} \omega_{\psi, q_K}(g, h)\phi(v).$$

(6) The space (V_K^8, q_K) admits a decomposition $(V_K^8, q_K) = (V^7, q) \oplus (V_K^1, q_K^1)$, where (V_K^1, q_K^1) is a one-dimensional quadratic space of discriminant K, and (V^7, q^7) is the split quadratic space of dimension 7 and discriminant 1. The associated Weil representations of $\widetilde{\mathrm{SL}}_2 \times \mathrm{SO}(V_K^1)$ and $\widetilde{\mathrm{SL}}_2 \times \mathrm{SO}(V^7)$, realized on $\mathcal{S}(V_K^1)$ and $\mathcal{S}(V^7)$, respectively, are denoted by ω_{ψ, q_K^1} and ω_{ψ, q^7}. The space $\omega_{\psi, q^7} \otimes \omega_{\psi, q_K^1}$ is a dense subspace in ω_{ψ, q_K} and one has

$$\theta_{\psi, q_K}(\phi_1 \otimes \phi_2)(g, (h_1, h_2)) = \theta_{\psi, q^7}(\phi_1)(g, h_1)\theta_{\psi, q_K^1}(\phi_2)(g, h_2) \tag{4.1}$$

for any $(h_1, h_2) \in \mathrm{SO}(V^7) \times \mathrm{SO}(V_K^1)$, $g \in \widetilde{\mathrm{SL}}_2$, $\phi_1 \in \omega_{\psi, q^7}$, and $\phi_2 \in \omega_{\psi, q_K^1}$.

(7) For an automorphic form $\varphi \in \mathcal{A}(\mathrm{SL}_2)$ and a Schwartz function $\phi \in \mathcal{S}(V_K^8(\mathbb{A}))$, the global theta lift $\theta_{\psi, q_K}(\phi, \varphi)$ is defined by

$$\theta_{\psi, q_K}(\phi, \varphi) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})} \theta_{\psi, q_K}(\phi)(g, h)\varphi(g)\, dg, \tag{4.2}$$

whenever it converges. This defines an automorphic form on $\overline{H}(\mathbb{A})$.

In the following discussion we shall omit subscripts and write ω and θ instead of ω_{ψ, q_K} and θ_{ψ, q_K} when there is no confusion.

4.2 Regularization of the Theta Lift

For arbitrary $f \in \Pi_B(\chi_K, 1/2)$ the integral in (4.2) does not converge, so a regularization of the integral is required.

When $K = F \times F$, in which case $\Pi_B(\chi_K, 1/2)$ is trivial, the regularization was detailed in [GG06]. Although the idea of regularization is the same for the case where K is a field, we repeat the construction for the convenience of the reader.

As a first step, we define an $\mathrm{SL}_2(\mathbb{A}) \times \overline{H}(\mathbb{A})$-submodule ω^0 of ω such that for $\phi \in \omega^0$ the function $\theta(\phi)(g,h)$ is rapidly decreasing as a function of $g \in \mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})$.

The function $\theta(\phi)(g)$ is rapidly decreasing whenever $\omega(g)\phi(0) = 0$ for all $g \in \mathrm{SL}_2(\mathbb{A})$. Fixing an Archimedean place v_0, define a map

$$\overline{T}\colon \omega_{v_0} \longrightarrow I_{B_{v_0}}(\chi_K, 3/2), \quad \overline{T}(\phi)(g) = \omega(g)\phi(0)$$

and put $\omega^0_{v_0} = \mathrm{Ker}(\overline{T})$. This allows us to define $\omega^0 = \omega^0_{v_0} \otimes (\otimes_{v \neq v_0} \omega_v)$. That is obviously an $\mathrm{SL}_2(\mathbb{A}) \times \overline{H}(\mathbb{A})$-module. Hence, the map $\theta\colon \omega^0 \otimes I_B(\chi_K, s) \to \mathcal{A}(\overline{H})$, given by

$$\theta(\phi, f)(h) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \theta(\phi)(g,h)\mathcal{E}_B(\chi_K, f, g, s)\, dg,$$

is well defined.

Recall that the center $\mathcal{Z}_{v_0}(\mathfrak{sl}_2)$ of the universal enveloping algebra $\mathcal{U}_{v_0}(\mathfrak{sl}_2)$ is isomorphic to $\mathbb{C}[\Delta]$, where Δ is the Casimir operator. The element Δ acts on $I_{B_{v_0}}(\chi_{K_{v_0}}, s)$ by the constant $s^2 - 1/4$. The element $z = \Delta - 2 \in \mathcal{Z}_{v_0}(\mathfrak{sl}_2)$ annihilates the representations $I_{B_{v_0}}(\chi_{K_{v_0}}, \pm 3/2)$ and acts by a non-zero constant $s^2 - 9/4$ on any $I_{B_{v_0}}(\chi_{K_{v_0}}, s)$ with $s \neq \pm 3/2$.

Clearly, z defines an $\mathrm{SL}_2(\mathbb{A}) \times \overline{H}(\mathbb{A})$-equivariant map from ω to itself. Moreover, the image is contained in ω^0 since z commutes with $\overline{T}$ and annihilates $I_B(\chi_{K_{v_0}}, 3/2)$. This allows us to extend the map θ from $\omega^0 \otimes I_B(\chi_K, s)$ to $\omega \otimes I_B(\chi_K, s)$, $s \neq \pm 3/2$ by

$$\theta^{\mathrm{reg}}(\phi, f) = \frac{1}{s^2 - 9/4} \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \theta(z\phi)(g,h)\mathcal{E}_B(\chi_K, f, g, s)\, dg.$$

The extension is unique for all $s \neq \pm 3/2$. Otherwise, having two possible extensions θ_1 and θ_2 of θ, we notice that $\theta_1 - \theta_2$ vanishes on $\omega^0 \otimes I_B(\chi_K, s)$ and hence defines an $\mathrm{SL}_2(F_{v_0})$-invariant functional on $I_B(\chi_{K_{v_0}}, 3/2) \otimes I_B(\chi_{K_{v_0}}, s)$ that must be zero.

4.3 The Regularized Theta Lift of $\Pi_B(\chi_K, 1/2)$

For an automorphic representation $\Pi \subset \mathcal{A}(\mathrm{SL}_2)$, we define the automorphic representation $\theta^{\mathrm{reg}}(\Pi)$ of $\overline{H}(\mathbb{A})$ to be generated by $\theta^{\mathrm{reg}}(\phi, f)$ as $\phi \in \omega_{\psi, q_K}$ and $f \in \Pi$.

Theorem 4.1 $\theta^{\mathrm{reg}}(\Pi_B(\chi_K, 1/2)) = \Pi_{\overline{Q}}$.

Proof Let $K = F \times F$. Then $\Pi_B(\chi_0, 1/2)$ is trivial, and the result follows from [GRS97a, Theorem 6.8].

Let K be a field. The proof of the theorem in this case will occupy the rest of this section. We start by considering $\theta^{\mathrm{reg}}(\phi, \mathcal{E}_B(\chi_K, f, g, s))$ for $\mathrm{Re}(s) \gg 0$.

Proposition 4.2 *Let* $\mathrm{Re}(s) \gg 0$. *For any* $\phi \in \omega^0$ *and* $f \in I_B(\chi_K, s)$, *one has*

$$\theta(\phi, \mathcal{E}_B(\chi_K, f, \cdot, s))(h) = \mathcal{E}_{\overline{Q}}(\mathcal{F}(\phi, f, s), h, s/3),$$

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where $\mathcal{F}(\phi, f, s)(h) = \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega(g,h)\phi(e_0)f(g,s)\,dg$, *where* e_0 *is a non-zero isotropic vector in* $V_K^8(F)$ *such that* $\overline{Q}(F)$ *stabilizes* $\mathrm{Span}\{e_0\}$*; it is defined in Section* 3.1 (11).

Remark 4.3 For a standard section $f(\cdot, s) \in I_B(\chi_K, s)$, the section $\mathcal{F}(\phi, f, s)$ is not standard, but holomorphic.

This follows by the standard unfolding technique:

$$\begin{aligned}
\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} & \theta^{16}(\phi)(g,h)\mathcal{E}_B(\chi_K, f, g, s)\,dg \\
&= \int_{B(F)\backslash \mathrm{SL}_2(\mathbb{A})} \sum_{v \in V_K^8(F)} \omega(g,h)\phi(v)f(g,s)\,dg \\
&= \int_{T(F)N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \sum_{\substack{v \in V_K^8(F),\\ q_K(v)=0}} \omega(g,h)\phi(v)f(g,s)\,dg.
\end{aligned}$$

The group $\overline{H}(F)$ acts on the set $\{v \in V_K^8(F) \mid q_K(v) = 0\}$. The only non-zero orbit is open. Let us choose the vector e_0 as its representative. The contribution of the zero orbit vanishes since $\phi \in \omega^0$. The stabilizer of e_0 satisfies

$$\begin{aligned}
\mathrm{Stab}_{\overline{H}(F)}(e_0) &= \mathrm{SO}(V_6^K)(F) \cdot \overline{V}(F) \subset \overline{Q}(F),\\
\mathrm{GL}_1(F) \cdot \mathrm{Stab}_{\overline{H}(F)}(e_0) &= \overline{Q}(F).
\end{aligned}$$

Thus the integral above equals

$$\begin{aligned}
\int_{T(F)N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} & \sum_{\gamma \in \overline{Q}(F)\backslash \overline{H}(F)} \sum_{t_1 \in \mathrm{GL}_1(F)} \omega(g, t_1\gamma h)\phi(e_0)f(g,s)\,dg \\
&= \int_{T(F)N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \sum_{\gamma \in \overline{Q}(F)\backslash \overline{H}(F)} \sum_{t \in T(F)} \omega(t^{-1}g, \gamma h)\phi(e_0)f(g,s)\,dg \\
&= \sum_{\gamma \in \overline{Q}(F)\backslash \overline{H}(F)} \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega(g, \gamma h)\phi(e_0)f(g,s)\,dg \\
&= \sum_{\gamma \in \overline{Q}(F)\backslash \overline{H}(F)} \mathcal{F}(\phi, f, s)(\gamma h),
\end{aligned}$$

as required.

It remains to check that $\mathcal{F}(\phi, f, s)$ belongs to $I_{\overline{Q}}(s/3)$. Indeed, for $(t, m) \in \overline{L}(\mathbb{A}) = \mathrm{GL}_1(\mathbb{A}) \times \mathrm{SO}(V_K^6)(\mathbb{A})$ it holds that

$$\begin{aligned}
\mathcal{F}(\phi, f, s)((t,m)h) &= \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega(g, (t,m)h)\phi(e_0)f(g)\,dg \\
&= \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega(g,h)\phi(t^{-1}e_0)f(g)\,dg.
\end{aligned}$$

By the formulas of the Schrödinger model, one has

$$\phi(t^{-1}e_0) = \omega(t^{-1},1)|t|^4\chi_K(t)\phi(e_0), \quad f(g,s) = \chi_K(t)|t|^{1+2s}f(t^{-1}g),$$
$$dg = |t|^{-2}d(t^{-1}g),$$

and hence

$$\mathcal{F}(\phi,f,s)((t,m)g) = |t|^{3+2s}\mathcal{F}(\phi,f,s)(g) = \delta_{\overline{Q}}^{s/3+1/2}(t)\mathcal{F}(\phi,f,s)(g),$$

as required.

Remark 4.4 For $\mathrm{Re}(s) \gg 0$, $\phi = \otimes_v \phi_v \in \omega^0$, and $f = \otimes_v f_v \in I_B(\chi_K, s)$, there is a factorization $\mathcal{F}(\phi,f,s)(h) = \prod_v \mathcal{F}_v(\phi_v, f_v, s)(h_v)$, where

$$\mathcal{F}_v(\phi_v, f_v, s)(h) = \int_{N(F_v)\backslash \mathrm{SL}_2(F_v)} \omega_v(g,h)\phi_v(e_0)f_v(g,s)\,dg.$$

Our goal is to define $\mathcal{F}(\phi,f,s)$ at $s = 1/2$ using analytic continuation. In order to do this, we consider the behaviour of $\mathcal{F}_v(\phi_v, f_v, s)(h_v)$.

Proposition 4.5 (i) *For any place v, the map $\mathcal{F}_v$ admits a holomorphic continuation and is non-zero for* $\mathrm{Re}(s) > -3/2$.

(ii) *Let K_v be either $F_v \times F_v$ or the unramified quadratic extension of F_v. We fix the normalized spherical vectors $\phi_v^0 \in \omega_v$ and $\underline{f}^0_{v,s} \in I_B(\chi_{K_v}, s)$. Then*

$$\mathcal{F}_v(\phi_v^0, \underline{f}^0_{v,s}, s) = \zeta_v(2s+3)\overline{f}^0_{v,s/3},$$

where $\overline{f}^0_{v,s}$ is the normalized spherical section of $I_{\overline{Q}_v}(s)$.

(iii) *For any place v, the map $\mathcal{F}_v\colon \omega_v \otimes I_{B,v}(\chi_{K_v}, 1/2) \to I_{\overline{Q}_v}(1/6)$ is surjective.*

(iv) *For $v = v_0$, the map $\mathcal{F}_v\colon \omega^0_{v_0} \otimes I_{B,v_0}(\chi_{K_{v_0}}, 1/2) \to I_{\overline{Q}_{v_0}}(1/6)$ is surjective.*

Proof (i) Recall that the section $f_{s,v}$ is standard, so that its restriction f_v to $\mathcal{K}_{2,v}$ does not depend on s. Using the Iwasawa decomposition, $\mathrm{SL}_2(F_v) = N(F_v)\cdot T(F_v)\cdot \mathcal{K}_2$, we can write $\mathcal{F}_v(\phi_v, f_v, s) = \int_{\mathcal{K}_{2_v}} L(\omega_v(k)\phi_v, s) f_v(k)\,dk$, where

$$L(\phi_v, s) = \int_{T(F_v)} \omega_v(t)\phi_v(e_0)\chi_{K_v}(t)\delta_B^{s+1/2}(t)\delta_B^{-1}(t)\,dt = \int_{F_v^\times} |t|^{2s+3}\phi_v(te_0)\,d^\times t.$$

Obviously, the operator $L(\cdot, s)$ is holomorphic for $\mathrm{Re}(s) > -3/2$ and does not vanish for any v. Thus, the operator $\mathcal{F}_v$ is also holomorphic for $\mathrm{Re}(s) > -3/2$.

We now show that $\mathcal{F}_v$ is non-zero for $\mathrm{Re}(s) > -3/2$. Note that for any

$$b \in B(F_v) \cap \mathcal{K}_{2,v}$$

holds $L(\omega_v(b)\phi_v, s) = \chi_{K_v}(b)L(\phi_v, s)$. For s with $\mathrm{Re}(s) > -3/2$, we choose ϕ_v such that $L(\phi_v, s) \neq 0$. We can choose f_v, whose support modulo $B_2(F_v) \cap \mathcal{K}_{2,v}$ is small enough, so that $\mathcal{F}_v(\phi_v, f_v, s) \neq 0$.

(ii) If $I_{B_v}(\chi_{K_v}, s)$ is an unramified representation with $\underline{f}^0_{v,s}$ the normalized spherical vector, then $\mathcal{F}_v(\phi^0_v, \underline{f}^0_v, s)$ is spherical. One checks that $L(\phi^0_v, s) = \zeta_v(2s+3)$ and hence $\mathcal{F}_v(\phi^0_v, \underline{f}^0_{v,s}, s) = \zeta_v(2s+3)\overline{f}^0_{v,s/3}$.

(iii) If $K_v = F_v \times F_v$, then the representation $I_{B_v}(\chi_{K_v}, 1/2) = I_{B_v}(\chi_0, 1/2)$, where χ_0 is the trivial character, is generated by the spherical vector and the image of the spherical data is not zero. The fact that $I_{\overline{Q}_v}(1/6)$ is generated by the spherical vector implies surjectivity. If K_v is a field, then $I_{\overline{Q}_v}(1/6)$ is irreducible, as shown in Theorem 3.3, and hence the non-vanishing of $\mathcal{F}_v$ implies its surjectivity.

(iv) The map $\mathcal{F}_{v_0}\colon \omega_{v_0} \otimes I_{B_{v_0}}(\chi_{K_{v_0}}, 1/2) \to I_{\overline{Q}_{v_0}}(1/6)$ is surjective and $\mathrm{SL}_2 \times \overline{H}$-equivariant. If $U = \mathcal{F}_{v_0}(\omega^0_{v_0} \otimes I_{B_{v_0}}(\chi_K, 1/2))$ is a proper subrepresentation of $I_{\overline{Q}}(1/6)$, then the map $\mathcal{F}_{v_0}$ factors to a non-zero SL_2-equivariant map

$$I_{B_{v_0}}(\chi_{K_{v_0}}, 3/2) \otimes I_{B_{v_0}}(\chi_{K_{v_0}}, 1/2) \to I_{\overline{Q}_{v_0}}(1/6)/U \neq 0,$$

which is impossible. Hence, $\mathcal{F}_{v_0}$, when restricted to $\omega^0_{v_0} \otimes I_{B_{v_0}}(\chi_{K_{v_0}}, 1/2)$, is surjective. ■

We define the holomorphic continuation of the operator $\mathcal{F}(\cdot, \cdot, s)$ for $\mathrm{Re}(s) > -1$. Let $\phi = \otimes \phi_v \in \omega^0$, $f = \otimes f_v \in I_B(\chi_K, s)$ be factorizable data and let S be a finite set of places of F such that K_v, ϕ_v, and f_v are unramified outside of S. Then the equality

$$\mathcal{F}(\phi, f, s)(h) = \zeta^S(2s+3)\Big[\prod_{v \notin S} \overline{f}^0_{v,s} \times \prod_{v \in S} \mathcal{F}_v(\phi_v, f_v, s)(h_v)\Big]$$

holds for $\mathrm{Re}(s) \gg 0$ which allows us to define $\mathcal{F}(\phi, f, 1/2)$. Moreover, the operator $\mathcal{F}(\cdot, \cdot, 1/2)\colon \omega^0 \otimes I_B(\chi_K, 1/2) \to I_{\overline{Q}}(1/6)$ is surjective.

Theorem 4.1 follows since, for $\phi \in \omega^0$ and $f \in I_B(\chi_K, 1/2)$, one has

$$\theta(\phi, \mathcal{E}_B(\chi_K, f, \cdot, 1/2)) = \mathcal{E}_{\overline{Q}}(\mathcal{F}(\phi, f, 1/2), \cdot, 1/6). \qquad ■$$

5 Proof of Theorem 1.1

In this section we prove the remaining direction of Theorem 1.1.

Theorem 5.1 *Let π be an irreducible cuspidal representation of $G_2(\mathbb{A})$ such that $\mathcal{L}^S(s, \pi, \mathfrak{st})$ has a pole at $s = 2$. Then $\mathrm{RS}_\psi(\pi) \neq 0$.*

Proof Since $\mathcal{L}^S(s, \pi, \mathfrak{st})$ has a pole at $s = 2$, it follows from Theorem 2.4 (ii) that either $\widehat{F}_\psi(\pi) = \{F \times F \times F\}$ or there exists a quadratic field extension K of F so that $F \times K \in \widehat{F}_\psi(\pi)$.

In the first case, $\mathrm{RS}_\psi(\pi) \neq 0$ by Theorem 2.4 (iii).

In the second case put $H = H_{F\times K}$ and $\overline{H} = \mathrm{SO}(V^8_K)$. We conclude from equations (1.1) and (1.2) that there exist $\varphi \in \pi$ and $\eta \in \Pi_P = \Pi_Q$, an automorphic form on $H(\mathbb{A})$, such that $\int_{G_2} \varphi(h)\eta(h)\,dh \neq 0$.

The embedding of G_2 into $\overline{H}$ factors through H. Hence, by Corollary 3.6, there exists $\overline{\eta} \in \Pi_{\overline{Q}}$ such that $\int_{G_2} \varphi(h)\overline{\eta}(h)\,dh \neq 0$.

Since the map $\Lambda_{\overline{Q}}(1/6)\circ\mathcal{F}(\cdot,\cdot,1/2)\colon \omega^0\otimes I_B(\chi_K,1/2)\to\Pi_{\overline{Q}}$ is surjective, it follows that there exist $\phi\in\omega^0$, $f\in I_B(\chi_K,1/2)$ such that

$$\int_{G_2(F)\backslash G_2(\mathbb{A})}\varphi(h)\Big(\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})}\theta_{\psi,q_K}(\phi)(g,h)\mathcal{E}_B(\chi_K,f,g,1/2)\,dg\Big)\,dh\neq 0.$$

Since φ is rapidly decreasing on G_2 and $\theta_{\psi,q_K}(\phi)$ is rapidly decreasing on SL_2, the integral converges absolutely and it is possible to change the order of integration. Hence

$$\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})}\Big(\int_{G_2(F)\backslash G_2(\mathbb{A})}\varphi(h)\theta_{\psi,q_K}(\phi)(g,h)\,dh\Big)\mathcal{E}_B(\chi_K,f,g,1/2)\,dg\neq 0$$

for some choice of data. In particular, the inner integral does not vanish.

By (4.1), there exist $\varphi\in\pi$, $\phi_1\in\omega_{\psi,q^7}$, and $\phi_2\in\omega_{\psi,q_K^1}$ such that for some $g\in\widetilde{\mathrm{SL}}_2$

$$\int_{G_2(F)\backslash G_2(\mathbb{A})}\varphi(h)\theta^{14}_{\psi,q^7}(\phi_1)(g,h)\,dh\cdot\theta^2_{\psi,q_K^1}(\phi_2)(g)\neq 0$$

and hence $RS_\psi(\pi)\neq 0$, as required. ■

Remark 5.2 The argument above can be visualized by the following see-saw diagram.

$$\begin{tikzcd}
\widetilde{\mathrm{SL}}_2\times\widetilde{\mathrm{SL}}_2 \arrow[drr, dash] & & \overline{H} \\
\Delta\widetilde{\mathrm{SL}}_2 \arrow[u, hook] \arrow[urr, dash] & & G_2\times\mathrm{SO}(V_K^1) \arrow[u, hook]
\end{tikzcd}$$

Remark 5.3 Assume that $\mathcal{L}^S(s,\pi,\mathfrak{st})$ has a simple pole at $s=2$ and $F\times K\in\widehat{F}_\psi(\pi)$. We can distinguish whether π is a weak lift from $\tau\boxtimes 1$ for cuspidal or for one-dimensional τ by studying the twisted $\mathcal{L}$-function $\mathcal{L}^S(s,\pi\boxtimes\chi_K,\mathfrak{st}\boxtimes\mathfrak{st})$, whose meromorphic continuation was also studied in [Seg17, Theorem 3.1]. Specifically, $\mathcal{L}^S(s,\pi\boxtimes\chi_K,\mathfrak{st}\boxtimes\mathfrak{st})$ has a simple pole at $s=2$ if and only if π is a weak lift of $(\chi_K\circ\det)\boxtimes 1$. If it is holomorphic, then π is a weak lift of $\tau\boxtimes 1$ for a cuspidal τ.

A Irreducibility of $I_Q(1/6)$ When K is a Non-Archimedean Field

We fix a finite place v of F such that K_v is a field and drop v from all notations. For any group G, we write just G for $G(F)$.

In [Wei03], the main object was a degenerate principal series associated with a parabolic subgroup with an Abelian unipotent radical of a split simply-connected simply-laced group. One uses the Fourier–Jacobi functor to determine its reducibility. In particular, for $K=F\times F$, the result of [Wei03], applied to $I_Q(s)$, shows that $I_Q(1/6)$ is of length 2 and its unique irreducible quotient is spherical. For a field K, the group $H_{F\times K}$ is not split and the restricted root system is not simply-laced and hence the results of [Wei03] cannot be applied directly. In this section, we prove that $I_Q(1/6)$ is irreducible using a variation of the arguments in [Wei03]. When the statements and their proofs can be repeated verbatim, we refer the reader to [Wei03].

A.1 The Fourier–Jacobi Functor

This functor takes smooth representations of H to the smooth representations of SL_2.

The unipotent radical U of P is a Heisenberg group with center Z and the polarization $U = A \cdot A' \cdot Z$, where

$$\begin{aligned} Z &= \{x_{[1,2,2]}(r) \mid r \in F\}, \\ A &= \{x_{[1,1,0]}(r_1)x_{[1,1,1]}(r_2)x_{[1,1,2]}(r_3) \mid r_1, r_3 \in F,\ r_2 \in K\}, \\ A' &= \{x_{[0,1,0]}(r_1)x_{[0,1,1]}(r_2)x_{[0,1,2]}(r_3) \mid r_1, r_3 \in F,\ r_2 \in K\}. \end{aligned}$$

By the Stone–von Neumann theory, for any additive character ψ of $Z \simeq \mathbb{G}_a$ there is unique irreducible representation ω_ψ of U with central character ψ. It can be realized as $\mathrm{ind}_{AZ}^{U}\psi$, where ψ is extended trivially on A. By restriction this space is isomorphic to the space of Schwartz functions $\mathcal{S}(A')$. The Weil representation can be extended to a representation ω_ψ of $\widetilde{Sp}(U/Z) \cdot U$.

Proposition A.1 *Fix an embedding $\varphi_{\alpha_1}\colon \mathrm{SL}_2 \hookrightarrow M' \hookrightarrow Sp(U/Z)$, where M' is the derived group of the Levi factor M of P. Then the image splits in $\widetilde{Sp(U/Z)}$. In particular, the Weil representation ω_ψ of U can be extended to a representation of $\mathrm{SL}_2 \cdot U$.*

Once this proposition is proved we can define

Definition A.2 The Fourier–Jacobi functor of a smooth representation π of H is defined by $\mathrm{FJ}_\psi(\pi) = \mathrm{Hom}_U(\omega_\psi, \pi_{Z,\psi})$, which is a smooth representation of SL_2.

Note that the functor is exact as a composition of two exact functors.

Proof of Proposition A.1 The splitting is provided by S. Kudla [Kud94], but to apply his result we need to fix certain isomorphisms.

Consider the four-dimensional quadratic space $\mathbb{V} = F \oplus K \oplus F$ equipped with a quadratic form

$$q_K(\vec{r}) = r_1 r_3 + Tr_{K/F}(r_2^2), \quad \vec{r} = (r_1, r_2, r_3) \in \mathbb{V}; \quad r_1, r_3 \in F, r_2 \in K.$$

Let $\mathbb{H} = X \oplus Y$ be a two-dimensional symplectic space with the standard isotropic polarization. In particular, we have $Sp(\mathbb{H}) \simeq \mathrm{SL}_2$.

The space $\mathbb{W} = \mathbb{V} \otimes \mathbb{H}$ is equipped with the natural symplectic form and the following polarization $\mathbb{W} = \mathbb{V} \otimes \mathbb{H} = \mathbb{V} \otimes X \oplus \mathbb{V} \otimes Y$. We denote by $H(\mathbb{W})$ the Heisenberg group associated with the space $\mathbb{W}$ and we identify

$$U \simeq H(\mathbb{W}), \quad U/Z \simeq \mathbb{W}, \quad A \simeq \mathbb{V} \otimes X, \quad A' \simeq \mathbb{V} \otimes Y.$$

Under this identification, the subgroup $\varphi_{\alpha_1}(\mathrm{SL}_2)$ of $Sp(U/Z)$ is identified with $Sp(\mathbb{H}) \hookrightarrow Sp(\mathbb{W})$.

The dual pair $Sp(\mathbb{H}) \times O(\mathbb{V})$ splits in $\widetilde{Sp}(\mathbb{W})$ and the splitting $S_{\mathbb{V},\mathbb{H}}$ is given by [Kud94]. By pullback, ω_ψ is a representation of $Sp(\mathbb{H}) \cdot H(\mathbb{W}) = \mathrm{SL}_2 \cdot U$. ∎

We note that, according to [Wei03, Proposition 3.1], FJ_ψ is independent of ψ ,hence we denote it by FJ.

Proposition A.3 *Let σ be an irreducible representation of H. Then $\mathrm{FJ}(\sigma) = 0$ if and only if σ is trivial.*

Proof If σ is trivial, then $\sigma_{Z,\psi} = 0$ and hence $\mathrm{FJ}_\psi(\sigma) = 0$. Conversely, if $\mathrm{FJ}(\sigma) = 0$, then $\sigma_{Z,\psi} = 0$ for any non-trivial character ψ of Z, *i.e.*, Z acts trivally on σ. The subgroup generated by the conjugates of Z is a normal subgroup of H. Since H modulo its finite center is simple, we deduce that σ is trivial on H. ∎

A.2 The Effect of the Fourier–Jacobi Functor on the Principal Series

Let B denote the Borel subgroup of SL_2 and let δ_B denote its modular character. The group B acts by conjugation on U normalizing AZ and commuting with Z. Hence every $b \in B$ defines an endomorphism $\mathrm{Ad}(b)$ of $\mathrm{ind}_{AZ}^U \psi$ by $(\mathrm{Ad}(b)\varphi)(u) = \varphi(bub^{-1})$. Any $b \in B$ has a form $b = t(a)\cdot n$, where $t(a) = \alpha_1^\vee(a)$, and n belongs to the unipotent radical of B.

Proposition A.4 $\mathrm{FJ}(I_Q(s)) = I_B(\chi_K, 3s)$.

Proof The proof is computational. We begin with the following lemma

Lemma A.5 (i) *For any $b = t(a)n \in B$, the endomorphism $\mathrm{Ad}(b) \circ \omega_\psi(b)$ of $\mathrm{ind}_{AZ}^U \psi$ is the scalar $\chi_K(a)|a|^2 = \chi_K(a)(\delta_Q^{1/2}\delta_B^{-1/2})(t(a))$.*

(ii) *There is an $\mathrm{SL}_2 \cdot U$-equivariant isomorphism*

$$J\colon I_B(\chi_K, 3s) \otimes \omega_\psi \cong \mathrm{ind}_{B\cdot AZ}^{\mathrm{SL}_2\cdot U}[\delta_B^{3s}(\delta_Q^{1/2}\delta_B^{-1/2}) \otimes \psi],$$

defined by $J(f \otimes \varphi)(mu) = f(m)\omega_\psi(m)\varphi(mum^{-1})$.

Proof (i) Similarly to [Wei03, Proposition 2.6], one shows that $\mathrm{Ad}(b) \circ \omega_\psi(b)$ acts on $\mathrm{ind}_{AZ}^U \psi$ by a character and hence is trivial on the unipotent radical of the Borel. To determine the character on the torus T of B, we use the Schrödinger module $S(\mathbb{V} \otimes Y)$ of ω_ψ. The formula of the action of $t(a)$ in this model appears in [Pra98, §1.1].

$$\begin{aligned}\mathrm{Ad}(t(a))\omega_\psi(t(a))\varphi(v) &= \omega_\psi(t(a))\varphi(a^{-1}v) = \chi_K(a)|a|^2\varphi(v)\\ &= \chi_K(a)(\delta_Q^{1/2}\delta_B^{-1/2})(t(a))\varphi(v).\end{aligned}$$

(ii) It follows from part (i) that $J(\varphi \otimes f)$ belongs to $\mathrm{ind}_{B\cdot AZ}^{\mathrm{SL}_2\cdot U}[\delta_B^{3s}(\delta_Q^{1/2}\delta_B^{-1/2}) \otimes \psi]$ for any $f \in I_B(\chi_K, 3s)$ and $\varphi \in \omega_\psi$. The bijectivity of the map J follows similarly to that in [Wei03, Theorem 4.3.1]. ∎

By [Wei03, Propositions 3.2, 4.2.3] we have an SL_2-equivariant isomorphism $\mathrm{FJ}(I_Q(s)) \otimes \omega_\psi \cong I_Q(s)_{Z,\psi} \cong I_Q^{w_0}(s)_{Z,\psi}$, where $w_0 = w[2132132]$ is the shortest representative of the coset containing the longest element in $W(M, T) = P\backslash H/B_H$ and $I_Q^{w_0}(s) = \{f \in I_Q(s) \mid \mathrm{Supp}(f) \subset Qw_0P\}$.

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There is a chain of $\mathrm{SL}_2 \cdot U$-equivariant isomorphisms:

$$I_Q^{w_0}(s)_{Z,\psi} \cong \mathrm{ind}_{B\cdot AZ}^{\mathrm{SL}_2\cdot U}[(\delta_Q^s|_B(\delta_Q^{1/2}\delta_B^{-1/2}))\otimes\psi]$$
$$\cong \mathrm{ind}_{B\cdot AZ}^{\mathrm{SL}_2\cdot U}[\delta_B^{3s}(\delta_Q^{1/2}\delta_B^{-1/2})\otimes\psi] \cong I_B(\chi_K, 3s)\otimes\omega_\psi.$$

The first isomorphism is realized via the map sending $f \in I_Q^{w_0}(s)_{Z,\psi}$ to

$$F(g) = \int_Z f(w_0 gz)\overline{\psi(z)}\,dz, \quad \forall g \in \mathrm{SL}_2 \cdot U.$$

The integral is convergent, since for any fixed g the function $f(w_0 gz)$ is compactly supported as a function of z. The second isomorphism follows from the fact that

$$\delta_Q(t(a)) = |a|^6 = \delta_B(t(a))^3,$$

and the third isomorphism is part (ii) of the lemma. Hence, $\mathrm{FJ}(I_Q(s))\otimes\omega_\psi = I_B(\chi_K, 3s)\otimes\omega_\psi$ and so $\mathrm{FJ}(I_Q(s)) = I_B(\chi_K, 3s)$, as required. ■

Finally, we can prove the irreducibility of $I_Q(1/6)$. Inspecting the Jacquet module $I_Q(1/6)_U$, it is easy to see that $I_Q(1/6)$ does not contain trivial constituents. Hence, by the exactness of FJ and Proposition A.3, the length of $I_Q(1/6)$ equals the length of $\mathrm{FJ}(I_Q(1/6)) = I_B(\chi_K, 1/2)$, which is an irreducible representation of SL_2.

B Structure of $I_{\overline{Q}}(1/6)$ at Archimedean Places

In this section, v is an Archimedean place, real or complex. We drop v from all notations. We consider the structure of the representation $I_{\overline{Q}}(1/6)$.

Proposition B.1 (i) *If $F = \mathbb{C}$ and $K = \mathbb{C}\times\mathbb{C}$, then $I_{\overline{Q}}(1/6)$ has length two and admits a unique irreducible quotient that is spherical.*

(ii) *If $F = \mathbb{R}$ and $K = \mathbb{R}\times\mathbb{R}$, then $I_{\overline{Q}}(1/6)$ has length three and admits a unique irreducible quotient that is spherical.*

(iii) *If $F = \mathbb{R}$ and $K = \mathbb{C}$, then $I_{\overline{Q}}(1/6)$ is spherical and irreducible.*

Proof We use the results of [Sah95] to establish the composition series of the representation $I_{\overline{Q}}(1/6)$. It is essential to present the relevant notations.

- Let $\mathfrak{h} = \mathrm{Lie}(\overline{H}(F))$, $\mathfrak{k} = \mathrm{Lie}(\mathcal{K})$, $\mathfrak{l} = \mathrm{Lie}(L(F))$, and fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ orthogonal to $\mathfrak{k}\cap\mathfrak{l}$. Also, let $\Theta_{\mathfrak{k}}$ be the Cartan involution of $\overline{H}(F)$ associated with $\mathcal{K}$ and denote by $\mathfrak{p}$ the (-1)-eigenspace of $\Theta_{\mathfrak{k}}$ in $\mathfrak{h}$ We fix a basis γ_1, γ_2 of the dual space to $\mathfrak{t}^*$ as in [Sah95, §0].
- The symmetric space $\overline{\mathcal{K}}/\overline{\mathcal{K}}\cap\overline{L}(F)$ is of rank two and its root system is of type D_2 for $F = \mathbb{R}$ and of type C_2 when $F = \mathbb{C}$.
- The half-sum of positive roots of the root system of $\mathfrak{k}$ relative to $\mathfrak{t}$ is given by $\rho = r_1\gamma_1 + r_2\gamma_2$.
- We consider the degenerate principal series $(\pi_t, I_{\overline{Q}^{op}}(t))$ as an $(\mathfrak{h}, \overline{\mathcal{K}})$-module. Since it is admissible, its structure is the same as of an $\overline{H}(F)$-module. The representation $I_{\overline{Q}^{op}}(1/6)$ is denoted in [Sah95] by $I(1)$ for $F = \mathbb{R}$ and $I(2)$ for $F = \mathbb{C}$. Since the parabolic subgroups $\overline{Q}^{\mathrm{op}}$ and $\overline{Q}$ are conjugate, one has $\widetilde{I}_{\overline{Q}}(1/6) \simeq I_{\overline{Q}^{op}}(1/6)$, and

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so the composition series of the latter determines the composition series of $I_{\overline{Q}}(1/6)$. The reducibility of the representation $I(t)$ is given in terms of the numbers r_i above.

• As a representation of $\overline{\mathcal{K}}$, $I(t)$ is isomorphic to $L^2(\overline{\mathcal{K}}/\overline{\mathcal{K}} \cap \overline{L}(F)$, which is, by [Hel84, Theorem V.4.3], a multiplicity-free sum of highest weight representations, called $\overline{\mathcal{K}}$-types. For each $\overline{\mathcal{K}}$-type, Sahi defined its rank and proved [Sah95, Theorem 3B] that two $\overline{\mathcal{K}}$-types belong to the same irreducible constituent of $I(t)$ if and only if they have the same rank.

Now we list the relevant numbers for the cases of interest and cite the relevant theorems from [Sah95].

Case 1: $F = \mathbb{C}, K = \mathbb{C}\times\mathbb{C}$. The group is $\mathrm{SO}(8,\mathbb{C})$. The root system of the symmetric space is of type C_2. One has $r_1 = 5/2$ and $r_2 = 1/2$. Theorem 3C implies that $I(2)$ has exactly two irreducible constituents, one of rank 1 that is the minimal representation denoted by Π_1, and another of rank 2 denoted by Π_2. Moreover, Π_1 is a spherical subrepresentation of $I(2)$.

Case 2: $F = \mathbb{R}, K = \mathbb{R}\times\mathbb{R}$. The group is $\mathrm{SO}(4,4)$. The root system of the symmetric space is of type D_2. One has $r_1 = 1, r_2 = 0$. Theorem 3C implies that the representation $I(1)$ has exactly three irreducible constituents of rank 1, 2^+ and 2^-, respectively, all of them unitarizable. The constituents of rank $2^\pm$ are quotients.

Case 3: $F = \mathbb{R}, K = \mathbb{C}$. The group is $\mathrm{SO}(5,3)$. The root system of the symmetric space is of type D_2. One has $r_1 = 1$ and $r_2 = 1/2$. The Theorem 3A implies that $I(1)$ is irreducible.

It remains to show that $I(2)$ in Case 1 and $I(1)$ in Case 2 are not direct sums of their constituents, *viz.* they have a unique irreducible subrepresentation.

Let us do that for Case 1. Case 2 is proven similarly.

The $\overline{\mathcal{K}}$ types are the highest weight representations $V_{(a_1,a_2)} = V_{a_1\gamma_1+a_2\gamma_2}$, where $a_1 \geq a_2 \geq 0$. Let $v_{(a_1,a_2)} = v_{a_1\gamma_1+a_2\gamma_2}$ denote the highest weight vector in $V_{(a_1,a_2)}$.

The roots $\pm\gamma_1$, $\pm\gamma_2$ are the extreme $\mathfrak{k}$-weights of $\mathfrak{p}$ and hence their weight-space, $\mathfrak{p}_{\pm\gamma_1}$ and $\mathfrak{p}_{\pm\gamma_2}$, are one-dimensional. We fix non-zero elements X_1, X_2, $\overline{X_1}$, and $\overline{X_1}$ of $\mathfrak{p}_{\gamma_1}, \mathfrak{p}_{\gamma_2}, \mathfrak{p}_{-\gamma_1}$, and $\mathfrak{p}_{-\gamma_2}$, respectively.

For any highest weight γ, $\pi_t(X_i)$ is a $\overline{\mathcal{K}}$-orthogonal projection from V_γ to $V_{\gamma+\gamma_i}$ sending v_γ to $c_i(\gamma,t)v_{\gamma+\gamma_i}$. Hence, $V_{\gamma+\gamma_i}$ occurs in $\pi_t(\mathfrak{h})V_\gamma$ if and only if $c_i(\gamma,t)v_{\gamma+\gamma_i} \neq 0$.

Similarly, $\pi_t(\overline{X_i})$ is a $\overline{\mathcal{K}}$-orthogonal projection from $V_{\gamma+\gamma_i}$ to V_γ sending $v_{\gamma+\gamma_i}$ to $d_i(\gamma,t)v_\gamma$. Hence, V_γ occurs in $\pi_t(\mathfrak{h})V_{\gamma+\gamma_i}$ if and only if $d_i(\gamma,t)v_{\gamma+\gamma_i} \neq 0$.

Since $V_{(1,0)}$ is of rank one [Sah95, p. 10], it is a constituent of Π_1. On the other hand, $V_{(1,1)}$ has rank two and hence it is a constituent of Π_2. Furthermore, $\pi_t(\overline{X_2})$ is a $\overline{\mathcal{K}}$-orthogonal projection from $V_{(1,1)}$ to $V_{(1,0)}$ that sends $v_{(1,1)}$ to $d_2(\gamma_1,\frac{1}{6})v_{(1,0)}$. From [Sah95, Theorem 1] it follows that $d_2(\gamma_1,\frac{1}{6})$ (denoted $d_2(\gamma_1,2)$ there) is non-zero and hence $V_{(1,0)}$ occurs in $\pi_{1/6}(\mathfrak{h})V_{\gamma+\gamma_i}$. We conclude that Π_2 generates $I(2)$ and hence cannot be a submodule.

Taking the contragredient, we conclude that $\widetilde{\Pi}_2$ is a subrepresentation and $\widetilde{\Pi}_1$ is the unique irreducible quotient of $I_{\overline{Q}}(1/6)$. In particular, $I_{\overline{Q}}(1/6)$ is generated by its spherical vector. ■

C The Normalized Eisenstein Series

Assume that G is a quasi-split, simply-connected and simple group defined over F. Let B be a Borel subgroup of G with maximal torus T containing a maximal split torus T_S. Let Φ denote the relative root system of G with respect to (B, T_S) with relative Weyl group $W = W(G, T_S)$. Let Φ^+ denote the associated set of positive roots of G with the set of simple roots Δ. The Weyl group W is generated by the simple reflections w_α with $\alpha \in \Delta$. For any $\alpha \in \Phi^+$, we denote by F_α the field of definition of the root α. Finally, the space $\mathfrak{a}_{\mathbb{C}}^* = X^*(T) \otimes \mathbb{C}$ can be identified with the space of totally unramified automorphic characters of $T(\mathbb{A})$.

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We consider the Eisenstein series $\mathcal{E}_B(\lambda, f_\lambda^0, g)$ corresponding to the normalized spherical section $f_\lambda^0 \in \operatorname{Ind}_B^G \lambda$ (normalized induction). As in equation (3.6), let

$$\mathcal{E}_B^\sharp(\lambda, g) = \Big[\prod_{\alpha\in\Phi^+} \zeta_{F_\alpha}(\langle\lambda, \alpha^\vee\rangle + 1) l_\alpha^+(\lambda) l_\alpha^-(\lambda)\Big] \mathcal{E}_B(\lambda, f_\lambda^0, g),$$

where $l_\alpha^\pm(\lambda) = \langle\lambda, \alpha^\vee\rangle \pm 1$.

Proposition C.1 *The normalized Eisenstein series $\mathcal{E}_B^\sharp(\lambda, g)$ is entire and W-invariant.*

Proof We start by proving that $\mathcal{E}_B^\sharp(\lambda, g)$ is indeed W-invariant. It suffices to prove it for a simple reflection w_β for some $\beta \in \Delta$, since W is generated by these reflections.

Indeed, applying the functional equation [MW95, IV.1.10]

$$\mathcal{E}_B(\lambda, f_\lambda^0, g) = \mathcal{E}_B(w_\beta \cdot \lambda, M(w_\beta, \lambda) f_\lambda^0, g),$$

the fact that

$$\prod_{\alpha\in\Phi^+} l_\alpha^+(w_\beta\cdot\lambda) l_\alpha^-(w_\beta\cdot\lambda) = \prod_{\alpha\in\Phi^+} l_\alpha^+(\lambda) l_\alpha^-(\lambda),$$

$$\prod_{\alpha\in\Phi^+} \zeta_{F_\alpha}(\langle w_\beta\cdot\lambda, \alpha^\vee\rangle + 1) = \frac{\zeta_{F_\beta}(-\langle\lambda, \beta^\vee\rangle + 1)}{\zeta_{F_\beta}(\langle\lambda, \beta^\vee\rangle + 1)} \prod_{\alpha\in\Phi^+} \zeta_{F_\alpha}(\langle\lambda, \alpha^\vee\rangle + 1),$$

and the functional equation $\zeta_{F_\alpha}(s) = \zeta_{F_\alpha}(1-s)$ for all $\alpha \in \Phi^+$, yields

$$\begin{aligned}\mathcal{E}_B^\sharp(w_\beta\cdot\lambda, g) &= \Big[\prod_{\alpha\in\Phi^+} \zeta_{F_\alpha}(\langle w_\beta\cdot\lambda, \alpha^\vee\rangle + 1) l_\alpha^+(w_\beta\cdot\lambda) l_\alpha^-(w_\beta\cdot\lambda)\Big] \mathcal{E}_B(w_\beta\cdot\lambda, f_{w_\beta\cdot\lambda}^0, g)\\ &= \frac{\zeta_{F_\beta}(-\langle\lambda, \beta^\vee\rangle + 1)}{\zeta_{F_\beta}(\langle\lambda, \beta^\vee\rangle + 1)} \Big[\prod_{\alpha\in\Phi^+} \zeta_{F_\alpha}(\langle\lambda, \alpha^\vee\rangle + 1) l_\alpha^+(\lambda) l_\alpha^-(\lambda)\Big]\\ &\quad \times \mathcal{E}_B(\lambda, M(w_\beta, w_\beta\cdot\lambda) f_{w_\beta\cdot\lambda}^0, g)\end{aligned}$$

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$$
\begin{aligned}
&= \frac{\zeta_{F_\beta}(-\langle\lambda,\beta^\vee\rangle+1)}{\zeta_{F_\beta}(\langle\lambda,\beta^\vee\rangle+1)}\Big[\prod_{\alpha\in\Phi^+}\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle+1)l_\alpha^+(\lambda)l_\alpha^-(\lambda)\Big]\\
&\quad\times\frac{\zeta_{F_\beta}(-\langle\lambda,\beta^\vee\rangle)}{\zeta_{F_\beta}(-\langle\lambda,\beta^\vee\rangle+1)}\mathcal{E}_B(\lambda,f_\lambda^0,g)\\
&=\Big[\prod_{\alpha\in\Phi^+}\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle+1)l_\alpha^+(\lambda)l_\alpha^-(\lambda)\Big]\mathcal{E}_B(\lambda,f_\lambda^0,g)=\mathcal{E}_B^\sharp(\lambda,g).
\end{aligned}
$$

We now turn to prove that $\mathcal{E}_B^\sharp(\lambda,g)$ is entire. It follows from the general theory of Eisenstein series that $\mathcal{E}_B^\sharp(\lambda,g)$ is entire if and only if its constant term along B is entire. We recall that

$$\mathcal{E}_B(\lambda,f_\lambda^0,t)_N=\sum_{w\in W}\Big(\prod_{\substack{\alpha\in\Phi^+\\ w^{-1}\cdot\alpha\notin\Phi^+}}\frac{\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle)}{\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle+1)}\Big)f^0_{w^{-1}\cdot\lambda}(t).$$

We write

$$F(\lambda,t)=\mathcal{E}_B^\sharp(\lambda,f_\lambda^0,t)_N=L(\lambda)\sum_{w\in W}F_w(\lambda)f^0_{w^{-1}\cdot\lambda}(t),$$

where $L(\lambda)=\prod_{\alpha\in\Phi^+}l_\alpha^+(\lambda)l_\alpha^-(\lambda)$ and

$$F_w(\lambda)=\Big(\prod_{\substack{\alpha\in\Phi^+\\ w^{-1}\cdot\alpha\in\Phi^+}}\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle+1)\Big)\Big(\prod_{\substack{\alpha\in\Phi^+\\ w^{-1}\cdot\alpha\notin\Phi^+}}\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle)\Big).$$

While $L(\lambda)$ is entire, $F_w(\lambda,t)$ can possibly have simple poles along the hyper-planes $H_\alpha^\epsilon=\{\lambda\in\mathfrak{a}_\mathbb{C}^*\mid\langle\lambda,\alpha^\vee\rangle=\epsilon\}$, $\alpha\in\Phi^+,\epsilon=-1,0,1$. To prove that the sum is holomorphic, we shall show the cancellation of all the poles, either by studying the contribution of different terms or by using the zeroes of $L(\lambda)$. Let

$$X=\bigcup_{\substack{\epsilon,\epsilon'\in\{-1,0,1\}\\ \alpha,\alpha'\in\Phi^+,\alpha\neq\alpha'}}(H_\alpha^\epsilon\cap H_{\alpha'}^{\epsilon'}),\qquad Y=\bigcup_{\substack{\epsilon\in\{-1,0,1\}\\ \alpha\in\Phi^+}}H_\alpha^\epsilon.$$

The set X is a closed subset of $\mathfrak{a}_\mathbb{C}^*$ of codimension 2. We first prove that $F(\lambda,t)$ is holomorphic on $\mathfrak{a}_\mathbb{C}^*\smallsetminus X$.

For any $w\in W$ and $\alpha\in\Phi^+$, the poles of $F_w(\lambda)$ along $H_\alpha^{\pm1}\smallsetminus X$ cancel by the zeroes of $L(\lambda)$ along these hyper-planes. Since $F(\lambda,t)=\sum_{w\in W}L(\lambda)F_w(\lambda)f^0_{w^{-1}\cdot\lambda}(t)$, the function $F(\lambda,t)$ is holomorphic on $\bigcup_{\alpha\in\Phi^+}H_\alpha^{\pm1}\smallsetminus X$.

By W-invariance, it is enough to show that $F(\lambda,t)$ is holomorphic along $H_\alpha^0\smallsetminus X$ for any simple root α since any root is W-conjugate to a simple root. Let us fix such α.

The function $F_w(\lambda)$ admits a simple pole along $H_\alpha^0\smallsetminus X$ for any $w\in W$. We shall show that $F_{w_\alpha w}(\lambda)+F_w(\lambda)$ is holomorphic along $H_\alpha^0\smallsetminus X$ for any $w\in W$.

Introduce a partition of the set of positive roots as $\Phi^+(w)\cup\Phi^-(w)$, where

$$\Phi^+(w)=\{\beta:\beta>0,w^{-1}\cdot\beta>0\},\quad \Phi^-(w)=\{\beta:\beta>0,w^{-1}\cdot\beta<0\}.$$

Without loss of generality, assume that $l(w_\alpha w)>l(w)$, where l is the length function on W. Hence, one has

$$\Phi^+(w_\alpha w)=w_\alpha(\Phi^+(w)\smallsetminus\{\alpha\}),\quad \Phi^-(w_\alpha w)=w_\alpha(\Phi^-(w))\cup\{\alpha\}.$$

Claim *For any $\lambda\notin Y$ it holds that $F_{w_\alpha w}(\lambda)=F_w(w_\alpha^{-1}\cdot\lambda)$.*

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Indeed, using the relations above we obtain

$$F_w(w_\alpha^{-1}\cdot\lambda) = \Big(\prod_{\beta\in\Phi^+(w)} \zeta_{F_\beta}(\langle\lambda, w_\alpha\cdot\beta^\vee\rangle+1)\Big)\Big(\prod_{\beta\in\Phi^-(w)} \zeta_{F_\beta}(\langle\lambda, w_\alpha\cdot\beta^\vee\rangle)\Big)$$
$$= \Big(\prod_{\beta\in\Phi^+(w_\alpha w)} \zeta_{F_\beta}(\langle\lambda,\beta^\vee\rangle+1)\Big)\,\zeta_{F_\alpha}(-\langle\lambda,\alpha^\vee\rangle+1)$$
$$\times\Big(\prod_{\beta\in\Phi^-(w_\alpha w)} \zeta_{F_\beta}(\langle\lambda,\beta^\vee\rangle)\Big)(\zeta_{F_\alpha}(\langle\lambda,\alpha^\vee\rangle))^{-1} = F_{w_\alpha w}(\lambda).$$

We use the claim to show the cancellation of the pole along H_α^0. The residue of F_w along the hyperplane H_α^0 is defined by $[\mathrm{Res}_{H_\alpha^0} F_w](\lambda') = \lim_{\lambda\to\lambda'}(\langle\lambda,\alpha^\vee\rangle)F_w(\lambda)$ for $\lambda'\in H_\alpha^0$. For any $\lambda\notin Y$, write $\lambda = s\alpha+\lambda'$, where $\lambda'\in H_\alpha^0$. Thus, $w_\alpha\cdot\lambda = -s\alpha+\lambda'$ and $\langle\lambda,\alpha^\vee\rangle = 2s$. Hence,

$$[\mathrm{Res}_{H_\alpha^0} F_{w_\alpha w}](\lambda') = \lim_{s\to 0} 2s\cdot F_w(-s\alpha+\lambda') = -\lim_{s\to 0} 2s\cdot F_w(s\alpha+\lambda') = -[\mathrm{Res}_{H_\alpha^0} F_w](\lambda')$$

and hence the poles of $F_w(\lambda)f^0_{w^{-1}\cdot\lambda}(t)$ and $F_{(w_\alpha w)}(\lambda)f^0_{(w_\alpha w)^{-1}\cdot\lambda}(t)$ along $H_\alpha^0\smallsetminus X$ cancel each other. Thus, we proved that $\mathcal{E}_B^\sharp(\lambda,t)_N$ is holomorphic on $\mathfrak{a}_\mathbb{C}^*\smallsetminus X$. By Hartogs' theorem, [Hör90, Theorem 2.3.2], $\mathcal{E}_B^\sharp(\lambda,t)_N$ is entire and so is $\mathcal{E}_B^\sharp(\lambda,t)$. ∎

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