

## ARTICLES

# SIMPLE MODEL OF ASYMMETRICAL BUSINESS CYCLES: INTERACTIVE DYNAMICS OF A LARGE NUMBER OF AGENTS WITH DISCRETE CHOICES

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A (jump) Markov process (generalized birth-and-death process) is used to model interactions of a large number of agents subject to field-type externalities. Transition rates are (nonlinear) functions of the composition of the population of agents classified by the choices they make. The model state randomly moves from one equilibrium to another, and exhibits asymmetrical oscillations (business cycles). It is shown that the processes can have several locally stable equilibria, depending on the degree of uncertainty associated with consequences of alternative choices. There is a positive probability of transition between any pair of such basins of attraction, and mean first-passage times between equilibria can be different when different pairs of basins are calculated.

**Keywords:** Business Cycles, Hysteresis, Multiple Equilibria, Uncertain Choices, Mean-Field Effects, Mean First-Passage Times

## 1. INTRODUCTION

In a macroeconomic model with several locally stable equilibria, the set of states of the model is partitioned into several subsets, each of which serves as a basin of attractions.<sup>1</sup> Once the state falls into one particular basin, it converges to the locally stable equilibrium state in the basin until it is disturbed out of the particular basin into another for some reason.

We construct a continuous-time model of a finite but large number of interacting agents in which the number of locally stable equilibria varies with the degree of uncertainty or ignorance when agents make their decisions. Multiple equilibria are produced by uncertainty associated with consequences of alternative choices. Depending on the degree of uncertainty associated with relative advantages of alternative choices, the model can produce one or more than one equilibrium state.

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Perceived relative advantage of alternative choices as the function of the fraction of agents of particular choice is crucial in determining macroeconomic behavior of such models.

We discuss a simple example by assuming that each agent has a binary choice, and that his choice is affected by the aggregate choices of all the agents which are represented by the fraction or proportion of agents having a given choice. We thus describe the aggregate state of the agents by the fraction or percentage of agents making the same decision. Becker (1974) calls these aggregate effects social influences or social demand. Group sentiments, group pressure, or field effects of Aoki and Miyahara (1993) and Aoki (1995a,b; 1996) also refer to this type of externalities in decision-making processes.

When the model possesses several locally stable equilibria, the equilibrium distribution assigns generally unequal probabilities to the basins. Hence, mean first-passage times between equilibria also will be different in general between different pairs of equilibria. With two basins of attraction the model will stay longer on average in one basin of attraction than in the other.<sup>2</sup>

Section 2 develops a generalized birth-and-death process with transition rates that depend on the population composition. We model how transition rates are affected by perceived relative advantages of alternative choices. Section 3 discusses asymmetrical cycles. After a short example in Section 4, the paper concludes with Section 5. In the Appendix, we discuss a simple two-state continuous-time Markov process to highlight the role of a potential barrier in causing asymmetrical dynamic behavior.

## 2. MODEL AND MACRODYNAMICS

### 2.1. Master Equation

This section shows that a large collection of interacting agents, each with binary choices, can produce multiple locally stable stationary states, called equilibria. This model is the same as the one in Aoki (1995b; 1996, Ch. 5) in its basic specification. This paper focuses on the relationship between the degree of uncertainty associated with alternative choices and the number of equilibria.

We fix the number  $N$  of agents, each of whom has two choices at any time.<sup>3</sup>

We classify agents by their choices. There are thus two types of agents. Let  $n(t)$  be the number of agents who have chosen decision (algorithm or alternative) 1 at time  $t$ . Consequently, the remaining  $N - n(t)$  of agents are using the other decision or alternative. The fraction of the population with choice 1 is therefore  $n/N$ . (We drop the time argument when convenient.) The population of  $N$  agents thus is partitioned into two subsets of agents.<sup>4</sup> In general, the number of choices may depend on  $N$ . Again, we do not discuss this more general cases for simpler exposition. A certain limiting case in which both  $N$  and  $K$  go to infinity is discussed in Aoki (1997).

Dynamics of continuous-time Markov processes are specified once we specify transition rates between states. The process is assumed to be time homogeneous,

and it does not execute an infinite number of jumps in an arbitrarily small interval of time. Because  $N$  is fixed, the number  $n$  of type-1 agents can be used as a scalar state variable. Here we allow transitions from  $n$  only to  $n \pm 1$  to keep our model simple.<sup>5</sup>

Given the composition of the population, we can describe only probabilities of a state switching to another state in a small interval of time, namely, we specify transition rates for the population. No statement on specific agents can be made in deterministic terms. In this sense, behavior of the collection of agents is described statistically, or probabilistically. Labels of agents, such as agent 1, 2, are mere convenience and have no intrinsic meanings. Permutations of labels of agents should have no effects on the analysis. Agents therefore are treated as exchangeable in the sense of probability theory. See Galambos (1988) or Kingman (1978). Only patterns of random partitions of  $N$  into two subsets matter. Representative agents are identical. States of exchangeable agents are conditionally independently and identically distributed (i.i.d.). See Rényi (1970, p. 315), Kingman (1978), or Galambos (1988, p. 306).

Imagine a situation in which merit, desirability, or cost of each choice is affected by the fraction of agents with the same choice in the population. This means that probabilities of agents switching their choices depend on the fraction or more generally on population composition by types or categories. We assume that the transition rates of agents between two choices depend (nonlinearly) on the fraction; i.e., we assume that the transition rates are functions of the fraction in addition to some of the more traditional macroeconomic variables such as prices, quantities, and/or interest rates. This is a straightforward way of incorporating into analysis nonprice variables that affects choices, such as congestion, fads, or information contagion.

Our process is therefore a generalized birth-and-death process in which the birth and death rates are some nonlinear functions of the fraction  $x = n/N$ .

We use  $l_n$  for the transition rate from state  $n$  to  $n - 1$ , and  $r_n$  from  $n$  to  $n + 1$ . The symbol  $l$  is for a left move on a line with  $0, 1, 2, \dots, N$  marked on it, and  $r$  is a rightward move. The former means that one agent of type 1 changes his mind and becomes type 2, and the latter is the converse.

Using this notation,  $p_n(t)$ , the probability of  $n$  agents using choice 1 at time  $t$ , is governed by an ordinary deterministic differential equation,

$$\frac{dp_n}{dt} = l_{n+1}p_{n+1}(t) + r_{n-1}p_{n-1}(t) - (r_n + l_n)p_n(t), \quad (1)$$

for  $n = 1, 2, \dots, N - 1$ , and

$$dp_0/dt = -r_0p_0 + l_1p_1$$

and

$$dp_N/dt = -l_Np_N + r_{N-1}p_{N-1}$$

as the boundary conditions. This is called the master equation. It is the differential equation for the probability flux as described by Kelly (1979, Ch. 1). See Aoki

(1996, p. 116) for further references on master equations. Basically, it is a backward Chapman-Kolmogorov equation as in Cox and Miller (1965).

Setting the right-hand side of (1) to zero, we see that two pairs of two terms individually vanish, i.e., the detailed balance condition holds [see Kelly (1979, Ch. 1)], that is,

$$l_{k+1}p_{k+1}^e = r_k p_k^e$$

for all  $k = 0, \dots, N-1$ , where  $p_k^e$  now denotes equilibrium (stationary) probability for state  $k$ . By regarding this as the first-order difference equation for  $p_k^e$ , the stationary or equilibrium probabilities are given by

$$p^e(n) = p^e(0) \prod_{k=1}^n \frac{r_{k-1}}{l_k}. \quad (2)$$

To proceed further we specify the transition rates more explicitly.

## 2.2. Transition Rates

The process of a collection of agents changing their minds asynchronously is modeled probabilistically in a manner similar to that of hazard functions introduced in reliability; see, for example, Cox and Miller (1965).

In the generalized birth-and-death process of this paper, we specify the transition rates to be

$$l_n = f(N)\mu x \eta_2(x)$$

and

$$r_n = f(N)\lambda(1-x)\eta_1(x).$$

The birth rate is  $\lambda$  and the death rate is  $\mu$ . In the classical model usually described in probability textbooks,  $f(N) = N$ , and  $\eta$ 's are one because agents are assumed to act independently, i.e., over a small interval of time, agents independently decide to switch their choices. The scale factor  $f(N)$  does not really matter because it can be absorbed into choice of unit of time. To simplify the equation, we take  $\lambda$  and  $\mu$  to be the same.

Dependence among agents is represented by nonconstant  $\eta$ 's. Because we postulate pervasive and some (intrinsic or extrinsic) sources of uncertainty in the economic or other environment surrounding the decision-making processes, nobody knows for certain if choice 1 is superior to choice 2.<sup>6</sup>

We thus set

$$\eta_1(x) = Z^{-1} e^{\beta g(x)}$$

and

$$\eta_2(x) = 1 - \eta_1(x)$$

with  $Z = e^{\beta g(x)} + e^{-\beta g(x)}$ , and where  $\beta$  is a parameter introduced to reflect degree of uncertainty, as we explain next.

To see heuristically that  $\eta_1(x)$  is the probability that choice 1 is better than choice 2, suppose that relative advantage conditional on the fraction  $x$  is a random variable,  $\xi$ , normally distributed with mean  $g(x)$  and variance  $\sigma^2$ . Thus,

$$Pr(\xi \geq 0) = \frac{1}{2}\{1 + \text{erf}(u)\},$$

with  $u = g(x)/\sqrt{2}\sigma(x)$  where erf is the error function

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-y^2} dy.$$

Using the approximate formula of Ingber (1982), the error function is approximated by  $\tanh(\kappa u)$  where  $\kappa = 2/\sqrt{\pi}$ . This approximation is surprisingly good, especially in the interval [0 1]. For example,  $\text{erf}(x) = \kappa(x - x^3/3 + x^5/10 - \dots)$ , while  $\tanh(\kappa x) = \kappa(x - x^3/2.36 + x^5/4.63 - \dots)$ . The above probability thus is given approximately by

$$(1 + e^{-2\kappa u})^{-1} = \frac{e^{\kappa u}}{e^{\kappa u} + e^{-\kappa u}}.$$

If we define the parameter

$$\beta^{-1} = \sqrt{2\pi}\sigma,$$

then we have the expression

$$Pr(\xi \geq 0) \approx e^{\beta g(x)} / Z = \eta_1(x),$$

where  $Z$  is the normalizing constant. Thus, if normal approximation to the differences in the return or utility is valid,  $\beta$  is inversely proportional to the standard deviation of the conditional mean of the difference. More generally, we may say that the smaller the values of  $\beta$ , the larger is the degree of uncertainty about the consequences of a particular choice. On the other hand, the larger the values of  $\beta$ , the smaller is the degree of uncertainty about consequences or implications of particular choices.

Alternatively, we may think of a logarithmic odds ratio, because

$$\ln \left[ \frac{\eta_1(x)}{\eta_2(x)} \right] = 2\beta g(x).$$

This ties the expression to those in the discrete-choice-model literature as in Anderson et al. (1993), even though this literature does not deal with dynamics.

Next, we express the right-hand side of (2) by introducing the potential,  $U$ , defined by

$$p^e(n) = Z^{-1} \exp[-\beta NU(n/N)],$$

where  $Z$  is the normalizing factor,  $\sum_k \exp[-\beta NU(k/N)]$ , assumed to be finite.

The function  $U$  is called potential because it depends only on the current state variable,  $n(t)$ , i.e.,  $n/N$  because  $N$  is fixed, and is independent of the path from the initial state to the current state. Equating the logarithms, we see that

$$-\beta NU(n/N) = \ln Z + \ln p^e(0) + \sum_{k=1}^n \ln\left(\frac{r_{k-1}}{l_k}\right).$$

Substitute the expressions for the transition rates involving  $\eta$ 's and use the approximation

$$\ln({}_N C_n) = NH(n/N) + O(1/N),$$

where  $H(p) = -p \ln p - (1 - p) \ln(1 - p)$  is the Shannon entropy of the distribution  $(p, 1 - p)$  obtained by approximating factorials by Stirling's formula [see Aoki (1996, pp. 56, 248)] to obtain the expression that ties the potential to  $g(x)$ , which refers to relative advantage of choice 1 over 2, and to the binomial coefficient, which represents the effects of random combinatorial structure embedded in the process. That is, how many ways can  $N$  agents be partitioned into two groups such that the fraction is fixed at  $x$ ?

$$U\left(\frac{n}{N}\right) = \frac{2}{N} \sum_k g\left(\frac{k}{N}\right) - \frac{1}{\beta} H\left(\frac{n}{N}\right) + O\left(\frac{1}{N}\right).$$

When  $N$  is large, we treat  $x$  as continuous and write the above as

$$U(x) = -2 \int^x g(z) dz - \frac{1}{\beta} H(x),$$

noting that  $dz \approx 1/N$ .

### 2.3. Macrodynamics

It has been shown by Aoki (1995a,b; 1996, p. 127) that macrobehavior of this model can be described by the equation for the mean of  $x$ ,<sup>7</sup>

$$d\phi/dt = (1 - \phi)\eta_1(\phi) - \phi\eta_2(\phi) = (1 - \phi)\eta_2\left(\frac{\eta_1}{\eta_2} - \frac{\phi}{1 - \phi}\right). \tag{3}$$

This is the macrodynamic equation for this model. The critical points of the dynamics are the zeros of the right-hand side,

$$\frac{\eta_1(\phi)}{\eta_2(\phi)} = \frac{\phi}{1 - \phi},$$

or substituting the expressions for  $\eta$ 's given above,

$$2\beta g(\phi) = \ln[\phi/(1 - \phi)]. \tag{4}$$

Local equilibrium points are asymptotically stable if

$$2\phi(1 - \phi)\beta g'(\phi) \leq 1. \quad (5)$$

Therefore, if  $g'$  is positive at a local equilibrium, then too large a value of  $\beta$  will cause it to become unstable.

Equations (4) and (5) clearly show the importance of  $\beta$  and  $g$  in determining the number of critical points.

This set of critical points is exactly those points at which the potential is stationary, or the equilibrium probabilities are stationary, because

$$\frac{dU(x)}{dx} = -2g(x) - \frac{1}{\beta} \frac{dH(x)}{dx}. \quad (6)$$

After substituting the expression for the derivative of the entropy term, we note that the zeros of (6) are the same as those of (4).

Equation (6) shows that, for large values of  $\beta$ , the minimum of the potential is nearly the same as the zeros of  $g$ , i.e., at  $x$  where the conditional mean of the differences of alternative choices are zero. However, for smaller values of  $\beta$  (larger  $\sigma$  values under normal-distribution approximation), then the potential may be minimized at points quite different from the zeros of  $g$ , because of entropy effects; i.e., random combinatorial effects represented by the binomial coefficient in the potential will cause the potential to be minimal at  $x$  at which the integral of  $g(x)$  is not minimal.

Next, we describe a situation in which the function  $g$  has two locally stable states and one locally unstable state in  $(0, 1)$ .

### 3. MEAN FIRST-PASSAGE TIMES AND ASYMMETRICAL CYCLES

Suppose that three critical points exist between 0 and 1, arranged as  $a \leq b \leq c$  of which the middle point is locally unstable, where we use  $a$  for short-hand notation for  $\phi_a$  and so on.

The method for calculating first-passage probabilities or mean first-passage time is well known, and is discussed by Cox and Miller (1965, Sect. 3.4), Parzen (1962, Ch. 6), Grimmett and Stirzaker (1992, Sect. 6.2), or van Kampen (1992, Ch. XII) to mention a few textbooks. From now on, we examine the solutions with the initial probability concentrated at state  $m$ .

To calculate the probability distribution of the random time to reach state  $c$  for the first time from the initial position  $m$ , we treat state  $c$  as an absorbing state. Define  $\tau_m$  be the mean first-passage time from state  $m$  to state  $c$ . It is governed by

$$\tau_m - \Delta t = (r_m \Delta t) \tau_{m+1} + (l_m \Delta t) \tau_{m-1} + \{1 - (l_m + r_m) \Delta t\} \tau_m,$$

for any  $m$  in  $\{a + 1, a + 2, \dots, c - 1\}$ , and where  $\Delta t$  is a small positive time interval.

This equation gives rise to the next second-order difference equation,

$$-1 = r_m(\tau_{m+1} - \tau_m) - l_m(\tau_m - \tau_{m-1}).$$

This equation can be conveniently solved by converting it into two first-order difference equations. The first one is defined by

$$\delta_m = \tau_{m+1} - \tau_m,$$

and the second one by

$$\delta_m = \frac{l_m}{r_m} \delta_{m-1} - \frac{1}{r_m}.$$

The two conditions needed to fix the solution of the second-order difference equation are

$$r_a \delta_a = -1,$$

because  $l_a$  is zero by the boundary condition, and

$$\tau_c = 0.$$

It is easy to verify by the mathematical induction on the index that

$$\delta_k = -\frac{1}{r_k p_k^e} (p_a^e + \cdots + p_k^e), \quad (7)$$

for  $k = a, a + 1, \dots$ , after we rewrite the ratio  $l_k/r_{k-1}$  as  $p_{k-1}^e/p_k^e$  by using the detailed balance condition displayed just before equation (2).

Summing the expression for  $\delta_k$ , we obtain the expression for the mean first-passage time from state  $m$  to state  $c$ , which we now denote as  $\tau(m \rightarrow c)$ ,  $m \leq c - 1$ ,

$$\tau(m \rightarrow c) = -\sum_{\mu=m}^{c-1} \delta_\mu.$$

By selecting  $a$  as the initial condition, we arrive at the expression for the mean first-passage time from state  $a$  to state  $c$  as

$$\tau(a \rightarrow c) = \sum_{k=a}^{c-1} \frac{1}{r_k p_k^e} \sum_{\mu=a}^k p_\mu^e. \quad (8)$$

Reasoning analogously in the case in which state  $a$  is treated as absorbing instead of  $c$ , we obtain the expression for  $\tau(c \rightarrow a)$  as well. In this case we solve the difference equation backward from  $c$  with the boundary condition  $\tau_a = 0$  and  $l_c \delta_{c-1} = 1$ . The resulting expression is

$$\tau(c \rightarrow a) = \sum_{k=a}^{c-1} \frac{1}{r_k p_k^e} \sum_{\mu=k+1}^c p_\mu^e.$$



Because these expressions are not transparent, we approximate (8) by recalling the expression for  $p_k^e$  in terms of the potential  $U(k/N)$ , and approximating sums by their maximum terms. See Aoki (1996, Sect. 2.2.3) on this approximation. For example, because the inner sum in the numerator  $\sum p_\mu^e \approx p_a^e$  and the denominator  $\sum \frac{1}{r_k p_k^e} \approx \frac{1}{r_b p_b^e}$ , we have an approximate expression for the mean first-passage time from state  $a$  to state  $c$  as

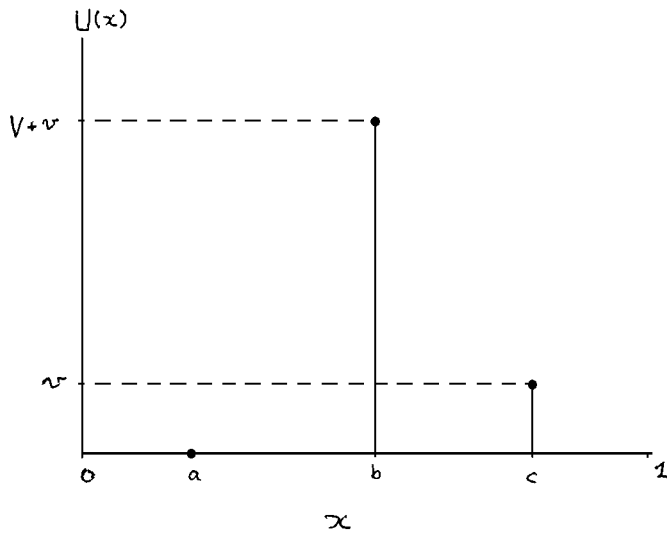
$$\tau(a \rightarrow c) \approx \text{const } e^{\beta N[U(b)-U(a)]}.$$

Note that  $U(b) - U(a)$  is the height of the potential barrier in going from state  $a$  to state  $c$ . Similarly

$$\tau(c \rightarrow a) \approx \text{const } e^{\beta N[U(b)-U(c)]}.$$

In going from state  $c$  to state  $a$ , the barrier has height  $U(b) - U(c)$  which is generally different from the barrier in the other direction.

Figure 1 is a diagram for a potential with two local minima at states  $a$  and  $c$  separated by a barrier at state  $b$ . It takes less time on the average to go from state  $c$  to state  $a$  than the other transition barrier from state  $a$  to state  $c$ , because the latter transition must overcome a higher barrier. See Appendix.



**FIGURE 1.** Schematic diagram of potential with two local equilibria at states  $a$  and  $c$ . The height of barrier is  $V$  in going from state  $c$  to state  $a$ , but is  $V + v$  in the opposite direction.

#### 4. MULTIPLICITY AND UNCERTAINTY

To illustrate the effects of decision uncertainty on the equilibrium distribution, consider an example with

$$g(x) = -89/3 - 72x + 400x^2 - 800x^3/3,$$

where  $0 \leq x \leq 1$ .

With a finite  $N$ ,  $x$  takes on values on a discrete set and the values of  $\beta$  for which the potential has two locally stable minima change somewhat with  $N$ . Numerical experiments show that, even for small values of  $N = 20$ , the potential exhibits unique minimum, or two locally stable minima or an unstable maximum depending on values of  $\beta$ . When  $\beta$  is too large, the condition of stability (5) is violated.

With  $N = 100$ ,  $U(x)$  has two local minima and one local maximum in  $(0, 1)$  for the range of  $\beta$  values approximately between 0.017 and 0.05. For values of  $\beta$  larger than 0.05, the minima are located at  $x = 1$  or at  $x = 0$ . These minima at  $x = 0$  or at  $x = 1$  indicate that all agents choose the same decision.

What is essential to realize is that the part of the potential independent of  $\beta$  does not have two local minima in the range of  $(0, 1)$  and that the entropy term by itself has only one maximum in this range as well. It is the combination of these two monotone functions that produce two locally stable minima for the right range of large uncertainty.

The four panels of Figure 2 plot potential  $U(n)$  as  $n$  ranges from 1 to 99, for  $\beta = 0.01, 0.017, 0.03$ , and  $0.07$ . For a small value of  $\beta$ , there is no clear-cut superior choice, and roughly equal numbers of agents change their minds from choice 1 to 2, and conversely. Hence an equilibrium near  $x = 0.5$  is locally stable. This is panel A. A much larger value of  $\beta$  means that one choice is perceived to be decidedly superior. Consequently, a large number of agents change their minds, responding even to tiny changes making the critical point in  $(0, 1)$  locally unstable, and  $x = 0$  and  $x = 1$  minimal points of the potential. This is shown in panel D. For values of  $\beta$  between these two extremes, just a right fraction of agents change their minds, and a small fraction and a large fraction of the population of agents are locally stably maintained in equilibrium. Panel B shows that two locally minimal equilibria are just developing. Panel C shows two local minima clearly.

In a pioneering paper, Kirman (1993) used a generalized birth-and-death process with transition rates

$$\eta_1(x) = \epsilon + (1 - \delta)x$$

and

$$\eta_2(x) = \epsilon + (1 - \delta)(1 - x)$$

in the notation of this paper. He chose  $\epsilon$  to be  $\alpha/N$  and  $\delta$  as  $2\alpha/N$ .<sup>8</sup> In other words,

$$\beta g(x) = \ln \left[ x + \frac{\alpha}{N}(1 - 2x) \right].$$

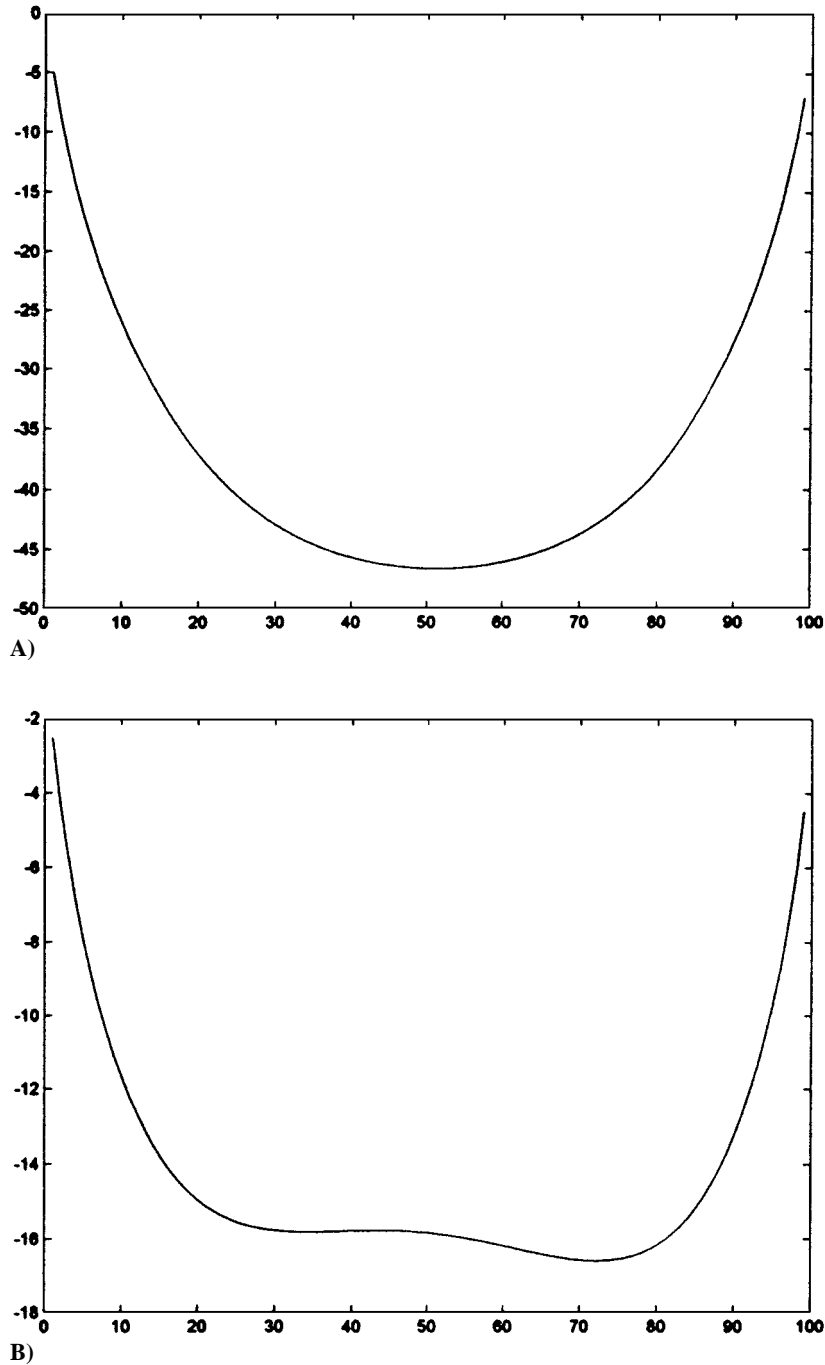
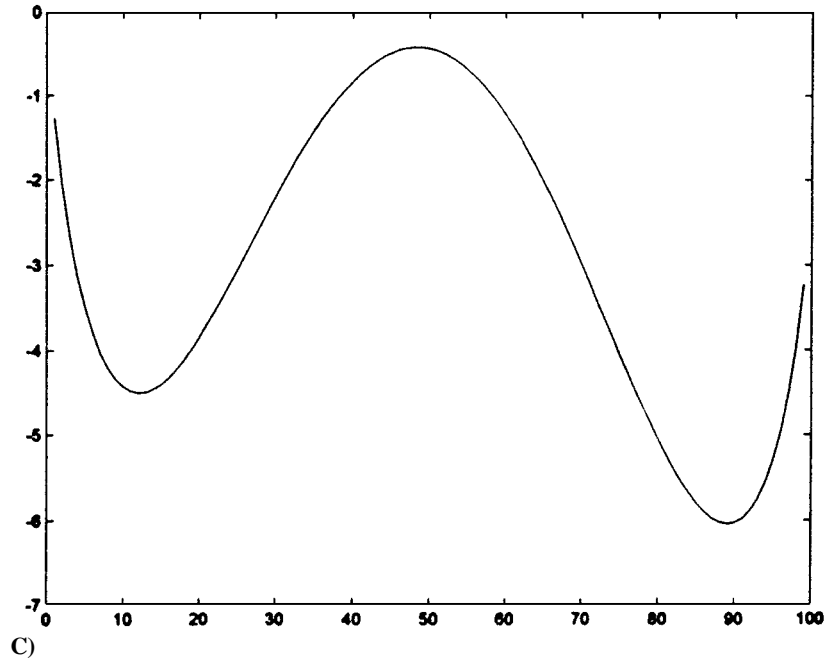
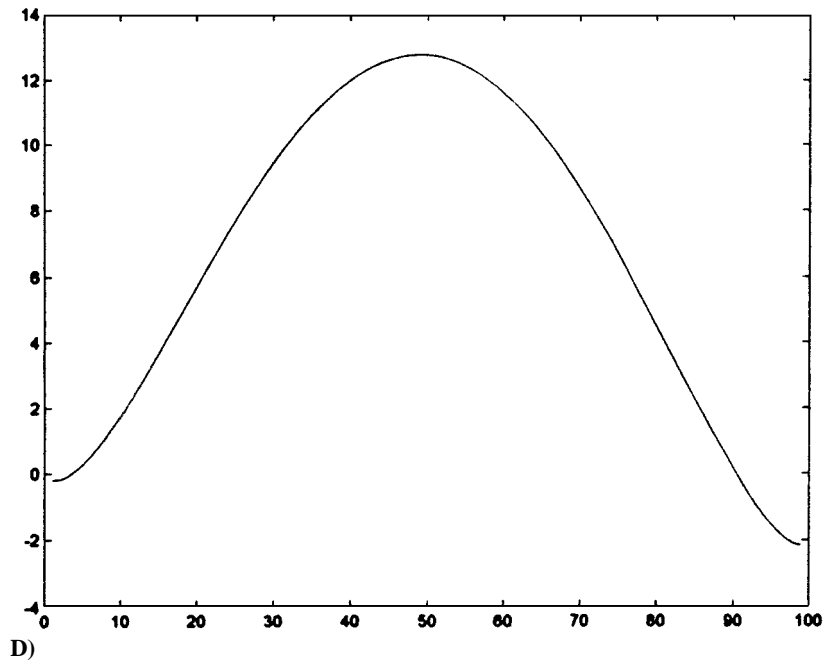


FIGURE 2. Plot of potential  $U(n/N)$  with  $N = 100$ : (A)  $\beta = 0.01$  with one local minimum near  $n = 50$ ; (B)  $\beta = 0.017$  with two local minima slightly developed; (C)  $\beta = 0.03$  with two well-defined local minima; (D)  $\beta = 0.07$  with minima at the end of the interval.



C)



D)

FIGURE 2. (Continued.)

His specification is such that  $\beta$  plays no role and there is only one critical point. His model thus does not exhibit the interplay of multiple equilibria and uncertainty cum degeneracy. In his Figure 1, panels A and C correspond to the equilibrium distribution with stable and unstable equilibrium, respectively, and panel C is a special case in which the equilibrium distribution is constant.

## 5. CONCLUDING DISCUSSION

To explain asymmetrical fluctuations in macroeconomic phenomena we have proposed a model with many agents who are partitioned into several types or categories, and the categorical composition of the population influences transition rates.

We have illustrated that, with uncertain benefits or costs of alternative choices facing agents, there may be zero, one, two, or more locally stable equilibria, depending on the mean of the distribution of the difference of the two alternative benefits, conditional on the composition of the population of the agents expressed as the fraction favoring a particular choice.

The model states are divided into several basins of attraction, each with locally stable attracting stationary state. In general, there is positive probability of transition from any one basin to another.

The role of uncertainty in the model is crucial. If there is too much uncertainty, the model may be stuck in one equilibrium for a long time. In this sense an alternative title of this paper could be “Uncertainty and Multiple Equilibria” or “A Prototype Model for Hysteresis in Macroeconomics” because removal of a shock that moved the model state from one basin to another does not mean that the model will return to the original state. We have shown that the heights of (potential) barriers in moving from one equilibrium to the other and in the opposite direction are generally different, and this is the reason for ratchet effects or hysteresis associated with asymmetrical fluctuations.

This prototype model can explain, for example, the type of persistence in the labor market discussed by Moene et al. (1997) in which firms have two employment policies and the composition of the firms by the policies affects the hiring costs, hence the possibility of two basins of attractions. The explanations of the hysteresis of this paper are thus different from those in the economic literature, for example, Blanchard and Summers (1986).

## NOTES

1. In discrete-time models, random maps of a set of finite points studied by Katz (1955) or Derrida and Flyvbjerg (1987) are of this type.

2. The model of this paper endogenously produces stationary or equilibrium probabilities. It is different from the models of Hamilton (1989) or Neftci (1984) which empirically fit time series associated with business cycles by estimating transition probabilities of associated Markov chains. For one thing, our model is of continuous-time Markov processes. For another, we model underlying agent interaction processes explicitly.

3. This assumption can be removed. We keep it to simplify our presentation.

4. In general, the number of choices may depend on  $N$ . Again, for simpler exposition, we do not discuss this more general case. A certain limiting case in which both  $N$  and  $K$  go to infinity is discussed by Aoki (1997).

5. More general transitions can be handled with no conceptual difficulty. See, for example, Karlin and Taylor (1975, p. 135).

6. Think, for example, of a proposal to undertake a large public work such as a reservoir, airport, or highway. No amount of feasibility, environmental impact, or economic multiplier studies will completely eliminate uncertainty associated with choices.

7. Briefly put, expand (1) in the inverse power of  $\sqrt{N}$  after substituting  $n = N\phi + \sqrt{N}\xi$ , where  $\phi$  is the mean of the random variable  $x = n/N$ . The equation for  $\phi$ , which is (3), separates from the rest.

8. Use of  $N - 1$  rather than  $N$  is immaterial in the limit as  $N$  goes to infinity.

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## APPENDIX

This appendix illustrates the role of potential barriers that are responsible for asymmetrical behavior in a simple model.

Let  $X(t)$  be a scalar-valued state variable of a model which is either  $s_1$  or  $s_2$ , or simply 1 or 2. To shorten the notation, we write  $Pr\{X(t) = 1\}$  as  $p_1(t)$  and, similarly,  $p_2(t)$ .

Because  $p_1(t) + p_2(t) = 1$  for all  $t \geq 0$ , we examine only probability  $p_2(t)$ . The dynamic equation for it can be immediately written as a backward Chapman-Kolmogorov equation,

$$dp_2(t)/dt = w_{1,2}p_1(t) - w_{2,1}p_2(t), \quad (\text{A.1})$$

where  $w_{1,2}$  is the transition rate from state 1 to state 2. It is the derivative of  $Pr[X(t) = 2 \mid X(0) = 1]$  with respect to time  $t$ ; i.e., the probability of moving from state 1 to state 2 in a small positive time interval  $\Delta t$  is equal to  $w_{1,2}\Delta t + o(\Delta t)$ , and similarly for  $w_{2,1}$ . We assume that the process is time homogenous and does not execute an infinite number of jumps in a small time interval.

In this example, we assume that  $w_{1,2} = e^{-\beta(V+v)}$ , and that  $w_{2,1} = e^{-\beta V}$ , where the parameter  $\beta$  is taken to be nonnegative. Even though we treat  $\beta$  simply as a parameter, we think of it as something that reflects the level of economic activity in models, or the level of uncertainty that pervades the model. We may think roughly that the level of economic activity and that of this pervasive uncertainty are inversely related. This form of specification for transition rates here is merely a convenient device. These transition rates reflect or model the fact that, to go from state 1 to 2, there is a barrier of height  $V + v$ , and from state 2 to state 1, the height of the barrier is  $V$ . We assume that  $v$  is (much) smaller than  $V$ . This form of specification for transition rates here is merely a convenient device. The assumed form of the transition rates implies that we measure the potential in the exponent of the Gibbs distribution from that of state 1, and assume that the value of the potential function at state 2 differs only slightly from the first by the amount  $v$ , but that these two stable equilibria are separated by a barrier of height  $V$ , as shown in Figure 1.

The stationary or equilibrium probability that the state variable  $X(t)$  lies in state 1, or in state 2, is obtained simply by setting the right-hand side to zero to define equilibrium probabilities in (A.1). Denote them by  $\pi_1$  and  $\pi_2$ , respectively. Notice that they satisfy the relation

$$\pi_1 w_{1,2} = \pi_2 w_{2,1},$$

which states that the probability flux of jumping from state 1 to state 2 and the reverse balance out in equilibrium. This relationship is an especially simple example of the detailed

balance condition. The detailed balance condition implies that the equilibrium distribution is a Gibbs distribution. See Kelly (1979) or Aoki (1996, Ch. 3) for example. The equilibrium probability is given by

$$\pi_2 = (1 + e^{\beta v})^{-1}.$$

Note that these equilibrium probabilities are independent of the height of the barrier,  $V$ . In equilibrium, the model is more likely to be in state 1 than in state 2, as  $\beta$  becomes larger.

Dropping the subscript 2, we rewrite the differential equation as

$$dp/dt = e^{-\beta(V+v)} - \gamma p = -\gamma(p - \pi_2), \quad (\text{A.2})$$

where  $\gamma = e^{-\beta V} + e^{-\beta(V+v)}$ . This probability monotonically approaches its equilibrium value. Although these equilibrium probabilities are independent of the height of the barrier,  $V$ , the time constant  $1/\gamma$  does depend on  $V$ . Lowering the value of  $\beta$  increases the probability of moving from one state to the other. This may be interpreted as a reflection of a higher level of economic activity or a lower degree of uncertainty in the model.

Now consider varying this parameter  $\beta$  over time so that the path to an equilibrium is the quickest possible. The value of the parameter  $\beta$  may be influenced by a policy maker to reduce economywide uncertainty, for example. Put differently, if the value of  $\beta$  can be manipulated, we ask how we are to hasten convergence to an equilibrium state. Such a  $\beta$  can be obtained by maximizing the rate of change of  $p$ , i.e., the right-hand side of (A.2) with respect to  $\beta$ ,

$$\frac{\partial}{\partial \beta} [e^{-\beta(V+v)} - \gamma p] = 0. \quad (\text{A.3})$$

For a simpler explanation, let us set the initial condition to zero,  $p(0) = 0$ . Then,

$$p(t) = \pi_2(1 - e^{-\gamma t}),$$

and the right-hand side of (A.2) becomes  $\exp\{-\beta(V+v) - \gamma t\}$ . Minimizing the exponent of this expression yields

$$V + v = -\frac{\partial \gamma}{\partial \beta} t$$

as a necessary and sufficient condition for maximizing the right-hand side of (A.2), i.e., (A.3). This expression becomes

$$t = \frac{e^{\beta V}}{e^{-\beta v} + V/(V+v)}$$

or

$$\beta V \approx \ln t$$

by noting that  $\beta$  becomes large as  $t$  increases. This result is interesting because of the similarity with the optimal annealing schedule in the simulated-annealing literature; see Kirkpatrick et al. (1983) and Kabashima and Shinomoto (1991).

One interpretation we make of this example is that under the best of circumstances, the approach to an equilibrium is sluggish, i.e., at most at a rate  $\ln t$ , and not exponential for stochastic dynamics with multiple equilibria.