Proceedings of the Royal Society of Edinburgh, **141A**, 127–154, *2011*

Compactly supported solutions for a semilinear elliptic problem in $\mathbb{R}^n$ with sign-changing function and non-Lipschitz nonlinearity

Qiuping Lu
Centro de Modelamiento Matemático, Av. Blanco Encalada 2120, Santiago, Chile (qlu@dim.uchile.cl)

(MS received 10 August 2009; accepted 1 April 2010)

For a sign-changing function $a(x)$ we consider the solutions of the following semilinear elliptic problem in $\mathbb{R}^n$ with $n \geqslant 3$:

$$-\Delta u = (\gamma a^+ - a^-)u^q + u^p, \qquad u \geqslant 0 \quad \text{and} \quad u \in \mathcal{D}(\mathbb{R}^n),$$

where $\gamma > 0$ and $0 < q < 1 < p < (n+2)/(n-2)$. Under an appropriate growth assumption on a^- at infinity, we show that all solutions are compactly supported. When $\Omega^+ = \{x \in \mathbb{R}^n \mid a(x) > 0\}$ has several connected components, we prove that there exists an interval on γ in which the solutions exist. In particular, if $a(x) = a(|x|)$, by applying the mountain-pass theorem there are at least two solutions with radial symmetry that are positive in Ω^+.

1. Introduction

For a locally Hölder-continuous and sign-changing function $a(x)$ in $\mathbb{R}^n$, we study the following elliptic problem in $\mathbb{R}^n$ with $n \geqslant 3$:

$$\left.\begin{aligned} -\Delta u &= a_\gamma(x)u^q + u^p \quad \text{in } \mathbb{R}^n, \quad 0 < q < 1 < p < \frac{n+2}{n-2}, \\ u &\geqslant 0 \qquad\qquad\qquad\quad \text{in } \mathbb{R}^n, \quad u \in \mathcal{D}(\mathbb{R}^n), \end{aligned}\right\} \tag{1.1}$$

where

$$\begin{aligned} a_\gamma(x) &= \gamma a^+(x) - a^-(x), \quad \gamma > 0, \\ a^+(x) &= \max(0, a(x)), \\ a^-(x) &= \max(0, -a(x)). \end{aligned}$$

The following assumption is also made on $a^-(x)$ throughout this paper:

$$0 < \liminf_{|x|\to\infty} a^-(x) \leqslant \limsup_{|x|\to\infty} a^-(x) < \infty.$$

We will discuss this assumption at the end of the paper.

By $\mathcal{D}(\mathbb{R}^n)$ we mean the completion of $C_0^\infty(\mathbb{R}^n)$ under the Dirichlet semi-norm,

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x\right)^{1/2}.$$

https://doi.org/10.1017/S0308210509001231 Published online by Cambridge University Press

Equations of this type (1.1) arise as stationary solutions to the degenerate reaction–diffusion equations introduced by Gurtin and MacCamy [17, 18] to model the evolution of a biological population (see also [2]). Throughout the paper, to emphasize the dependence on λ, equation (1.1) is often referred to as $(1.1)_\gamma$ (the subscript γ is omitted if no confusion arises).

Let

$$\begin{aligned} \Omega^+ &= \{x \in \mathbb{R}^n \mid a(x) > 0\}, \\ \Omega^{0+} &= \{x \in \mathbb{R}^n \mid a(x) \geqslant 0\}, \\ \Omega^- &= \{x \in \mathbb{R}^n \mid a(x) < 0\}. \end{aligned}$$

Since $a(x)$ is sign-changing and $\liminf_{|x|\to\infty} a^-(x) > 0$, Ω^+ and Ω^{0+} are not empty and bounded. The important feature of this equation is that it not only combines a non-Lipschitz nonlinearity u^q with a sign-changing coefficient $a(x)$ but also exhibits a combination of concave and convex nonlinearities in Ω^+. Such 'concave plus convex' nonlinearities in a bounded domain have been studied by Ambrosetti *et al*. [6] (see also [1, 14]), so we expect some similar results. In particular, we hope to prove the multiplicity of solutions by using variational methods.

It was originally observed by Schatzman [21] that solutions could vanish on large sets and in fact that, under appropriate hypotheses on $a(x)$, there exist solutions with compact support. We show more solutions as follows.

Theorem 1.1. *Every weak solution of (1.1) is a compactly supported classical solution.*

The sublinear term u^q, $0 < q < 1$, is essential for this phenomenon to occur. If, instead, we consider the same equation (1.1) with $q \geqslant 1$, then a simple application of the classical strong maximum principle shows that a non-negative solution must be strictly positive in $\mathbb{R}^n$, so the existence of compactly supported solutions would be impossible.

In [2] a similar equation $-\Delta u = a(x)u^q + b(x)u^p$ with $b(x) \leqslant 0$ was studied, and it was shown that all of the non-negative solutions in $\mathcal{D}(\mathbb{R}^n)$ have compact support. Moreover, the size of the support of these solutions is controlled by $a(x)$ and does not depend on any particular solution. In contrast with [2], the size of the support of solutions to (1.1) cannot be controlled. In order to understand why the solutions for the case $b(x) \equiv 1$ are different from those in the case $b(x) \leqslant 0$, we begin by recalling an important result of [12]. In [12] it was proved that the equation $-\Delta v = v^p - v^q$ in $\mathbb{R}^n$ has a unique compactly supported radial solution. This suggests that (1.1) could have a solution whose support lies completely in Ω^-. Indeed, consider the following special example.

Example 1.2. Let $\Omega^+ \subset\subset B(0, r)$ and $a(x) \equiv -1$ in $\mathbb{R}^n - B(0, r)$ for some $r > 0$. Again, from [12], we may construct arbitrarily many solutions of (1.1) by gluing together the compactly supported solutions of $-\Delta v = v^p - v^q$ in disjoint balls in $\mathbb{R}^n - B(0, r)$ (see figure 1).

We also study the structure of the solution set of (1.1) in case the favourable domain Ω^+ has several components. We make the following assumption on Ω^+.

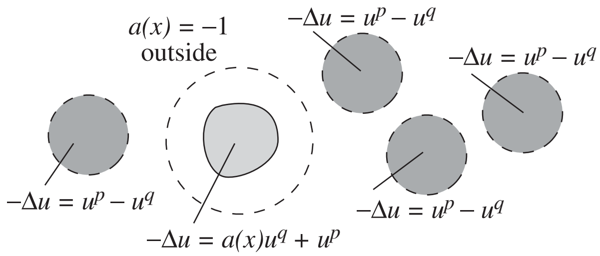

Figure 1. $a(x)$ as in example 1.2.

ASSUMPTION 1.3. Ω^+ has $k < \infty$ connected components with $\Omega^+ = \bigcup_{i=1}^k \Omega_i^+$ and each connected component Ω_i^+ satisfies an interior ball condition.

Set $M = \{1, 2, 3, \dots, k\}$. Under assumption 1.3, for any solution $u(x)$ of (1.1), by Hopf's lemma it is easy to see that solution $u(x)$ is either positive in Ω_i^+ or completely vanishes in Ω_i^+ for any $i \in M$. To organize the set of solutions of $(1.1)_\gamma$ according to the pattern of their supports we define the following classes of solutions.

DEFINITION 1.4.

(i) For any non-empty $I \subset M$, denote by $S_{I,\gamma}$ the class of solutions of $(1.1)_\gamma$ that are positive in $\Omega_I^+ = \bigcup_{i \in I} \Omega_i^+$.

(ii) $N_{I,\gamma}$ denotes the set $\{u \in S_{I,\gamma} \mid u \equiv 0 \text{ in } \Omega^+ - \Omega_I^+\}$.

When $\gamma > 0$ is small, we show in the following theorem that there exists a 'small' solution which is the minimal solution of $(1.1)_\gamma$ in $S_{I,\gamma}$, but for large γ there is no solution at all.

THEOREM 1.5. *For any non-empty* $I \subset M$*, there exists* $0 < \Gamma_I < \infty$ *such that*

(i) $S_{I,\gamma} \neq \varnothing$ *when* $0 < \gamma \leqslant \Gamma_I$ *and* $S_{I,\gamma} = \varnothing$ *when* $\gamma > \Gamma_I$*,*

(ii) $S_{I,\gamma}$ *has a minimal element* $u_{I,\gamma}$ *for all* $0 < \gamma \leqslant \Gamma_I$*,*

(iii) $\|u_{I,\gamma}\|_{L^\infty(\mathbb{R}^n)} \to 0$ *as* $\gamma \to 0^+$.

Note that the existence of a solution in $S_{I,\gamma}$ at the endpoint $\gamma = \Gamma_I$ is not trivial, and it is the result of *a priori* estimates for the family of minimal solutions $u_{I,\gamma}$ as $\gamma \to \Gamma_I^-$. It is an 'extremal solution' of the family of minimal solutions, and similar results in a bounded domain have been obtained [4]. In addition, Cabré [10] studied extremal solutions for certain autonomous equations in bounded domains and showed that extremal solutions exist for stable solution families, even for nonlinearities with super-linear growth, for which the usual Palais–Smale-type compactness results fail. As in [1, 4] we may view this existence theorem as a bifurcation result in the parameter γ. It is expected that the family of solutions will bifurcate from the trivial solution at $\gamma = 0$ and that the extremal value Γ_I will be a sort of turning point in a bifurcation curve. The difficulty with making this precise for $(1.1)_\gamma$ is that the linearization is singular at $u = 0$, so standard continuation methods [13] do not apply.

Assuming more on $a(x)$, we also obtain an existence result for $N_{I,\gamma}$ for small γ.

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Definition 1.6. We say that $a(x)$ is *admissible* if assumption 1.3 holds and

(i) Ω^{0+} also has exactly k connected components with $\Omega^{0+} = \bigcup_{i=1}^{k} \Omega_i^{0+}$,

(ii) $\Omega_i^{+} \subset \Omega_i^{0+}$ for $i \in M$ and $\operatorname{dist}(\Omega_i^{0+}, \Omega_j^{0+}) > 0$ for $i \neq j$.

Theorem 1.7. *If $a(x)$ is admissible, there exists $\gamma_0 > 0$ such that $N_{I,\gamma} \neq \varnothing$ for $0 < \gamma \leqslant \gamma_0$.*

Unlike the results in [2], the elements in $N_{I,\gamma}$ are not unique. In fact, there are at least two elements in $N_{M,\gamma} = S_{M,\gamma}$. To study multiplicity of solutions, we adopt a variational framework for this problem. As mentioned in [1], variational analysis of solutions in $N_{I,\gamma}$, $I \neq M$, is difficult since these solutions have infinite-dimensional negative spaces associated to them (see remark 4.3). Therefore, we will only consider the solutions $u \in S_{M,\gamma}$, that is, $u(x) > 0$ in all of Ω^{+}. For convenience we denote by $\Gamma = \Gamma_M$ and U_γ the minimal solution in $S_{M,\gamma}$ for $0 < \gamma \leqslant \Gamma$.

As the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$ is not compact, we always expect the Palais–Smale condition to be an important issue in variational problems posed on $\mathbb{R}^n$. To illustrate how compactness may break down for these specific problems we return to example 1.2, for which the solution space itself is non-compact. The strategy we use here to eliminate this loss of compactness is to consider $a(x)$ with radial symmetry, and to restrict our attention to the class of radial functions. A forthcoming paper [3] will present some existence and multiplicity results in non-radial settings. Therefore, we restrict the functional space to be radial and assume $a(x) = a(|x|)$. Consider the Banach space

$$H_r = \left\{ v \in \mathcal{D}(\mathbb{R}^n) \;\middle|\; v \text{ is radial and } \int_{\mathbb{R}^n} |v|^{q+1} \, \mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|v\|_{H_q^1} = \left(\int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}x \right)^{1/2} + \left(\int_{\mathbb{R}^n} |v|^{q+1} \, \mathrm{d}x \right)^{1/(q+1)}.$$

Define the energy functional $I_\gamma \colon H_r \to \mathbb{R}$ associated with $(1.1)_\gamma$ as

$$I_\gamma(v) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}x - \frac{1}{q+1} \int_{\mathbb{R}^n} a_\gamma (v^+)^{q+1} \, \mathrm{d}x - \frac{1}{p+1} \int_{\mathbb{R}^n} (v^+)^{p+1} \, \mathrm{d}x.$$

From [24] we see that I_γ is C^1 from H_r to $\mathbb{R}$. Since $a(x) = a(|x|)$, the minimal element U_γ in $S_{M,\gamma}$ is radial. Hence, we study the following minimization problem in a convex constraint set:

$$\inf\{I_\gamma(v) \mid v \in Y\} \quad \text{and} \quad Y = \{v \in H_r \mid 0 \leqslant v \leqslant U_\Gamma \text{ almost everywhere (a.e.)}\}.$$

As in lemma 4.1, the infimum is attained at some function in Y, say v_γ, and $v_\gamma \in S_{M,\gamma}$. We show in the following theorem that v_γ is actually a local minimizer of I_γ in the H_r topology.

Theorem 1.8. *If $a(x) = a(|x|)$, for $0 < \gamma < \Gamma$, v_γ is a local minimizer for I_γ in H_r; that is, there exists $\delta > 0$ such that*

$$I_\gamma(v_\gamma) \leqslant I_\gamma(v) \quad \textit{for all } v \in H_r \textit{ with } \|v - v_\gamma\|_{H_r} < \delta.$$

Recall that Brezis and Nirenberg [9] first observed that minimization in the C^1-topology (for example, the sub- and super-solution construction above) yields minima in the weaker H^1-topology for a large class of subcritical elliptic variational problems (see also [5] for remarks on supercritical problems).

Given that we have a local minimizer of I_γ for $\gamma \in (0, \Gamma)$, we expect a second solution by using the celebrated mountain-pass theorem of Ambrosetti and Rabinowitz [20].

THEOREM 1.9. *If $a(x) = a(|x|)$, for $0 < \gamma < \Gamma$, (1.1)$_\gamma$ has at least two radially symmetric solutions in $S_{M,\gamma}$.*

Denoting the mountain-pass solution by V_γ, we could not rule out the possibility that $\operatorname{supp}(v_\gamma) \cap \operatorname{supp}(V_\gamma - v_\gamma) = \varnothing$, which means that V_γ and v_γ may coincide in the region Ω^+. The forthcoming paper [3] will present some results on this subject. This paper is organized as follows. We prove theorem 1.1 and part of theorem 1.5 in § 2. The other part of theorem 1.5 and theorem 1.7 are proved in §§ 3 and 4. In § 5 we prove theorem 1.8 and theorem 1.9. At the end of the paper we also discuss the boundedness assumption that we made for $a^-(x)$ at ∞. We would like to mention that there is a forthcoming paper [19] in which we deal with problem $(1.1)_\gamma$ in the case when $p = (n+2)/(n-2)$ and we show that there basically exist two solutions both in radial and non-radial settings.

2. Compact support and minimal solution

In this section we first prove theorem 1.1. The method used here is derived from the approach of Cortázar *et al.* [12] on the constant-coefficient equation $-\Delta u = u^p - u^q$. The regularity of solutions of (1.1) follows from standard bootstrap arguments (see [23, appendix B]) and standard elliptic theory [16]. Let $u(x)$ be a solution of (1.1). For any ball $B(x, 1) \subset B(x, 2)$, $x \in \mathbb{R}^n$, we have the following lemma.

LEMMA 2.1. *There exists a continuous function $h\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ with $h(0) = 0$ such that*

$$\|u\|_{L^\infty(B(x,1))} \leqslant Kh(\|u\|_{H^1(B(x,2))}).$$

The function h depends on q, p and n and the constant K depends on q, p, n and $\|a_\gamma\|_{L^\infty(B(x,2))}$.

Proof. This is a simple application of lemma 2.1 of [12], and we should mention that the assumption that $\limsup_{|x|\to\infty} a^-(x) < \infty$ is very important for this. □

LEMMA 2.2. *We have $\lim_{|x|\to\infty} u(x) = 0$.*

Proof. Since $u \in \mathcal{D}(\mathbb{R}^n)$ for $\varepsilon > 0$, there exists $R_1 > 0$, which depends on ε, such that

$$\|u\|_{\mathcal{D}(\mathbb{R}^n - B(0,R_1))} + \|u\|_{L^{2^*}(\mathbb{R}^n - B(0,R_1))} < \varepsilon.$$

Hence, for $x \in \mathbb{R}^n - \overline{B(0, R_1 + 3)}$, we have $B(x, 1) \subset B(x, 2) \subset \mathbb{R}^n - \overline{B(0, R_1)}$. From lemma 2.1 we obtain

$$|u(x)| \leqslant \|u\|_{L^\infty(B(x,1))} \leqslant Kh(\|u\|_{H^1(B(x,2))}).$$

https://doi.org/10.1017/S0308210509001231 Published online by Cambridge University Press

Note that $\|u\|_{H^1(B(x,2))}$ is controlled by $\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^n-B(0,R_1))}$ and $\|u\|_{L^{2^*}(\mathbb{R}^n-B(0,R_1))}$. Since $h(t)$ is continuous and $h(0)=0$, this lemma is proved. □

Now we give the proof of theorem 1.1.

Proof. Define two functions $f(s)$, $F(s)\colon \mathbb{R}^+\to\mathbb{R}$ as

$$f(s)=s^p-cs^q \quad\text{and}\quad F(s)=\frac{1}{p+1}s^{p+1}-c\frac{1}{q+1}s^{q+1},$$

where $c=\frac{1}{2}\liminf_{|x|\to\infty}a^-(x)$. Let $B>0$ be a constant such that $B^{p-q}=cq/p$. It is easy to see that $f(s)$ is strictly decreasing in the range $[0,B]$. Because of the choice of c and $\lim_{|x|\to\infty}u(x)=0$, there exists $R_1>R$ such that

$$a^-(x)\geqslant c \quad\text{and}\quad u(x)<B \quad\text{for all } x\in\mathbb{R}^n-B(0,R_1).$$

Let $w(r)$ be the function defined implicitly by

$$\int_{w(r)}^{B}\frac{\mathrm{d}s}{\sqrt{-F(s)}}=\sqrt{2}r.$$

It is easy to see that $w(r)$ satisfies

$$w''(r)+f(w(r))=0 \quad\text{in } (0,A),$$

where A is given by

$$\sqrt{2}A=\int_0^B\frac{\mathrm{d}s}{\sqrt{-F(s)}}.$$

Moreover, $w(r)$ is a decreasing function in r that satisfies

$$w(0)=B \quad\text{and}\quad w(A)=w'(A)=w''(A)=0.$$

Therefore, by defining $w(r)\equiv 0$ for $r\in[A,\infty)$, we obtain a non-increasing solution of

$$w''(r)+f(w(r))=0 \quad\text{in } (0,\infty)$$

with $w(0)=B$ and $\operatorname{supp}(w)=[0,A]$.

Finally, let $V(x)=w(|x|-R_1)$. Then we have

$$\begin{aligned}\Delta V-cV^q+V^p&\leqslant 0 &&\text{in } \mathbb{R}^n-\overline{B(0,R_1)},\\ V&=B &&\text{on } \partial(\mathbb{R}^n-B(0,R_1)).\end{aligned}$$

Note that, for u, we have

$$\begin{aligned}\Delta u-a^-u^q+u^p&=0 &&\text{in } \mathbb{R}^n-\overline{B(0,R_1)},\\ u&<B &&\text{on } \partial(\mathbb{R}^n-B(0,R_1)).\end{aligned}$$

By subtracting them, we have

$$-\Delta(V-u)\geqslant V^p-cV^q+a^-(x)u^q-u^p \quad\text{for } x\in(\mathbb{R}^n-\overline{B(0,R_1)}).$$

Claim 2.3. *$V\geqslant u\geqslant 0$ for $x\in\mathbb{R}^n-B(0,R_1)$.*

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Otherwise there exists $x_0 \in \mathbb{R}^n - \overline{B(0, R_1)}$ such that $u(x_0) > V(x_0)$, which implies that $V - u$ attains a global minimal value at some point in $\mathbb{R}^n - \overline{B(0, R_1)}$. We may assume that $V - u$ achieves minimal value at x_0. Then

$$\begin{aligned}
0 &\geqslant -\Delta(V-u)(x_0) \\
&\geqslant V^p(x_0) - cV^q(x_0) + a^-(x_0)u^q(x_0) - u^p(x_0) \\
&\geqslant V^p(x_0) - cV^q(x_0) + a^-(x_0)u^q(x_0) - u^p(x_0) + cu^q(x_0) - cu^q(x_0) \\
&\geqslant (V^p(x_0) - cV^q(x_0)) - (u^p(x_0) - cu^q(x_0)) + (a^-(x_0) - c)u^q(x_0) \\
&> 0.
\end{aligned}$$

This a contradiction. So $V \geqslant u \geqslant 0$ for $x \in (\mathbb{R}^n - B(0, R_1))$, which implies that u has compact support. □

Now we turn to the existence of a minimal element in $S_{I,\gamma}$ if it is not empty. We have the following theorem, which is the second part of theorem 1.5.

THEOREM 2.4. *Under assumption 1.3 and $I \neq \varnothing$, if $S_{I,\gamma} \neq \varnothing$, there exists a minimal element $u_{I,\gamma}$ in $S_{I,\gamma}$.*

The subscript γ is not important for the remainder of this section and we therefore drop it. Let $\bar{S}_I$ and $\bar{N}_I$ be the corresponding set of S_I and N_I for the following equation:

$$-\Delta v = a(x)v^q, \quad v \in \mathcal{D}(\mathbb{R}^n), \ v \geqslant 0. \tag{2.1}$$

Since $\liminf_{|x|\to\infty} a^-(x) > 0$, from [2], all solutions of (2.1) have compact support, the set $\bar{S}_I \neq \varnothing$ and $\bar{S}_I$ has a minimal element denoted by $\underline{u}_I$. Moreover, $\bar{N}_I$ has a unique element.

LEMMA 2.5. *Under assumption 1.3, if $S_I \neq \varnothing$, then $u \geqslant \underline{u}_I$ for any $u \in S_I$.*

Proof. Since $S_I \neq \varnothing$, pick any $u \in S_I$. Then there exists $J \subset M$ such that $I \subset J$ and $u \in N_J$. By the sub-supersolution method and the uniqueness in $\bar{N}_J$ we have $\underline{u} \leqslant u$, where $\underline{u}$ is the unique element in $\bar{N}_J$. Since $I \subset J$, $\underline{u} \in \bar{S}_I$. Therefore, $u \geqslant \underline{u} \geqslant \underline{u}_I$. □

We also need existence and uniqueness results for the equation

$$-\Delta v + a^-(x)v^q = a^+(x)h^q + h^p \quad \text{in } \mathbb{R}^n \text{ and } v \geqslant 0 \text{ in } \mathbb{R}^n, \tag{2.2}$$

where $h(x)$ is non-negative, smooth and compactly supported in $\mathbb{R}^n$.

LEMMA 2.6. *Equation (2.2) has a unique compactly supported solution.*

Proof. For $R > 0$, let us consider the Dirichlet boundary-value problem

$$-\Delta v + a^-(x)v^q = a^+(x)h^q + h^p \quad \text{in } B(0,R) \quad \text{and} \quad v = 0 \text{ on } \partial B(0,R).$$

Since h is non-negative, 0 is a subsolution to this problem. We also find that

$$\bar{v} = \int_{\mathbb{R}^n} \Phi(x-y)(a^+(y)h^q(y) + h^p)\,\mathrm{d}y$$

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satisfies

$$-\Delta\bar{v} = a^+h^q + h^p \geqslant a^+h^q + h^p - a^-\bar{v}^q \quad \text{in } \mathbb{R}^n,$$

where Φ is the fundamental solution of the Laplace equation, so $\bar{v}$ is a supersolution. By the sub-supersolution method [23] there exist a non-zero solution to this Dirichlet boundary problem, $v \in H_0^1(B(0,R))$, and $0 \leqslant v \leqslant \bar{v}$. Since h and a^- are Hölder continuous, this solution v is classical. A simple comparison argument shows that v is unique.

Next we show that when R is sufficiently large, v is compactly supported in $B(0,R)$. Indeed, since h is compactly supported, $\bar{v} \to 0$ uniformly as $|x| \to \infty$. Hence, there exists $R_1 > 0$ such that for $R > R_1$,

$$h(x) = 0, \qquad a^-(x) \geqslant c \quad \text{and} \quad v(x) \leqslant \bar{v} < B \quad \text{for all } x \in B(0,R) - B(0,R_1),$$

where B and c are chosen as in the proof of theorem 1.1. Following the proof of theorem 1.1, we construct a supersolution V and make the comparison in $B(0,R) - B(0,R_1)$ to show that $V \geqslant v$ when R_1 is sufficiently large. Since V is compactly supported, v is also a solution of (2.2).

The uniqueness is also an easy consequence of comparison. Suppose that there are two compactly supported smooth solutions v_1 and v_2. They satisfy

$$-\Delta v_1 + a^-v_1^q = a^+(x)h^q + h^p \quad \text{and} \quad -\Delta v_2 + a^-v_2^q = a^+(x)h^q + h^p \quad \text{in } \mathbb{R}^n.$$

Subtracting them, we have $-\Delta(v_1 - v_2) + a^-(x)(v_1^q - v_2^q) = 0$ in $\mathbb{R}^n$. We now multiply both sides by $(v_1 - v_2)$ and integrate over $\mathbb{R}^n$. Since they are compactly supported, we have

$$\int_{\mathbb{R}^n} |\nabla(v_1 - v_2)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a^-(v_1^q - v_2^q)(v_1 - v_2)\,\mathrm{d}x = 0.$$

So we must have $v_1 = v_2$. □

Now we start the monotone iteration process, using the minimal element in $\bar{S}_I$ as the starting point. Consider the following iteration problem:

$$-\Delta u_{n+1} + a^-u_{n+1}^q = a^+u_n^q + u_n^p \text{ in } \mathbb{R}^n, \qquad u_{n+1} \geqslant 0 \text{ in } \mathbb{R}^n, \tag{2.3}$$

where $u_1 = \underline{u}_I$ is the minimal element in $\bar{S}_I$.

Lemma 2.7. *Under assumption 1.3, u_n is well defined and compactly supported. Moreover, $u_{n+1} \geqslant u_n$ for all n.*

Proof. From lemma 2.6, u_n is well defined and compactly supported. Now we want to show that $u_2 \geqslant u_1$. u_1 and u_2 satisfy the following equations:

$$-\Delta u_1 + a^-u_1^q = a^+u_1^q \quad \text{and} \quad -\Delta u_2 + a^-u_2^q = a^+u_1^q + u_1^p \quad \text{in } \mathbb{R}^n.$$

By subtracting them, we obtain $-\Delta(u_1 - u_2) + a^-(u_1^q - u_2^q) = -u_1^p \leqslant 0$ in $\mathbb{R}^n$. Multiplying both sides by $(u_1 - u_2)^+$ and integrating over $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} |\nabla(u_1 - u_2)^+|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a^-(u_1^q - u_2^q)(u_1 - u_2)^+ \,\mathrm{d}x \leqslant 0,$$

which implies $(u_1 - u_2)^+ = 0$; that is, $u_2 \geqslant u_1$ in $\mathbb{R}^n$. The proof is completed by the standard induction process, which we omit. □

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LEMMA 2.8. *Under assumption 1.3, if $S_I \neq \varnothing$, then $u_n \leqslant u$ for any $u \in S_I$.*

Proof. Take any $u \in S_I$. From lemma 2.5, $u \geqslant u_1$. By the standard induction process, which is very similar to the previous one, we complete the proof. □

Finally, we are ready to prove theorem 2.4.

Proof. Taking any $u \in S_I$, the above lemmas show that u_n is increasing in n and $u_n \leqslant u$. Let $u_I = \lim_{n\to\infty} u_n$, then $u_I \leqslant u$. We only need to prove that u_I is a solution of (1.1).

Indeed, u_n is uniformly bounded above by u, which is compactly supported. From equation (2.3), we obtain that $\|u_n\|_{C^{1,\alpha}(\mathbb{R}^n)}$ is uniformly bounded, so, by the Arzela–Ascoli compactness theorem, u_n uniformly converges to u_I. Moreover, $u_n \rightharpoonup u_I$ weakly in $\mathcal{D}(\mathbb{R}^n)$. Now, taking any function $\phi \in C_0^\infty(\mathbb{R}^n)$, multiplying both sides of equation (2.3) by ϕ and integrating over $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} \nabla u_{n+1} \nabla\phi \,\mathrm{d}x + \int_{\mathbb{R}^n} a^- u_{n+1}^q \phi \,\mathrm{d}x = \int_{\mathbb{R}^n} a^+ u_n^q \phi \,\mathrm{d}x + \int_{\mathbb{R}^n} u_n^p \phi \,\mathrm{d}x.$$

Passing to the limit, we have

$$\int_{\mathbb{R}^n} \nabla u_I \nabla\phi \,\mathrm{d}x + \int_{\mathbb{R}^n} a^- u_I^q \phi \,\mathrm{d}x = \int_{\mathbb{R}^n} a^+ u_I^q \phi \,\mathrm{d}x + \int_{\mathbb{R}^n} u_I^p \phi \,\mathrm{d}x,$$

which implies that u_I is a solution of equation (1.1) in the weak sense, by standard bootstrap arguments [23] and elliptic theory [16], u_I is a classical solution. □

We want to mention that if $a(x) = a(|x|)$, the minimal element u_I in S_I is radial.

3. Existence for $S_{I,\gamma}$ and $N_{I,\gamma}$

In this section we first show the existence of $(1.1)_\gamma$ in $S_{I,\gamma}$. The idea is very simple and has already appeared in the proof of lemma 2.6. Namely, we find a global supersolution for $(1.1)_\gamma$ in $S_{I,\gamma}$, which is positive in $\mathbb{R}^n$ and uniformly goes to zero at infinity. It is obvious that this supersolution is also a supersolution of the following Dirichlet boundary-value problem:

$$-\Delta u = a_\gamma(x)u^q + u^p \quad \text{in } B(0,R),\ u \in H_0^1(B(0,R)),\ u \geqslant 0, \tag{3.1}$$

for any $R > 0$. For large R, we show that this boundary-value problem has a compactly supported solution in $B(0,R)$, which of course is a solution to $(1.1)_\gamma$ in $S_{I,\gamma}$.

First, let us define, for non-empty $I \in M := \{1, 2, \dots, k\}$ (recall that k denotes the number of connected components of Ω^+),

$$\Gamma_I \equiv \sup\{\gamma > 0 \mid S_{I,\gamma} \neq \varnothing \text{ for } (1.1)_\gamma\}.$$

LEMMA 3.1. *Under assumption 1.3, Γ_I is finite.*

Proof. Otherwise, for each Ω_i^+, $i \in I$, take a small ball B_i such that $B_i \subset\subset \Omega_i^+$. Let φ_i and λ_i, respectively, be the first positive eigenvalue and eigenfunction of the following problem:

$$-\Delta\varphi_i = \lambda_i \varphi_i \text{ in } B_i \quad \text{and} \quad \varphi_i = 0 \text{ on } \partial B_i.$$

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Multiplying both sides of $(1.1)_\gamma$ with φ_i and integrating over B_i, we obtain

$$\int_{B_i}(-\Delta u)\varphi_i\,\mathrm{d}x = \int_{B_i} a_\gamma u^q\varphi_i\,\mathrm{d}x + \int_{B_i} u^p\varphi_i\,\mathrm{d}x = \int_{B_i}\gamma a_i^+ u^q\varphi_i\,\mathrm{d}x + \int_{B_i} u^p\varphi_i\,\mathrm{d}x.$$

But

$$\int_{B_i}(\Delta\varphi_i u - \Delta u\varphi_i)\,\mathrm{d}x = \int_{\partial B_i}\left(\frac{\partial\varphi_i}{\partial n}u - \frac{\partial u}{\partial n}\varphi_i\right)\mathrm{d}S = \int_{\partial B_i}\frac{\partial\varphi_i}{\partial n}u\,\mathrm{d}S \leqslant 0,$$

where n is the outer unit normal vector of ∂B_i. Therefore, we have

$$\lambda_i\int_{B_i} u\varphi_i\,\mathrm{d}x = \int_{B_i} -\Delta\varphi_i u\,\mathrm{d}x \geqslant \int_{B_i} -\Delta u\varphi_i\,\mathrm{d}x = \int_{B_i} a_i^+ u^q\varphi_i\,\mathrm{d}x + \int_{B_i} u^p\varphi_i\,\mathrm{d}x,$$

i.e.

$$\int_{B_i}(\lambda_i u - \gamma a_i^+ u^q - u^p)\varphi_i\,\mathrm{d}x \geqslant 0.$$

Let $\underline{a} = \inf_{x\in\bigcup_{i\in I} B_i} a(x)$, then $\underline{a} > 0$. We obtain

$$\int_{B_i}(\lambda_i u - \gamma\underline{a}u^q - u^p)\varphi_i\,\mathrm{d}x \geqslant 0 \quad \text{for } i \in I,\ \gamma > 0.$$

By assumption, u is positive in $\Omega_I^+ = \bigcup_{i\in I}\Omega_i^+$, but

$$\lambda_i t - \gamma\underline{a}t^q - t^p = t^q(\lambda_i t^{1-q} - \gamma\underline{a} - t^{p-q}) < 0 \quad \text{for all } t > 0$$

if γ is sufficiently large, so this is a contradiction. We must have $\Gamma_I < \infty$. □

Since $\Gamma_I < \infty$, we shall prove that $\Gamma_I > 0$. Recall that it is shown in [11] that the non-negative smooth solutions of

$$\Delta v + v^{(n+2)/(n-2)} = 0 \quad \text{in } \mathbb{R}^n$$

with $n \geqslant 3$ are of the form

$$v(x) = \frac{[n(n-2)\lambda^2]^{(n-2)/4}}{(\lambda^2 + |x-x^0|^2)^{(n-2)/2}},$$

where $\lambda > 0$ and $x^0 \in \mathbb{R}^n$. Note that

$$v(x) = \frac{[n(n-2)\lambda^2]^{(n-2)/4}}{(\lambda^2 + |x-x^0|^2)^{(n-2)/2}} \leqslant \frac{[n(n-2)]^{(n-2)/4}}{\lambda^{(n-2)/2}} \equiv c(\lambda).$$

Pick $\lambda > 0$ so that $c(\lambda) = 1$ and fix some $x^0 \in \Omega^+$, with this special v denoted by V. Let $\liminf_{|x|\to\infty} a^- = a_\infty$. Then we have the following lemma.

LEMMA 3.2. *There exists* $\gamma_* > 0$ *so that*

$$-\Delta(MV) \geqslant a_\gamma(x)(MV)^q + (MV)^p$$

is always true for $\gamma \leqslant \gamma_*$ *and some* $M > 0$.

Proof. Let $a^\infty = \sup\{a(x) \mid x \in \mathbb{R}^n\}$ and B^+ be a ball including Ω^+ with centre x^0 such that

$$\inf\{a^-(x) \mid x \in \mathbb{R}^n - B^+\} > \tfrac{1}{2}a_\infty.$$

This can be done because $\liminf_{|x|\to\infty} a^- = a_\infty$. Let $K = \inf\{V(x) \mid x \in B^+\}$. When the radius of the ball B^+ tends to infinity, K goes to zero.

For some suitable positive constant M and small γ, to show

$$-\Delta(MV) \geqslant a_\gamma(x)(MV)^q + (MV)^p,$$

it is equivalent to show

$$MV^{(n+2)/(n-2)} \geqslant a_\gamma(x)(MV)^q + (MV)^p \quad \text{in } B^+,$$
$$MV^{(n+2)/(n-2)} \geqslant a_\gamma(x)(MV)^q + (MV)^p \quad \text{in } \mathbb{R}^n - B^+.$$

First we study the part in $\mathbb{R}^n - B^+$, where we need to obtain

$$M^{1-q}V^{(n+2)/(n-2)-q} \geqslant a_\gamma + (MV)^{p-q}.$$

But in $\mathbb{R}^n - B^+$, $a_\gamma = -a^- < -\frac{1}{2}a_\infty$ and $V \leqslant 1$, we have

$$-\tfrac{1}{2}a_\infty + M^{p-q} \geqslant a_\gamma + (MV)^{p-q}.$$

Choose M such that $0 < M < (\frac{1}{2}a_\infty)^{1/(p-q)}$. We obtain

$$M^{1-q}V^{(n+2)/(n-2)-q} > 0 \geqslant -\tfrac{1}{2}a_\infty + M^{p-q} \geqslant a_\gamma + (MV)^{p-q} \quad \text{in } \mathbb{R}^n - B^+.$$

Therefore, for $0 < M < (\frac{1}{2}a_\infty)^{1/(p-q)}$, we have

$$MV^{(n+2)/(n-2)} \geqslant a_\gamma(x)(MV)^q + (MV)^p \quad \text{in } \mathbb{R}^n - B^+.$$

Next we study the part in B^+, where we need to obtain

$$MV^{(n+2)/(n-2)} \geqslant a_\gamma(x)(MV)^q + (MV)^p.$$

In B^+, we know

$$MV^{(n+2)/(n-2)} \geqslant MK^{(n+2)/(n-2)},$$
$$\gamma a^\infty M^q + M^p \geqslant a_\gamma(x)(MV)^q + (MV)^p.$$

Therefore, we only need to show

$$MK^{(n+2)/(n-2)} \geqslant \gamma a^\infty M^q + M^p.$$

https://doi.org/10.1017/S0308210509001231 Published online by Cambridge University Press

Letting $A = a^{\infty} K^{-(n+2)/(n-2)}$ and $B = K^{-(n+2)/(n-2)}$, we need to show that

$$M^{1-q} \geqslant \gamma A + BM^{p-q}, \quad \text{i.e. } M^{1-q} - BM^{p-q} - \gamma A \geqslant 0.$$

We know that

$$\max\{t^{1-q} - Bt^{p-q} - \gamma A\} > 0 \iff (\gamma A)^{p-1} B^{1-q} < \frac{(p-1)^{p-1}(1-q)^{1-q}}{(p-q)^{p-q}},$$

and the maximal value is achieved at

$$t_B = \left[\frac{(1-q)}{B(p-q)}\right]^{1/(p-1)}.$$

As mentioned at the beginning of the proof, a large radius of B^+ means small K. In turn, B is large and t_B is small. So we choose large B^+ such that

$$0 < t_B < (\tfrac{1}{2} a_\infty)^{1/(p-q)}.$$

Take γ_* such that

$$(\gamma_* A)^{p-1} B^{1-q} = \frac{(p-1)^{p-1}(1-q)^{1-q}}{2(p-q)^{p-q}},$$

and choose $M = t_B$. For this choice of M we have, for $\gamma \leqslant \gamma_*$,

$$-\Delta(MV) \geqslant a_\gamma (MV)^q + (MV)^p \quad \text{in } \mathbb{R}^n.$$

□

Remark 3.3. Note that we can choose M somewhere between zero and t_B depending on γ so that, when $\gamma \to 0$, M also goes to zero.

The following theorem proves the first part of theorem 1.5 except for the existence at Γ_I.

Theorem 3.4. *Under assumption 1.3, we have* $0 < \Gamma_I < \infty$.

Proof. We only need to show that $\Gamma_I > 0$. Indeed, for any $R > 0$, from the previous lemma, MV is a supersolution for the Dirichlet boundary-value problem (3.1), which is

$$-\Delta u = a_\gamma(x) u^q + u^p \quad \text{in } B(0,R),\ u \in H_0^1(B(0,R)),\ u \geqslant 0,$$

where $\gamma \leqslant \gamma_*$. Because of the sublinear term we can always find an arbitrarily small subsolution supported in each of Ω_i^+, $i \in I$, (for details see [2, 7]). By the sub-supersolution method, this Dirichlet boundary-value problem has a solution $u_R \leqslant MV$. Since $\lim_{|x|\to\infty} MV = 0$, we can adopt the same argument as that used in the proof of lemma 2.6 to show that u_R is compactly supported in $B(0,R)$ for large R. Therefore, for large R, u_R is also a solution of $(1.1)_\gamma$ in $S_{I,\gamma}$, which means that $\Gamma_I > 0$. □

Recall that the minimal element in $S_{I,\gamma}$ is denoted as $u_{I,\gamma}$. The following proposition is also part of theorem 1.5.

Proposition 3.5. *$u_{I,\gamma}$ is increasing in γ; that is*

$$u_{I,\gamma_1} \leqslant u_{I,\gamma_2} \quad \textit{for } 0 < \gamma_1 < \gamma_2 < \Gamma_I.$$

Moreover, $\lim_{\gamma \to 0^+} \|u_{I,\gamma}\|_{L^\infty(\mathbb{R}^n)} = 0$.

Proof. It is easy to see that u_{I,γ_2} acts naturally as a supersolution for $(1.1)_{\gamma_1}$. Noting that u_{I,γ_2} has compact support, with proper small subsolution which is supported at each Ω_i^+ for $i \in I$, $(1.1)_{\gamma_1}$ has a compactly supported solution u in S_{I,γ_1} such that $u \leqslant u_{I,\gamma_2}$ by the sub-supersolution method. Since u_{I,γ_1} is the minimal element in S_{I,γ_1}, we have $u_{I,\gamma_1} \leqslant u_{I,\gamma_2}$. From remark 3.3 and the fact that $u_{I,\gamma}$ is the minimal element in $S_{I,\gamma}$, we have $\lim_{\gamma \to 0^+} \|u_{I,\gamma}\|_{L^\infty(\mathbb{R}^n)} = 0$. □

To this end, theorem 1.5 is proved except for the existence at the 'end point' Γ_I. Next we are going to prove theorem 1.7, which addresses the existence in $N_{I,\gamma}$ as in [2] when $a(x)$ is admissible.

Taking $c > 0$, which is chosen later, let

$$F(s) = \int_0^s \left(t^p - \frac{c}{n+1} t^q\right) \mathrm{d}t \quad \text{and} \quad \sigma = \left(\frac{c}{n+1}\frac{q}{p}\right)^{1/(p-q)}.$$

Let $e \in (0, \sigma]$, to be chosen later, and denote

$$\delta = \frac{1}{\sqrt{2}} \int_0^e \frac{\mathrm{d}s}{\sqrt{-F(s)}}.$$

We have the following lemma.

Lemma 3.6. *Letting $B = \{x \in \mathbb{R}^n \mid |x| < \delta\}$, the equation*

$$-\Delta v = v^p - cv^q \textit{ in } B \quad \textit{and} \quad v = e \textit{ on } \partial B$$

has a unique classical solution $\bar{u}$ such that $\bar{u}(0) = 0$ and $0 \leqslant \bar{u}(x) \leqslant e$ in B.

Proof. The uniqueness result is a simple matter of comparison. We are going to use the sub-supersolution method to show the existence. First we construct the supersolution. Let $w(r)$ be the function defined implicitly by

$$\int_{w(r)}^e \frac{\mathrm{d}s}{\sqrt{-F(s)}} = \sqrt{2} r.$$

It is easy to see that $w(r)$ satisfies

$$w''(r) + w^p(r) - \frac{c}{n+1} w^q(r) = 0 \quad \text{in } (0, \delta),$$

where δ is given as above. $w(r)$ is a decreasing function in r, $w(0) = e$ and $w''(\delta) = w'(\delta) = w(\delta) = 0$.

Now let $V(r) = w(\delta - r)$. Then $V(0) = V'(0) = V''(0) = 0$, $V(\delta) = e$ and $V(r)$ is increasing in $[0, \delta]$. Moreover, V satisfies

$$V''(r) + V^p(r) - \frac{c}{n+1} V^q(r) = 0 \quad \text{in } (0, \delta).$$

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Hence, for $r \leqslant \delta$, we have

$$V'(r) = \int_0^r V''(s)\,\mathrm{d}s = \int_0^r \frac{c}{n+1}V^q(s) - V^p(s)\,\mathrm{d}s \leqslant \left(\frac{c}{n+1}V^q(r) - V^p(r)\right)r.$$

A simple calculation shows that

$$\Delta V(r) = V''(r) + \frac{n-1}{r}V'(r) \leqslant cV^q(r) - V^p(r).$$

Therefore, V satisfies

$$-\Delta V \geqslant V^p - cV^q \text{ in } B(0,\delta) \quad \text{and} \quad V = e \text{ on } \partial B(0,\delta),$$

which implies that V is a supersolution. It is easy to see that 0 is a subsolution, so, by the sub-supersolution method, we have a solution $\bar{u}$ such that $0 \leqslant \bar{u} \leqslant V \leqslant e$ and $\bar{u}(0) = V(0) = 0$. □

We are now ready to give the proof of theorem 1.7.

Proof. By assumption, $a(x)$ is admissible and $\operatorname{dist}(\Omega_i^{0+}, \Omega_j^{0+}) > 0$ for any $i \neq j$. Letting $\bar{\delta} = \inf_{i\neq j} \operatorname{dist}(\Omega_i^{0+}, \Omega_j^{0+})$, we have $\bar{\delta} > 0$.

Picking R large enough that $\Omega^{0+} \subset\subset B(0,R)$ and denoting

$$C_i = \{x \in B(0, R+3\bar{\delta}) \mid \operatorname{dist}(x, \Omega_i^{0+}) \leqslant \tfrac{1}{16}\bar{\delta}\},$$

it is easy to see that $C_i \cap C_j = \varnothing$ for any $i \neq j$. Let $C = \bigcup_{i\in M} C_i$. We define

$$N = \{x \in B(0, R+2\bar{\delta}) \mid \operatorname{dist}(x, \Omega^{0+}) \geqslant \tfrac{1}{4}\bar{\delta}\}.$$

For any $x \in N$, $\overline{B(x, \bar{\delta}/16)} \cap C_i = \varnothing$ for any $i \in M$. Finally, letting

$$\underline{a} = \inf_{x\in B(0,R+3\bar{\delta})-C} a^-(x),$$

we have $\underline{a} > 0$. For the constants c and e used in lemma 3.6, let $c = \underline{a}$, then

$$\sigma = \left(\frac{\underline{a}}{n+1}\frac{q}{p}\right)^{1/(p-q)}.$$

Let

$$\delta_1 = \frac{1}{\sqrt{2}}\int_0^\sigma \frac{\mathrm{d}s}{\sqrt{-F(s)}}.$$

We make the following choice for e: if $\delta_1 > \bar{\delta}/16$, choose suitable e so that $\delta = \bar{\delta}/16$, and if $\delta_1 \leqslant \bar{\delta}/16$, choose e to be σ. The purpose of this choice is to make sure that $B(x,\delta) \cap C = \varnothing$ for any $x \in N$. Recall that $u_{M,\gamma}$ is the minimal element in $S_{M,\gamma}$. Since $\lim_{\gamma\to 0^+} \|u_{M,\gamma}\|_{L^\infty(\mathbb{R}^n)} = 0$, there exists $\gamma_0 > 0$ so that $\|u_{M,\gamma}\|_{L^\infty(\mathbb{R}^n)} < e$ for $\gamma \leqslant \gamma_0$.

Claim 3.7. *If $\gamma \leqslant \gamma_0$, $u_{M,\gamma}(x) = 0$ for any $x \in N$.*

In fact, taking $x \in N$, consider the following equation:

$$-\Delta v(y) = v^p(y) - a^-(y)v^q(y) \text{ in } B(x,\delta) \quad \text{and} \quad v = u_{M,\gamma} \text{ on } \partial B(x,\delta). \tag{3.2}$$

A simple comparison argument shows that this problem has a unique classical solution $v \leqslant e$, so $v = u_{M,\gamma}$. But, from lemma 3.6, the unique solution $\bar{u}$ of the problem

$$-\Delta v(y) = v^p(y) - cv^q(y) \text{ in } B(x,\delta) \quad \text{and} \quad v = e \text{ on } \partial B(x,\delta)$$

is a supersolution for equation (3.2). Since 0 is a subsolution, by the sub-supersolution method and uniqueness, we have $0 \leqslant u_{M,\gamma} \leqslant \bar{u}$ in $B(x,\delta)$. Since $\bar{u} \leqslant e$ and $\bar{u}(x) = 0$, $u_{M,\gamma} = 0$ for $x \in N$.

Since $u_{M,\gamma}$ is the minimal element in $S_{M,\gamma}$, then $u_{M,\gamma}$ vanishes outside of $B(0, R+2\bar{\delta})$. It is therefore easy to see that the support of $u_{M,\gamma}$ consists of k disjoint components, and its restriction to each component gives k compactly supported solutions of $(1.1)_\gamma$. By taking an appropriate union we can construct an element of N_I for any choice of $I \subset M$. This concludes the proof of theorem 1.7. □

4. Existence for $S_{I,\gamma}$ at Γ_I

So far, we have established an interval of existence for $(1.1)_\gamma$, $\gamma \in (0, \Gamma_I)$, in the class $S_{I,\gamma}$, where $I \subset M$ indicates the components of Ω^+ in which these solutions must be positive. Now we assert that a solution of class $S_{I,\gamma}$ must exist at the endpoint of the maximal interval of existence, $\gamma = \Gamma_I$. This is the 'extremal solution' for this family [10].

First we introduce the Banach space

$$H_q^1 = \left\{ v \in \mathcal{D}(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} |v|^{q+1}\,\mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|v\|_{H_q^1} = \left(\int_{\mathbb{R}^n} |\nabla v|^2\,\mathrm{d}x \right)^{1/2} + \left(\int_{\mathbb{R}^n} |v|^{q+1}\,\mathrm{d}x \right)^{1/(q+1)}.$$

Define the energy functional $I_\gamma \colon H_q^1 \to \mathbb{R}$ associated with $(1.1)_\gamma$ as

$$I_\gamma(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2\,\mathrm{d}x - \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+(v^+)^{q+1}\,\mathrm{d}x + \frac{1}{q+1}\int_{\mathbb{R}^n} a^-(v^+)^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n} (v^+)^{p+1}\,\mathrm{d}x.$$

It is a standard fact that I_γ is a C^1-functional on H_q^1 [24].

LEMMA 4.1. *Suppose that $\bar{u} \in N_{I,\bar{\gamma}}$ for some $\bar{\gamma} > 0$. Then $N_{I,\gamma}$ admits an element u_γ for every $0 < \gamma \leqslant \bar{\gamma}$. Moreover, $u_\gamma \leqslant \bar{u}$ and $I_\gamma(u_\gamma) < 0$.*

Proof. For $0 < \gamma \leqslant \bar{\gamma}$, $\bar{u}$ is a supersolution for the equation $(1.1)_\gamma$ and 0 is a subsolution. We consider the following minimization problem in a convex constraint set:

$$\inf\{I_\gamma(v) \mid v \in X\} \quad \text{and} \quad X = \{v \in H_q^1 \mid 0 \leqslant v \leqslant \bar{u} \text{ a.e.}\}.$$

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Note that $\bar{u}$ has compact support so, following [23], the infimum is achieved at some $u_\gamma \in X$ and $(\phi, I'_\gamma(u_\gamma)) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, and by routine regularity arguments, u_γ is a solution to $(1.1)_\gamma$. Since $u_\gamma \in X$, it vanishes on the components $\Omega^+ - \bigcup_{i\in I} \Omega_i^+$. It remains to show that u_γ does not vanish in Ω_i^+ for each $i \in I$.

Claim 4.2. *u_γ does not vanish in Ω_i^+ for each $i \in I$.*

Indeed, suppose, for some $i \in I$, that $u_\gamma \not> 0$ in Ω_i^+. Then the strong maximum principle and Hopf's lemma imply that $u_\gamma \equiv 0$ over $\overline{\Omega_i^+}$. Choose a ball $B \subset\subset \Omega_i^+$ and ϕ with $0 \leqslant \phi \in C_0^\infty(B)$. Hence, for small positive t, $(u_\gamma + t\phi) \in X$ and

$$I_\gamma(u_\gamma + t\phi) = I_\gamma(u_\gamma) + I_\gamma(t\phi) < I_\gamma(u_\gamma),$$

since

$$I_\gamma(t\phi) = \tfrac{1}{2}t^2 \int_B |\nabla\phi|^2 \,\mathrm{d}x - \frac{1}{q+1}t^{q+1}\gamma \int_B a^+\phi^{q+1}\,\mathrm{d}x - \frac{t^{p+1}}{p+1}\int_B \phi^{p+1}\,\mathrm{d}x < 0$$

for sufficiently small t. This contradicts the fact that u_γ is the infimum of I_γ over X. So we must have $u_\gamma \in N_{I,\bar{\gamma}}$. Also, note that $I_\gamma(t\phi) < 0$ for sufficiently small t, and thus $I_\gamma(u_\gamma) < 0$. □

Remark 4.3. Given the variational formulation of the problem as an infimum, it is natural to ask whether the solutions obtained by lemma 4.1 are local minima of I_γ in any sense. Note that this cannot be the case when $I \neq M$. Indeed, following the arguments used in the last part of the proof, we can decrease the value of I_γ near such a solution by small perturbations in each Ω_j^+, where $j \notin I$. So the existence of a second solution in the classes $N_{I,\gamma}$ remains an open question.

Corollary 4.4. *For $0 < \gamma < \Gamma_I$, $I_\gamma(u_{I,\gamma}) < 0$, where $u_{I,\gamma}$ is the minimum element in $S_{I,\gamma}$.*

Proof. We apply lemma 4.1 with $\bar{u} = u_{I,\gamma}$, $\bar{\gamma} = \gamma$ and some $J \subset M$ such that $I \subset J$ and $u_{I,\gamma} \in N_{J,\gamma}$. Hence, by lemma 4.1 we obtain a solution $u_\gamma \in S_{I,\gamma}$ such that

$$I_\gamma(u_\gamma) < 0 \quad \text{and} \quad 0 \leqslant u_\gamma \leqslant u_{I,\gamma}.$$

Since $u_{I,\gamma}$ is the minimal element in $S_{I,\gamma}$, we must have $u_\gamma = u_{I,\gamma}$. □

In order to show the existence at Γ_I, we need to show some estimates.

Lemma 4.5. *$\|u_{I,\gamma}\|_{H_q^1} + \|u_{I,\gamma}\|_{L^{p+1}(\mathbb{R}^n)}$ is uniformly bounded.*

Proof. We use the equation $-\Delta u_{I,\gamma} = a_\gamma u_{I,\gamma}^q + u_{I,\gamma}^p$. Multiplying both sides of this equation by $u_{I,\gamma}$ and integrating over $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} |\nabla u_{I,\gamma}|^2 \,\mathrm{d}x = \gamma \int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1}\,\mathrm{d}x - \int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1}\,\mathrm{d}x + \int_{\mathbb{R}^n} u_{I,\gamma}^{p+1}\,\mathrm{d}x. \tag{4.1}$$

From the above corollary, we have $I_\gamma(u_{I,\gamma}) < 0$; that is

$$\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_{I,\gamma}|^2 \,\mathrm{d}x + \frac{1}{q+1}\int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1}\,\mathrm{d}x < \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1}\,\mathrm{d}x + \frac{1}{p+1}\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1}\,\mathrm{d}x. \tag{4.2}$$

Putting (4.1) into (4.2), we obtain

$$\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n} a^- u_{I,\gamma}^{q+1}\,\mathrm{d}x + \left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1}\,\mathrm{d}x < \gamma\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1}\,\mathrm{d}x.$$

Since $1/(q+1) > \frac{1}{2} > 1/(p+1)$, from the above inequality we have

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n} u_{I,\gamma}^{p+1}\,\mathrm{d}x < \gamma\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1}\,\mathrm{d}x. \tag{4.3}$$

Since a^+ is compactly supported, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} a^+ u_{I,\gamma}^{q+1}\,\mathrm{d}x &\leqslant \|a^+\|_{L^\infty(\mathbb{R}^n)}\int_{\mathrm{supp}(a^+)} u_{I,\gamma}^{q+1}\,\mathrm{d}x\\ &\leqslant C(a^+)\|a^+\|_{L^\infty(\mathbb{R}^n)}\|u_{I,\gamma}\|_{L^{p+1}(\mathbb{R}^n)}^{q+1},\end{aligned} \tag{4.4}$$

where $C(a^+)$ is some constant depending on a^+ and Ω^+. Putting this back into (4.3), we find that

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_{I,\gamma}\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \leqslant C(a^+)\gamma\left(\frac{1}{q+1}-\frac{1}{2}\right)\|a^+\|_{L^\infty(\mathbb{R}^n)}\|u_{I,\gamma}\|_{L^{p+1}(\mathbb{R}^n)}^{q+1}.$$

Therefore, we have

$$\|u_{I,\gamma}\|_{L^{p+1}(\mathbb{R}^n)}^{p-q} \leqslant C(a^+)\gamma\left(\frac{1}{q+1}-\frac{1}{2}\right)\|a^+\|_{L^\infty(\mathbb{R}^n)}\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1},$$

which implies that $\|u_{I,\gamma}\|_{L^{p+1}(\mathbb{R}^n)}$ is uniformly bounded. Plugging this and (4.4) into (4.2), we conclude that $\|\nabla u_{I,\gamma}\|_{L^2(\mathbb{R}^n)}$ and $\|u_{I,\gamma}\|_{L^{q+1}(\mathbb{R}^n)}$ are uniformly bounded. □

Now we are ready to complete the proof of theorem 1.5.

Proof. Picking an increasing sequence $\{\gamma_n\}$ with limit Γ_I, from lemma 4.5,

$$\|u_{I,\gamma_n}\|_{H_q^1} + \|u_{I,\gamma_n}\|_{L^{p+1}(\mathbb{R}^n)}$$

is uniformly bounded. Hence, there exists $u_{\Gamma_I} \in H_q^1$ such that

$$u_{I,\gamma_n} \rightharpoonup u_{\Gamma_I} \quad \text{weakly in } \mathcal{D}(\mathbb{R}^n),\ L^{p+1}(\mathbb{R}^n) \text{ and } L^{q+1}(\mathbb{R}^n).$$

Moreover, $u_{I,\gamma_n} \to u_{\Gamma_I}$ a.e. in $\mathbb{R}^n$. From proposition 3.5 we know that u_{I,γ_n} is increasing in n, so by the monotone convergence theorem

$$u_{I,\gamma_n} \to u_{\Gamma_I} \quad \text{strongly in } L^{p+1}(\mathbb{R}^n) \text{ and } L^{q+1}(\mathbb{R}^n). \tag{4.5}$$

We know that u_{I,γ_n} satisfies the equation $-\Delta u_{I,\gamma_n} = a_{\gamma_n} u_{I,\gamma_n}^q + u_{I,\gamma_n}^p$. So, taking any $\varphi \in C_0^\infty(\mathbb{R}^n)$, multiplying both sides of the equation by φ and integrating over $\mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^n} \nabla u_{I,\gamma_n}\nabla\varphi\,\mathrm{d}x = \int_{\mathbb{R}^n} a_{\gamma_n} u_{I,\gamma_n}^q\varphi + \int_{\mathbb{R}^n} u_{I,\gamma_n}^p\varphi.$$

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By (4.5), passing to the limit on n, we have

$$\int_{\mathbb{R}^n} \nabla u_{\Gamma_I} \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^n} a_{\Gamma_I} u_{\Gamma_I}^q \varphi + \int_{\mathbb{R}^n} u_{\Gamma_I}^p \varphi.$$

Therefore, u_{Γ_I} is a weak solution of $(1.1)_{\Gamma_I}$. By routine regularity arguments, u_{Γ_I} is a classical solution. □

COROLLARY 4.6. *u_{Γ_I} is the minimal element in S_{I,Γ_I}, i.e. $u_{\Gamma_I} = u_{I,\Gamma_I}$.*

Proof. From above S_{I,Γ_I} is not empty. Picking any $U \in S_{\Gamma_I}$, we just need to apply lemma 4.1 to equation $(1.1)_\gamma$ with $\bar{u} = U$, $\bar{\gamma} = \Gamma_I$ and some $J \subset M$ such that $I \subset J$ and $U \in N_{\Gamma_J}$. We obtain a solution u_γ to $(1.1)_\gamma$ such that $u_\gamma \in S_{I,\gamma}$ and we also have $U \geqslant u_\gamma \geqslant u_{I,\gamma}$. Since $\lim_{\gamma \to \Gamma_I^-} u_{I,\gamma} = u_{\Gamma_I}$, we have $U \geqslant u_{\Gamma_I}$. □

For later, denote Γ_M by Γ, denote $u_{M,\gamma}$ by U_γ and denote u_{M,Γ_M} by U_Γ. We conclude this section with a simple result.

COROLLARY 4.7. *Assume that $a(x) = a(|x|)$, then $U_\gamma(x) = U_\gamma(|x|)$ for $0 < \gamma \leqslant \Gamma$.*

5. Second solution in $S_{M,\gamma}$

In this section we are going to show the existence of a second solution in $S_{M,\gamma}$ for $0 < \gamma < \Gamma$. The embedding $H_q^1(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$ is not compact and the compactly supported solution of $-\Delta v = v^p - v^q$ in $\mathbb{R}^n$ by Cortázar *et al.* [12] poses a difficulty for proving the compactness of the Palais–Smale sequence. So we assume $a(x) = a(|x|)$ in this section and restrict the functional space to be radial. Consider the Banach space

$$H_r = \left\{ v \in \mathcal{D}(\mathbb{R}^n) \;\middle|\; v \text{ is radial and } \int_{\mathbb{R}^n} |v|^{q+1} \, \mathrm{d}x < \infty \right\}$$

endowed with the same norm as in H_q^1. It is obvious that I_γ is a C^1 functional on H_r.

Recall that U_γ represents the minimal element in $S_{M,\gamma}$ for $0 < \gamma \leqslant \Gamma$. Consider the following minimization problem in a convex constraint set:

$$\inf\{I_\gamma(v) \mid v \in Y\} \quad \text{and} \quad Y = \{v \in H_r \mid 0 \leqslant v \leqslant U_\Gamma \text{ a.e.}\}. \tag{5.1}$$

As in lemma 4.1, the infimum is attained at some radial function in Y, say v_γ. By the principle of symmetric criticality, $v_\gamma \in S_{M,\gamma}$.

Now we are going to show that v_γ is actually a local minimizer of I_γ in H_r. Pick $R > 0$ so that $\operatorname{supp}(U_\Gamma) \subset\subset B(0,R)$ and $\Omega^{0+} \subset\subset B(0,R)$. Let $H_r^1(B(0,R))$ be the subspace of $H^1(B(0,R))$, which contains radially symmetric functions.

LEMMA 5.1. *For $\gamma \in (0,\Gamma)$, v_γ is a local minimizer for I_γ in $H_r^1(B(0,R))$; that is, there exists $\delta > 0$ such that*

$$I_\gamma(v_\gamma) \leqslant I_\gamma(v) \quad \textit{for all } v \in H_r^1(B(0,R)) \textit{ with } \|v - v_\gamma\|_{H_r^1(B)} < \delta.$$

https://doi.org/10.1017/S0308210509001231 Published online by Cambridge University Press

Proof. We already know that

$$I_\gamma(v_\gamma) = \inf\{I_\gamma(v) \mid v \in Y\}.$$

Since $\operatorname{supp}(U_\Gamma) \subset\subset B(0,R)$, we find out that

$$I_\gamma(v_\gamma) = \inf\{I_\gamma(v) \mid v \in H_r^1(B(0,R)) \text{ and } 0 \leqslant v \leqslant U_\Gamma\}.$$

We follow the same proof as that used in proposition 5.2 of [1] to complete the proof. It is worth pointing out that there is an extra assumption on Ω^{0+} in the proof of proposition 5.2 of [1], which is the following.

ASSUMPTION 5.2. Ω^{0+} has $m < \infty$ connected components with $\Omega^{0+} = \bigcup_{i=1}^m \Omega_i^{0+}$, and $\Omega_i^{0+} \cap \Omega^+ \neq \varnothing$ for every $i = 1, \dots, m$.

We do not need this extra assumption because U_Γ is the minimal element in $S_{M,\Gamma}$. Indeed, if there is one connected component of Ω^{0+}, say Ω_j^{0+}, such that $\Omega_j^{0+} \cap \Omega^+ \equiv \varnothing$, we must have $a(x) \equiv 0$ in Ω_j^{0+}. Moreover, either $U_\Gamma(x) \equiv 0$ in Ω_j^{0+} or one of the connected components of $\operatorname{supp}(U_\Gamma)$ includes Ω_j^{0+} and some connected component of Ω^+, which means that any connected component of $\operatorname{supp}(U_\Gamma)$ has to include one of the connected components of Ω^+. This fact helps to lift the extra assumption on Ω^+ and completes the proof. □

LEMMA 5.3. *For* $\gamma \in (0, \Gamma)$, v_γ *is also a local minimizer for* I_γ *in* H_r.

Proof. From lemma 5.3 there exists $\delta > 0$ such that

$$I_\gamma(v_\gamma) \leqslant I_\gamma(v) \quad \text{for all } v \in H_r^1(B) \text{ with } \|v - v_\gamma\|_{H_r^1(B)} < \delta.$$

There exists $\delta_1 > 0$ such that

$$\|v - v_\gamma\|_{H_r^1(B)} < \delta \quad \text{if } \|v - v_\gamma\|_{H_r} < \delta_1.$$

Now, by density, taking any symmetric function $v \in C_0^\infty(\mathbb{R}^n) \cap H_r$ with $\|v - v_\gamma\|_{H_r} < \delta_1$ and noting that $\Omega^{0+} \subset\subset B(0,R)$, we have

$$\begin{aligned}
I_\gamma(v) &= \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 \,\mathrm{d}x - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma (v^+)^{q+1} \,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n} (v^+)^{p+1} \,\mathrm{d}x \\
&= \frac{1}{2}\int_{B(0,R)} |\nabla v|^2 \,\mathrm{d}x - \frac{1}{q+1}\int_{B(0,R)} a_\gamma (v^+)^{q+1} \,\mathrm{d}x - \frac{1}{p+1}\int_{B(0,R)} (v^+)^{p+1} \,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n - B(0,R)} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{q+1}\int_{\mathbb{R}^n - B(0,R)} a^- (v^+)^{q+1} \,\mathrm{d}x \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^n - B(0,R)} (v^+)^{p+1} \,\mathrm{d}x \\
&\geqslant I_\gamma(v_\gamma) + \frac{1}{2}\int_{\mathbb{R}^n - B(0,R)} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{q+1}\int_{\mathbb{R}^n - B(0,R)} a^- (v^+)^{q+1} \,\mathrm{d}x \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^n - B(0,R)} (v^+)^{p+1} \,\mathrm{d}x.
\end{aligned}$$

Denote $\inf_{x\in R^n-B(0,r)} a^-$ by c. We obtain

$$I_\gamma(v) \geqslant I_\gamma(v_\gamma) + \frac{1}{2}\int_{\mathbb{R}^n-B(0,R)} |\nabla v|^2\,\mathrm{d}x + \frac{c}{q+1}\int_{\mathbb{R}^n-B(0,R)} (v^+)^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n-B(0,R)} (v^+)^{p+1}\,\mathrm{d}x.$$

Let

$$V = \begin{cases} v(R) & x\in B(0,R), \\ v & x\in \mathbb{R}^n - B(0,R). \end{cases}$$

Then $V\in H_r$. So we have

$$I_\gamma(v) - I_\gamma(v_\gamma) \geqslant \frac{1}{2}\int_{\mathbb{R}^n} |\nabla V|^2\,\mathrm{d}x + \frac{c}{q+1}\int_{\mathbb{R}^n-B(0,R)} (V^+)^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n-B(0,R)} (V^+)^{p+1}\,\mathrm{d}x.$$

Claim 5.4.

$$E(V) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla V|^2\,\mathrm{d}x + \frac{c}{q+1}\int_{\mathbb{R}^n-B} (V^+)^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n-B} (V^+)^{p+1}\,\mathrm{d}x \geqslant 0$$

when δ_1 is sufficiently small.

Indeed, by using Hölder's inequality and denoting

$$d = \frac{n+2-p(n-2)}{n+2-q(n-2)},$$

we have

$$\int_{\mathbb{R}^n-B(0,R)} |V^+|^{p+1}\,\mathrm{d}x \leqslant \|V^+\|^{d(q+1)}_{L^{q+1}(\mathbb{R}^n-B(0,R))}\|V^+\|^{2^*(1-d)}_{L^{2^*}(\mathbb{R}^n-B(0,R))}. \tag{5.2}$$

Since $d+(1-d)n/(n-2)>1$, there exist $\alpha>1$ and $\beta>1$ such that

$$\frac{1}{\alpha}+\frac{1}{\beta}=1, \qquad \bar{\alpha}=d\alpha(q+1)>q+1 \quad \text{and} \quad \bar{\beta}=\beta(1-d)2^*>2.$$

Hence, from (5.2) and Young's inequality, we obtain

$$\int_{\mathbb{R}^n-B(0,R)} |V^+|^{p+1}\,\mathrm{d}x \leqslant \frac{1}{\alpha}\|V^+\|^{\bar{\alpha}}_{L^{q+1}(\mathbb{R}^n-B(0,R))} + \frac{1}{\beta}\|V^+\|^{\bar{\beta}}_{L^{2^*}(\mathbb{R}^n-B(0,R))}.$$

From the above inequality and the Sobolev inequality, we find that

$$E(v) \geqslant \frac{C(n)}{2}\|V^+\|^2_{L^{2^*}(\mathbb{R}^n)} + \frac{c}{q+1}\|V^+\|^{q+1}_{L^{q+1}(\mathbb{R}^n-B(0,R))} - \frac{1}{\alpha(p+1)}\|V^+\|^{\bar{\alpha}}_{L^{q+1}(\mathbb{R}^n-B(0,R))} - \frac{1}{\beta(p+1)}\|V^+\|^{\bar{\beta}}_{L^{2^*}(\mathbb{R}^n-B(0,R))}$$

$$\geqslant \frac{C(n)}{2}\|V^+\|^2_{L^{2^*}(\mathbb{R}^n-B(0,R))} + \frac{c}{q+1}\|V^+\|^{q+1}_{L^{q+1}(\mathbb{R}^n-B(0,R))} - \frac{1}{\alpha(p+1)}\|V^+\|^{\bar\alpha}_{L^{q+1}(\mathbb{R}^n-B(0,R))} - \frac{1}{\beta(p+1)}\|V^+\|^{\bar\beta}_{L^{2^*}(\mathbb{R}^n-B(0,R))}.$$

Since $\bar\alpha > q+1$ and $\bar\beta > 2$, for sufficiently small δ_1, we obtain

$$E(v) \geqslant \frac{C(n)}{2}\|V^+\|^2_{L^{2^*}(\mathbb{R}^n-B(0,R))} + \frac{c}{q+1}\|V^+\|^{q+1}_{L^{q+1}(\mathbb{R}^n-B(0,R))} - \frac{1}{\alpha(p+1)}\|V^+\|^{\bar\alpha}_{L^{q+1}(\mathbb{R}^n-B(0,R))} - \frac{1}{\beta(p+1)}\|V^+\|^{\bar\beta}_{L^{2^*}(\mathbb{R}^n-B(0,R))} \geqslant 0.$$

Therefore, we have $I_\gamma(v) - I_\gamma(v_\gamma) \geqslant 0$ for sufficiently small δ_1; that is, v_γ is a local minimizer in H_r. □

From lemma 5.3 we know that v_γ is a local minimizer for the energy functional

$$I_\gamma = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2\,\mathrm{d}x - \frac{1}{q+1}\int_{\mathbb{R}^n} a_\gamma (v^+)^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n} (v^+)^{p+1}\,\mathrm{d}x, \quad v \in H_r.$$

It is easy to see that $I_\gamma(t\varphi) \to -\infty$ as $t \to \infty$ for some positive radially symmetric $\varphi \in C_0^\infty(\mathbb{R}^n)$. So we have a mountain-pass structure. We expect to find a second solution in the form $u = v_\gamma + v$ with $v \geqslant 0$. If u solves the problem $(1.1)_\gamma$, then v should solve

$$-\Delta v = a_\gamma[(v_\gamma + v)^q - v_\gamma^q] + [(v_\gamma + v)^p - v_\gamma^p].$$

Set

$$\begin{aligned} h(x,v) &= a_\gamma[(v_\gamma + v^+)^q - v_\gamma^q] + [(v_\gamma + v^+)^p - v_\gamma^p],\\ H(x,v) &= \int_0^v h(x,s)\,\mathrm{d}s\\ &= \int_0^v a_\gamma[(v_\gamma + s^+)^q - v_\gamma^q] + [(v_\gamma + s^+)^p - v_\gamma^p]\,\mathrm{d}s\\ &= \frac{1}{q+1}a_\gamma[(v_\gamma + v^+)^{q+1} - v_\gamma^{q+1}] - a_\gamma v_\gamma^q v^+\\ &\quad + \frac{1}{p+1}[(v_\gamma + v^+)^{p+1} - v_\gamma^{p+1}] - v_\gamma^p v^+. \end{aligned}$$

For $v \in H_r$, define the functional

$$J_\gamma(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2\,\mathrm{d}x + \frac{1}{q+1}|v|^{q+1} - \frac{1}{q+1}(v^+)^{q+1} - H(x,v)\,\mathrm{d}x.$$

By some calculations, we reach

$$J_\gamma(v) = I_\gamma(v_\gamma + v^+) - I_\gamma(v_\gamma) + \tfrac{1}{2}\|\nabla v^-\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{q+1}\|v^-\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}.$$

LEMMA 5.5. *There exists* $\delta_1 > 0$ *such that* $J_\gamma(v) \geqslant J_\gamma(0) = 0$ *when* $\|v\|_{H_r} < \delta_1$.

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Proof. From the above calculations, we have

$$J_\gamma(v) = I_\gamma(v_\gamma + v^+) - I_\gamma(v_\gamma) + \tfrac{1}{2}\|\nabla v^-\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{q+1}\|v^-\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}.$$

The result follows from lemma 5.3. □

LEMMA 5.6. *For $\gamma > 0$, there exists a radial function $\varphi \in C_0^\infty(R^n)$ with $\varphi \gneqq 0$ and $T > 0$ such that $J_\gamma(T\varphi) < 0$.*

Proof. Taking a radial function $\varphi \in C_0^\infty(R^n)$ with $\varphi \gneqq 0$ such that the support of φ is separated from the support of v_γ, we have

$$J_\gamma(T\varphi) = I_\gamma(T\varphi) = T^2 \int \tfrac{1}{2}|\nabla\varphi|^2 - T^{q+1}\int \frac{a_\gamma(x)}{q+1}|\varphi|^{q+1} - T^{p+1}\int \frac{1}{p+1}|\varphi|^{p+1} < 0$$

for sufficiently large T, since $q < 1 < p$. □

The next lemma shows that the Palais–Smale sequence is bounded.

LEMMA 5.7. *Suppose that $0 < \gamma < \Gamma$, $\{v_n\}$ is a sequence in H_r such that $J_\gamma(v_n) \to c_\gamma$ and $J'_\gamma(v_n) \to 0$. Then $\{v_\gamma + v_n^+\}$ is uniformly bounded in H_r.*

Proof. First, noting that $J'_\gamma(v_n)v_n^- = -(\|\nabla v_n^-\|^2_{L^2(\mathbb{R}^n)} + \|v_n^-\|^{q+1}_{L^{q+1}(\mathbb{R}^n)})$, we have

$$\begin{aligned}\|\nabla v_n^-\|^2_{L^2(\mathbb{R}^n)} + \|v_n^-\|^{q+1}_{L^{q+1}(\mathbb{R}^n)} &\leqslant \|J'_\gamma(v_n)\|(\|\nabla v_n^-\|_{L^2(\mathbb{R}^n)} + \|v_n^-\|_{L^{q+1}(\mathbb{R}^n)}) \\ &\leqslant \|J'_\gamma(v_n)\|(\|\nabla v_n^-\|^2_{L^2(\mathbb{R}^n)} + \|v_n^-\|^{q+1}_{L^{q+1}(\mathbb{R}^n)} + O(1)) \\ &\leqslant o(1)(\|\nabla v_n^-\|^2_{L^2(\mathbb{R}^n)} + \|v_n^-\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}) + o(1).\end{aligned}$$

Hence, we derive that

$$(1 - o(1))(\|\nabla v_n^-\|^2_{L^2(\mathbb{R}^n)} + \|v_n^-\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}) \leqslant o(1),$$

which implies $v_n^- \to 0$ in H_r.

Therefore, we may take $u_n = v_\gamma + v_n^+$. Then we obtain

$$I_\gamma(u_n) \to I_\gamma(v_\gamma) + c_\gamma \quad \text{and} \quad I'_\gamma(u_n) \to 0.$$

Since $I_\gamma(v_\gamma) < 0$, we have

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_n|^2 \,\mathrm{d}x + \frac{1}{q+1}\int_{\mathbb{R}^n} a^- u_n^{q+1}\,\mathrm{d}x \\ - \frac{\gamma}{q+1}\int_{\mathbb{R}^n} a^+ u_n^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n} u_n^{p+1}\,\mathrm{d}x < c_\gamma. \qquad (5.3)\end{aligned}$$

We also have

$$\begin{aligned}I'_\gamma(u_n)u_n &= \int_{\mathbb{R}^n} |\nabla u_n|^2\,\mathrm{d}x + \int_{\mathbb{R}^n} a^- u_n^{q+1}\,\mathrm{d}x - \gamma\int_{\mathbb{R}^n} a^+ u_n^{q+1}\,\mathrm{d}x - \int_{\mathbb{R}^n} u_n^{p+1}\,\mathrm{d}x \\ &= o(1)\|u_n\|_{H_r}.\end{aligned}$$

Pick θ such that $2 < \theta < p+1$. Then

$$\frac{1}{p+1} < \frac{1}{\theta} < \frac{1}{2} < \frac{1}{q+1}.$$

From the above, we obtain

$$\frac{1}{\theta}\int_{\mathbb{R}^n} |\nabla u_n|^2 \,\mathrm{d}x + \frac{1}{\theta}\int_{\mathbb{R}^n} a^- u_n^{q+1} \,\mathrm{d}x - \frac{\gamma}{\theta}\int_{\mathbb{R}^n} a^+ u_n^{q+1} \,\mathrm{d}x - \frac{1}{\theta}\int_{\mathbb{R}^n} u_n^{p+1} \,\mathrm{d}x = o(1)\|u_n\|_{H_r}. \qquad (5.4)$$

Subtracting (5.3) from (5.4), we obtain

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\left(\int_{\mathbb{R}^n} |\nabla u_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a^- u_n^{q+1} \,\mathrm{d}x\right) + \left(\frac{1}{\theta} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} u_n^{p+1} \,\mathrm{d}x \leqslant \gamma\left(\frac{1}{q+1} - \frac{1}{\theta}\right)\int_{\mathbb{R}^n} a^+ u_n^{q+1} \,\mathrm{d}x + c_\gamma + o(1)\|u_n\|_{H_r}.$$

From Young's inequality, we have

$$a^+ u_n^{q+1} = \left(\frac{1}{\varepsilon}a^+\right)(\varepsilon u_n^{q+1}) < \frac{q+1}{p+1}(\varepsilon u_n^{q+1})^{(p+1)/(q+1)} + \frac{p-q}{p+1}\left(\frac{1}{\varepsilon}a^+\right)^{((p+1)/(q+1))^*},$$

where $((p+1)/(q+1))^*$ is the dual of $(p+1)/(q+1)$ and ε is small enough that

$$\Gamma\frac{q+1}{p+1}\varepsilon^{(p+1)/(q+1)}\left(\frac{1}{q+1} - \frac{1}{\theta}\right) \leqslant \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p+1}\right).$$

Pick $C > 0$ such that

$$\Gamma\frac{p-q}{p+1}\left(\frac{1}{q+1} - \frac{1}{\theta}\right)\int_{\mathbb{R}^n}\left(\frac{1}{\varepsilon}a^+\right)^{((p+1)/(q+1))^*} \leqslant C.$$

Overall, we reach

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\left(\int_{\mathbb{R}^n} |\nabla u_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a^- u_n^{q+1} \,\mathrm{d}x\right) + \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} u_n^{p+1} \,\mathrm{d}x \leqslant c_\gamma + C + o(1)\|u_n\|_{H_r}. \qquad (5.5)$$

CLAIM 5.8. *There exists small positive* $\eta < \min\{\frac{1}{2}a_\infty, \frac{1}{2}\}$ *and a constant* $C_1 > 0$ *such that*

$$\int_{\mathbb{R}^n} |\nabla u_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a^- u_n^{q+1} \,\mathrm{d}x + C_1 \geqslant \eta(\|\nabla u_n\|^2_{L^2(\mathbb{R}^n)} + \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}).$$

Indeed, since $\liminf_{|x|\to\infty} a^- = a_\infty$, there exists $r_1 > 0$ such that $a^-(x) \geqslant \frac{1}{2}a_\infty$ for $x \in R^n - B(0, r_1)$. Now we see that

$$\begin{aligned}(1-\eta)\|\nabla u_n\|^2_{L^2(\mathbb{R}^n)} &\geqslant \tfrac{1}{2}\|\nabla u_n\|^2_{L^2(\mathbb{R}^n)} \\ &\geqslant \tfrac{1}{2}C(n)\|u_n\|^2_{L^{2^*}(\mathbb{R}^n)} \\ &\geqslant \tfrac{1}{2}C(n)\|u_n\|^2_{L^{2^*}(B(0,r_1))}\end{aligned}$$

and

$$\eta\|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^n)} - \int_{\mathbb{R}^n} a^- u_n^{q+1}\,\mathrm{d}x \leqslant \eta\|u_n\|^{q+1}_{L^{q+1}(B(0,r_1))} \leqslant \eta C(r_1)\|u_n\|^{q+1}_{L^{2^*}(B(0,r_1))},$$

where $C(n)$ is the best Sobolev constant and $C(r_1)$ is a constant depending on r_1. When η is small and C_1 is large, we conclude that

$$\begin{aligned}(1-\eta)\|\nabla u_n\|^2_{L^2(\mathbb{R}^n)} + C_1 &\geqslant \tfrac{1}{2}C(n)\|u_n\|^2_{L^{2^*}(B(0,r_1))} + C_1 \\ &\geqslant \tfrac{1}{2}C(n)\|u_n\|^{q+1}_{L^{2^*}(B(0,r_1))} \\ &\geqslant \eta C(r_1)\|u_n\|^{q+1}_{L^{2^*}(B(0,r_1))} \\ &\geqslant \eta\|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^n)} - \int_{\mathbb{R}^n} a^- u_n^{q+1}\,\mathrm{d}x.\end{aligned}$$

From (5.5) and the above claim, enlarging the constant C, we obtain

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{\theta}\right)\eta(\|\nabla u_n\|^2_{L^2(\mathbb{R}^n)} + \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}) \\ &\qquad\leqslant c_\gamma + C + o(1)(\|\nabla u_n\|_{L^2(\mathbb{R}^n)} + \|u_n\|_{L^{q+1}(\mathbb{R}^n)}) \\ &\qquad\leqslant c_\gamma + C + o(1)(\|\nabla u_n\|^2_{L^2(\mathbb{R}^n)} + \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}) \\ &\qquad\leqslant c_\gamma + C + o(1)(\|\nabla u_n\|^2_{L^2(\mathbb{R}^n)} + \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^n)}),\end{aligned}$$

which implies that $\|\nabla u_n\|_{L^2(\mathbb{R}^n)}$ and $\|u_n\|_{L^{q+1}(\mathbb{R}^n)}$ are uniformly bound. Going back to (5.5), we have $\|u_n\|_{L^{p+1}(\mathbb{R}^n)}$ also uniformly bounded. □

The most important fact about the radial function in H_r is the following lemma due to Strauss [22] (see also [8]).

Lemma 5.9 (Strauss [22]). *H_r compactly embeds in $L^{p+1}(\mathbb{R}^n)$ for $1 < p < 2^* - 1$.*

We can now prove the compactness of the Palais–Smale sequence using this lemma.

Lemma 5.10. *Suppose that $0 < \gamma < \Gamma$, $\{v_n\}$ is a sequence in H_r such that $J_\gamma(v_n) \to c$ and $J'_\gamma(v_n) \to 0$. Then $\{v_n\}$ contains a strongly convergent subsequence in H_r. Moreover, if $v_n \to v_0 \geqslant 0$, then $u_0 = v_\gamma + v_0$ is a solution to (1.1)$_\gamma$.*

Proof. In view of lemma 5.7, taking $u_n = v_n^+ + v_\gamma$, we have

$$I_\gamma(u_n) \to I_\gamma(v_\gamma) + c_\gamma \quad \text{and} \quad I'_\gamma(u_n) \to 0.$$

Again from lemma 5.7, we have that $\|\nabla u_n\|_{L^2} + \|u_n\|_{L^{q+1}} + \|u_n\|_{L^{p+1}}$ is uniformly bounded. So, from lemma 5.9, restricting to a subsequence if necessary, there exists $u_0 \in H_r$ such that

$$u_n \rightharpoonup u_0 \text{ weakly in } H_r \quad \text{and} \quad u_n \to u_0 \text{ strongly in } L^{p+1}(\mathbb{R}^n).$$

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By the weak and strong convergences it is easy to see that u_0 is a solution to equation $(1.1)_\gamma$. Hence, u_0 has compact support and $I'_\gamma(u_0) = 0$. Now we obtain

$$\begin{aligned}(I'_\gamma(u_n) - I'_\gamma(u_0))(u_n - u_0) &= \int_{\mathbb{R}^n} |\nabla(u_n - u_0)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a^-(u_n^q - u_0^q)(u_n - u_0)\,\mathrm{d}x \\ &\quad - \gamma \int_{\mathbb{R}^n} a^+(u_n^q - u_0^q)(u_n - u_0)\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^n} (u_n^p - u_0^p)(u_n - u_0)\,\mathrm{d}x \\ &\to 0. \end{aligned} \tag{5.6}$$

Since $u_n \to u_0$ strongly in $L^{p+1}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} (u_n^p - u_0^p)(u_n - u_0)\,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^n} a^+(u_n^q - u_0^q)(u_n - u_0)\,\mathrm{d}x \to 0.$$

Therefore, (5.6) reduces to

$$\int_{\mathbb{R}^n} |\nabla(u_n - u_0)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} a^-(u_n^q - u_0^q)(u_n - u_0)\,\mathrm{d}x \to 0,$$

which implies that $u_n \to u_0$ in $\mathcal{D}(\mathbb{R}^n)$.

CLAIM 5.11. *$u_n \to u_0$ strongly in $L^{q+1}(\mathbb{R}^n)$.*

Indeed, we know that u_0 is a solution to equation $(1.1)_\gamma$ and has compact support. Hence, take a ball B centred at the origin such that $\Omega^{0+} \subset\subset B$ and $\operatorname{supp}(u_0) \subset\subset B$. Since

$$\int_{\mathbb{R}^n} a^-(u_n^q - u_0^q)(u_n - u_0)\,\mathrm{d}x \to 0,$$

we have

$$\int_{\mathbb{R}^n - B} a^-(u_n^q - u_0^q)(u_n - u_0)\,\mathrm{d}x \to 0,$$

that is

$$\int_{\mathbb{R}^n - B} a^- u_n^{q+1}\,\mathrm{d}x \to 0.$$

Noting that $\liminf_{|x|\to\infty} a^- > 0$, we obtain

$$\int_{\mathbb{R}^n - B} u_n^{q+1}\,\mathrm{d}x \to 0,$$

which implies

$$u_n \to u_0 \quad \text{strongly in } L^{q+1}(\mathbb{R}^n)$$

by the fact that $u_n \to u_0$ strongly in $L^{p+1}(\mathbb{R}^n)$. Therefore, $u_n \to u_0$ strongly in H_r. □

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Now, for fixed γ, consider the following set:

$$S_\gamma = \{\sigma \in C([0,1], H_r) \mid \sigma(0) = 0 \text{ and } \sigma(1) = T\varphi\},$$

where φ from lemma 5.6. Let $c_\gamma = \inf_{\sigma \in S_\gamma} \max_{s\in[0,1]} J_\gamma(\sigma(s))$. From lemma 5.5, we see that $J_\gamma(v) \geqslant 0$ with $\|v\|_{H_r} < \delta_1$. Therefore, $c_\gamma \geqslant 0$.

THEOREM 5.12. *Suppose that $c_\gamma = 0$ and that there exists $\eta_\gamma > 0$ such that, for any $\rho \in [0, \eta_\gamma]$,*

$$\inf\{J_\gamma(v) \mid \|v\|_{H_r} = \rho\} = 0.$$

Then, for each $\rho \in (0, \eta_\gamma)$, problem (1.1)$_\gamma$ has a solution with $\|u - v_\gamma\|_{H_r} = \rho$.

Proof. For any fixed $\rho \in (0, \eta_\gamma)$, the set $F = \partial B(0, \rho)$ in H_r satisfies the hypothesis of theorem 1 of [15]. Theorem (1.bis) of [15] asserts the existence of a solution for each $\rho \in (0, \eta_\gamma)$ with the compactness of the Palais–Smale sequence. □

Here is the proof for theorem 1.9.

Proof. If there exists some $\rho < \delta_1$ such that $\inf\{J_\gamma(v) \mid \|v\|_{H_r} = \rho\} > 0$, we have $c_\gamma > 0$. By the mountain-pass theorem of Ambrosetti and Rabinowitz, there exists a solution V_γ of (1.1)$_\gamma$ with $J_\gamma(V_\gamma) > 0$, i.e. $I_\gamma(V_\gamma) > I_\gamma(v_\gamma)$, which implies that V_γ is different from v_γ.

If this is not the case, but $c_\gamma > 0$, we still have the same result as above.

If not, and $c_\gamma = 0$, then, for all $\rho \in [0, \delta_1)$, we have $\inf\{J_\gamma(v) \mid \|v\|_{H_r} = \rho\} = 0$, then, from theorem 5.12, we see that there are infinitely many solutions of (1.1)$_\gamma$. □

To conclude, we will discuss the assumption

$$0 < \liminf_{|x|\to\infty} a^-(x) \leqslant \limsup_{|x|\to\infty} a^-(x) < \infty$$

that we made throughout this paper.

First, taking $\liminf_{|x|\to\infty} a^-(x) > 0$, from the proof of theorem 1.1 we can see that this ensures that the solution u of (1.1)$_\gamma$ with $\lim_{|x|\to\infty} u(x) = 0$ has compact support. Now we give an example, in which $\lim_{|x|\to\infty} u(x) = 0$ and $\lim_{|x|\to\infty} a^-(x) = 0$ but one solution does not have compact support, so we know that the compactness of the support of a^+ is not enough to guarantee the compactness of the support of solutions of (1.1)$_\gamma$.

Let us pick a locally Hölder-continuous and sign-changing function $c(x)$ and assume that $\operatorname{supp}(c^+(x)) \subset\subset B(0,1)$ and $1 \leqslant c^-(x) \leqslant 2$ for $|x| \geqslant 2$. We make $\|c^+\|_{L^\infty(B(0,1))}$ so small that the first eigenvalue of the operator $-\Delta v - c(x)v$ in $H^1(\mathbb{R}^n)$ is positive, i.e. there exists a positive constant $\mu > 0$ such that, for any $v \in H^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla v|^2 - c(x)v^2 \,\mathrm{d}x \geqslant \mu \int_{\mathbb{R}^n} v^2 \,\mathrm{d}x.$$

This is always possible because $\|\nabla v\|_{L^2(\mathbb{R}^n)} \geqslant C_1\|v\|_{L^{2^*}(\mathbb{R}^n)} \geqslant C_2\|v\|_{L^2(B(0,2))}$, where C_1 and C_2 are two positive constants depending on n. Therefore, if we choose $\|c^+\|_{L^\infty(B(0,1))} \leqslant \frac{1}{2}C_2^2$, then we have, for any $v \in H^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla v|^2 - c(x)v^2 \,\mathrm{d}x \geqslant \min(1, \tfrac{1}{2}C_2^2) \int_{\mathbb{R}^n} v^2 \,\mathrm{d}x.$$

https://doi.org/10.1017/S0308210509001231 Published online by Cambridge University Press

Furthermore, assuming that $c(x)$ is radial, i.e. $c(x) = c(|x|)$, consider the energy functional

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 - c(x)v^2 \, \mathrm{d}x,$$

which acts upon

$$Y = \left\{ v \in H_r \;\middle|\; \int_{\mathbb{R}^n} |v|^{p+1} \, \mathrm{d}x = 1 \right\},$$

where $H_r = \{v \in H^1(\mathbb{R}^n) | v(x) = v(|x|)\}$. Since the operator $-\Delta v - c(x)v$ in $H^1(\mathbb{R}^n)$ has a positive first eigenvalue, we let $0 < \varepsilon < \frac{1}{2}\mu$ be such that $(1-\varepsilon)\mu - \varepsilon c(x) \geqslant \frac{1}{2}\mu$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla v|^2 - c(x)v^2 \, \mathrm{d}x &\geqslant \varepsilon \int_{\mathbb{R}^n} |\nabla v|^2 - c(x)v^2 \, \mathrm{d}x + (1-\varepsilon)\mu \int_{\mathbb{R}^n} v^2 \, \mathrm{d}x \\ &\geqslant \varepsilon \int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}x + \tfrac{1}{2}\mu \int_{\mathbb{R}^n} v^2 \, \mathrm{d}x \\ &\geqslant \varepsilon \int_{\mathbb{R}^n} |\nabla v|^2 + v^2 \, \mathrm{d}x.\end{aligned}$$

Therefore, $-\Delta v - c(x)v$ is coercive on H_r. By lemma 5.9 and standard minimization arguments [23], we obtain a positive radial solution v to the equation $-\Delta v - c(x)v = v^p$, and it is well known that $v(r)$ tends to zero at infinity very fast, like exponential decay. Simply letting $a(x) = c(x)v^{1-q}$, we are back to the form of $(1.1)_\gamma$ and v is a positive solution, which is what we are trying to find. It should be mentioned that we can still obtain solutions with compact support even though $\lim_{|x|\to\infty} a^-(x) = 0$ if the speed of a^- going to zero at infinity is slow compared with the speed of solution u going to zero at infinity. We will address this problem in a forthcoming paper.

Secondly, $\limsup_{|x|\to\infty} a^-(x) < \infty$. As mentioned earlier, this assumption is very important for lemma 2.1 to hold, although we have found that theorem 1.9 continues to hold even without this assumption. By allowing $\limsup_{|x|\to\infty} a^-(x) = \infty$ we can still find two solutions of $(1.1)_\gamma$ with radial symmetry. We will present this result in a forthcoming paper.

Acknowledgements

The author is supported by Proyecto Fondecyt Posdoctorado no. 3100050 and the Richard Fuller Memorial Scholarship at McMaster University.

References

1 S. Alama. Semilinear elliptic equations with sublinear indefinite non-linearities. *Adv. Diff. Eqns* **4** (1999), 813–842.
2 S. Alama and Q. Lu. Compactly supported solutions to stationary degenerate diffusion equations. *J. Diff. Eqns* **246** (2009), 3214–3240.
3 S. Alama and Q. Lu. Multiple compactly supported solutions to stationary degenerate diffusion equations with concave and convex nonlinearities. (In preparation.)
4 S. Alama and G. Tarantello. On semilinear elliptic equations with indefinite nonlinearities. *Calc. Var. PDEs* **1** (1993), 439–475.
5 S. Alama and G. Tarantello. Some remarks on C^1 versus H^1 minimizers. *C. R. Acad. Sci. Paris Sér. I* **319** (1994), 1165–1169.

6 A. Ambrosetti, H. Brezis and G. Cerami. Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Analysis* **122** (1994), 519–543.

7 C. Bandle, M. A. Pozio and A. Tesei. The asymptotic behavior of the solutions of degenerate parabolic equation. *Trans. Am. Math. Soc.* **303** (1987), 487–501.

8 H. Berestycki and P. L. Lions. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Ration. Mech. Analysis* **82** (1983), 313–345.

9 H. Brezis and L. Nirenberg. H^1 versus C^1 local minimizers. *C. R. Acad. Sci. Paris Sér. I* **317** (1993), 465–472.

10 X. Cabré. Extremal solutions and instantaneous complete blow-up for elliptic and parabolic problems. In *Perspectives in nonlinear partial differential equations*, Contemporary Mathematics, vol. 446, pp. 159–174 (Providence, RI: American Mathematical Society, 2007).

11 W.-X. Chen and C. Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **63** (1991), 615–622.

12 C. Cortázar, M. Elgueta and P. Felmer. On a semilinear elliptic problem in $\mathbb{R}^n$ with a non-Lipschitzian non-linearity. *Adv. Diff. Eqns* **1** (1996), 199–218.

13 M. Crandall and P. H. Rabinowitz. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Ration. Mech. Analysis* **58** (1975), 207–218.

14 D. G. De Figueiredo, J.-P. Gossez and P. Ubilla. Local superlinearity and sublinearity for indefinite semilinear elliptic problems. *J. Funct. Analysis* **199** (2003), 452–476.

15 N. Ghoussoub and D. Preiss. A general mountain pass principle for locating and classifying critical points. *Annales Inst. H. Poincaré Analyse Non Linéaire* **6** (1989), 321–330.

16 D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order* (Springer, 1977).

17 M. E. Gurtin and R. C. MacCamy. On the diffusion of biological populations. *Math. Biosci.* **33** (1977), 35–49.

18 M. E. Gurtin and R. C. MacCamy. Product solutions and asymptotic behavior for age-dependent, dispersing populations. *Math. Biosci.* **62** (1982), 157–167.

19 Q. Lu. Compactly supported solutions for semilinear elliptic problem in $\mathbb{R}^n$ involving critical Sobolev exponents and non-Lipschitz nonlinearity. (In preparation.)

20 P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, vol. 65 (Providence, RI: American Mathematical Society, 1986).

21 M. Shatzman. Stationary solutions and asymptotic behavior of a quasilinear degenerate parabolic equation. *Indiana Univ. Math. J.* **33**, (1984), 1–30.

22 W. A. Strauss. Existence of solitary waves in higher dimensions. *Commun. Math. Phys.* **55** (1977), 149–162.

23 M. Struwe. *Variational methods* (Springer, 1990).

24 V. C. Zelati and P. H. Rabinowitz. Homoclinic type solutions for a semilinear elliptic PDE on R^n. *Commun. Pure Appl. Math.* **45** (1992), 1217–1269.

(*Issued 25 February 2011*)