INEQUALITIES FOR SOME MONOTONE MATRIX FUNCTIONS

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I. Introduction. Let V denote a unitary vector space with inner product (x, y). A self-adjoint linear map $T: V \to V$ is *positive* (*positive definite*) if $(Tx, x) \geqq 0$ $((Tx, x) > 0)$ for all $x \neq 0$ in V. We write $S \geqq T$ $(S > T)$ if S and T are self-adjoint and $S - T \geqq 0$ $(S - T > 0)$. If U is a unitary vector space, a function f: $\mathrm{Hom}(V, V) \to \mathrm{Hom}(U, U)$ is *monotone* if $S \geqq T$ implies that $f(S) \geqq f(T)$. If both U and V are taken to be the n-dimensional unitary space C^n of n-tuples of complex numbers with standard inner product, then f is a *monotone matrix function*, a notion introduced for a more restrictive class of functions by Löwner (**3**) which has important applications in pure and applied mathematics. For orientation we refer the reader to (**1**), where several interesting examples of monotone and related functions are displayed in detail.

Let H denote a permutation group of degree n and let $M: H \to \mathrm{Hom}(U, U)$ denote a representation of H as unitary linear mappings on a unitary vector space U of dimension s. Let $A = [a_{ij}]$ denote an n-square, positive semi-definite hermitian matrix and consider the generalized matrix function M_A: $U \to U$ of Schur (**11**) defined by $M_A \equiv \sum_{\sigma \in H} M(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$. Schur proved that M_A is positive and is, in fact, positive definite when A is positive definite. Moreover, he proved the beautiful inequality

$$\mathrm{tr}(M_A) = \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \geqq s \det A \tag{1}$$

for the trace (tr) of M_A, where χ is the character of M and "det" is the familiar determinant function. A considerably more general result was announced in (**5**), where the positivity of M_A and the inequality (1) are direct consequences of the Cauchy-Schwarz inequality in the tensor product of U with the n-fold tensor product of C^n. These multilinear techniques continue to be a fruitful source of inequalities for the generalized matrix functions $d_\chi^H(A) \equiv \mathrm{tr}(M_A)$.

THEOREM 1. *Regarded as a function from* $\mathrm{Hom}(C^n, C^n)$ *to* $\mathrm{Hom}(U, U)$, M_A *is monotone on the set of positive semi-definite hermitian matrices.*

COROLLARY 1. *If* $A \geqq B \geqq 0$, *then* $d_\chi^H(A) \geqq d_\chi^H(B) \geqq 0$.

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One can employ the preceding result to generate inequalities for the matrix functions $d_\chi^H(A)$. Generally speaking, given a class of positive A we seek a positive B for which all A satisfy either $A \geqq B$ or $B \geqq A$. It is then desirable that $d_\chi^H(B)$ be easy to bound or to compute. The following assertion is an example.

THEOREM 2. *Let $A = [a_{ij}]$ denote an n-square, positive semi-definite hermitian matrix whose row sums $r_1, \ldots, r_n$ satisfy $\sum_{i=1}^n r_i \equiv r \neq 0$. Let H denote a permutation group of degree $m \leqq n$ and order h and let χ be the identity character of degree 1 on H. Let $A_1, \ldots, A_N$ denote the $N \equiv \binom{n}{m}$ principal m-square submatrices of A and let $p_m(|r_1|, \ldots, |r_n|)$ be the mth weighted elementary symmetric function of the moduli of $r_1, \ldots, r_n$. Then*

$$\sum_{i=1}^{N} d_\chi^H(A_i)/N \geqq (h/r^m)p_m^2(|r_1|, \ldots, |r_n|). \tag{2}$$

The inequality (2) is strict unless A has rank 1 or unless A has fewer than m non-zero rows. If A has rank 1, then equality holds in (2) if and only if A has exactly m non-zero rows, $m = n$, or all non-zero row sums of A have equal modulus.

The preceding inequality is proved for the case $m = n$ in (**4**; **7**). A second generalization, in terms of the completely symmetric functions (**2**) of the moduli of the row sums of A, is available as follows.

THEOREM 3. *Let A be as in Theorem 2 and let $B_1, \ldots, B_N$ denote the set of all $N \equiv \binom{n+m-1}{m}$ m-square principal submatrices of A, where repeated occurrences of row and column subscripts are permitted. Then*

$$\sum_{i=1}^{N} d_\chi^H(B_i)/N \geqq (h/r^m)q_m^2(|r_1|, \ldots, |r_n|), \tag{3}$$

where q_m is the mth weighted completely symmetric function and $\chi \equiv 1$. Equality holds in (3) if and only if A has rank 1 and all row sums $r_1, \ldots, r_n$ of A have equal modulus.

Proofs of these theorems as well as detailed statements of further results require a rather comprehensive discussion of multilinear methods. However, once the proper setting is established, the proofs follow readily from well-known properties of inner product spaces.

II. Tensor products of unitary spaces. Let $f: V_1 \times \ldots \times V_r \to P$ denote an r-linear mapping from the cartesian product of vector spaces $V_1, \ldots, V_r$ over a field F to a vector space P over F. If the pair (P, f) satisfies:

(i) the linear closure of the range of f is P, i.e., $\langle \operatorname{rng} f \rangle = P$;

(ii) for any r-linear $\mu: V_1 \times \ldots \times V_r \to Q$, Q any vector space over F, there is a linear $\lambda: P \to Q$ for which $\mu = \lambda f$ (universal factorization property),

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then (P, f) is a *tensor product of vector spaces* $V_1, \ldots, V_r$. A diagram conveniently illustrates the mappings.

$$\begin{tikzcd} V_1 \times \ldots \times V_r \arrow[r, "f"] \arrow[dr, "\mu"'] & P \arrow[d, "\lambda"] \\ & Q \end{tikzcd} \tag{4}$$

It is easy to show that the tensor product of vector spaces is unique up to isomorphism; we adopt the familiar notation $\otimes_{i=1}^{r} V_i = V_1 \otimes \ldots \otimes V_r$ for P and call its elements *tensors*. If an element in $V_1 \otimes \ldots \otimes V_r$ is the image under f of an r-tuple $(v_1, \ldots, v_r)$, we call this element a *decomposable* or *pure* tensor and denote it by $v_1 \otimes \ldots \otimes v_r$. By (i), $V_1 \otimes \ldots \otimes V_r$ contains a basis of pure tensors.

Let the field F be the field C of complex numbers and suppose that each of $V_1, \ldots, V_r$ is a unitary vector space. It is well-known (**8**) that the inner products in $V_1, \ldots, V_r$ together induce an inner product in $\otimes_{i=1}^{r} V_i$ which for decomposable tensors satisfies

$$(u_1 \otimes \ldots \otimes u_r, v_1 \otimes \ldots \otimes v_r) = \prod_{i=1}^{r} (u_i, v_i), \tag{5}$$

where the possibly different inner products are identified only by their operands. In fact, (5) can be taken as the definition of the induced inner product for, since $\langle \operatorname{rng} f \rangle = \otimes_{i=1}^{r} V_i$, the inner product (5) can be extended to all of $\otimes_{i=1}^{r} V_i$ by conjugate-bilinear extension.

Linear mappings on $V_1, \ldots, V_r$ induce linear mappings on $\otimes_{i=1}^{r} V_i$ in a natural way. Let $T_i \colon V_i \to V_i$, $i = 1, \ldots, r$, be linear mappings. In the diagram (4) set $\mu(v_1, \ldots, v_r) = f(T_1 v_1, \ldots, T_r v_r)$, clearly an r-linear mapping. Thus, for $P = Q = \otimes_{i=1}^{r} V_i$, the linear $\lambda \colon \otimes_{i=1}^{r} V_i \to \otimes_{i=1}^{r} V_i$ satisfying $f(T_1 v_1, \ldots, T_r v_r) = \lambda f(v_1, \ldots, v_r)$ is called the *tensor product of mappings* $T_1, \ldots, T_r$, and we write $T_1 \otimes \ldots \otimes T_r$ for λ. Let $S_i \colon V_i \to V_i$ and $T_i \colon V_i \to V_i$, $i = 1, \ldots, r$, be linear. Then the tensor product of mappings satisfies the rule of composition,

$$(S_1 \otimes \ldots \otimes S_r)(T_1 \otimes \ldots \otimes T_r) = (S_1 T_1 \otimes \ldots \otimes S_r T_r). \tag{6}$$

Thus, if each of $T_1, \ldots, T_r$ is invertible, then $T_1 \otimes \ldots \otimes T_r$ is invertible. Moreover, if each of $T_1, \ldots, T_r$ is normal, unitary, self-adjoint, or positive relative to the respective inner products in $V_1, \ldots, V_r$, then the mapping $T_1 \otimes \ldots \otimes T_r$ inherits the corresponding property relative to the induced inner product (5). Let $\dim V_i = n_i$, $i = 1, \ldots, r$. If $\lambda_{i1}, \ldots, \lambda_{in_i}$ are the eigenvalues of T_i, then it is clear that the eigenvalues of $T_1 \otimes \ldots \otimes T_r$ are the $\prod_{i=1}^{r} n_i$ homogeneous products $\lambda_{1\omega_1} \ldots \lambda_{r\omega_r}$, $1 \leqq \omega_i \leqq n_i$, $i = 1, \ldots, r$.

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Symmetry classes of tensors have proven useful in the study of the generalized matrix functions $d_\chi^H(A)$. If H is any permutation group of degree m and order h and χ is any character of degree 1 of H to a field F of characteristic exceeding h, then an m-linear function f: $V \times \ldots \times V \to P$ from the m-fold cartesian product of a vector space V over F to a vector space P over F is *symmetric with respect to H and χ* if for every m-tuple $(v_1, \ldots, v_m)$ and every $\sigma \in H$, we have that

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \chi(\sigma)f(v_1, \ldots, v_m). \tag{7}$$

If the pair (P, f) satisfies (i) and every m-linear μ symmetric with respect to H and χ satisfies (ii) (universal factorization property with respect to H and χ), then (P, f) is a *symmetry class of tensors over V associated with H and χ*. Uniqueness of symmetry classes follows from that property for tensor product, and we write $V_\chi^m(H)$ in this case for P. In case H is the identity group, then $V_\chi^m(H)$ coincides with $\otimes^m V$, the m-fold tensor product of V.

A most convenient method of representing symmetry classes is through the medium of symmetry operators on $\otimes^m V$. In the diagram (4) let $P = Q = \otimes^m V$ and set $\mu(v_1, \ldots, v_m) = f(v_{\phi(1)}, \ldots, v_{\phi(m)})$, where $\phi = \sigma^{-1}$ and $\sigma \in H$. We denote the linear map λ by $P(\sigma)$, a *permutation operator* on $\otimes^m V$. $P(\sigma)$ is a unitary operator relative to the induced inner product (5) and, as is easily shown, satisfies $P(\sigma\tau) = P(\sigma)P(\tau)$. Thus, the set of $P(\sigma)$, $\sigma \in H$, form a unitary representation of H as linear operators on $\otimes^m V$. If $M(\sigma)$ is any other unitary representation of H to a unitary space U, then the mappings $M(\sigma) \otimes P(\sigma)$ are a unitary representation of H in the unitary space $U \otimes (\otimes^m V)$. Let the linear mapping T_M: $U \otimes (\otimes^m V) \to U \otimes (\otimes^m V)$ be defined by

$$T_M = (1/h)\textstyle\sum_{\sigma\in H} M(\sigma) \otimes P(\sigma), \tag{8}$$

where h is the order of H. Now T_M is hermitian relative to the induced inner product for

$$\begin{aligned} T_M^* &= ((1/h)\textstyle\sum_{\sigma\in H} M(\sigma) \otimes P(\sigma))^* = (1/h)\textstyle\sum_{\sigma\in H} M^*(\sigma) \otimes P^*(\sigma) \\ &= (1/h)\textstyle\sum_{\sigma\in H} M(\sigma^{-1}) \otimes P(\sigma^{-1}) = (1/h)\textstyle\sum_{\sigma\in H} M(\sigma) \otimes P(\sigma) = T_M. \end{aligned}$$

If $\tau \in H$, then

$$\begin{aligned} (M(\tau) \otimes P(\tau))(1/h)\textstyle\sum_{\sigma\in H} M(\sigma) \otimes P(\sigma) &= (1/h)\textstyle\sum_{\sigma\in H} M(\tau\sigma) \otimes P(\tau\sigma) \\ &= (1/h)\textstyle\sum_{\sigma\in H} M(\sigma) \otimes P(\sigma). \end{aligned}$$

Hence,

$$\begin{aligned} T_M^2 &= ((1/h)\textstyle\sum_{\tau\in H} M(\tau) \otimes P(\tau))((1/h)\textstyle\sum_{\sigma\in H} M(\sigma) \otimes P(\sigma)) \\ &= (h/h^2)\textstyle\sum_{\sigma\in H} M(\sigma) \otimes P(\sigma) = T_M \end{aligned}$$

so that T_M is idempotent. Thus, T_M is a projection, a positive operator.

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For applications to symmetry classes we restrict ourselves to representations M of degree 1. We are thus led to the *symmetry operator*

$$T_\chi = (1/h)\sum_{\sigma\in H}\chi(\sigma)P(\sigma)$$

from $\otimes^m V$ to itself. It is not difficult to show that rng $T_\chi = V_\chi^m(H)$. For pure tensors $T_\chi(v_1 \otimes \ldots \otimes v_m)$ we write $v_1 * \ldots * v_m$.

It is customary to index the elements of a basis of $\otimes^m V$ using the sequence set $\Gamma_{m,n}$ consisting of all n^m sequences $\omega = (\omega_1, \ldots, \omega_m)$, $1 \leqq \omega_i \leqq n$, $i = 1, \ldots, m$, and to order the basis of $\otimes^m V$ using the lexicographical ordering of the sequence set $\Gamma_{m,n}$. If $e_1, \ldots, e_n$ is a (orthonormal) basis of the n-dimensional (unitary) space V, then $e_\omega^\otimes = e_{\omega_1} \otimes \ldots \otimes e_{\omega_m}$, $\omega \in \Gamma_{m,n}$, is a (orthonormal) basis of $\otimes^m V$ (**8**). We can use the fact that rng $T_\chi = V_\chi^m(H)$ to derive certain combinatorial properties of sequence sets which determine which sequences of $\Gamma_{m,n}$ index elements of a basis for $V_\chi^m(H)$ (**7**). If $\sigma \in S_m$ and $\omega \in \Gamma_{m,n}$ let ω^σ denote the sequence $(\omega_{\sigma(1)}, \ldots, \omega_{\sigma(m)})$. Define a binary relation $\sim$ on $\Gamma_{m,n}$: $\alpha \sim \beta$ if and only if there is a $\sigma \in H$ for which $\alpha^\sigma = \beta$. The relation $\sim$ is in fact an equivalence relation. If we represent each of the resulting classes by its lexicographical minimum sequence we call this system of distinct representatives $\Delta_{m,n}{}^H$. If $\omega \in \Delta_{m,n}{}^H$, denote by H_ω that subgroup of H which fixes ω and call its order $\nu(\omega)$. The character χ of H is a character of H_ω which satisfies $\sum_{\sigma\in H_\omega}\chi(\sigma) = \nu(\omega)$ or 0 according as χ is or is not identically 1 on H_ω. We denote by $\bar{\Delta}_{m,n}{}^H$ the class of all sequences $\omega \in \Delta_{m,n}{}^H$ for which the character χ is identically 1 on H_ω. When the context permits, we denote the above sequence sets by Γ, Δ, and $\bar{\Delta}$. We have the following result (**7**). Let $e_1, \ldots, e_n$ be a (orthonormal) basis of the (unitary) space V of dimension n. Then the (unitary) space $V_\chi^m(H)$ has a (orthonormal) basis $(h/\nu(\omega))^{1/2}e_\omega^* = (h/\nu(\omega))^{1/2}e_{\omega_1} * \ldots * e_{\omega_m}$, $\omega \in \bar{\Delta}$, of decomposable tensors. For any pure tensors in $V_\chi^m(H)$ over the unitary space V, the induced inner product satisfies

$$(9)\qquad (u_\omega^*, v_\gamma^*) = (u_{\omega_1} * \ldots * u_{\omega_m}, v_{\gamma_1} * \ldots * v_{\gamma_m}) = (1/h)d_\chi{}^H([(u_{\omega i}, v_{\gamma_j})]),$$

a result basic to the proofs of many inequalities for generalized matrix functions (**6**).

A useful and important subset of Hom$(V_\chi^m(H), V_\chi^m(H))$ is the class of linear mappings induced by linear mappings in Hom(V, V). In the diagram (4) set $r = m$, $V_i = V$, $i = 1, \ldots, m$, and $P = Q = V_\chi^m(H)$. Assume that f is symmetric with respect to H and χ, and set $\mu(v_1, \ldots, v_m) = f(Tv_1, \ldots, Tv_m)$. We write $K(T)$ for the linear λ: $V_\chi^m(H) \to V_\chi^m(H)$ satisfying

$$Tv_1 * \ldots * Tv_m = \lambda v_1 * \ldots * v_m$$

and call $K(T)$ the *mth induced transformation associated with H and χ*. The composition rule (6) provides the formula $K(ST) = K(S)K(T)$. Indeed, $K(T)$ is just the restriction of the mth Kronecker power $T \otimes \ldots \otimes T \equiv \Pi^m(T)$: $\otimes^m V \to \otimes^m V$ to the symmetry class $V_\chi^m(H) =$ rng T_χ. Accordingly,

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properties of T such as invertible, normal, unitary, self-adjoint, or positive each carry over to $K(T)$. In fact, we have the following result.

THEOREM 4. *Let S: $V \to V$ and T: $V \to V$ be self-adjoint mappings on the unitary space V satisfying $S \geqq T \geqq 0$. Then $K(S)$: $V_\chi^m(H) \to V_\chi^m(H)$ and $K(T)$: $V_\chi^m(H) \to V_\chi^m(H)$ satisfy $K(S) \geqq K(T) \geqq 0$.*

Hence, $K(T)$ considered as a function

$$K: \operatorname{Hom}(V, V) \to \operatorname{Hom}(V_\chi^m(H), V_\chi^m(H)),$$

in addition to being multiplicative, is a monotone function of T.

A companion of the mth induced transformation is its matrix representation. Let $A = [a_{ij}]$ denote the matrix representation of a linear map T from an n-dimensional vector space V to itself relative to the ordered basis $e_1, \ldots, e_n$. Then it is easy to show that the matrix representation of the mth induced map $K(T)$: $V_\chi^m(H) \to V_\chi^m(H)$ relative to the induced basis $(h/\nu(\omega))^{1/2}e_\omega{}^*$, $\omega \in \bar{\Delta}$, ordered lexicographically, is the matrix whose entry in row α and column β is $(1/(\nu(\alpha)\nu(\beta))^{1/2})d_\chi^H(A^{\mathrm{T}}[\beta|\alpha])$, where $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ are in $\bar{\Delta}$, and $A[\alpha|\beta]$ denotes the m-square matrix whose i, j entry is $a_{\alpha_i\beta_j}$, $i, j = 1, \ldots, m$. The superscript T denotes matrix transpose whose presence in the above formula is made necessary by the well-known perversity of indices in matrix notation. We call this matrix $K(A)$ the *mth induced matrix of A associated with H and χ*. Suppose, for example, that A^{T} is the Gram matrix of a set of vectors $v_1, \ldots, v_n$ in a unitary space V. Then $K^{\mathrm{T}}(A)$ is the Gram matrix of the decomposable tensors $(h/\nu(\omega))^{1/2}v_\omega{}^*$, $\omega \in \bar{\Delta}$, in $V_\chi^m(H)$.

The most familiar example of a symmetry class of tensors is the *mth Grassman space* $\wedge^m V$ whose decomposable tensors are usually written $v_1 \wedge \ldots \wedge v_m$. Here $H = S_m$, the full symmetric group of degree m, and $\chi = \epsilon$, the alternating character on S_m. If V is unitary, the induced inner product (9) satisfies $(u_1 \wedge \ldots \wedge u_m, v_1 \wedge \ldots \wedge v_m) = (1/m!)\det[(u_i, v_j)]$. The sequence set Δ is the set $G_{m,n}$ of all $\binom{n+m-1}{m}$ non-decreasing sequences of length m chosen from $1, \ldots, n$ while $\bar{\Delta}$ is the set $Q_{m,n}$ of all $\binom{n}{m}$ strictly increasing sequences in $G_{m,n}$. We note in passing that for any group H and character χ, $G_{m,n}$ and $Q_{m,n}$ are subsets of Δ and $\bar{\Delta}$, respectively. A linear map T induces in $\wedge^m V$ a linear map $C_m(T)$ called the *mth compound of T*. When $\chi \equiv 1$ on S_m, $V_\chi^m(H)$ is called the *mth* symmetric space which we denote by $V^{(m)}$ with decomposable tensors written simply $v_1 \ldots v_m$. For unitary V, the induced inner product (9) satisfies $(u_1 \ldots u_m, v_1 \ldots v_m) = (1/m!)\operatorname{per}[(u_i, v_j)]$, where "per" is the permanent function. For $V^{(m)}$ both Δ and $\bar{\Delta}$ coincide with the sequence set $G_{m,n}$. The mth induced map here is often called the *mth induced power of T* and is usually denoted by $P_m(T)$. The corresponding induced matrices $C_m(A)$ and $P_m(A)$ when A is an incidence matrix of a certain combinatorial configuration were studied by Ryser (**9**; **10**). Much of the current interest in inequalities for matrix functions can be traced to these efforts.

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III. Proofs. Before proving Theorem 1 we need two results the first of which is immediate from definitions.

LEMMA 1. *Let S: $V \to V$ and T: $V \to V$ be self-adjoint linear mappings on a unitary space V satisfying $S \geqq T$. If X: $V \to V$ is any linear mapping, then $X^*SX \geqq X^*TX$.*

We now assert our second preliminary result.

LEMMA 2. *Let $V_1, \ldots, V_r$ be unitary vector spaces and let S_i: $V_i \to V_i$ and T_i: $V_i \to V_i$ denote self-adjoint linear mappings satisfying $S_i \geqq T_i \geqq 0$, $i = 1, \ldots, r$, relative to the respective inner products. Then, relative to the induced inner product,*

$$S_1 \otimes \ldots \otimes S_r \geqq T_1 \otimes \ldots \otimes T_r \geqq 0.$$

Proof. The right-hand side of the identity

$$S_1 \otimes S_2 - T_1 \otimes T_2 \equiv (S_1 - T_1) \otimes S_2 + T_1 \otimes (S_2 - T_2)$$

is clearly positive. The proof now follows by induction on r.

We proceed to the proof of Theorem 1. Let $S \geqq T \geqq 0$ for S: $V \to V$ and T: $V \to V$, where V is an n-dimensional unitary vector space. By Lemma 2, the mth Kronecker powers of S and T satisfy $\Pi^m(S) \geqq \Pi^m(T) \geqq 0$. Set $T_M = (1/h)\sum_{\sigma \in H} M(\sigma) \otimes P(\sigma)$ and let I_U denote the identity map on U. By Lemmas 1 and 2,

$$T_M^*(I_U \otimes \Pi^m(S))T_M \geqq T_M^*(I_U \otimes \Pi^m(T))T_M \geqq 0.$$

However, $T_M = T_M^* = T_M^2$. Hence, for decomposable tensors $x = x_1 \otimes \ldots \otimes x_m$ in $\otimes^m V$ and $u \in U$,

$$(T_M(I_U \otimes \Pi^m(S))u \otimes x, u \otimes x) \geqq (T_M(I_U \otimes \Pi^m(T))u \otimes x, u \otimes x) \geqq 0.$$

Alternatively, with $\phi = \sigma^{-1}$, we have

$$\Big(\sum_{\sigma \in H} M(\sigma)u \otimes Sx_{\phi(1)} \otimes \ldots \otimes Sx_{\phi(m)}, u \otimes x_1 \otimes \ldots \otimes x_m\Big)$$
$$\geqq \Big(\sum_{\sigma \in H} M(\sigma)u \otimes Tx_{\phi(1)} \otimes \ldots \otimes Tx_{\phi(m)}, u \otimes x_1 \otimes \ldots \otimes x_m\Big) \geqq 0.$$

Hence,

$$\Big(\Big(\sum_{\sigma \in H} M(\sigma) \prod_{i=1}^{m} (Sx_i, x_{\sigma(i)})\Big)u, u\Big) \geqq \Big(\Big(\sum_{\sigma \in H} M(\sigma) \prod_{i=1}^{m} (Tx_i, x_{\sigma(i)})\Big)u, u\Big) \geqq 0. \tag{10}$$

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We set $m = n$, $V = C^n$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be positive semi-definite hermitian matrices and set $S = A^{\mathrm{T}}$ and $T = B^{\mathrm{T}}$. Let $x_i = e_i = (\delta_{i1}, \ldots, \delta_{in})^{\mathrm{T}}$, $i = 1, \ldots, n$, the standard basis of C^n. Then $S \geqq T \geqq 0$ implies, by (10), that

$$\left(\left(\sum_{\sigma \in H} M(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}\right)u, u\right) \geqq \left(\left(\sum_{\sigma \in H} M(\sigma) \prod_{i=1}^{n} b_{i\sigma(i)}\right)u, u\right) \geqq 0$$

which is the assertion of Theorem 1.

The preceding argument also proves Theorem 4. Let $s = \dim U = 1$. $K(S)$ is just the restriction of $\Pi^m(S)$ to $\operatorname{rng} T_\chi = V_\chi{}^m(H)$. Thus, $K(S)$ satisfies $T_\chi \Pi^m(S) T_\chi = \Pi^m(S) T_\chi = K(S)$. Before proving Theorem 2 we prove the following lemma.

LEMMA 3. *Let $A = [a_{ij}]$ denote an n-square positive semi-definite hermitian matrix whose row sums r_i satisfy $\sum_{i=1}^n r_i \equiv r \neq 0$. Then the n-square matrix $R = [r_i \bar{r}_j / r]$ of rank 1 is positive semi-definite hermitian and $A \geqq R$.*

Proof. R is a hermitian matrix when r is real and is positive if r is positive. Let e denote the column vector in C^n each of whose entries is 1. Then $r \neq 0$ implies that $r = e^{\mathrm{T}} A e = (Ae, e) > 0$. Now let the positive semi-definite hermitian matrix $A = [a_{ij}] = [(x_i, x_j)]$ represent the Gram matrix of the set of vectors $x_1, \ldots, x_n$ in C^n. Let P denote the n-square matrix whose ith row vector is x_i^{T}, $i = 1, \ldots, n$, and let Q be the n-square matrix each of whose column vectors is the vector $u = x_1 + \ldots + x_n$. Denote by J_n the positive semi-definite hermitian matrix of rank 1 each of whose entries is $1/n$. We observe that $PP^* = A$, that $Q^*Q = rnJ_n = QQ^*$, and that PQ has r_i as its i, j entry. We claim that the n-square identity matrix I_n satisfies $I_n \geqq J_n$. Since $J_n e = e$, a unitary matrix U exists for which $U^* J_n U = E_{11}$, the n-square matrix with 1 in the 1, 1 entry and zeros elsewhere. Clearly, $I_n \geqq E_{11}$. Lemma 1 now proves the claim. Finally, we compute using Lemma 1

$$rnA = rnPI_nP^* \geqq rnPJ_nP^* = PQQ^*P^* = PQ(PQ)^* = n[r_i \bar{r}_j].$$

The inequality (2) of Theorem 2 is now easy. $A \geqq R$ implies that $A[\omega|\omega] \geqq R[\omega|\omega]$ for every choice of $\omega = (\omega_1, \ldots, \omega_m) \in \Gamma_{m,n}$. However, $d_\chi{}^H(R[\omega|\omega]) = (h/r^m)\prod_{i=1}^m |r_{\omega_i}|^2$ when $\chi \equiv 1$. Thus, by Corollary 1,

$$d_\chi{}^H(A[\omega|\omega]) \geqq (h/r^m) \prod_{i=1}^{m} |r_{\omega_i}|^2. \tag{11}$$

Summing over all $\omega \in Q_{m,n}$ yields

$$\sum_{\omega \in Q_{m,n}} d_\chi{}^H(A[\omega|\omega]) \geqq (h/r^m) \sum_{\omega \in Q_{m,n}} \prod_{i=1}^{m} |r_{\omega_i}|^2. \tag{12}$$

The function $f(t) = t^2$ is everywhere strictly convex so that

$$\binom{n}{m} \sum_{\omega \in Q_{m,n}} \prod_{i=1}^{m} |r_{\omega_i}|^2 \geqq \left(\sum_{\omega \in Q_{m,n}} \prod_{i=1}^{m} |r_{\omega_i}| \right)^2 = \left(\binom{n}{m} p_m(|r_1|, \ldots, |r_n|)\right)^2; \tag{13}$$

(12) and (13) together imply (2). In order that equality hold in (12), equality must hold for every $\omega \in Q_{m,n}$ in (11). The vector $u = x_1 + \ldots + x_n$ is

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non-zero since its length is $r^{1/2}$. Hence, the tensor $u * \ldots * u/||u * \ldots * u||$ is non-zero and of unit length (**7**, Lemma 2.4). An easy calculation using (9) yields

$$|(x_\omega^*, u * \ldots * u/||u * \ldots * u||)|^2 = hd_\chi^H(R[\omega|\omega])$$

for every $\omega \in Q_{m,n}$ so that the inequality (11) is just the Cauchy-Schwarz inequality in $V_\chi^m(H)$

$$||x_\omega^*||^2 \geqq |(x_\omega^*, u * \ldots * u/||u * \ldots * u||)|^2. \tag{14}$$

Equality holds in (14) if and only if $x_\omega^* = 0$ or x_ω^* and $u * \ldots * u$ are multiples. Again by (**7**, Lemma 2.4), $x_\omega^* = 0$ if and only if some $x_{\omega_i} = 0$, and x_ω^* is a multiple of $u * \ldots * u$ if and only if each x_{ω_i} is a multiple of u, $i = 1, \ldots, m$. In other words, equality holds in (11) if and only if $A[\omega|\omega]$ has a zero row or $A[\omega|\omega]$ has rank 1. If for each $\omega \in Q_{m,n}$, $A[\omega|\omega]$ has a zero row, then fewer than m of the vectors x_i, hence, fewer than m rows of A, are non-zero. Conversely, if fewer than m rows of A are non-zero, then every principal submatrix $A[\omega|\omega]$ of A has a zero row. Clearly, if each $A[\omega|\omega]$ has rank at most 1, then A has rank at most 1. On the other hand, suppose that A has rank 1. Then $A = [c_i \bar{c}_j]$. An easy calculation shows that both sides of (12) equal $h\sum_{\omega \in Q_{m,n}} \prod_{i=1}^m |c_{\omega_i}|^2$. Finally, equality holds in (13) if and only if all the non-zero products $\prod_{i=1}^m |r_{\omega_i}|^2$, $\omega \in Q_{m,n}$, are equal. This is the case if and only if A has exactly m non-zero rows, $m = n$, or all the non-zero $|r_i|$, $i = 1, \ldots, n$, are equal.

The proof of the inequality (3) of Theorem 3 follows from the proof of (2). However, instead of summing over the $\binom{n}{m}$ sequences $\omega \in Q_{m,n}$ we sum over the $\binom{n+m-1}{m}$ sequences of $G_{m,n}$. The proof in the cases of equality in (3) can be argued in a manner similar to that for (2).

The proof of Corollary 1 for a character χ of degree 1 and, hence, Theorems 2 and 3 can also be obtained from the formula (9) and the mth induced mapping $K(S)$: $V_\chi^m(H) \rightarrow V_\chi^m(H)$. Set $V = C^n$ and $\chi \equiv 1$. Let $S = A^{\mathrm{T}}$ and $T = B^{\mathrm{T}}$. Then $A \geqq B \geqq 0$ implies that $S \geqq T \geqq 0$ whence, by Theorem 4, $K(S) \geqq K(T)$. Let $\omega = (\omega_1, \ldots, \omega_m) \in G_{m,n}$ and let $e_1, \ldots, e_n$ denote the standard basis of C^n. Then using (9), we have that

$$\begin{aligned} d_\chi^H(A[\omega|\omega]) = d_\chi^H([(Se_{\omega_i}, e_{\omega_j})]) &= h(K(S)e_\omega^*, e_\omega^*) \geqq h(K(T)e_\omega^*, e_\omega^*) \\ &= d_\chi^H([(Te_{\omega_i}, e_{\omega_j})]) = d_\chi^H(B[\omega|\omega]). \end{aligned}$$

Set $m = n$ and $\omega = (1, \ldots, n)$ to obtain Corollary 1. Of course, Theorem 1 provides this result immediately.

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