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Universal Entire Functions That Define Order Isomorphisms of Countable Real Sets

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Abstract. In 1895, Cantor showed that between every two countable dense real sets, there is an order isomorphism. In fact, there is always such an order isomorphism that is the restriction of a universal entire function.

1 Introduction

In 1895, Cantor proved that every two countable dense sets of reals are order isomorphic. The same year, Stäckel [17] showed that if A is a countable and B a dense subset of $\mathbb{C}$, then there exists a non-constant entire function that maps A *into* B. He also claimed the corresponding result if A is a countable real set and B is a dense real set.

The following striking result was published by Franklin in 1925.

Theorem 1.1 ([11]) *Let A and B be countable dense subsets of $\mathbb{R}$. Then, there exists an analytic function $f : \mathbb{R} \to \mathbb{R}$ that restricts to an order isomorphism of A onto B.*

Unfortunately, the proof invoked the statement that the uniform limit of analytic functions is analytic, which is false, as one can see, for example, from the Weierstrass approximation theorem. Fortunately, Franklin's theorem follows from a more general result of Burke [7].

In this paper, we present the following two extensions of Franklin's Theorem.

Theorem 1.2 *Let A and B be countable dense subsets of $\mathbb{R}$ and $0 < m \le M < +\infty$. Let Φ be an entire function such that, for $x \in \mathbb{R}$, $\Phi(x) \in \mathbb{R}$ and $m \le \Phi'(x) \le M$. Then there is an entire function f of the form*

$$f(z) = \Phi(z) + \sum_{j=1}^{\infty} \lambda_j H_j(z),$$

$$H_1 = 1, \quad H_j(z) = e^{-\Phi^2(z)} \prod_{k=1}^{j-1} \big(\Phi(z) - \Phi(\alpha_k)\big) \text{ for } j = 2, 3, \dots,$$

such that $f(\mathbb{R}) = \mathbb{R}$, f restricts to an order isomorphism of A onto B, and $f'(x) > 0$, for $x \in \mathbb{R}$, so the mapping $f : \mathbb{R} \to \mathbb{R}$ is bianalytic. An example of such a function Φ is the function $\Phi(z) = z$ and, for this example, it is possible to choose the λ_j's such that the function f is of finite order of growth.

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We present this improvement of Franklin's theorem, first, because of the error in Franklin's proof, but mainly because the functions we obtain can be modified to prove the following.

Theorem 1.3 *Let A and B be countable dense subsets of* $\mathbb{R}$. *Then there exists a universal entire function F of the form*

$$F(z) = z + \sum_{j=1}^{\infty} \Lambda_j(z) H_j(z),$$

$$H_1 = 1, \quad H_j(z) = e^{-z^2} \prod_{k=1}^{j-1} (z - \alpha_k) \text{ for } j = 2, 3, \ldots,$$

such that $F(\mathbb{R}) = \mathbb{R}$, *F restricts to an order isomorphism of A onto B, and* $F'(x) > 0$, *for* $x \in \mathbb{R}$.

Here, by a universal entire function, we mean an entire function having the remarkable property that its translates are dense in the space of all entire functions. The existence of a universal entire function was proved in 1929 by George Birkhoff [5].

In higher dimensions, the following result, was proved by Morayne [14] in 1987 for $\mathbb{R}^n$ and $\mathbb{C}^n$ and by Rosay and Rudin [16] in 1988, with a different proof for $\mathbb{C}^n$.

Theorem 1.4 (Morayne [14], Rosay and Rudin [16]) *Let A and B be countable dense subsets of* $\mathbb{C}^n$ *(respectively,* $\mathbb{R}^n$*),* $n > 1$. *Then there is a measure preserving biholomorphic mapping of* $\mathbb{C}^n$ *(respectively, bianalytic mapping of* $\mathbb{R}^n$*), that maps A onto B.*

This theorem appears to be stronger than Franklin's Theorem, however the proof of Theorem 1.4 fails for $n = 1$. Moreover, for $n = 1$, all measure-preserving automorphisms are of the form $z \mapsto az + b$, $|a| = 1$, so the only automorphic images of a set A are the sets $aA + b$.

Erdős [10] asked whether, given countable dense subsets A and B of $\mathbb{C}$, there exists an entire function f that maps A onto B, *cf.* Stäckel [17]. Maurer [13] gave an affirmative answer. In this context, there are the following two interesting results of Barth and Schneider [2, 3], the second of which improves the result of Maurer.

Theorem 1.5 *Let A and B be countable dense subsets of* $\mathbb{R}$. *Then there exists an entire transcendental function* f *such that* $f(z) \in B$ *if and only if* $z \in A$.

Theorem 1.6 *Let A and B be countable dense subsets of* $\mathbb{C}$. *Then there exists an entire function* f, *such that* $f(z) \in B$ *if and only if* $z \in A$.

Although the next result is not directly on the topic of the present paper, we consider it worth mentioning, perhaps as a distant cousin.

Theorem 1.7 ([15]) *Let A and B be countable dense subsets of the Hilbert cube* $H = [0,1]^{\mathbb{N}}$. *Then, for every* $\epsilon > 0$, *there is a measure preserving homeomorphism* f *of H that maps A onto B, and* $\rho(f, \mathrm{id}) < \epsilon$, *where* ρ *is a distance on the set of continuous mappings* $H \to H$ *and* id *is the identity mapping.*

In recent years, Maxim Burke obtained deep results of a nature similar to ours and Franklin's [6–8]. I thank the referees for pointing out an important error in an earlier version.

2 Proof of Theorem 1.2

Proof In proving Theorem 1.2 we shall also show that f can be further required to map a preassigned point $a \in A$ to a preassigned point $b \in B$. The desired function f will be of the form

$$f(z) = \lim_{n\to\infty} f_n(z) = \Phi(z) + \sum_{j=1}^{\infty} \lambda_j H_j(z),$$

where $f_n(z) = \Phi(z) + \sum_{j=1}^{n} \lambda_j H_j(z)$. The purpose of the H_j's, which we will momentarily define, is to recursively make adjustments to obtain more of the desired mapping properties, without losing those properties that we have previously assured.

Recalling that by hypothesis, $\Phi'(x) \geq m$ for $x \in \mathbb{R}$, let $\{\epsilon_n\}$ be a sequence of positive numbers whose sum is less than m and let $\{r_n\}$ be a strictly increasing sequence of positive numbers such that $r_n \to +\infty$. We shall construct an enumeration (α_n) of A, an enumeration (β_n) of B, and a real sequence (λ_n) such that, taking the functions H_j of the form

$$H_1 = 1, \quad H_j(z) = e^{-\Phi(z)^2} \prod_{k=1}^{j-1} (\Phi(z) - \Phi(\alpha_k)) \text{ for } j = 2, 3, \dots,$$

we shall have that, for $n = 1, 2, \dots,$

$$f_n(\alpha_j) = \beta_j, \quad j = 1, \dots, n, \tag{2.1}$$

$$\lambda_1 = \beta_1 - \alpha_1, \quad \text{and} \quad |\lambda_n H_n(z)| < \epsilon_n \text{ if } |z| \leq r_n, n > 1, \tag{2.2}$$

$$|\lambda_n H_n'(x)| < \epsilon_n, \quad \text{if } x \in \mathbb{R}, \tag{2.3}$$

$$f_n(\mathbb{R}) \subset \mathbb{R}. \tag{2.4}$$

From the second condition, f will be an entire function, and the third condition will allow us to differentiate this series term by term on $\mathbb{R}$. On $\mathbb{R}$ we have

$$f'(x) \geq m + \sum_{j=1}^{\infty} \lambda_j H_j'(x) > m - \sum_{j=1}^{\infty} \epsilon_j > 0.$$

Hence, $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and, consequently, injective. Moreover, since $f(x) \to \pm\infty$ as $x \to \pm\infty$, the function $f : \mathbb{R} \to \mathbb{R}$ is surjective and, consequently, bianalytic.

Choose $a_1 \in A$ and $b_1 \in B$. Now we shall choose the sequences $\{\alpha_n\}, \{\beta_n\}$, and $\{\lambda_n\}$. First, we choose enumerations $\{a_n\}$ and $\{b_n\}$ of A and B. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ will be rearrangements of $\{a_n\}$ and $\{b_n\}$ chosen recursively. Set $\alpha_1 = a_1$, $\beta_1 = b_1$, $\lambda_1 = \beta_1 - \Phi(\alpha_1)$. We have defined α_1, λ_1, β_1, and hence also h_1, f_1, and H_2. Note that $f_1(a_1) = f_1(\alpha_1) = \beta_1 = b_1$. Let β_2 be the first b_j not equal to $\beta_1 = b_1$. In fact $\beta_2 = b_2$.

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Suppose we have chosen

- distinct members $\alpha_1, \dots, \alpha_{2n-1}$ of the sequence $\{a_i\}$ such that, for each $k = 1, \dots, n$, α_{2k-1} is the first a_i not previously chosen;
- distinct $\beta_1, \dots, \beta_{2n}$ from the sequence $\{b_j\}$ such that, for each $k = 1, \dots, n$, β_{2k} is the first b_j not previously chosen;
- real numbers $\lambda_1, \dots, \lambda_{2n-1}$, such that conditions (2.1)–(2.3) are satisfied.

We shall now choose α_{2n}, λ_{2n}, α_{2n+1}, λ_{2n+1}, β_{2n+1}, and $\beta_{2(n+1)}$.

Since H_{2n} is locally bounded and Φ' is bounded on $\mathbb{R}$, we can choose $\eta > 0$ such that (2.2) and (2.3) hold for H_{2n} and $|\lambda| < \eta$. Now it follows from (2.3) that $f'_{2n-1}(x) \geq m - \sum_{j=1}^{\infty} \epsilon_j > 0$ for all $x \in \mathbb{R}$. Thus the function f_{2n-1} is surjective on $\mathbb{R}$. Choose a number x_n such that $f_{2n-1}(x_n) = \beta_{2n}$. We claim that $H_{2n}(x_n) \neq 0$. To see this, suppose, to obtain a contradiction, that $H_{2n}(x_n) = 0$. Then $x_n = \alpha_j$, for some $j = 1, \dots, 2n-1$, and $f_{2n-1}(\alpha_j) = \beta_{2n}$. However, $f_{2n-1}(\alpha_j) = \beta_j$. Thus, $\beta_{2n} = \beta_j$. But this contradicts the choice of β_{2n} as being distinct from β_j, for $j < 2n$. Since $H_{2n}(x_n) \neq 0$, the function

$$\lambda(x) = \frac{\beta_{2n} - f_{2n-1}(x)}{H_{2n}(x)}$$

is defined and continuous in a neighbourhood I of x_n, with $\lambda(x_n) = 0$; by choosing I smaller we can also have that $|\lambda(x)| < \eta$ for $x \in I$. By the density of A there is some $\alpha_{2n} \in A \setminus \{\alpha_1, \dots, \alpha_{2n-1}\}$ in I. Write $\lambda_{2n} = \lambda(\alpha_{2n})$. Then we have that $|\lambda_{2n}| < \eta$ and $f_{2n-1}(\alpha_{2n}) + \lambda_{2n} H_{2n}(\alpha_{2n}) = \beta_{2n}$. We have established (2.1)–(2.3) for $2n$.

The choice of α_{2n+1} is easy. We choose the first of the a_j different from $\alpha_1, \dots, \alpha_{2n}$ and call it α_{2n+1}.

Since $H_{2n+1}(\alpha_{2n+1}) \neq 0$, the linear function

$$\beta(\lambda) = f_{2n}(\alpha_{2n+1}) + \lambda H_{2n+1}(\alpha_{2n+1})$$

is non-constant. Hence $J_n = \{\beta(\lambda) : |\lambda| < \epsilon\}$ is a non-empty open interval and, since B is dense, we can choose an element of $(B \cap J_n) \setminus \{\beta_1, \dots, \beta_{2n}\}$, which we call β_{2n+1}. The element β_{2n+1} by definition has the form $\beta_{2n+1} = \beta(\lambda)$ for a certain λ, with $|\lambda| < \epsilon$. We denote this λ by λ_{2n+1}. If ϵ is sufficiently small, then λ_{2n+1} satisfies (2.2) and (2.3). Condition (2.1) is satisfied by the choice we have just made for β_{2n+1} and λ_{2n+1}.

For $\beta_{2(n+1)}$ we choose the first of the b_j's different from $\beta_1, \dots, \beta_{2n+1}$.

The construction of the sequences (α_n), (λ_n), and (β_n), and hence also of the the entire function f is complete. This concludes the proof of Theorem 1.2, except for the final statement regarding the existence of such a function having finite order of growth.

To obtain a function of finite order, we take as Φ the function $\Phi(z) = z$. Then at each step of the proof, in addition to making sure that conditions (2.1)–(2.3) are satisfied, we can choose λ_n smaller so that the condition

$$|\lambda_n| \prod_{k=1}^{n-1} |z - \alpha_k| < e^{|z|}/2^n, \quad \text{for all } z \in \mathbb{C},$$

is also satisfied. It is straightforward to check that the function

$$f(z) = z + e^{-z^2} \sum_{n=1}^{\infty} \lambda_n \prod_{k=1}^{n-1} (z - \alpha_k)$$

so obtained is indeed of finite order. ∎

3 Approximation by Entire Functions

For a set $S \subset \mathbb{C}$ we denote by S^0 the interior of S. We say that a function $f : S \to \mathbb{C}$ is holomorphic on S if there is an open neighbourhood U of S and a holomorphic function F on U such that $F = f$ on S. We denote by $H(S)$ the class of functions holomorphic on S and by $A(S)$ the class of functions continuous on S and holomorphic on S^0. The extended complex plane is denoted by $\overline{\mathbb{C}}$. Let $E \subset \mathbb{C}$ be symmetric with respect to the real axis. We shall say that a function $f : E \to \mathbb{C}$ defined on such a set E is symmetric with respect to the real axis if $f(\overline{z}) = \overline{f(z)}$, for $z \in E$.

A compact set $K \subset \mathbb{C}$ is a *Mergelyan set* if every $f \in A(K)$ can be uniformly approximated by polynomials.

Theorem 3.1 *A compact set $K \subset \mathbb{C}$ is a Mergelyan set if and only if $\mathbb{C} \setminus K$ is connected. Moreover, if K is symmetric with respect to the real axis, $f \in A(K)$ and $f(\overline{z}) = \overline{f(z)}$, $z \in K$, the approximating polynomials can be taken with real coefficients.*

Proof To verify the last statement, which is not part of the original Mergelyan Theorem, suppose K is symmetric with respect to the real axis, $f \in A(K)$ and $f(\overline{z}) = \overline{f(z)}$, $z \in K$. Let p_n, $n = 1, 2, \dots,$ be a sequence of polynomials that converges uniformly to f and set $q_n(z) = (p_n(z) + \overline{p_n(\overline{z})})/2$. Then the sequence of polynomials q_n also converges uniformly to f and, moreover, have real coefficients. ∎

For a topological vector space X, we denote by X^* the continuous dual space. The following Walsh-type lemma on simultaneous approximation and interpolation is due to Frank Deutsch.

Lemma 3.2 ([9]) *Let X be a locally convex topological complex vector space and Y a dense subspace. Then if $x \in X$, U is a neighbourhood of 0, and $L_1, \dots, L_n \in X^*$, there is a $y \in Y$ such that $y \in x + U$ and $L_j(y) = L_j(x)$, for $j = 1, \dots, n$.*

Let E be a closed set that is starlike with respect to the origin. For $f : E \to \mathbb{C}$, we shall write $f \in A^1(E)$, if $f \in H(E^0)$ and f has a radial derivative at each point of E, which by abuse of notation, we shall also denote as f', such that f' is continuous on E. We note that there are many other meanings assigned to the notation $A^1(E)$ in the literature.

Lemma 3.3 *Let $K \subset \mathbb{C}$ be compact and starlike with respect to the origin, let E be a Mergelyan set disjoint from K, and set $Q = K \cup E$. Suppose $f \in A(Q)$, $f|_K \in A^1(K)$, and $z_1, \dots, z_n$ are distinct points in K. Then for every $\epsilon > 0$, there is a polynomial p*

such that

$$|p-f|_Q<\epsilon,\ |p'-f'|_K<\epsilon,$$
$$p(z_j)=f(z_j),\ \ p'(z_j)=f'(z_j),\quad j=1,\dots,n.$$

If Q is symmetric with respect to the real axis, the z_j are real numbers, and $f(\overline{z})=\overline{f(z)}$ for $z\in Q$, we can take p with real coefficients.

Proof Without loss of generality, we can assume that $f(0)=0$. Choose a number $r>1$ such that $r>|z|$, for all $z\in K$. By Theorem 3.1, there is a polynomial q, for which $|q(z)-f'(z)|<\epsilon/r<\epsilon$, for all $z\in K$ and $|q(z)-f(z)|<\epsilon$ for all $z\in E$. Consider the polynomial $p(z)=\int_0^z q(\zeta)\,d\zeta$, $z\in K$. Thus,

$$|p-f|_K<\epsilon,\quad |p'-f'|_K<\epsilon,\quad |q-f|_E<\epsilon. \tag{3.1}$$

Let U_K and U_E be open simply connected (but not necessarily connected) neighbourhoods of K and E, respectively, with disjoint closures, and define $g\in H(U_K\cup U_E)$ by setting $g=p$ on U_K and $g=q$ on U_E. Then by Runge's theorem and the fact that local uniform convergence implies local uniform convergence of derivatives, there is a polynomial $\widetilde{p}$ such that

$$|\widetilde{p}-p|_K<\epsilon,\quad |\widetilde{p}'-p'|_K<\epsilon,\quad |\widetilde{p}-q|_E<\epsilon.$$

Together with (3.1) we see that the polynomials are dense in $X=A^1(K)\cap A(E)$, endowed with the canonical norm. In addition, point evaluation at points of Q and point derivation at points of K are continuous linear functionals on X. Thus, Lemma 3.2 implies the main part of the result. The last part of the lemma follows as in the proof of Theorem 3.1 above. ∎

A *chaplet* is a closed set which is the union of an infinite family of disjoint closed discs that is locally finite, *i.e.*, each compact set meets at most finitely many members of the family. Henceforth, E will denote a chaplet that is disjoint from the real axis and is symmetric with respect to the real axis. Thus, E is the union of an infinite, but locally finite, family of disjoint closed discs E_n^+ in the open upper half-plane and their reflections E_n^- in the open lower half-plane. We shall suppose that the radii of the $E_n^\pm$ tend to infinity. We shall also suppose that these discs are ordered and separated in the following sense. There is a sequence $r_n>0$, $r_n\nearrow\infty$, such that $E_n^\pm$ is contained in the annulus $r_n<|z|<r_{n+1}$, for each n. A chapelet E having all of these properties will be called a *special chaplet.*

Lemma 3.4 *Let K be a closed disc centered at the origin, E be a special chaplet disjoint from K, and set $F=K\cup\mathbb{R}\cup E$. Let ϵ be a positive continuous function on $\mathbb{C}$. Then, for every function $g\in A(F)$ such that $g(\overline{z})=\overline{g(z)}$ and $g\in C^1(K\cup\mathbb{R})$, there exists an entire function Λ such that $\Lambda(\overline{z})=\overline{\Lambda(z)}$ and*

$$|\Lambda(z)-g(z)|<\epsilon(z)\ \forall z\in F,\quad |\Lambda'(z)-g'(z)|<\epsilon(z)\ \forall z\in K\cup\mathbb{R}.$$

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Proof We can consider the radius r_1 of K as the first member of a sequence $\{r_n\}$ of separating radii for the chaplet E. For $n = 1, 2, \dots,$ set

$$\overline{D_n} = \{z : |z| \le r_n\},$$
$$S_n = [-r_{n+1}, -r_n] \cup \overline{D_n} \cup [r_n, r_{n+1}].$$

Without loss of generality, we can assume that $\epsilon(z) = \epsilon(|z|)$ and that $\epsilon(r)$ is strictly decreasing on $[0, +\infty)$. Let $\epsilon_1, \epsilon_2, \dots,$ be a strictly decreasing sequence of positive numbers, such that

$$\epsilon_n < \min_{z \in K_n \cup E_n} \epsilon(z) = \epsilon(r_{n+1}) \quad \text{and} \quad \sum_{k=n+1}^{\infty} \epsilon_k < \epsilon_n, \quad n = 1, 2, \dots.$$

The compact sets S_n are starlike with respect to the origin and symmetric with respect to the real axis. The union E_n of the two closed discs $E_n^{\pm}$ is a Mergelyan set disjoint from S_n. Each compact set $Q_n = S_n \cup E_n$ satisfies the hypotheses of Lemma 3.3; we shall recursively define corresponding functions f_n.

Set $f_1 = g$. By Lemma 3.3, there is a polynomial p_1, with real coefficients, such that

$$|p_1 - f_1|_{Q_1} < \epsilon_2, \quad |p_1' - f_1'|_{S_1} < \epsilon_2,$$

and

$$p_1(\pm r_2) = g(\pm r_2), \quad p_1'(\pm r_2) = g'(\pm r_2).$$

Set $p_0 = p_1$ and suppose, for $n \ge 1$ and $k = 1, \dots, n-1$, we already have polynomials p_k, with real coefficients, such that, for

$$f_k(z) = \begin{cases} p_{k-1}(z) & z \in \overline{D}_k, \\ g(z) & z \in [-r_{k+1}, -r_k] \cup [r_k, r_{k+1}], \\ g(z) & z \in E_k, \end{cases}$$

we have

$$|p_k - f_k|_{Q_k} < \epsilon_{k+1}, \quad |p_k' - f_k'|_{S_k} < \epsilon_{k+1},$$

and

$$p_k(\pm r_{k+1}) = g(\pm r_{k+1}), \quad p_k'(\pm r_{k+1}) = g'(\pm r_{k+1}).$$

We define $f_n \in A^1(Q_n)$, by setting

$$f_n(z) = \begin{cases} p_{n-1}(z) & z \in \overline{D}_n, \\ g(z) & z \in [-r_{n+1}, -r_n] \cup [r_n, r_{n+1}], \\ g(z) & z \in E_n. \end{cases}$$

By Lemma 3.3, there is a polynomial p_n, with real coefficients, such that

$$|p_n - f_n|_{Q_n} < \epsilon_{n+1}, \quad |p_n' - f_n'|_{S_n} < \epsilon_{n+1},$$

and

$$p_n(\pm r_{n+1}) = g(\pm r_{n+1}), \quad p_1'(\pm r_{n+1}) = g'(\pm r_{n+1}).$$

By induction, the polynomials p_n are now defined for all $n = 1, 2, \dots.$

Fix positive integers $k < m < n$. On $\overline{D}_k$, we have

$$|p_n(z) - p_m(z)| \le \sum_{j=m}^{n-1} |p_{j+1}(z) - p_j(z)| < \sum_{j=m}^{n-1} \epsilon_{j+1}$$

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and, since $\sum \epsilon_j$ is convergent, the sequence p_n is uniformly Cauchy on each $\overline{D}_k$ and hence converges uniformly on compact subsets to an entire function Λ. Of course, we also have that $p_n' \to \Lambda'$ uniformly on compact subsets. Since all of the p_n have real coefficients, $\Lambda(\overline{z}) = \overline{\Lambda(z)}$.

Fix $z \in E$. Then $z \in E_m$, for some m and $g(z) = f_m(z)$. Choose $n > m$ such that $|\Lambda(z) - p_n(z)| < \epsilon_{m+1}$. Then

$$|\Lambda(z) - g(z)| \le |\Lambda(z) - p_n(z)| + |p_n(z) - f_m(z)| \le \epsilon_{m+1} + \sum_{m+1}^{n+1} \epsilon_k < \epsilon_m < \epsilon(z).$$

It remains to show that Λ has the desired approximation properties on $K \cup \mathbb{R}$, namely $|\Lambda(z) - g(z)| < \epsilon(z)$ and $|\Lambda'(z) - g'(z)| < \epsilon(z)$, for all $z \in K \cup \mathbb{R}$. Fix $z \in K \cup \mathbb{R}$. Let $m = m_z$ be the first m such that $z \in K \cup [-r_{m+1}, r_{m+1}]$. Noting that $g = f_1$ on S_1 and $g = f_m$ on $[-r_{m+1}, -r_m] \cup [r_m, r_{m+1}]$, we have

$$\begin{aligned}|p_n(z) - g(z)| = |p_n(z) - f_m(z)| &\le |p_m(z) - f_m(z)| + \sum_{k=m}^{n-1} |p_{k+1}(z) - p_k(z)| \\ &\le \epsilon_{m+1} + \sum_{k=m}^{n-1} \epsilon_{k+2} < \sum_{k=m+1}^{\infty} \epsilon_k < \epsilon_m < \epsilon(x).\end{aligned}$$

Thus, $|\Lambda(z) - g(z)| < \epsilon(z)$, for all $z \in K \cup \mathbb{R}$. The proof that $|\Lambda'(z) - g'(z)| < \epsilon(x)$, for all $z \in K \cup \mathbb{R}$, is completely analogous. ∎

Lemma 3.5 *Let K be a closed disc centered at the origin, let E be a special chaplet disjoint from K, let $g \in A(E)$ with $g(\overline{z}) = \overline{g(z)}$, let ϵ be a positive continuous function on $\mathbb{C}$. Then there is an entire function Λ with $\Lambda(\overline{z}) = \overline{\Lambda(z)}$ such that*

$$|\Lambda(z) - g(z)| < \epsilon(z),\ z \in E, \quad \max\{|\Lambda(z)|, |\Lambda'(z)|\} < \epsilon(z),\ z \in K \cup \mathbb{R}.$$

Moreover, if we are given a real number α, there is such a function Λ_α for which $\Lambda_\alpha(\alpha) \ne 0$.

Proof The first part is merely an instance of Lemma 3.4, obtained by putting $g = 0$ on $K \cup \mathbb{R}$.

For the second part, suppose we are given a real number α. Let U be a neighbourhood of $K \cup \mathbb{R}$ disjoint from E and symmetric with respect to $\mathbb{R}$ and consider a function $\gamma \in C^1(U)$ that is of compact support and equal to 1 in a neighbourhood of $K \cup \{\alpha\}$ and such that $\overline{\gamma}(z) = \gamma(\overline{z})$. We obtain the second assertion of the lemma from Lemma 3.4 by putting $g = \delta\gamma$ on $K \cup \mathbb{R}$ for a sufficiently small real δ. ∎

4 Proof of Theorem 1.3

By Theorem 1.2, there is a function of the form

$$f(z) = z + \sum_{j=1}^{\infty} \lambda_j H_j(z), \tag{4.1}$$
$$H_1 = 1, \quad H_j(z) = e^{-z^2} \prod_{k=1}^{j-1} (z - \alpha_k) \text{ for } j = 2, 3, \dots,$$

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such that $f(\mathbb{R}) = \mathbb{R}$, f restricts to an order isomorphism of A onto B and $f'(x) > 0$, for $x \in \mathbb{R}$, so the mapping $f : \mathbb{R} \to \mathbb{R}$ is bianalytic. All that might be lacking to satisfy the conclusion of Theorem 1.3 is universality.

To obtain a function F of the form

$$F(z) = z + \sum_{n=1}^{\infty} \Lambda_n(z)H_n(z) = \lim_{n\to\infty} F_n(z)$$

satisfying the conclusion of Theorem 1.3, we shall imitate the proof of Theorem 1.2. Namely, appropriate arrangements $\{\alpha_n\}$ of A, $\{\beta_n\}$ of B, and $\{\Lambda_n\}$ will be chosen recursively. The only difference between the function f of Theorem 1.2 and the function F of Theorem 1.3, is that the coefficients λ_n of f are numbers, whereas the coefficients Λ_n of F are functions.

Let $\{r_n\}$ be the sequence used in the proof of Theorem 1.2 to obtain the function (4.1). Let $\{K_n\}$ be the associated exhaustion, $K_n = \{z : |z| \le r_n\}$, and let $A_n = \{z : r_n < |z| < r_{n+1}\}$ be the associated annuli. We can assume that $r_{n+1} - r_n \nearrow +\infty$ and we use this to define a special chaplet E, whose closed discs $E_n^{\pm} = E(c_n^{\pm}, \rho_n)$ of centers $c_n^{\pm}$ and radii ρ_n lie in the annuli A_n. Since E is a special chaplet $\rho_n \to +\infty$, but since $r_{n+1} - r_n \nearrow +\infty$, we can also assume that $\Re(c_n^{\pm} - \rho_n) \to +\infty$ and that there is a $\delta > 0$ such that $E(c_n^{\pm}, \rho_n + \delta)$ is also contained in A_n. Denoting by $\overline{D}_n$ the closed disc centered at the origin of radius ρ_n, we have $E_n^{\pm} = \overline{D}_n + c_n^{\pm}$.

Let p_n, $n = 1, 2, \ldots$, be a sequence of all the polynomials whose coefficients have both real and imaginary parts rational. Since these polynomials are dense in the space of entire functions, an entire function will be universal, providing its translates approximate each p_n. We shall assume that each polynomial of the sequence occurs infinitely often in the sequence.

Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\epsilon_1 < 1$ and for each $n > 1$ we have $\epsilon_n < \min\{1/(n-1)^2, \epsilon_{n-1}/2\}$ and so $\sum_{k>n} \epsilon_k < \epsilon_n$. For $n = 1, 2 \ldots$, let Λ_n be an entire function, such that

$$\Lambda_n(x) \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{4.2}$$

$$|\Lambda_n(z)H_n(z)| < \epsilon_n, \quad z \in K_n \cup \mathbb{R}, \tag{4.3}$$

$$|\Lambda_n(z)H_n(z)| < \epsilon_k, \quad z \in E_k^+, k > n, \tag{4.4}$$

$$|z + \Lambda_n(z)H_n(z) - p_n(z - c_n)| < \epsilon_n, \quad z \in E_n^+, \tag{4.5}$$

$$\max\{|\Lambda_n(x)H_n'(x)|, |\Lambda_n'(x)H_n(x)|\} < \epsilon_n/2, \quad x \in \mathbb{R}, \tag{4.6}$$

Then the function (4.1) is again an entire function for which $F(\mathbb{R}) \subset \mathbb{R}$ and, for $z \in E_n^+$,

$$\begin{aligned} |F(z) - p_n(z - c_n)| &< |z + \Lambda_n(z)H_n(z) - p_n(z - c_n)| + \sum_{k\ne n} |\Lambda_k(z)H_k(z)| \\ &\le \Big(\sum_{k<n} |\Lambda_k(z)H_k(z)| + \epsilon_n\Big) + \sum_{k>n} |\Lambda_k(z)H_k(z)| \\ &< \big((n-1)\epsilon_n + \epsilon_n\big) + \sum_{k>n} \epsilon_k < n\epsilon_n + \epsilon_n < \frac{1}{n} + \frac{1}{n^2} < \frac{2}{n}. \end{aligned}$$

Thus, $|F(z) - p_n(z - c_n)| < \frac{2}{n}, z \in E_n^+$. Equivalently,

$$|F(z + c_n) - p_n(z)| < \frac{2}{n}, \quad z \in \overline{D}_n.$$

Hence, translates of F approximate all polynomials, which means that the entire function F is universal.

Since $|\Lambda_n(x)H_n(x)| < \epsilon_n$, for $x \in \mathbb{R}$, we have $|x - F(x)| \leq 1$, for $x \in \mathbb{R}$. Thus $F(x) \to \pm\infty$, as $x \to \pm\infty$. It follows from (4.6) that $F'(x) > 0$, for $x \in \mathbb{R}$ so $F : \mathbb{R} \to \mathbb{R}$ is bianalytic.

Two properties of F remain to be established to complete the proof of Theorem 1.3: that F exists and that F restricts to an order isomorphim $A \to B$. We have indeed shown that if the Λ_n's satisfy (4.2)–(4.6), then F is well defined and has all of the desired properties except possibly that of being an order isomorphism $A \to B$. We shall now show, by essentially the same recursive argument as in the proof of Theorem 1.2, that appropriate Λ_j's indeed exist. The recursion will also yield the order isomorphism $F : A \to B$, thus completing the proof of Theorem 1.3.

The recursive choice of the sequences α_n, β_n, and Λ_n is the same (with the help of Section 3) as in the proof of Theorem 1.2, but there are technical difficulties that justify our presenting this part of the proof again in this new context.

Choose enumerations $\{a_n\}$ and $\{b_n\}$ of A and B. Set

$$\alpha_1 = a_1. \quad \beta_1 = b_1, \quad \Lambda_1 = \beta_1 - \alpha_1.$$

We have defined α_1, Λ_1, β_1 and hence also F_1 and H_2. Note that $F_1(\alpha_1) = \beta_1$. Let β_2 be the first b_j not equal to β_1. In fact $\beta_2 = b_2$. Suppose we have chosen

- distinct members $\alpha_1, \dots, \alpha_{2n-1}$ of the sequence $\{a_i\}$, such that, for each $k = 1, \dots, n$, α_{2k-1} is the first a_i not previously chosen;
- distinct $\beta_1, \dots, \beta_{2n}$ from the sequence $\{b_j\}$, such that, for each $k = 1, \dots, n$, β_{2k} is the first b_j not previously chosen;
- entire functions $\Lambda_1, \dots, \Lambda_{2n-1}$, such that conditions (4.2)–(4.6) and

$$F_n(\alpha_k) = \beta_k, \quad k = 1, \dots, 2k - 1 \tag{4.7}$$

are satisfied.

We shall now choose α_{2n}, Λ_{2n}, α_{2n+1}, Λ_{2n+1}, β_{2n+1}, and $\beta_{2(n+1)}$.
Set

$$g(z) = \begin{cases} 0 & z \in \overline{D}_{2n} \cap \bigcup_{k>2n} E_k^{\pm} \cup \mathbb{R}, \\ \left(p_{2n}(z - c_{2n}^+) - z\right)/H_{2n}(z) & z \in E_{2n}^+, \\ \overline{g(\overline{z})} & z \in E_{2n}^-. \end{cases}$$

It follows from Lemma 3.5 that there is an entire function Λ that satisfies the conditions (4.2), (4.3), (4.4), (4.5), and (4.6), with n replaced by $2n$ and Λ_n replaced by Λ.

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That is,

$$\Lambda(x) \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{4.8}$$

$$|\Lambda(z)H_{2n}(z)| < \epsilon_{2n}, \quad z \in K_{2n} \cup \mathbb{R}, \tag{4.9}$$

$$|\Lambda(z)H_n(z)| < \epsilon_k, \quad z \in E_k^+, k > 2n, \tag{4.10}$$

$$|z + \Lambda(z)H_{2n}(z) - p_{2n}(z - c_{2n})| < \epsilon_{2n}, \quad z \in E_{2n}^+, \tag{4.11}$$

$$\max\{|\Lambda(x)H_{2n}'(x)|, |\Lambda'(x)H_{2n}(x)|\} < \epsilon_{2n}/2, \quad x \in \mathbb{R}, \tag{4.12}$$

For $(z, \lambda) \in \mathbb{C} \times \mathbb{R}$, set

$$G(z,\lambda) = z + \sum_{k=1}^{2n-1} \Lambda_k(z)H_k(z) + \lambda\Lambda(z)H_{2n}(z) = F_{2n-1}(z) + \lambda\Lambda(z)H_{2n}(z).$$

Choose $\epsilon > 0$ so small that (4.11) holds with Λ replaced by $\lambda\Lambda$ for each λ in the interval $1 - \epsilon < \lambda < 1$ and choose such a λ_0. It follows from (4.9) together with (4.3), which holds for $k = 1, \dots, 2n-1$ by the induction hypothesis, that $|x - G(x, \lambda_0)| < 1$ for all $x \in \mathbb{R}$, so $G(\cdot, \lambda_0) : \mathbb{R} \to \mathbb{R}$ is surjective. In particular, there is a point $x_0 \in \mathbb{R}$ such that $G(x_0, \lambda_0) = \beta_{2n}$. Since

$$\frac{\partial G}{\partial \lambda}(x_0, \lambda_0) > 1 - \sum_{k=1}^{\infty} \epsilon_k > 0,$$

it follows from the implicit function theorem that there is a continuous function $\ell(x)$ in an interval $(x_0 - \delta, x_0 + \delta)$ such that $\ell(x_0) = \lambda_0$ and $G(x, \ell(x)) = \beta_{2n}$, for all x in this interval. Since A is dense in $\mathbb{R}$, we can choose a point $\alpha_{2n} \in A$, which is in this interval and is sufficiently close to x_0 that $1 - \epsilon < \ell(\alpha_{2n}) < 1$. Now set $\Lambda_{2n} = \ell(\alpha_{2n})\Lambda$. Since $0 < \ell(\alpha_{2n} < 1$, the function Λ_{2n} satisfies (4.8), (4.9), (4.10), and (4.12). Moreover, from the choice of ϵ, it follows that Λ_{2n} also satisfies (4.11). We have verified that Λ_{2n} satisfies conditions (4.2), (4.3), (4.4), (4.5), and (4.6). Also, condition (4.7) is satisfied by Λ_{2n}, since

$$F_{2n}(\alpha_{2n}) = F_{2n-1}(z) + \Lambda_{2n}(z)H_{2n}(z) = G\big(\alpha_{2n}, \ell(\alpha_{2n})\big) = \beta_{2n}.$$

The choice of α_{2n+1} is easy. We choose the first of the a_j different from $\alpha_1, \dots, \alpha_{2n}$ and call it α_{2n+1}. The construction of the corresponding Λ_{2n+1} and β_{2n+1} is very similar to the construction of Λ_{2n} and α_{2n} that we just completed. Indeed, it follows from Lemma 3.5 that there is an entire function Λ that satisfies the conditions (4.2), (4.3), (4.4), (4.5), and (4.6), with n replaced by $2n+1$ and Λ_n replaced by Λ and with the additional property that $\Lambda(\alpha_{2n+1}) \neq 0$.

For $(z, \lambda) \in \mathbb{C} \times \mathbb{R}$, set

$$G(z,\lambda) = z + \sum_{k=1}^{2n} \Lambda_k(z)H_k(z) + \lambda\Lambda(z)H_{2n+1}(z) = F_{2n}(z) + \lambda\Lambda(z)H_{2n+1}(z).$$

Since $\Lambda(\alpha_{2n+1})H_{2n+1}(\alpha_{2n+1}) \neq 0$, the linear function

$$\beta(\lambda) = G(\alpha_{2n+1}, \lambda) = F_{2n}(\alpha_{2n+1}) + \lambda\Lambda(\alpha_{2n+1})H_{2n+1}(\alpha_{2n+1})$$

is non-constant. Hence $J_n = \{\beta(\lambda) : |\lambda| < \epsilon\}$ is a non-empty open interval and, since B is dense, we can choose an element of $(B \cap J_n) \setminus \{\beta_1, \dots, \beta_{2n}\}$that we call β_{2n+1}. By definition, the element β_{2n+1} has the form $\beta_{2n+1} = \beta(\lambda)$ for a certain λ, with $|\lambda| < \epsilon$.

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We denote this λ by λ_{2n+1} and set $\Lambda_{2n+1} = \lambda_{2n+1}\Lambda$. If ϵ is sufficiently small, then Λ_{2n+1} satisfies the conditions (4.2), (4.3), (4.4), (4.5), and (4.6). Condition (4.7) is satisfied by the choice we have just made for β_{2n+1} and Λ_{2n+1}.

For $\beta_{2(n+1)}$ we choose the first of the b_j's different from $\beta_1, \dots, \beta_{2n+1}$.

The construction of the sequences (α_n), (λ_n), and (β_n), and hence also of the the entire function F, is complete. This concludes the proof of Theorem 1.3.

5 Universality and Linear Dynamics

By an argument anticipating those of the present paper, G. D. Birkhoff [5] established the existence of universal entire functions. It turned out that universality is generic. That is, *most* entire functions are universal. More precisely, the family of universal entire functions is residual (it is of Baire category II and its complement is of category I) in the space of all entire functions. However, the situation for order isomorphisms between countable dense subsets of the reals is quite the opposite. Let $\mathcal{E}$ denote the space of entire functions, let $\mathcal{E}_R$ be the "real" entire functions, that is, the entire functions that map reals to reals. And let $\mathcal{E}_\rightarrow$ be the space of functions in $\mathcal{E}_R$ whose restrictions to the reals are non-decreasing. Then $\mathcal{E}_R$ is a closed nowhere-dense subset of $\mathcal{E}$ and $\mathcal{E}_\rightarrow$ is a closed nowhere-dense subset of $\mathcal{E}_R$. Thus, the class of universal entire functions constructed here is, within the space of all entire functions, "topologically thin" *i.e.*, of first Baire category, whereas the space of all universal entire functions is "thick," *i.e.*, of second category. In other words, the "hard analysis" driving our constructions cannot be replaced by "soft" methods.

Although most entire functions are universal, no explicit example is known. The only known function that has a universality property in the sense of Birkhoff (universality of translations) is the Riemann zeta-function! It is not entire, but as close to entire as possible. It has only one pole and that pole is simple. More precisely, the spectacular universality theorem of Voronin [18] states that vertical translates of $\zeta(z)$ "frequently" approach all functions holomorphic in the strip $1/2 < \Re z < 1$ having no zeros. Moreover, Bagchi [1] showed that the Riemann Hypothesis is equivalent to the possibility of approximating the function $\zeta(z)$ itself in this fashion by its own translates (a sort of almost periodicity). Bagchi established this formulation of the Riemann Hypothesis in the language of topological dynamics.

Universality in the sense of the present paper is also connected to the burgeoning field of linear dynamics in which the concept of universality has evolved within the past thirty or so years. Linear dynamics is a fusion of the (usually nonlinear) study of dynamical systems with the theory of linear operators on topological vector spaces. In this setting, Birkhoff's universality theorem becomes the statement that translation operators on the space of entire functions are hypercyclic. An operator on a linear topological space is called cyclic if there is a vector whose orbit under the operator's iterates has dense linear span. Hypercyclic means that the orbit itself is dense.

Recently the subject of linear dynamics has attained enough maturity to justify two recent books written by accomplished young researchers [4,12]. It is possible that the results and methods of the present paper, given their intrinsically non-soft nature, might be of interest to researchers in this burgeoning area.

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