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SMALL DRIFT LIMIT THEOREMS FOR RANDOM WALKS

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Abstract

We show analogs of the classical arcsine theorem for the occupation time of a random walk in $(-\infty, 0)$ in the case of a small positive drift. To study the asymptotic behavior of the total time spent in $(-\infty, 0)$ we consider parametrized classes of random walks, where the convergence of the parameter to 0 implies the convergence of the drift to 0. We begin with shift families, generated by a centered random walk by adding to each step a shift constant $a > 0$ and then letting a tend to 0. Then we study families of associated distributions. In all cases we arrive at the same limiting distribution, which is the distribution of the time spent below 0 of a standard Brownian motion with drift 1. For shift families this is explained by a functional limit theorem. Using fluctuation-theoretic formulae we derive the generating function of the occupation time in closed form, which provides an alternative approach. We also present a new form of the first arcsine law for the Brownian motion with drift.

Keywords: Random walk; transient; occupation time; arcsine law; small drift; limit distribution

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1. Introduction

For the classical symmetric random walk with ± 1 steps it is well known that the three random variables 'time spent on the positive axis', 'position of the first maximum', and 'last exit from zero' are identically distributed and (suitably normalized) asymptotically arcsine-distributed. Here the norming factor is the length of the time interval the random walk has been observed, so that the limiting statements refer to 'relative' times.

Consider now a classical random walk with drift $\delta \neq 0$. Clearly the same 'relative' variables can be studied. The asymptotic distribution of the random variable (fraction of) time spent in $(-\infty, \alpha]$ has been determined by Takács [20], by applying a functional limit theorem.

But if $\delta \neq 0$ there is also another, 'absolute' perspective. If, for example, $\delta > 0$ for a general random walk, it is clear that $Z(\delta) =$ 'number of visits in $(-\infty, 0)$' is almost surely finite, and that $Z(\delta) \to \infty$ in probability as $\delta \searrow 0$. One may ask if $Z(\delta)$, after multiplication with some deterministic function $a(\delta)$, has a nondegenerate limit distribution. In this paper we aim to answer these and related questions for random walks in the heavy-traffic regime, i.e. when the drift converges to 0. In all cases the limiting distribution for the occupation time in $(-\infty, 0)$,

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properly rescaled, turns out to have the density

$$p(t) = 2\frac{\phi(\sqrt{2t})}{\sqrt{2t}} - 2\Phi(-\sqrt{2t}), \qquad t > 0, \tag{1.1}$$

where ϕ and Φ are the density and the distribution function of $N(0, 1)$, respectively.

The distribution of the occupation time in $(-\infty, 0)$ of Brownian motion with positive drift also has density (1.1), and in Section 2 we begin with related results for Brownian motion. We show, for example, that the distribution of the time of the last exit from 0 of Brownian motion with drift during a finite-time interval is composed of the arcsine and a truncated exponential distribution. In Section 2 we derive the limiting occupation time distribution for shift families generated from a centered random walk by adding to each step a shift constant $a > 0$ and then letting a tend to 0. The proof that (1.1) gives the asymptotic distribution is based on Donsker's invariance principle. In Section 4 we give the key fluctuation-theoretic formulae for the distribution for the occupation time in $(-\infty, 0)$ for general random walks.

The arcsine law and its ramifications are a classical topic but there are always recent contributions, for example, some new explicit distributions [14], new proofs [8], or asymptotic considerations [15]. Interesting results on the number of visits to one point by skipfree random walks and related questions can be found in [4]. The problem considered in this paper is also connected to the heavy traffic approximation problem in queueing theory, in which the growth of the all-time maximum of $S_n - na$ (where S_n is the nth partial sum of independent and identically distributed (i.i.d.) random variables with mean 0) is studied as $a \searrow 0$. In the queueing context this is equivalent to the growth of the steady-state waiting time in a GI/G/1 system when the traffic load tends to 1. This question was first posed by Kingman (see [12]) and was investigated by many authors (see, e.g. [3], [13], [16], [17], [19].)

2. Occupation times and last exit from 0 for Brownian motion with drift

We start by presenting two results on occupation times for Brownian motion with positive drift $\delta > 0$ and variance σ^2, one known and one new. Let B_t be a standard Brownian motion and $X_t = \sigma B_t + \delta t$.

Lemma 2.1. (i) *Let $z > 0$ and $T_z = \inf\{t \geq 0 \colon X_t \geq z\}$ be the first time when X_t reaches level z. Then T_z has the Laplace transform*

$$\ell_{T_z}(s) = \mathbb{E}\exp(-sT_z) = \exp\biggl(-\frac{z}{\sigma^2}(\sqrt{\delta^2 + 2\sigma^2 s} - \delta)\biggr).$$

(ii) *Let $V_0 = V_0(\delta) = \int_0^\infty \mathbf{1}_{(-\infty,0)}(X_t)\,\mathrm{d}t$ be the total time that X_t spends below* 0. *Then V_0 has the Laplace transform*

$$\ell_{V_0}(s) = \mathbb{E}\mathrm{e}^{-sV_0} = \frac{2\delta}{\delta + \sqrt{\delta^2 + 2\sigma^2 s}}.$$

Proof. Proofs for (i) and (ii) (for $\sigma^2 = 1$) can be found in [11] and [9], respectively. Note that $(\delta^2/2\sigma^2)V_0$ has the Laplace transform $2/(1 + \sqrt{1+s})$. We call A a generic random variable with this Laplace transform. □

The density of A is given by (1.1). To see this, note that $1/\sqrt{1+s}$ is the Laplace transform of the gamma distribution $\Gamma_{1,1/2}$, which has density

$$\gamma_{1,1/2}(t) = \mathbf{1}_{(0,\infty)}(t)\frac{\mathrm{e}^{-t}}{\sqrt{\pi t}}.$$

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Therefore, $[1-(1/\sqrt{1+s})]/s$ is the Laplace transform of $1-\Gamma_{1,1/2}(t)=\int_t^\infty \gamma_{1,1/2}(x)\,\mathrm{d}x$. The equality

$$\frac{1}{1+\sqrt{1+s}}=\frac{1}{\sqrt{1+s}}-\frac{1}{s}\left(1-\frac{1}{\sqrt{1+s}}\right)$$

now yields density (1.1).

For $z\ge 0$, let $V_z=\int_0^\infty \mathbf{1}_{(-\infty,z)}(X_t)\,\mathrm{d}t$ the total time the process spends below z. Then the obvious decomposition (obtained by conditioning on T_Z) $V_z=T_z+V_0'$ (where V_0' is independent of T_z) yields the following lemma.

Lemma 2.2. *It holds that V_z has the Laplace transform*

$$\ell_{V_z}(s)=\mathbb{E}(\mathrm{e}^{-sV_z})=\ell_{T_z}(s)\ell_{V_0}(s).$$

The density and distribution function are given in [9].

We focus in the sequel on the time spent on the negative axis, but it is also of interest to look at the other classical arcsine variable, i.e. the time of the last exit from 0. Here we determine its distribution. Let $\delta\in\mathbb{R}\setminus\{0\}, \sigma^2=1$, so that $X_t=B_t+\delta t$, and consider $W=\sup\{t\in[0,1]\colon X_t=0\}$, the last time X_t visits 0 in $[0,1]$.

Recall that for $\delta=0$, i.e. for the standard Brownian motion, the standard arcsine distribution (which has density $\mathbf{1}_{(0,1)}(t)(1/\pi\sqrt{t(1-t)})$ and distribution function $(2/\pi)\arcsin(\sqrt{t})$ on $[0,1]$) is the distribution of the last exit time from 0 in the interval $[0,1]$.

The distribution of W turns out to have a nice representation in terms of the standard arcsine distribution and a truncated exponential distribution. As this result seems new, we provide a proof.

Theorem 2.1. *We have $W\overset{\mathrm{D}}{=}C\cdot\min\{1,D_\delta\}$, where C and D_δ are independent, C is arcsine-distributed, and D_δ is $\exp(\delta^2/2)$-distributed. We denote by '$\overset{\mathrm{D}}{=}$' equality in distribution. The moments of W are given by*

$$\mathbb{E}W^k=\binom{2k}{k}\frac{1}{2^{2k}}\int_0^1 ky^{k-1}\mathrm{e}^{-\delta^2y/2}\,\mathrm{d}y,\qquad k\ge 1.$$

Proof. We use a random walk approximation in the style of Takács [20]. Let $Y_1,Y_2,\dots$ be i.i.d. with

$$\mathbb{P}(Y_i=1)=p=\frac{1}{2}+\frac{\delta}{2\sqrt{n}},\qquad \mathbb{P}(Y_i=-1)=q=1-p$$

(p and q depend on n, but this is suppressed in the notation), and partial sums $S_0=0$, $S_k=\sum_{i=1}^k Y_i$.

It it easy to see that the processes $X^{(n)}$ defined by

$$X^{(n)}(t)=\frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor},\qquad 0\le t\le 1,$$

converge in distribution to $X=(X_t)_{t\in[0,1]}$ in $D[0,1]$.

Furthermore, the last-exit time from 0 is continuous in the Skorokhod topology on $D[0,1]$ on a set of P_X-measure 1, and

$$T_n=\sup\{t\in[0,1]\colon X^{(n)}(t)=0\}=\frac{1}{n}\max\{0\le k\le n\colon S_k=0\}=:\frac{M_n}{n}.$$

Then it suffices to show that $M_N/N\to C\cdot\min\{1,D_\delta\}$ as $N\to\infty$.

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Since $1/\sqrt{1-4pqz^2}$ and $(\sqrt{1-4pqz^2})/(1-z)$ are the generating functions for the sequences of probabilities $\mathbb{P}(S_n = 0)$ and $\mathbb{P}(S_1 \neq 0, \dots, S_n \neq 0)$, respectively, the generating function of M_N is

$$\begin{aligned}
\mathbb{E}t^{M_N} &= \sum_{k=0}^{N} t^k \mathbb{P}(S_k = 0, S_{k+1} \neq 0, \dots, S_N \neq 0) \\
&= \sum_{k=0}^{N} t^k \mathbb{P}(S_k = 0)\mathbb{P}(S_1 \neq 0, \dots, S_{N-k} \neq 0) \\
&= [z^N] \frac{1}{\sqrt{1-4pqt^2z^2}} \frac{\sqrt{1-4pqz^2}}{1-z} \\
&= [z^N] \frac{1}{\sqrt{1-4pqt^2z^2}} \frac{\sqrt{1-4pqz^2}}{1-z^2}(1+z).
\end{aligned}$$

Here and in what follows $[z^N]f(z)$ denotes the coefficient of $[z^N]$ in the Taylor expansion of the function $f(z)$ around 0. Thus, the generating functions for $N = 2n+1$ and $N = 2n$ are identical and it is enough to consider even N. Let $N = 2n$ be even (and $n > \delta^2$) and $U_n = M_N/2$. Then the generating function of U_n is

$$\mathbb{E}t^{U_n} = [z^{2n}] \frac{1}{\sqrt{1-4pqtz^2}} \frac{\sqrt{1-4pqz^2}}{1-z^2} = [z^n] \frac{1}{\sqrt{1-4pqtz}} \frac{\sqrt{1-4pqz}}{1-z}$$

so that the kth factorial moment $u_{k,n} = \mathbb{E}(U_n(U_n-1)\cdots(U_n-k+1))$ of U_n is given by

$$\begin{aligned}
u_{k,n} &= k!\,(-1)^k \binom{-1/2}{k} (4pq)^k [z^{n-k}] \frac{1}{(1-4pqz)^k(1-z)} \\
&= k(-1)^k \binom{-1/2}{k} (4pq)^k [z^{n-k}] \frac{1}{(1-z)} \int_0^\infty x^{k-1} \mathrm{e}^{-(1-4pqz)x}\,\mathrm{d}x \\
&= (-1)^k \binom{-1/2}{k} (4pq)^k \int_0^\infty kx^{k-1} \mathrm{e}^{-x} \bigg(\sum_{j=0}^{n-k} \frac{(4pqx)^j}{j!} \bigg)\,\mathrm{d}x.
\end{aligned}$$

Now denote by $\mathrm{Po}(\lambda)$ a random variable having the Poisson distribution with parameter λ. As $4pq = 1 - (\delta^2/2n)$, we obtain

$$\begin{aligned}
\int_0^\infty kx^{k-1} \mathrm{e}^{-x} \bigg(\sum_{j=0}^{n-k} \frac{(4pqx)^j}{j!} \bigg)\,\mathrm{d}x &= \int_0^\infty kx^{k-1} \mathrm{e}^{-x(1-4pq)} \mathbb{P}(\mathrm{Po}(4pqx) \leq n-k)\,\mathrm{d}x \\
&= n^k \int_0^\infty ky^{k-1} \mathrm{e}^{-\delta^2 y/2} \mathbb{P}\big(\mathrm{Po}\big(\big(n - \tfrac{1}{2}\delta^2\big)y\big) \leq n-k\big)\,\mathrm{d}y.
\end{aligned}$$

By the central limit theorem,

$$\mathbb{P}\big(\mathrm{Po}\big(\big(n - \tfrac{1}{2}\delta^2\big)y\big) \leq n-k\big) \to \begin{cases} 1 & \text{for } 0 \leq y < 1, \\ \frac{1}{2} & \text{for } y = 1, \\ 0 & \text{for } y > 1, \end{cases}$$

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so that, for every k, we have

$$\frac{u_{n,k}}{n^k} \to (-1)^k \binom{-1/2}{k} \int_0^1 k y^{k-1} \mathrm{e}^{-y\delta^2/2}\, \mathrm{d}y.$$

Hence, $\mathbb{E}T_N^k/N^k$ tends to the same limit. Finally,

$$\mathbb{E}C^k = (-1)^k \binom{-1/2}{k} = \binom{2k}{k} \frac{1}{2^{2k}}$$

and integration by parts shows that $\int_0^1 k y^{k-1} \mathrm{e}^{-y\delta^2/2}\, \mathrm{d}y = \mathbb{E} \min\{1, D_\delta^k\}$. Thus, all moments of T_N/N converge to the corresponding moments of $C \cdot \min\{1, D_\delta\}$. Since the distribution of $C \cdot \min\{1, D_\delta\}$ is clearly determined by its moments, both assertions follow. □

Remark 2.1. As an immediate consequence of the scaling properties of Brownian motion, we see that the distribution of

$$W_T = \sup\{t \le T : \sigma B_t + \delta t = 0\}$$

is the same as that of $C \cdot \min\{T, D_{\delta/\sigma}\}$. The time of the last 0 of $\sigma B_t + \delta t$ in the interval $[0, \infty)$ is, thus, distributed as $C \cdot D_{\delta/\sigma}$, which is the gamma distribution with parameters $\delta^2/2\sigma^2$ and $\frac{1}{2}$.

Remark 2.2. Clearly, V_0 (the occupation time on the negative axis) is stochastically smaller than W_∞ (the last exit time from 0), and the results above quantify this precisely. For example, we find that

$$\mathbb{E}(V_0) = \frac{\sigma^2}{2\delta^2} = \frac{1}{2}\mathbb{E}(W_\infty).$$

Remark 2.3. Last-exit times of Brownian motion from moving boundaries have been studied intensively, and more complicated expressions for the density of the last-exit time from a linear boundary were derived in [10] and [18]. The representation in (2.1) appears to be new, as it is not mentioned in the encyclopedic monograph [2]. For the density of the sojourn time, found by Takács, by a random walk limit, two ‘purely Brownian’ explanations were given in [6]. It is natural to ask for such an explanation for the representation in (2.1).

3. Limit of occupation times for shifted random walks

In this section we consider a shifted random walk. Specifically, let $(X_{\delta,1}, X_{\delta,2}, \dots)$ be a parametrized sequence of i.i.d. random variables with $\mathbb{E}(X_{\delta,i}) = 0$ and $\operatorname{var}(X_{\delta,i}) = \sigma^2(\delta) \in (0, \infty)$. Let $\delta > 0$ and

$$Y_i^\delta = X_{\delta,i} + \delta, \qquad S_{\delta,n} = \sum_{i=1}^n X_{\delta,i}, \qquad S_n^\delta = \sum_{i=1}^n Y_i^\delta.$$

We are interested in the occupation time $Z_0^\delta = \sum_{i=1}^\infty \mathbf{1}_{(-\infty,0)}(S_n^\delta)$.Throughout this section we assume that $\sigma^2(\delta) \to \sigma^2 > 0$ as $\delta \to 0$ and that the following Lindeberg-type condition holds: for every $\varepsilon > 0$,

$$\lim_{\delta\to 0} \int_{|\delta X_{\delta,1}| > \varepsilon} X_{\delta,1}^2 \, \mathrm{d}\mathbb{P} = 0. \tag{3.1}$$

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These conditions are chosen such that for the triangular array with the variables

$$Z_{\delta,k} = \frac{\delta}{\sigma(\delta)} X_{\delta,k}, \qquad k = 1, \dots, \left\lfloor \frac{1}{\delta^2} \right\rfloor,$$

the central limit theorem holds: indeed, $\operatorname{var}(Z_{\delta,1}) = \delta^2$ and the Lindeberg condition for this triangular array reads as

$$\begin{aligned} \lim_{\delta\to 0} \frac{1}{\delta^2} \int_{|Z_{\delta,1}|>\varepsilon\delta^2\lfloor 1/\delta^2\rfloor} Z_{\delta,1}^2 \,\mathrm{d}\mathbb{P} &= \lim_{\delta\to 0} \frac{1}{\sigma^2(\delta))} \int_{|\delta X_{\delta,1}|>\varepsilon\sigma(\delta)\delta^2\lfloor 1/\delta^2\rfloor} X_{\delta,1}^2 \,\mathrm{d}\mathbb{P} \\ &= 0 \quad \text{for every } \varepsilon > 0, \end{aligned}$$

which is clearly true under the conditions above.

We use similar ideas as Prohorov [16], who proved the following.

Theorem 3.1. (See Prohorov [16].) *In the situation above, let* $M^\delta = \min\{S_n^\delta : n \ge 0\}$. *Then*

$$\mathbb{P}(\delta M^\delta > x) \to \mathrm{e}^{-2x/\sigma^2} \quad \textit{for all } x > 0.$$

In [16], the maximum in the case of negative drift was considered instead of M^δ. The result had been proved earlier by Kingman [12] under the assumption of the existence of an exponential moment.

The following lemma will be needed to obtain tightness bounds.

Lemma 3.1. *In the situation above, let* $z \ge 0$ *and let* $\delta_k > 0$ *be a sequence of positive numbers satisfying* $\sup_{k\ge 1} \sigma^2(\delta_k) < \infty$. *Then, for every* $\varepsilon > 0$, *we can find a* T *such that, for all* k,

$$\mathbb{P}\Big(\sup_{n\ge T/\delta_k^2} (|S_{\delta_k,n}| - n\delta_k) \ge -\frac{z}{\delta_k} \Big) < \varepsilon.$$

Proof. First consider a sequence S_n of partial sums of an arbitrary i.i.d. sequence (X_i) with $\mathbb{E}(X_1) = 0$ and $\operatorname{var}(X_1) = \sigma^2$. Let $a, b > 0$, $Na > b$ and consider the event $E_N = \{\sup_{n\ge N}(|S_n| - na) \ge -b\}$. Clearly,

$$\begin{aligned} E_N &= \bigcup_{j=0}^{\infty} \Big\{ \max_{2^j N \le n < 2^{j+1} N} (|S_n| - na) \ge -b \Big\} \\ &\subseteq \bigcup_{j=0}^{\infty} \Big\{ \max_{2^j N \le n < 2^{j+1} N} |S_n| \ge 2^j Na - b \Big\} \\ &\subseteq \bigcup_{j=0}^{\infty} \Big\{ \max_{n \le 2^{j+1} N} |S_n| \ge 2^j Na - b \Big\}. \end{aligned}$$

By Kolmogorov's inequality,

$$\mathbb{P}\Big(\max_{n\le 2^{j+1}N} |S_n| \ge 2^j Na - b \Big) \le \frac{2^{j+1} N\sigma^2}{(2^j Na - b)^2}.$$

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Now set $N = T/\delta^2$, $a = \delta$, $b = z/\delta$, and $X_i = X_{\delta,i}$. It follows that

$$\mathbb{P}\Big(\sup_{n\geq T/\delta^2}(|S_{\delta,n}| - n\delta) \geq -\frac{z}{\delta}\Big) \leq \sum_{j=0}^{\infty}\frac{2^{j+1}T\sigma^2(\delta)}{(2^jT - z)^2}.$$

The bound on the right-hand side depends on δ only via $\sigma^2(\delta)$ and can clearly be made arbitrarily small (under the assumptions above). □

Corollary 3.1. *In the situation above, let* $z \geq 0$ *and let* $\delta_k > 0$ *be a sequence of positive numbers satisfying* $\sup_{k\geq 1}\sigma^2(\delta_k) < \infty$. *Then, for every* $\varepsilon > 0$, *one can find a* T *such that, for all* k,

$$\mathbb{P}\Big(\min_{n\geq T/\delta_k^2}\delta_k(S_{\delta_k,n} + n\delta_k) \leq z\Big) < \varepsilon.$$

Theorem 3.2. *We have*

$$\frac{\delta^2}{2\sigma^2(\delta)}Z_0^\delta \to A \quad \textit{in distribution as } \delta \searrow 0.$$

Proof. By the remark following Lemma 2.1 it suffices to show that $\delta^2 Z_0^\delta \to V_0$ in distribution, where V_0 is the distribution of the time the process $X_t = \sigma B_t + t$ spends below 0.

Let $T > 0$ and consider the sequence of processes

$$U^\delta(t) = \delta\sum_{i=1}^{\lfloor t/\delta^2\rfloor} Y_i^\delta, \qquad 0 \leq t \leq T.$$

By Donsker's limit theorem (in the version for triangular arrays, see, e.g. [1, p. 147]), the sequence $U^\delta \to \sigma B + \mathrm{id}$ in distribution in $D[0, T]$, where $\sigma B + \mathrm{id}$ denotes the Brownian motion with variance σ^2 and drift 1, i.e. with coordinate variables $\sigma B_t + t$. For any bounded Borel function v on $[0, T]$, the functional $x \mapsto \int_0^T v(x_t)\,\mathrm{d}t$ on $D[0, T]$ is Skorokhod-measurable and continuous except on a set of B-measure 0 (see, e.g. [1, p. 247]). Thus,

$$\begin{aligned}\delta^2\mathrm{card}\Big(\Big\{n\colon S_n^\delta < 0,\ 1 \leq n \leq \frac{T}{\delta^2}\Big\}\Big) &= \int_0^{\delta^2\lfloor T/\delta^2\rfloor}\mathbf{1}_{(-\infty,0)}(U^\delta(t))\,\mathrm{d}t\\ &\to \int_0^T \mathbf{1}_{(-\infty,0)}(X_t)\,\mathrm{d}t \quad \text{as } \delta \searrow 0\end{aligned}$$

in distribution and we will be done if we can justify the interchange of the limits $T \to \infty$ and $\delta \searrow 0$. Let $\delta_k > 0$ be a sequence decreasing to 0 and let $\varepsilon > 0$. By Corollary 3.1, we can find an N such that $\mathbb{P}(\min_{n\geq N/\delta_k^2} S_n^{\delta_k} \leq 0) < \varepsilon$ for all k. Thus,

$$\lim_{T\to\infty}\sup_{k\geq 1}\mathbb{P}\Big(\min_{n\geq T/\delta_k^2} S_n^{\delta_k} \leq 0\Big) = 0 \tag{3.2}$$

and the assertion follows since, by the monotone convergence theorem,

$$\lim_{T\to\infty}\int_0^T \mathbf{1}_{(-\infty,0)}(X_t)\,\mathrm{d}t = \int_0^\infty \mathbf{1}_{(-\infty,0)}(X_t)\,\mathrm{d}t.$$ □

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Remark 3.1. A related discussion can be found in [19]. In that paper, Shneer and Wachtel derived an extension of Kolmogorov's inequality and treated the maximum of random walks with negative drift and step-size distributions attracted to a stable law of index $\alpha \in (1, 2]$. In the case of finite variance ($\alpha = 2$) they already remarked that their results (including, in particular, the crucial relation (3.2)) remain valid if the conditions assumed above hold.

Remark 3.2. Assume that the X_i are independent with $\mathbb{E}(X_i) = 0$ and variances $\operatorname{var}(X_i) = \sigma_i^2$ and satisfy Lindeberg's condition. Let $s_i^2 = \sum_{k=1}^{i} \sigma_k^2$. Then the step processes $X_n(t)$ which jump to the value S_i/s_n at time s_i^2/s_n^2 converge weakly to a standard Brownian in $D[0, 1]$ (by Prohorov's extension of Donsker's theorem). One may thus expect that they exhibit a similar limiting behavior.

Finally, replacing 0 by z/δ and repeating the steps in the proof of 3.2 yields the following.

Theorem 3.3. *In the situation above, let* $z > 0$ *and* $Z_z^\delta = \sum_{n=1}^{\infty} \mathbf{1}_{(-\infty,z)}(S_n^\delta)$. *Then* $\delta^2 Z_{z/\delta}^\delta \to V_z$ *in distribution, where the Laplace transform of* V_z *is given in Lemma 2.2 with* $\delta = 1$.

If z here depends on δ such that $\delta z(\delta) \to 0$ as $\delta \to 0$, we have the following proposition.

Proposition 3.1. *In the situation above, let* $(z(\delta))$ *be a sequence of positive numbers with* $z(\delta) = \mathrm{o}(1/\delta)$ *and* $\sup_\delta z(\delta) < \infty$. *Then*

$$\delta^2 Z_{z(\delta)}^\delta \to V_0 = 2\sigma^2 A \quad \textit{as } \delta \to 0.$$

Proof. Clearly V_0 is stochastically smaller than any distributional limit of $\delta^2 Z_{z(\delta)}^\delta$ (because Z_0^δ is stochastically smaller than Z_y^δ for $y \geq 0$); furthermore, $V_y = T_y + V_0$ is stochastically smaller than V_z for $y \leq z$. Let $\varepsilon > 0$ and $C = \sup_\delta z(\delta)$, then $C < \infty$ and $\delta^2 Z_{\varepsilon C/\delta}^\delta \to V_{\varepsilon C}$ in distribution as $\delta \to 0$ (by Theorem 3.3). Since $Z_{z(\delta)}^\delta = Z_{\varepsilon z(\delta)/\varepsilon}^\delta$ is stochastically smaller than $Z_{\varepsilon C/\delta}^\delta$ for $\delta \leq \varepsilon$, any distributional limit of $\delta^2 Z_{z(\delta)}^\delta$ is stochastically smaller than $V_{\varepsilon C}$. Thus, the distributional limit exists and is equal to V_0. □

We close this section with an application of Theorem 3.2 in a frequently encountered situation.

Example 3.1. (*Expectation shift in exponential families.*) Let U be a nonconstant real random variable such that the moment generating function

$$m(s) = \mathbb{E}\mathrm{e}^{sU}$$

is finite in an open interval I around 0, and $E(U) = m'(0) = 0$, $\operatorname{var}(Y) = \sigma^2$.

For $p \in I$, let U_p have the 'associated' distribution with moment generating function $m_p(s) = m(p + s)/m(p)$, clearly U_p has expectation $\mathbb{E}(U_p) = m'(p)/m(p)$ and variance $\sigma^2(p) = (m''(p)m(p) - (m'(p))^2)/m(p)^2$.

Let $Z_0(p)$ denote the random variable 'time spent in $(-\infty, 0)$' by the random walk generated by i.i.d. variables with distribution U_p. Then

$$\frac{(\mathbb{E}(U_p))^2}{2\sigma^2(p)} Z_0(p) \to A \quad \text{in distribution for } p \searrow 0.$$

Proof. It is well known that $s \mapsto \log m(s)$ is strictly convex on I, thus, $p \mapsto m'(p)/m(p) = \mathbb{E}(U_p)$ is strictly increasing. Thus, we may parameterize the distributions by $\delta(p) = \mathbb{E}(U_p)$.

We have $\delta(p) \searrow 0$ for $p \searrow 0$ and $\sigma^2(p) \to \sigma^2$ as $p \searrow 0$. Let $X_{\delta(p)} = U_p - \mathbb{E}(U_p)$ and $Y^{\delta(p)} = X_{\delta(p)} + \delta(p) = U_p$. Then the Lindeberg condition (3.1) is satisfied, since by Chebyshev's inequality

$$\int_{|\delta(p)X_{\delta(p)}|>\varepsilon} X^2_{\delta(p)}\, \mathrm{d}\mathbb{P} \le \frac{\delta^2(p)\sigma^2(p)}{\varepsilon^2}$$

and the claim follows from Theorem 3.2. □

4. The fluctuation theoretic approach

The topics investigated here belong to the fluctuation theory of random walks. We recall some basic facts, which will be used in the sequel and can, for example, be found in [7, Section XII.7].

We consider a random walk $(S_n)_{n\ge 1}$, i.e. a sequence of partial sums of i.i.d. random variables and let $R = \inf\{n \ge 1 \colon S_n < 0\}$ and $W = \inf\{n \ge 1 \colon S_n \ge 0\}$ be the lengths of the first strictly descending and weakly ascending ladder epochs of the random walk, respectively. We denote by $r(z)$ and $a(z)$ the corresponding probability generating functions and set $\mu = \mathbb{E}W$. The occupation time of interest is $Z_0 = \sum_{n=1}^{\infty} \mathbf{1}_{(-\infty,0)}(S_n)$.

Theorem 4.1. (Sparre Andersen.) *For* $|z| < 1$,

$$\frac{1}{1-r(z)} = \exp\biggl\{\sum_{n=1}^{\infty} \frac{z^n}{n}\mathbb{P}(S_n < 0)\biggr\}, \qquad \frac{1}{1-a(z)} = \exp\biggl\{\sum_{n=1}^{\infty} \frac{z^n}{n}\mathbb{P}(S_n \ge 0)\biggr\}.$$

An immediate consequence is the factorization theorem.

Theorem 4.2. (Duality.) *For* $|z| < 1$,

$$(1-r(z))(1-a(z)) = 1-z.$$

It follows from the factorization theorem that $W(R)$ has a finite expected value if and only if $R(W)$ is defective, and that the relations $\mathbb{E}(R)\mathbb{P}(W = \infty) = 1$ and $\mathbb{E}(W)\mathbb{P}(R = \infty) = 1$ hold.

At the combinatorial heart of fluctuation theory is the 'Sparre Andersen transformation' (made explicit by Feller and refined by Bizley and Joseph) given in [7, Section XII.8, Lemma 3].

Lemma 4.1. *Let $x_1, \dots, x_n$ be real numbers with exactly $k \ge 0$ negative partial sums $s_{i_1}, \dots, s_{i_k}$, where $i_1 > \dots > i_k$. Write $x_{i_1}, \dots, x_{i_k}$ followed by the remaining x_i in their original order. (If $k = 0$, the sequence remains unchanged). The transformation thus defined is invertible, and the first (absolute) minimum of the partial sums of the new arrangement occurs at the kth place.*

Clearly this extends to *infinite sequences* with exactly k negative partial sums: just apply the bijection above to an initial section large enough to contain all the negative partial sums, and leave the rest unchanged.

In the following formulae we express the generating function of Z_0 in terms of $r(z)$ or $a(z)$, respectively.

Theorem 4.3. *We have*

$$\mathbb{E}z^{Z_0} = \frac{1-r(1)}{1-r(z)} = \frac{1}{\mu}\frac{1-a(z)}{1-z} = \exp\biggl\{-\sum_{k=1}^{\infty}(1-z^k)\frac{\mathbb{P}(S_k < 0)}{k}\biggr\}. \tag{4.1}$$

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Proof. According to Lemma 4.1, for each sequence $x_1, x_2, \dots$ with exactly k negative partial sums there corresponds (by a finite reordering) a unique sequence with a first (absolute) minimum at the kth place. The partial sums $s_0 = 0, s_1, s_2, \dots$ of the rearranged sequence consist of a first part $s_0, s_1, \dots, s_k$ and a second part $s_{k+1}, s_{k+2}, \dots$ such that the partial sums satisfy $s_i > s_k$ for $i \le k$ and $s_i - s_k \ge 0$ for $i > k$. For a random walk the joint distribution of the X_i is invariant under finite permutations, and the two parts are independent. The first part has probability

$$\mathbb{P}(0 > S_k, S_1 > S_k, \dots, S_{k-1} > S_k) = \mathbb{P}(S_1 < 0, \dots, S_k < 0)$$

(by reversing the order of the variables), the second part has probability

$$\mathbb{P}(S_{k+1} - S_k \ge 0, S_{k+2} - S_k \ge 0, \dots) = \mathbb{P}(S_1 \ge 0, S_2 \ge 0, \dots) = 1 - r(1).$$

This yields the first equation of (4.1). The second equation follows immediately from the factorization identity $(1 - a(z))(1 - r(z)) = 1 - z$ (recall Theorem 4.2) and the third equation from Sparre Andersen's theorem. □

In some cases $r(z)$ can be computed in closed form, and the asymptotics of Z_0 can be obtained from an explicit formula. An example is the normal random walk. Let the i.i.d. steps X_i be $N(\delta, \sigma^2)$-distributed. Here we only assume that $\delta \ne 0$, i.e. we consider the cases of positive and negative δ simultaneously and let $d := |\delta|$, $q := \delta^2/2\sigma^2$.

Example 4.1. For the normal random walk, we have the following:

(i) that

$$r(z) = 1 - (1 - z)^{1/2} \exp\biggl(\operatorname{sign}(\delta) \frac{d^2}{\pi \sigma^2} \int_0^1 \int_0^\infty \frac{\mathrm{e}^{-d^2(y^2+x^2)/2\sigma^2}}{1 - z\mathrm{e}^{-d^2(y^2+x^2)/2\sigma^2}} \, \mathrm{d}y \, \mathrm{d}x\biggr);$$

(ii) $qZ_0 \to A$ in distribution as $\delta^2/\sigma^2 \searrow 0$, $\delta \searrow 0$;

(iii) $r(\mathrm{e}^{-qs})^{1/\sqrt{q}} \to \mathrm{e}^{-(\sqrt{1+s}-1)}$ as $q \searrow 0$, $\delta \nearrow 0$.

Note that here σ^2 may vary with δ, it is only essential that $\delta/\sigma \to 0$.

Proof. Directly from Sparre Andersen's theorem, we find that

$$\begin{aligned}
\log\biggl(\frac{1}{1 - r(z)}\biggr) &= \sum_{n=1}^{\infty} \frac{z^n}{n} \mathbb{P}(S_n < 0) \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n} \int_{-\infty}^{-n\delta} \frac{1}{\sqrt{2n\pi\sigma^2}} \mathrm{e}^{-x^2/2n\sigma^2} \, \mathrm{d}x \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n} \biggl(\frac{1}{2} - \operatorname{sign}(\delta) \int_0^{nd} \frac{1}{\sqrt{2n\pi\sigma^2}} \mathrm{e}^{-x^2/2n\sigma^2} \, \mathrm{d}x\biggr).
\end{aligned}$$

Hence,

$$1 - r(z) = (1 - z)^{1/2} \exp(\operatorname{sign}(\delta) G(z)),$$

where

$$G(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^{nd} \frac{1}{\sqrt{2n\pi\sigma^2}} \mathrm{e}^{-x^2/2n\sigma^2} \, \mathrm{d}x.$$

We have

$$\int_0^{nd} \frac{1}{\sqrt{2n\pi\sigma^2}} \mathrm{e}^{-x^2/2n\sigma^2}\,\mathrm{d}x = \int_0^d \sqrt{\frac{n}{2\pi\sigma^2}} \mathrm{e}^{-ny^2/2\sigma^2}\,\mathrm{d}y$$
$$= \frac{n}{\pi\sigma^2}\int_0^d\int_0^\infty \mathrm{e}^{-n(y^2+x^2)/2\sigma^2}\,\mathrm{d}y\,\mathrm{d}x$$
$$= \frac{nd^2}{\pi\sigma^2}\int_0^1\int_0^\infty \mathrm{e}^{-nd^2(y^2+x^2)/2\sigma^2}\,\mathrm{d}y\,\mathrm{d}x$$

and, therefore,

$$G(z) = \frac{d^2}{\pi\sigma^2}\int_0^1\int_0^\infty \frac{\mathrm{e}^{-d^2(y^2+x^2)/2\sigma^2}}{1 - z\mathrm{e}^{-d^2(y^2+x^2)/2\sigma^2}}\,\mathrm{d}y\,\mathrm{d}x,$$

proving (i). Note that $G(z)$ depends only on the ratio $q = d^2/2\sigma^2$. Fix $s > 0$. Setting $z = \mathrm{e}^{-qs}$, we obtain, for $q \searrow 0$ (by dominated convergence),

$$G(\mathrm{e}^{-qs}) = \frac{2}{\pi}\int_0^1\int_0^\infty \frac{q\mathrm{e}^{-q(y^2+x^2)}}{1 - \mathrm{e}^{-q(s+y^2+x^2)}}\,\mathrm{d}y\,\mathrm{d}x$$
$$\to \frac{2}{\pi}\int_0^1\int_0^\infty \frac{1}{s + y^2 + x^2}\,\mathrm{d}y\,\mathrm{d}x$$
$$= \log\biggl(\frac{1+\sqrt{1+s}}{\sqrt{s}}\biggr).$$

From this (ii) and (iii) follow easily. □

It is of methodological interest to have also a purely fluctuation-theoretic proof of Theorem 3.2, i.e. a proof which does not rely on the 'functional limit theorem' approach used above. The anonymous referee suggested the following alternative derivation of Theorem 3.2 based on Theorem 4.3. Assume the conditions introduced in Section 3.

Theorem 4.4. (Equivalent to Theorem 3.3.) *We have*

$$\frac{\delta^2}{2\sigma^2(\delta)} Z_0^\delta \to A \quad \textit{in distribution as } \delta \searrow 0.$$

Proof. In principle, we follow the line of argument used for a similar proof in [19] (after Equation (11) there). Let $\varepsilon > 0$ and split the series in the exponent of the right-hand side of (4.1) into three parts:

$$\sum_{k=1}^{\infty} = \sum_{k=1}^{\varepsilon/\delta^2} + \sum_{\varepsilon/\delta^2}^{T/\delta^2} + \sum_{T/\delta^2}^{\infty} = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 .$$

Let $s > 0$ and set $z = \mathrm{e}^{-s\delta^2/2\sigma^2(\delta)}$. We consider the different sums separately, starting with $\sum_1$:

$$\sum_{k=0}^{\varepsilon/\delta^2} (1 - z^k)\frac{\mathbb{P}(S_k^\delta < 0)}{k} \le \frac{s\delta^2}{2\sigma^2(\delta)} \sum_{k=0}^{\varepsilon/\delta^2} \mathbb{P}(S_k^\delta < 0) \le \frac{s\varepsilon}{2\sigma^2(\delta)}.$$

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Furthermore, $\mathbb{P}(S_k^\delta < 0) = \mathbb{P}(\sum_{j=1}^k X_{\delta,j} < -k\delta) \le \sigma^2(\delta)/(k\delta^2)$ by Chebyshev's inequality. Therefore, we obtain, for $\varepsilon > \delta^2$,

$$\sum_{k\ge\varepsilon/\delta^2} \frac{(1-z^k)}{k}\mathbb{P}(S_k^\delta < 0) \le \frac{\sigma^2(\delta)}{\delta^2}\sum_{k\ge\varepsilon/\delta^2}\frac{1}{k^2} \le \frac{\sigma^2(\delta)}{\delta^2}\int_{\varepsilon/\delta^2}^\infty \frac{1}{(x-1)^2}\,\mathrm{d}x = \frac{\sigma^2(\delta)}{\varepsilon-\delta^2}.$$

Since $\sigma(\delta) \to \sigma^2 \in (0,\infty)$ as $\delta \to 0$, there is a δ_0 such that $2\delta_0^2 < \varepsilon$ and $\sigma^2(\delta)$ is bounded for $\delta \le \delta_0$. Without loss of generality, we assume in the sequel that $\delta \le \delta_0$. Then $\sum_3$ can be made arbitrarily small by a suitable choice of T, and $\sum_2 \le 2C/\varepsilon$ for a suitable constant C.

For $\sum_2$ we use the asymptotic normality of $\delta S^\delta_{t/\delta^2}$ (which is implied by the Lindeberg condition; see the beginning of Section 3), i.e.

$$\mathbb{P}(\delta S_k^d < 0) \to \mathbb{P}(N(t,\sigma^2 t) < 0) = \Phi\biggl(-\sqrt{\frac{t}{\sigma^2}}\biggr) \quad \text{as } \delta\to 0,\ k\delta^2 \to t$$

(uniformly for $t \in [\varepsilon, T]$), and by the dominated convergence, we conclude that

$$\sum\nolimits_2 \to \int_\varepsilon^T \frac{1-\mathrm{e}^{-t/2\sigma^2}}{t}\Phi\biggl(-\sqrt{\frac{t}{\sigma^2}}\biggr)\,\mathrm{d}t.$$

Letting $\varepsilon \to 0$, $T \to \infty$, we finally arrive at

$$\mathbb{E}\exp\biggl(-s\frac{\delta^2}{2\sigma^2}\biggr) \to \exp\biggl\{-\int_0^\infty \frac{1-\exp(-su)}{u}\Phi(-\sqrt{2u})\,\mathrm{d}u\biggr\}. \tag{4.2}$$

Evaluating the integral completes the proof. Avoiding the calculation, it suffices to note that the right-hand side of (4.2) is independent of the underlying distribution of the random walk so that one can look at the example of the normal random walk computed above, which leads to the conclusion that the right-hand side of (4.2) is equal to $2/(1+\sqrt{1+s})$. □

The advantage of this proof is that it is essentially unchanged when generalized to the α-stable case ($1 < \alpha < 2$)—the main difficulties (the corresponding estimates for these cases) can be overcome using Equation (6) in [19].

We close this section with a few remarks on the simple random walk taking step +1 with probability $p > \frac{1}{2}$ and step -1 with probability $q = 1-p$. It is well known that in this example

$$r(z) = \frac{1-\sqrt{1-4pqz^2}}{2pz},$$

so that a quick calculation shows that

$$\mathbb{E}z^{Z_0} = \frac{1-r(1)}{1-r(z)} = \frac{(p-q)(1+\sqrt{1-4pqz^2})}{p(1-2z^2+\sqrt{1-4pqz^2})}$$

and $2(p-\frac{1}{2})^2 Z_0 \to A$ in distribution as $p \searrow \frac{1}{2}$.

Remark 4.1. Let $T_0(p) = \sup\{n \ge 0\colon S_n^{(0)} = 0\}$ the time of the last return to the origin. In the symmetric case $p = \frac{1}{2}$ the walk is persistent and $T_0(\frac{1}{2}) = \infty$ almost surely. In the transient case $p > \frac{1}{2}$, $T_0(p)$ has the generating function

$$h(z) = \frac{p-q}{\sqrt{1-4pqz^2}}.$$

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A short computation yields that $\frac{1}{2}(p-q)^2 T_0(p)$ converges in distribution as $p \searrow \frac{1}{2}$, the limiting distribution having the Laplace transform $1/\sqrt{1+s}$, i.e. being the $\Gamma_{1,1/2}$ distribution with density $\gamma_{1,1/2}(t)$ as above.

Remark 4.2. Let $N_0(p)$ denote the number of 0s of the random walk. Then

$$\mathbb{P}(N_0(p) = r, T_0(p) = 2n) = \frac{r}{n-r}\binom{2n-r}{n} 2(pq)^n$$

and $(\delta N_0(p), \frac{1}{2}\delta^2 T_0(p))$ converges weakly to the distribution with density

$$f(y,t) = \mathbf{1}_{(0,\infty)}(y)\, \mathbf{1}_{(0,\infty)}(t) \frac{y}{2t} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-(y^2/4t)-t}.$$

In particular, $\delta N_0(p)$ is asymptotically $\exp(1)$. For the symmetric random walk, let $N_{0,2n}$ denote the number of 0s up to time $2n$. A classical theorem of Chung–Hunt [5] states that $\sqrt{2/n}N_{0,2n}$ is asymptotically distributed as $|N(0,1)|$. All these results show that deviations from the symmetric random walk become clearly visible after $n \approx \delta^{-2}$ steps. While characteristics such as the positive sojourn time and the last-exit time from 0 are in both cases of approximately the same size their distributions differ. For the last-exit time from 0 a precise description is given in Theorem 2.1.

Apparently the distribution of A occurs naturally as a limit of occupation times for random walks with drift. It is well known (see, e.g. Section XIV.3 in [7]) that the deeper reason for the frequent occurrence of the (generalized) arcsine distributions lies in their intimate connection to distribution functions with regularly varying tails. The same explanation applies here. In the case of zero drift the distribution functions of the ladder epochs are attracted to the standard positive stable distribution of index $\frac{1}{2}$ and the positive (negative) sojourn times are asymptotically arcsine-distributed. In the cases with small drift (and finite variance) the ladder epochs are attracted to an associated distribution of this stable distribution, and, therefore, the positive (negative) sojourn times have asymptotically the distribution of A.

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