Canad. Math. Bull. Vol. **14** (1), 1971

SOME REMARKS CONCERNING CONTRACTION MAPPINGS

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The following result is proved in [1, p. 6].

THEOREM 1. *Let X be a complete metric space, and let T and T_n $(n=1, 2, \ldots)$ be contraction mappings of X into itself with the same Lipschitz constant $k<1$, and with fixed points u and u_n respectively. Suppose that $\lim_{n\to\infty} T_n(x)=T(x)$ for every $x \in X$. Then $\lim_{n\to\infty} u_n=u$.*

The next result was established by Singh and Russell [4].

THEOREM 2. *Let (X, d) be a complete ε-chainable metric space, and let T_n $(n=1, 2, \ldots)$ be mappings of X into itself, and suppose that there is a real number k with $0 \le k<1$ such that $d(x, y)<\varepsilon \Rightarrow d(T_n(x), T_n(y)) \le kd(x, y)$ for all n. If u_n $(n=1, 2, \ldots)$ are the fixed points for T_n and $\lim_{n\to\infty} T_n(x)=T(x)$ for every $x \in X$, then T has a unique fixed point u and $\lim_{n\to\infty} u_n=u$.*

The aim of this note is to generalize Theorem 1 as well as Theorem 2.

Part 1. We begin with

THEOREM 3. *Let X be a complete metric space with metric d, and let $T: X \to X$ be a function with the following property:*

(1) $$d(T(x), T(y)) \le ad(x, T(x))+bd(y, T(y))+cd(x, y), \quad x, y \in X,$$

where a, b, c are nonnegative and satisfy $a+b+c<1$. Then T has a unique fixed point.

Note that $a=b=0$ yields Banach's fixed point theorem, while $a=b$, $c=0$ yields Kannan's fixed point theorem, mentioned in [5, p. 406]. Of course, we may assume always that $a=b$, but this is not essential.

Proof. Take any point $x \in X$ and consider the sequence $\{T^n(x)\}$. Putting $x=T^n(x)$, $y=T^{n-1}(x)$ in (1) we obtain for $n \ge 1$,

$$d(T^{n+1}(x), T^n(x)) \le ad(T^n(x), T^{n+1}(x))+bd(T^{n-1}(x), T^n(x)) + cd(T^n(x), T^{n-1}(x)).$$

Hence

$$d(T^{n+1}(x), T^n(x)) \le pd(T^n(x), T^{n-1}(x)),$$

where $p=(b+c)/(1-a)$. Note that $p<1$. It follows that $d(T^{n+1}(x), T^n(x)) \le p^n d(x, T(x))$, and that for any $m>n$, $d(T^m(x), T^n(x)) \le p^n d(x, T(x))/(1-p)$. Thus $\{T^n(x)\}$

is a Cauchy sequence and therefore $T^n(x) \to z$. Now we will show that $T(z)=z$. It is sufficient to prove that $T^{n+1}(x) \to T(z)$.

Indeed we have, taking $x=T^n(x)$, $y=z$ in (1),

$$\begin{aligned} d(T^{n+1}(x), T(z)) &\le ad(T^{n+1}(x), T^n(x))+bd(T(z), z)+cd(T^n(x), z) \\ &\le ad(T^{n+1}(x), T^n(x))+bd(T^{n+1}(x), T(z))+bd(T^{n+1}(x), z)+cd(T^n(x), z) \\ &\le ap^n d(T(x), x)+bd(T^{n+1}(x), T(z))+bd(T^{n+1}(x), z)+cd(T^n(x), z). \end{aligned}$$

Hence

$$d(T^{n+1}(x), T(z)) \le (ap^n d(T(x), x)+bd(T^{n+1}(x), z)+cd(T^n(x), z))/(1-b) \to 0.$$

Finally we prove that there is only one fixed point. Let x, y be two fixed points. Then

$$d(x, y) = d(T(x), T(y)) \le ad(x, x)+bd(y, y)+cd(x, y) = cd(x, y).$$

Were $d(x, y)$ nonzero, we would have $1 \le c$, a contradiction.

To see that this theorem is stronger than Banach's and Kannan's theorems, consider the following example: $X=[0,1]$, $T(x)=x/3$ for $0 \le x<1$ and $T(1)=\frac{1}{6}$. T does not satisfy Banach's condition because it is not continuous at 1. Kannan's condition also cannot be satisfied because $d(T(0), T(\frac{1}{3}))=\frac{1}{2}(d(0, T(0))+d(\frac{1}{3},T(\frac{1}{3})))$. But it satisfies condition (1) if we put $a=\frac{1}{6}$, $b=\frac{1}{9}$, $c=\frac{1}{3}$ (these are not the smallest possible values).

Using this result we obtain

THEOREM 4. *Let X be a complete metric space, and let T_n ($n=1, 2,\ldots$) be mappings of X into itself satisfying* (1) *with the same constants a, b, c, and with fixed points u_n. Suppose that a mapping T of X into itself can be defined by $T(x)=\lim_{n\to\infty} T_n(x)$. Then $u=\lim_{n\to\infty} u_n$ is the unique fixed point of T.*

Proof. Since d is a continuous function of both its variables we immediately see that T satisfies (1) and therefore has a unique fixed point u. Now

$$\begin{aligned} d(u_n, u) = d(T_n(u_n), T(u)) &\le d(T_n(u_n), T_n(u))+d(T_n(u), T(u)) \\ &\le ad(u_n, T_n(u_n))+bd(u, T_n(u))+cd(u_n, u)+d(T_n(u), T(u)). \end{aligned}$$

Hence

$$d(u_n, u) \le [(b+1)d(T_n(u), T(u))]/(1-c).$$

The result follows.

Part 2. Suppose the nonnegative function $k(x, y)$ satisfies the following conditions:

(a) $k(x, y)=k(d(x, y))$
(b) $k(d)<1$ for any $d>0$
(c) $k(d)$ is a monotonically decreasing function of d.

https://doi.org/10.4153/CMB-1971-024-9 Published online by Cambridge University Press

Rakotch proved the following result [2, p. 463].

THEOREM 5. *If $T: X \to X$ where X is a complete metric space, satisfies*

$$d(T(x), T(y)) \leq k(x, y)d(x, y), \quad x, y \in X, \quad x \neq y \tag{2}$$

then T has a unique fixed point.

Using this we obtain

THEOREM 6. *Let X be a complete metric space and let $T_n (n=1, 2, \ldots)$ be contraction mappings of X into itself with the same $k(x, y)$, and with fixed points u_n. If T can be defined by $T(x) = \lim_{n\to\infty} T_n(x)$, $x \in X$, then T has a unique fixed point u and $\lim_{n\to\infty} u_n = u$.*

Proof. By the continuity of d, T satisfies (2), and therefore has a unique fixed point u. Now

$$\begin{aligned} d(u_n, u) = d(T_n(u_n), T(u)) &\leq d(T_n(u_n), T_n(u)) + d(T_n(u), T(u)) \\ &\leq k(u_n, u)d(u_n, u) + d(T_n(u), T(u)). \end{aligned}$$

Hence

$$d(u_n, u) \leq [d(T_n(u), T(u))]/(1-k(u_n, u)).$$

Let $\varepsilon > 0$ be given, and denote $k(\varepsilon)$ by p. We can find an N such that for $n > N$, $d(T_n(u), T(u)) < (1-p)\varepsilon$. Take any $n > N$. We intend to show that $d(u_n, u) < \varepsilon$. If $d(u_n, u) < \varepsilon$, there is nothing to prove. If $d(u_n, u) \geq \varepsilon$ then $k(u_n, u) \leq p$. Therefore

$$d(u_n, u) \leq [d(T_n(u), T(u))]/(1-p) < \varepsilon.$$

The proof is complete.

We state now another result due to Rakotch [3].

THEOREM 7. *Let T be a mapping of a complete ε-chainable metric space into itself, and suppose there is a function $k(x, y)$, satisfying* (a), (b), (c), *such that $d(x, y) < \varepsilon \Rightarrow d(T(x), T(y)) \leq k(x, y)d(x, y)$, $x \neq y$, where d is the metric of the space. Then T has a unique fixed point.*

This theorem enables us to present a generalization of Theorem 2.

THEOREM 8. *Let (X, d) be a complete ε-chainable metric space, and let T_n $(n=1, 2, \ldots)$ be mappings of X into itself, and suppose that there is a nonnegative function $k(x, y)$ which satisfies* (a), (b), (c) *such that $d(x, y) < \varepsilon \Rightarrow d(T_n(x), T_n(y)) \leq k(x, y)d(x, y)$, $x \neq y$, for all n. If u_n are the fixed points of T_n and $\lim_{n\to\infty} T_n(x) = T(x)$ for every $x \in X$, then T has a unique fixed point u and $\lim_{n\to\infty} u_n = u$.*

Proof. We define a new metric for our space by $d_\varepsilon(x, y) = \inf \sum_{i=1}^{p} d(x_{i-1}, x_i)$, where the infimum is taken over all ε-chains $x_0, x_1, \ldots, x_p$ joining $x_0 = x$ and $x_p = y$. (X, d_ε) is a complete metric space. By a part of the proof of the previous theorem [3, p. 56] we have $d_\varepsilon T_n(x), T_n((y)) \leq k'(x, y)d_\varepsilon(x, y)$, $x \neq y$, where k' satisfies (a), (b), (c). Since $d(x, y)=d_\varepsilon(x, y)$ when $d(x, y)<\varepsilon$, T_n converges to T with respect to d_ε too. By Theorem 6, T has a unique fixed point u, and $\lim_{n\to\infty} d_\varepsilon(u_n, u)=0$. Since $d(x, y)\leq d_\varepsilon(u_n, u)$, we obtain the desired result.

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