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CORRIGENDA

ON A CLASSIFICATION OF THE FUNCTION FIELDS OF ALGEBRAIC TORI

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There are some errors in Theorems 3.3 and 4.2 in [2]. In this note we would like to correct them.

1) In Theorem 3.3 (and [IV]), the condition (1) must be replaced by the following one;

(1) *Π is* (i) *a cyclic group,* (ii) *a dihedral group of order* $2m$, *m odd,* (iii) *a direct product of a cyclic group of order q^f, q an odd prime, $f \geqq 1$, and a dihedral group of order $2m$, m odd, where each prime divisor of m is a primitive $q^{f-1}(q-1)$-th root of unity modulo q^f, or* (iv) *a generalized quaternion group of order $4m$, m odd, where each prime divisor of m is congruent to* 3 *modulo* 4.

Further replace the condition (1′) in p. 96 by the following one:

(1′) *Π is* (i′) *a cyclic group, or* (ii′) *a direct product of a cyclic group of order n, n odd, $n \geqq 1$, and a group with generators ρ, τ and relations $\rho^m = \tau^{2^d} = 1$ and $\tau^{-1}\rho\tau = \rho^{-1}$, m odd, $d \geqq 1$, where each rational prime dividing m is a prime in $Z[\zeta_{n2^d}]$.*

If the unit group $U(Z/n2^dZ)$ is not cyclic, then any rational prime is not prime in $Z[\zeta_{n2^d}]$. This observation shows that (1) is equivalent to (1′).

Now, let Π be a metacyclic group as in (ii′). Denote by σ an element of Π of order nm and put $\mu = \sigma\tau^2$. Let $m'|m$ $(m' > 1)$, $n'|n$ and $0 \leqq d' \leqq d-1$, and put $b = n'm'2^{d'}$. Suppose that m' is not a prime power. Then we see that $Z[\zeta_b] = Z[\mu]/(\Phi_b(\mu))$ is unramified over $Z[\zeta_{n'2^{d'}}, \zeta_{m'} + \zeta_{m'}^{-1}]$. Since

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$Z\Pi/(\Phi_b(\mu))$ is a crossed product of $Z[\zeta_b]$ and a cyclic group of order 2, this shows that $Z\Pi/(\Phi_b(\mu))$ is a maximal, separable $Z[\zeta_{n'2^{d'}}, \zeta_{m'} + \zeta_{m'}^{-1}]$-order in $Q\Pi/(\Phi_b(\mu))$.

Noting this fact, the implication $(1') \Rightarrow (2)$ can be proved along the same line as in [2]. The implication $(2) \Rightarrow (3)$ is evident. Hence we have only to prove the implication $(3) \Rightarrow (1')$.

Assume that Π does not satisfy the condition (1′). Now we will prove that $T(\Pi)$ is not a finite group. By virtue of (1.5) and (2.3), it suffices to show this in the case where every Sylow subgroup of Π is cyclic and $i(\Pi) \leqq 2$. If $T(\Pi)$ is a finite group, then, for any normal subgroup Π' of Π, $T(\Pi/\Pi')$ is a finite group. Therefore we may suppose that

(*) Π is a metacyclic group with generators σ, τ and relations $\sigma^{np} = \tau^{2^d} = 1$, $\tau^{-1}\sigma^n\tau = \sigma^{-n}$ and $\sigma^p\tau = \tau\sigma^p$, where $d \geqq 1$, n is an odd integer and p is an odd prime with $(p, n) = 1$ which is not a prime in $Z[\zeta_{n2^d}]$.

The case $d = 1$. Write $b = np$, and let $\Lambda = Z\Pi/(\Phi_b(\sigma))$. Then Λ is a trivial crossed product of $Z[\zeta_b]$ and $\langle\tau\rangle$. Let $R = Z[\zeta_b] = Z[\sigma]/(\Phi_b(\sigma))$ and $R_0 = Z[\zeta_n, \zeta_p + \zeta_p^{-1}]$, and let $\mathfrak{A} = (\zeta_p - 1) \subsetneqq R$. Both R and $\mathfrak{A}$ can be regarded as Λ-modules, and we have $\Lambda \underset{(t)}{\sim} 0$, $R \underset{(t)}{\sim} 0$, $\mathfrak{A} \underset{(t)}{\sim} 0$ and $\Lambda \cong R \oplus \mathfrak{A}$ as Λ-modules. Since p is not a prime in $Z[\zeta_n]$, we can find an ambiguous prime ideal $\mathfrak{P}$ of R such that $\mathfrak{A} \subsetneqq \mathfrak{P}$. By localizing Λ, R, $\mathfrak{A}$ and $\mathfrak{P}$ at $\mathfrak{P} \cap R_0$, it can be shown that the genus of $\mathfrak{P}$ is different from those of R and $\mathfrak{A}$. We note that, if $T \in S_\Pi$, $\Lambda T \cong R^{(u)} \oplus \mathfrak{A}^{(v)}$ for some u, $v \geqq 0$. Now suppose that $(\mathfrak{P}^*)^{(j)} \underset{(t)}{\sim} 0$ for $j > 0$. Then there is an exact sequence

$$0 \longrightarrow S' \longrightarrow S \longrightarrow \mathfrak{P}^{(j)} \longrightarrow 0$$

of Π-modules with $S', S \in S_\Pi$. Tensoring this with Λ over $Z\Pi$ and eliminating the torsion parts, we get $\mathfrak{P}^{(j)} \oplus \Lambda S' \cong \Lambda S$ and so $\mathfrak{P}^{(j)} \oplus R^{(u)} \oplus \mathfrak{A}^{(v)} \cong R^{(u')} \oplus \mathfrak{A}^{(v')}$ for some $u, v, u', v' \geqq 0$, which is a contradiction. This shows that $(\mathfrak{P}^*)^{(j)} \underset{(t)}{\not\sim} 0$ for any $j > 0$. Thus $T(\Pi)$ is not finite.

The case $d \geqq 2$. We first assume that $n = 1$. As is easily seen, p is not a prime in $Z[i]$ if and only if $p \equiv 1 \bmod 4$, and, for $d \geqq 3$, p is not a prime in $Z[\zeta_{2^d}]$. We now write $\mu = \sigma\tau^2$. Suppose that $p \equiv 1 \bmod 4$, and let $\Lambda = Z\Pi/(\Phi_{2p}(\mu))$. Then $\Lambda = Z[\zeta_p, \tau']$ where $\tau'^2 = -1$ and $\tau'^{-1}\zeta_p\tau' = \zeta_p^{-1}$, and $R_0 = Z[\zeta_p + \zeta_p^{-1}]$ is the center of Λ. Since $\Lambda/(\zeta_p - 1) = F_p[i] = F_p \oplus F_p$ and $R_0/(\zeta_p + \zeta_p^{-1} - 2) = F_p$, Λ is a non-maximal, hereditary R_0-order in $Q\Lambda$. Let $\mathfrak{M}$ be a maximal ideal of Λ containing $\zeta_p - 1$. Then the genus of $\mathfrak{M}$

is different from that of Λ. Note that, for $T \in \boldsymbol{S}_\Pi$, $\Lambda T \cong \Lambda^{(u)}$ for some $u \geqq 0$. Using this fact we see that $(\mathfrak{M}^*)^{(j)} \underset{(\iota)}{\not\longrightarrow} 0$ for any $j > 0$, which shows that $T(\Pi)$ is not finite. Suppose that $p \equiv 3 \bmod 4$ and $d = 3$, and let $\Lambda = \boldsymbol{Z}\Pi/(\Phi_{4p}(\mu))$. Then $\Lambda = \boldsymbol{Z}[\zeta_p, i, \tau']$ where $\tau'^2 = i$ and $\tau'^{-1}\zeta_p\tau' = \zeta_p^{-1}$, and $R_0 = \boldsymbol{Z}[\zeta_p + \zeta_p^{-1}, i]$ is the center of Λ. Since $\Lambda/(\zeta_p - 1) = \boldsymbol{F}_p[\zeta_8] = \boldsymbol{F}_{p^2} \oplus \boldsymbol{F}_{p^2}$ and $R_0/(\zeta_p + \zeta_p^{-1} - 2) = \boldsymbol{F}_p[i] = \boldsymbol{F}_{p^2}$, Λ is a non-maximal, hereditary R_0-order in $\boldsymbol{Q}\Lambda$. Note that, for $T \in \boldsymbol{S}_\Pi$, we have $\Lambda T \cong \Lambda^{(u)}$ for some $u \geqq 0$. Then, in the same way as in the case $p \equiv 1 \bmod 4$, we can show that $T(\Pi)$ is not finite.

Next, we assume that $n > 1$. We only need to consider the case where $p \equiv 3 \bmod 4$ and $d = 2$. If p is not a prime in $\boldsymbol{Z}[\zeta_n]$, then $T(\Pi/\langle\tau^2\rangle)$ is not finite as shown in the case $d = 1$, and so $T(\Pi)$ is not finite. Hence we may assume that p is a prime in $\boldsymbol{Z}[\zeta_n]$. Write $\mu = \sigma\tau^2$ and let $\Lambda = \boldsymbol{Z}\Pi/(\Phi_{2np}(\mu))$. Then $\Lambda = \boldsymbol{Z}[\zeta_n, \zeta_p, \tau']$ where $\tau'^2 = -1$, $\tau'^{-1}\zeta_n\tau' = \zeta_n$ and $\tau'^{-1}\zeta_p\tau' = \zeta_p^{-1}$, and $R_0 = \boldsymbol{Z}[\zeta_n, \zeta_p + \zeta_p^{-1}]$ is the center of Λ. We see that $\Lambda/(\zeta_p - 1)$ $\boldsymbol{F}_p[\zeta_n, i] = \boldsymbol{F}_p[\zeta_n] \oplus \boldsymbol{F}_p[\zeta_n]$ and $R_0/(\zeta_p + \zeta_p^{-1} - 2) = \boldsymbol{F}_p[\zeta_n]$. This shows that Λ is a non-maximal, hereditary R_0-order in $\boldsymbol{Q}\Lambda$. Therefore, along the same line as in the case $n = 1$, it can be shown that $T(\Pi)$ is not finite. This completes the proof of $(3) \Rightarrow (1')$.

The implication $(1') \Leftrightarrow (3)$ can also be proved by Theorem 3.1 in [1]. But Dress' result does not immediately show the implication $(1') \Rightarrow (2)$.

The argument on p. 96 in [2] is incorrect for non-cyclic groups. A detailed and rectified proof of the implication $(1') \Rightarrow (2)$ will be given in a more general form in a forthcoming paper.

2) In Theorem 4.2, the condition (1) must be replaced by the following one:

(1) *Π is one of the following groups*: (i) *a cyclic group of order n where for every $n'|n$ any prime ideal of $\boldsymbol{Z}[\zeta_{n'}]$ containing n is principal.* (ii) *a dihedral group of order $2m$, m odd, where for every $m'|m$ any prime ideal of $\boldsymbol{Z}[\zeta_{m'} + \zeta_{m'}^{-1}]$ containing m is principal.* (iii) *a direct product of a cyclic group of order q^f, q an odd prime, $f \geqq 1$, and a dihedral group of order $2m$, m odd, where any prime divisor of m is a primitive $q^{f-1}(q-1)$-th root of unity modulo q^f, for every $1 \leqq f' \leqq f$ any prime ideal of $\boldsymbol{Z}[\zeta_{q^{f'}}]$ containing 2 is principal, and for every $0 \leqq f' \leqq f$ and every $m'|m$ any prime ideal of $\boldsymbol{Z}[\zeta_{q^{f'}}, \zeta_m + \zeta_{m'}^{-1}]$ containing qm is principal.* (iv) *a generalized quaternion group of order $4m$, m odd, where any prime divisor of m is con-*

gruent to 3 *modulo* 4 *and for every* $m'|m$ *any prime ideal of* $\boldsymbol{Z}[\zeta_{m'} + \zeta_{m'}^{-1}]$ *containing* $2m$ *is generated by a totally positive element.*

It should be noted that, for a finite group Π satisfying the condition (1) in the part 1), the converse of (4.1), (1) is true. Then we can prove Theorem 4.2 in the same way as in [2].

References

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